§2. Iterability

In this section we shall sketch a proof that if $\mathcal{T}$ is a k-maximal iteration tree on $\mathfrak{C}_k(\mathcal{N}_\xi)$, and $\mathcal{T} \restriction \lambda$ is simple for all limit $\lambda < lh\mathcal{T}$, and $lh\mathcal{T}$ is a limit ordinal, then for all sufficiently large κ, $V^{\mathrm{Col}(\omega,\kappa)} \models \mathcal{T}$ has a cofinal, wellfounded branch. There are several other iterability facts we shall need, and actually we shall not prove even this one in this section, since we shall make some simplifying assumptions on $\mathcal{T}$. The reader seeking full detail and generality will find it in §9. The reader who would like to see the main ideas in our iterability proof, while avoiding full detail and generality, can content himself with this section.

In this paper, we shall diverge slightly from the terminology of [FSIT] regarding iteration trees. By an *iteration tree* we mean a system $\mathcal{T}$ obeying all the conditions required in the definition of §5 of [FSIT] except possibly the increasing length condition. That is, we do not require $\alpha < \beta \Rightarrow lh\ E^{\mathcal{T}}_\alpha < lh\ E^{\mathcal{T}}_\beta$. Iteration trees in the sense of [FSIT] we call *normal.* (We note that even in a normal tree $\mathcal{T}$, $E^{\mathcal{T}}_\alpha$ may not be applied to the earliest possible model. This last requirement is part of k-maximality.) Although the trees which arise in comparison processes are all normal and k-maximal for some $k \leq \omega$, we must cover more than such trees in our proof that K^c is iterable. This is because the proof of the Dodd-Jensen lemma (5.3 of [FSIT]) involves non-normal trees.

We shall say that $\mathcal{T}$ is simple iff for all sufficiently large κ, $V^{\mathrm{Col}(\omega,\kappa)} \models \mathcal{T}$ has at most one cofinal wellfounded branch. (This diverges slightly from the terminology of [FSIT].) We shall need a relative of this notion.

Definition 2.1. *Let $\mathcal{T}$ be a tree on $\mathcal{M}$ of limit length, and $\alpha \in OR$. We say $\mathcal{T}$ is α-short iff for all sufficiently large κ*

$$\begin{aligned} V^{Col(\omega,\kappa)} \models\ & \mathcal{T} \text{ has no cofinal branch } b \text{ such that} \\ & \alpha \text{ is isomorphic to an initial segment of } OR^{\mathcal{M}^{\mathcal{T}}_b}. \end{aligned}$$

(Here $\mathcal{M}^{\mathcal{T}}_b = \lim_{\alpha \in b} \mathcal{M}^{\mathcal{T}}_\alpha$.)

The next two lemmas come from the uniqueness theorem of §2 of [IT]. See also Theorem 6.1 of [FSIT]. Their formulation also owes a lot to work of Woodin, and to the Π^1_2 mouse condition of §5 of [IT].

Lemma 2.2. *Let $\mathcal{M}$ be 1-small and $\mathcal{T}$ an iteration tree on $\mathcal{M}$, and let $\lambda < lh\ \mathcal{T}$. Then for some $\alpha \in OR$, $\mathcal{T} \restriction \lambda$ is α-short.*

Proof. Assume not. Let

$$\begin{aligned} \delta &= \sup\{lh\ E^{\mathcal{T}}_\beta \mid \beta < \lambda\}, \\ \vec{E} &= \bigcup_{\beta<\lambda} \dot{E}^{\mathcal{M}^{\mathcal{T}}_\beta} \restriction lh\ E^{\mathcal{T}}_\beta, \end{aligned}$$

(so that $\vec{E}=$ common value of $\dot{E}^{\mathcal{M}_b^{\mathcal{T}}} \restriction \delta$, for all cofinal, possibly generic, branches b of $\mathcal{T} \restriction \lambda$). Our hypothesis implies that $\forall \alpha \in$ OR, there are possibly generic cofinal branches $b \neq c$ of $\mathcal{T} \restriction \lambda$ such that α is in the wellfounded parts of both $\mathcal{M}_b$ and $\mathcal{M}_c$. Hence so is $L_\alpha[\vec{E}]$. By §2 of [IT] or 6.1 of [FSIT], $L[\vec{E}] \models \delta$ is Woodin. But now $\vec{E}= \dot{E}^{\mathcal{M}_\lambda^{\mathcal{T}}} \restriction \delta$, and $E_\lambda^{\mathcal{T}}$ has length $\geq \delta$. It follows that $\mathcal{M}_\lambda^{\mathcal{T}}$ is not 1-small, a contradiction. □

We do not get from 2.2 that $\mathcal{T}$ itself is α-short for some $\alpha \in$ OR. But the proof of 2.2 gives at once:

Lemma 2.3. *Let $\mathcal{T}$ be an iteration tree of limit length on a premouse $\mathcal{M}$. Then either $\mathcal{T}$ is α-short for some $\alpha \in OR$, or there is a proper class inner model with a Woodin cardinal.*

The next lemma explains the importance of these uniqueness facts in our proof of iterability. It shows that the existence of a "bad" tree on $\mathcal{M}$ reflects to the existence of a bad countable tree on a countable $\mathcal{N} \preceq \mathcal{M}$. Part (a) of the lemma is due to Woodin and the author independently; part (b) is due essentially to Woodin.

Let us use "putative iteration tree" for a system having all the properties of an iteration tree, except that its last model, if it has one, may be illfounded.

Lemma 2.4. *Let $\mathcal{T}$ be a putative iteration tree on a 1-small premouse $\mathcal{M}$ such that $\mathcal{T} \restriction \lambda$ is simple for all $\lambda < lh\ \mathcal{T}$. Suppose that either* (a) $(\mathcal{T}, \mathcal{M})^\sharp$ *exists, or* (b) *There is no proper class inner model with a Woodin cardinal. Suppose also that either $\mathcal{T}$ has a last, illfounded model, or $\mathcal{T}$ has limit length and for all sufficiently large κ, $V^{Col(\omega,\kappa)} \models \mathcal{T}$ has no cofinal wellfounded branch.*

Then there is a countable $\mathcal{N} \preceq \mathcal{M}$ and a countable putative iteration tree $\mathcal{U}$ on $\mathcal{N}$ such that $\mathcal{U} \restriction \lambda$ is simple for all $\lambda < lh\ \mathcal{U}$ and either $\mathcal{U}$ has a last, illfounded model, or $\mathcal{U}$ has limit length but no cofinal wellfounded branch.

Proof. We give the proof under hypothesis (b). We also assume $lh\ \mathcal{T}$ is a limit ordinal, and leave the contrary case to the reader.

By 2.3 we can fix $\alpha \in$ OR such that for all $\lambda \leq lh\ \mathcal{T}$, $\mathcal{T} \restriction \lambda$ is α-short. Let θ be large enough that $\mathcal{T}, \mathcal{M}$, $\alpha \in V_\theta$ and V_θ satisfies a reasonable fragment of ZFC. Lowenheim-Skolem gives us a countable transitive H and embedding

$$\pi : H \rightarrow V_\theta$$

such that for some $(\mathcal{N}, \mathcal{U}, \bar{\alpha})$,

$$\pi((\mathcal{N}, \mathcal{U}, \bar{\alpha})) = (\mathcal{M}, \mathcal{T}, \alpha).$$

So $\mathcal{N} \preceq \mathcal{M}$ via $\pi \restriction \mathcal{N}$. Also, $\mathcal{U}$ is a countable iteration tree on $\mathcal{N}$ of limit length. Now $H \models \mathcal{U} \restriction \lambda$ is $\bar{\alpha}$ short, for all $\lambda \leq lh\ \mathcal{U}$. But this notion is sufficiently absolute that $\mathcal{U} \restriction \lambda$ truly is $\bar{\alpha}$ short, for all $\lambda \leq lh\ \mathcal{U}$. [Let x be a real which is a member of $H^{\mathrm{Col}(\omega,\kappa)}$, κ sufficiently large, and codes $(\mathcal{N},\mathcal{U},\bar{\alpha})$. Being $\bar{\alpha}$ short is a Π^1_1 property of x, and so absolute to $H^{\mathrm{Col}(\omega,\kappa)}$.]

Now $H \models \mathcal{U} \restriction \lambda$ is simple, $\forall \lambda < lh\, \mathcal{U}$. Since $\mathcal{U} \restriction \lambda$ is $\bar{\alpha}$ short and $\bar{\alpha} \in H$, this is absolute, and thus $\mathcal{U} \restriction \lambda$ truly is simple for all $\lambda < lh\, \mathcal{U}$.

Similarly, if $\mathcal{U}$ has a cofinal wellfounded branch b, then $\mathrm{OR}^{\mathcal{M}_b} < \bar{\alpha}$, and thus $\mathcal{U}$ has a cofinal wellfounded branch in $H^{\mathrm{Col}(\omega,\kappa)}$ where $\kappa = (\mathrm{card}(\mathcal{N})) \cdot \mathrm{card}(\mathcal{U}) \cdot \bar{\alpha})^H$. So $\mathcal{U}$ has no cofinal wellfounded branch.

The proof under hypothesis (a) is similar. Where in case (b) we used α-shortness and Π^1_1 absoluteness, we use $(\mathcal{T},\mathcal{M})^\sharp$ and Π^1_2 absoluteness. [We have $\pi : (\mathcal{U},\mathcal{N})^{\sharp^H} \to (\mathcal{T},\mathcal{M})^\sharp$, and this guarantees $(\mathcal{U},\mathcal{N})^{\sharp^H} = (\mathcal{U},\mathcal{N})^\sharp$. That in turn implies $H[G]$ is correct for Π^1_2 statements about x, where x is a real coding $(\mathcal{U},\mathcal{N})$ in $H[G]$, and G is generic $/H$ for $\mathrm{Col}(\omega,\kappa)$, $\kappa = (\mathrm{card}\ (\mathcal{U}) \cdot \mathrm{card}(\mathcal{N}))^H$.] □

Remark. If $\kappa = \mathrm{card}(\mathcal{T}) \cdot \mathrm{card}(\mathcal{M})$, then $\forall \theta > \kappa$, $\mathcal{T}$ has a cofinal wellfounded branch in $V^{\mathrm{Col}(\omega,\theta)}$ iff $\mathcal{T}$ has a cofinal wellfounded branch in $V^{\mathrm{Col}(\omega,\kappa)}$. In particular, if $\omega = \mathrm{card}(\mathcal{T}) \cdot \mathrm{card}(\mathcal{M})$, $\mathcal{T}$ has such a branch in $V^{\mathrm{Col}(\omega,\theta)}$ iff $\mathcal{T}$ has such a branch in V.

We are ready to state the main result of this section, which concerns the iterability of countable elementary submodels of K^c and its levels. Although we could prove that such structures are iterable with respect to arbitrary trees, to do so would add a layer of notational fog to the proof for normal trees. We shall therefore prove just the iterability we need, which is iterability with respect to *linear compositions of normal trees.* We call these trees *almost normal.* More precisely, suppose $\langle \mathcal{T}_\alpha \mid \alpha < \beta \rangle$ is a sequence of normal trees such that $\mathcal{T}_0$ is on $\mathcal{M}$, $\mathcal{T}_{\alpha+1}$ is on the last model $\mathcal{M}^{\mathcal{T}_\alpha}_{\nu_\alpha}$ of $\mathcal{T}_\alpha$ for all $\alpha + 1 < \beta$, and $\mathcal{T}_\lambda$ is on the direct limit of the $\mathcal{M}^{\mathcal{T}_\alpha}_{\nu_\alpha}$, for $\alpha < \lambda$, if $\lambda < \beta$ is a limit. We can form an iteration tree $\mathcal{U}$ by "laying the $\mathcal{T}_\alpha$'s end-to-end". We say $\mathcal{U}$ is generated by $\langle \mathcal{T}_\alpha \mid \alpha < \beta \rangle$, and call a tree $\mathcal{U}$ generated in this way almost normal. Such a composition $\mathcal{U}$ will generally not be maximal, even if the $\mathcal{T}_\alpha$'s are maximal, since maximality requires going back to the earliest possible model. We say $\mathcal{U}$ is *almost k-maximal* iff $\mathcal{T}_0$ is k-maximal, and $\forall \gamma < \beta$ ($\mathcal{T}_\gamma$ is j-maximal, where $j = \deg^{\mathcal{T}_\alpha}(\mathcal{M}^{\mathcal{T}_\alpha}_{\nu_\alpha})$ for all sufficiently large $\alpha < \gamma$.

Theorem 2.5. *Let $\mathcal{P} \preceq \mathfrak{C}_k(\mathcal{N}_\theta)$ for some k,θ, and $\mathcal{P}$ be countable. Let $\mathcal{T}$ be a countable, almost normal, almost k-maximal putative iteration tree on $\mathcal{P}$ such that $\mathcal{T} \restriction \lambda$ is simple for all $\lambda < lh\ \mathcal{T}$. Then either $\mathcal{T}$ has successor length, and its last model is wellfounded, or $\mathcal{T}$ has limit length, and $\mathcal{T}$ has a cofinal wellfounded branch.*

Sketch of Proof. We shall give the proof in a special case which highlights the new ideas.

The simplifying assumptions we make are: $\mathcal{T}$ is normal and k-maximal, and

(1) $\mathcal{T}$ has length ω,

(2) $k = \omega$; moreover $\mathcal{P}$ is passive and $\mathcal{P} \models ZF^-$, and there is no dropping on $\mathcal{T}$,

(3) Letting $\mathcal{P}_i$ be the ith model of $\mathcal{T}$, and ν_i the sup of the generators of $E_i^{\mathcal{T}}$,

$$\mathcal{P}_i \models \quad \text{strength}(E_i^{\mathcal{T}}) \geq \nu_i\,,$$

moreover, ν_i is a limit ordinal.

As ν_i is a limit ordinal, $\mathcal{J}_\beta^{\mathcal{P}_i}$ is either type I or type III, where $\beta = lh\ E_i^{\mathcal{T}}$. This implies that ν_i is a cardinal of $\mathcal{J}_\beta^{\mathcal{P}_i}$, and therefore, by our strength assumption in (3), ν_i is a cardinal of $\mathcal{P}_i$.

We have made assumptions (2) and (3) in part to avoid any need for "resurrection" (cf. §12 of [FSIT]) in the construction to follow.

By (2), $\rho_\omega(\mathcal{P}_i) = \mathrm{OR}^{\mathcal{P}_i}$ for all $i \in \omega$, all ultrapowers on $\mathcal{T}$ are Σ_ω (satisfy the full Łos theorem), yet are formed using functions which belong to the model in question. It may seem that these assumptions just resurrect the "coarse structure" setting of [IT], but in fact they do not. For one thing, we don't have $\mathcal{T} \in \mathcal{P}_0$.

Because $\rho_\omega(\mathcal{P}) = \mathrm{OR}^{\mathcal{P}}$, $\mathfrak{C}_\omega(\mathcal{N}_\theta) = \mathcal{N}_\theta$. Fix an elementary $\pi : \mathcal{P} \to \mathcal{N}_\theta$. We shall show that there is a cofinal branch b of $\mathcal{T}$ and elementary $\sigma : \mathcal{P}_b \to \mathcal{N}_\theta$ such that

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\pi"] \arrow[dr, "i_{0b}"'] & \mathcal{N}_\theta \\ & \mathcal{P}_b \arrow[u, "\sigma"'] \end{tikzcd}$$

commutes.

Let $\mathcal{U}$ be the tree of attempts to build such a branch b and embedding σ. More precisely, let $\tau : \mathcal{P} \to Q$ be elementary; we shall define a tree $\mathcal{U} = \mathcal{U}(\tau, Q)$ which tries to build (b, σ) such that $\sigma : \mathcal{P}_b \to Q$ and $\tau = \sigma \circ i_{0,b}^{\mathcal{T}}$.

Fix an enumeration

$$t : \omega \overset{\text{onto}}{\longrightarrow} \bigcup_{i \in \omega} (\{i\} \times \mathcal{P}_i)$$

such that $t^{-1}((e, x))$ is infinite for all (e, x) such that $x \in \mathcal{P}_e$. We then put $(\langle e_0, \ldots, e_k\rangle, \langle y_0, \ldots, y_k\rangle) \in \mathcal{U}$ iff

(a) $e_0 T e_1 T \cdots T e_k$

(b) $(\mathcal{P}_{e_k}, x_0, \ldots, x_k) \equiv (Q, y_0, \ldots, y_k)$, where $\forall\ i \leq k$

(i) $t(i) = (e, x)$ for $e \in [0, e_i]_T$ implies $x_i = i_{e,e_k}^{\mathcal{T}}(x)$,
(ii) $t(i) = (e, x)$ for $e \notin [0, e_i]_T$ implies $x_i = \emptyset$,
(iii) $t(i) = (0, x)$ implies $y_i = \tau(x)$.

Remark. $\langle x_0, \ldots, x_k\rangle$ in (b) is determined by $\langle e_0, \ldots, e_k\rangle$.

Let $\mathcal{U} = \mathcal{U}(\tau, Q)$ and $i \in \omega$, and suppose we are given an embedding $\sigma : \mathcal{P}_i \to Q$ such that $\tau = \sigma \circ i^{\mathcal{T}}_{0,i}$. From σ we get an initial segment of a branch of $\mathcal{U}(\tau, Q)$; let

$$p(i, \sigma, \tau, Q) = (\langle e_0, \ldots, e_k \rangle, \langle y_0, \ldots, y_k \rangle)$$

where $\langle e_0, \ldots, e_k \rangle$ is the increasing enumeration of $[0, i]_T$ and letting $\langle x_0, \ldots, x_k \rangle$ come from $\langle e_0, \ldots, e_k \rangle$ as in the definition of $\mathcal{U}(\tau, Q)$, we have $y_j = \sigma(x_j)$ for all $j \leq k$.

Theorem 2.5 is proved if we show that $\mathcal{U}(\pi, \mathcal{N}_\theta)$ has an infinite branch. Let us assume otherwise toward a contradiction.

By "coarse premouse" we mean a premouse in the sense of [IT]; that is, a structure $\mathcal{M} = (M, \in, \delta)$, where M is transitive which is power-admissible and satisfies choice, the full collection schema for domains $\subseteq V_\delta^{\mathcal{M}}$, and the full separation schema. We also require that ${}^\omega M \subseteq M$, and that δ be inaccessible in M. Write $\delta^{\mathcal{M}} = \delta$.

Let

$$\mathbb{C} = \langle \mathcal{N}_\xi \mid \xi < \gamma \rangle \qquad (\gamma \leq \Omega)$$

be the construction done in §1. Notice that $\mathbb{C}$ is definable from no parameters over V_Ω. (Here γ is the first place $< \Omega$ where the construction breaks down, if any, and $\gamma = \Omega$ otherwise. Thus $\theta < \gamma$.) It follows that if $\mathcal{R} = (R, \in, \delta)$ is any coarse premouse, then $\mathbb{C}^{\mathcal{R}}$ makes sense: we interpret the definition of $\mathbb{C}$ inside $V_\delta^{\mathcal{R}}$.

If $\mathcal{R}$ is a coarse premouse, then a *cutoff point* of $\mathcal{R}$ is an ordinal ξ such that $\delta^{\mathcal{R}} < \xi < \mathrm{OR}^{\mathcal{R}}$ and $(V_\xi^{\mathcal{R}}, \in, \delta^{\mathcal{R}})$ is a coarse premouse.

We now define by induction on i triples $(\pi_i, Q_i, \mathcal{R}_i)$ with the following properties:

(1) $\mathcal{R}_i$ is a coarse premouse,

(2) Q_i is an "N-model" of the construction $\mathbb{C}^{\mathcal{R}_i}$, moreover $\pi_i : \mathcal{P}_i \to Q_i$ elementarily,

(3) for all $j < i$, $\mathcal{R}_j$ agrees with $\mathcal{R}_i$, through $\pi_j(\nu_j)$,

(Recall that $\nu_j = \nu(E_j^{\mathcal{T}})$ = strict sup of the generators of $E_j^{\mathcal{T}}$. We say coarse premice $\mathcal{R}$ and $\mathcal{S}$ agree through η iff $V_\eta^{\mathcal{R}} = V_\eta^{\mathcal{S}}$.)

(4) for all $j < i$, $\pi_i \restriction \nu_j = \pi_j \restriction \nu_j$; moreover Q_j agrees with Q_i through $\pi_j(\nu_j)$,

(Ordinary, "fine" premice Q and $\mathcal{R}$ agree through η iff $\mathcal{J}_\eta^Q = \mathcal{J}_\eta^{\mathcal{R}}$.)

(5) Let $\mathcal{U} = \mathcal{U}(\pi_i \circ i^{\mathcal{T}}_{0,i}, Q_i)$ and

$$p = p(i, \pi_i, \pi_i \circ i^{\mathcal{T}}_{0,i}, Q_i)\,.$$

(So $\mathcal{U}$ is a tree in $\mathcal{R}_i$ and p is a node of $\mathcal{U}$.) Then $\mathcal{U}$ is wellfounded, and the order type of the set of cutoff points of $\mathcal{R}_i$ is at least $|p|_{\mathcal{U}}$,

(6) if $i > 0$, then $\mathcal{R}_i \in \mathcal{R}_{i-1}$.

Clause (6) gives us the desired contradiction.

Base step: Set

$$\begin{aligned}\pi_0 &= \pi\,,\\ Q_0 &= \mathcal{N}_\theta\,.\end{aligned}$$

By assumption, $\mathcal{U} = \mathcal{U}(\pi_0, Q_0)$ is wellfounded. Set

$$\mathcal{R}_0 = (V_\xi, \in, \Omega)$$

where ξ is the $|\mathcal{U}|$th ordinal $\alpha > \Omega$ such that $(V_\alpha, \in, \Omega)$ is a coarse premouse. Our applicable induction hypotheses, namely (1), (2), and (5), clearly hold.

Inductive step. We are given $\langle(\pi_j, Q_j, \mathcal{R}_j) \mid j \leq i\rangle$. Let $j = T\text{-pred }(i+1)$, and set

$$Q'_{i+1} = \mathrm{Ult}_\omega(Q_j, \pi_i(E_i^T))\,.$$

$Q_j \models ZF^-$, so the ultrapower is formed using functions belonging to Q_j.

Notice that the ultrapower makes sense. For set $\bar\kappa = \text{crit } E_i^T$ and $\kappa = \pi_i(\bar\kappa)$. Let $E = \pi_i(E_i^T)$. The rules for iteration trees guarantee $\bar\kappa < \nu_j$. Induction hypothesis (4) states that $\pi_i \upharpoonright \nu_j = \pi_j \upharpoonright \nu_j$; thus $\kappa < \pi_j(\nu_j)$. But $\pi_j(\nu_j)$ is a cardinal of Q_j, and Q_i agrees with Q_j through $\pi_j(\nu_j)$. Thus $P(\kappa)^{Q_j} \subseteq Q_i$, and the ultrapower makes sense. (We may have subsets of κ in Q_i but not Q_j; $\pi_j(\nu_j)$ may not be a cardinal of Q_i. So E may measure more sets than necessary.)

Let $\sigma : \mathcal{P}_{i+1} \to Q'_{i+1}$ be given by the shift lemma:

$$\sigma([a, f]^{\mathcal{P}_j}_{E_i^T}) = [\pi_i(a), \pi_j(f)]^{Q_j}_E\,.$$

We have that σ is well defined and elementary, that Q'_{i+1} agrees with Q_i through all $\eta < lh\ E$, that $\sigma \upharpoonright lh\ E_i^T = \pi_i \upharpoonright lh\ E_i^T$, and that $\sigma \circ i^T_{j,i+1} = i_E \circ \pi_j$ where $i_E : Q_j \to Q'_{i+1}$ is the canonical embedding.

The following little lemma will be useful.

Lemma 2.6. *Suppose $\mathcal{J}_\beta^{\mathcal{N}_\eta}$ is an initial segment of $\mathcal{N}_\eta$ such that*

$$\forall\kappa < \omega\beta[(\mathcal{N}_\eta \models \kappa \quad \textit{is a cardinal}) \Leftrightarrow (\mathcal{J}_\beta^{\mathcal{N}_\eta} \models \kappa \quad \textit{is a cardinal})]\,;$$

then there is a $\xi \leq \eta$ such that $\mathcal{J}_\beta^{\mathcal{N}_\eta} = \mathcal{N}_\xi$.

Proof. We may assume $\omega\beta < \mathrm{OR} \cap \mathcal{N}_\eta$.

Let $\xi \leq \eta$ be least such that $\mathcal{J}_\beta^{\mathcal{N}_\eta}$ is an initial segment of $\mathcal{N}_\xi$. Clearly, if ξ is a limit then $\mathcal{J}_\beta^{\mathcal{N}_\eta} = \mathcal{N}_\xi$, and we are done. Therefore we suppose $\xi = \tau+1$. We also suppose $\mathcal{J}_\beta^{\mathcal{N}_\eta} \neq \mathcal{N}_{\tau+1}$, and this implies that $\mathcal{J}_\beta^{\mathcal{N}_\eta}$ is an initial segment of $\mathcal{M}_\tau$. Since $\mathcal{J}_\beta^{\mathcal{N}_\eta}$ is not an initial segment of $\mathcal{N}_\tau$, and $\mathcal{M}_\tau = \mathfrak{C}_\omega(\mathcal{N}_\tau)$, we have from the proof of Theorem 8.1 of [FSIT] that $\rho_\omega(\mathcal{N}_\tau) < \mathrm{OR}^{\mathcal{N}_\tau}$ and $\mathcal{N}_\tau \models \rho_\omega(\mathcal{N}_\tau)^+$ exists, moreover $\omega\beta$ is strictly larger than $(\rho_\omega(\mathcal{N}_\tau)^+)^{\mathcal{N}_\tau}$. Now let $\delta = \inf\{\rho_\omega(\mathcal{N}_\theta) \mid \tau \leq \theta < \eta\}$, so that $\delta \leq \rho_\omega(\mathcal{N}_\tau)$ and $(\delta^+)^{\mathcal{N}_\tau} \leq (\rho_\omega(\mathcal{N}_\tau)^+)^{\mathcal{N}_\tau} < \omega\beta$. Then $(\delta^+)^{\mathcal{N}_\tau}$ is a cardinal of $\mathcal{M}_\tau$ (by §8 of [FSIT]), and thus it is a cardinal of $\mathcal{J}_\beta^{\mathcal{N}_\eta}$. On the other hand, the defining property of δ guarantees $(\delta^+)^{\mathcal{N}_\tau}$ is not a cardinal of $\mathcal{N}_\eta$. This contradicts the hypothesis of 2.6. □

Since E is on the Q_i sequence, $E = E_\beta^{Q_i}$ where $\beta = lh\ E$. Let $Q_i = \mathcal{N}_\eta^{\mathcal{R}_i}$. Lemma 2.6 then gives us a $\xi \leq \eta$ such that $\mathcal{J}_\beta^{Q_i} = \mathcal{N}_\xi^{\mathcal{R}_i}$. (We apply 2.6 within $\mathcal{R}_i$. Our simplifying assumptions tell us that $\pi_i(\nu_i)$ is the largest cardinal of $\mathcal{J}_\beta^{\mathcal{N}_\eta}$, and $\pi_i(\nu_i)$ remains a cardinal in $\mathcal{N}_\eta$. Thus 2.6 applies.)

Remark. Without our simplifying assumptions we don't get that $\pi_i(\nu_i)$ is a cardinal of Q_i, and therefore there may be no ξ such that $\mathcal{J}_\beta^{Q_i} = \mathcal{N}_\xi^{\mathcal{R}_i}$. At this point in the general argument we need to resurrect a background extender for E by inverting certain collapses.

If we were in the situation of [FSIT] we would now have a "background extender" F for E such that $\mathrm{Ult}(\mathcal{R}_j, F)$ makes sense. We would let $\mathcal{R}'_{i+1} = \mathrm{Ult}(\mathcal{R}_j, F)$, and then take $\mathcal{R}_{i+1}$ to be the collapse of a suitable hull of a suitable cutoff point of $\mathcal{R}'_{i+1}$. Q_{i+1} would be the image of Q'_{i+1} under collapse. Now, however, we have no such F (after all, $V_{\kappa+1}^{\mathcal{R}_j} = V_{\kappa+1}^{\mathcal{R}_i}$, so F would be a full extender in $\mathcal{R}_i$). So instead we get a suitable background certificate (N, F) for E in $\mathcal{R}_i$. Since N is large enough, and in particular $V_\kappa^{\mathcal{R}_j} = V_\kappa^{\mathcal{R}_i} \subseteq N$, we can take an analogue of the hull producing $\mathcal{R}_{i+1}$ and Q_{i+1} "almost everywhere" below κ. We get $\mathcal{R}_{i+1}(\bar{u})$, $Q_{i+1}(\bar{u})$ for F_b are $\bar{u}$, for a suitable b. We then let $\mathcal{R}_{i+1} = [b, \lambda\bar{u}\mathcal{R}_{i+1}(\bar{u})]_F^N$ and $Q_{i+1} = [b, \lambda\bar{u}Q_{\imath+1}(\bar{u})]_F^N$.

Let

$$\mathcal{A} = \pi_i'' \bigcup_{n<\omega} P([\bar{\kappa}]^n)^{\mathcal{P}_i} = \pi_j'' \bigcup_{n<\omega} P([\bar{\kappa}]^n)^{\mathcal{P}_j}.$$

Since $\mathcal{A}$ is a countable subset of $\mathcal{R}_i$, $\mathcal{A} \in \mathcal{R}_i$ and is countable in $\mathcal{R}_i$. Also $\mathcal{A} \subseteq \bigcup_{n<\omega} P([\kappa]^n)^{\mathcal{N}_\xi^{\mathcal{R}_i}}$, where $\mathcal{N}_\xi^{\mathcal{R}_i} = \mathcal{J}_\beta^{Q_i}$. E is the last extender of $\mathcal{N}_\xi^{\mathcal{R}_i}$, so we can let

$$\mathcal{R}_i \models (N, F) \text{ is an } \mathcal{A}\text{-certificate for } \mathcal{N}_\xi^{\mathcal{R}_i}.$$

Since ${}^\omega\mathrm{Ult}(N, F) \subseteq \mathrm{Ult}(N, F)$, $\pi_i \restriction \nu_i \in \mathrm{Ult}(N, F)$. Let us pick an F support b and functions $\lambda\bar{u} \cdot \pi_i(\bar{u})$, $\lambda\bar{u} \cdot \nu(\bar{u})$ such that

$$\begin{aligned} \pi_i \restriction \nu_i &= [b, \lambda\bar{u} \cdot \pi_i(\bar{u})]_F^N, \\ \pi_i(\nu_i) &= [b, \lambda\bar{u} \cdot \nu(\bar{u})]_F^N. \end{aligned}$$

We may assume that $\pi_i(\bar{u})$, $\nu(\bar{u}) \in V_\kappa^N$ for all $\bar{u}$.

Claim. For F_b a.e. $\bar{u}$, there are in V_κ^N: a coarse premouse $\mathcal{R}$, an "$\mathcal{N}$ model" Q of $\mathbb{C}^{\mathcal{R}}$, and an elementary embedding $\pi : \mathcal{P}_{i+1} \to Q$ such that

(1) $V_{\nu(\bar{u})}^{\mathcal{R}} = V_{\nu(\bar{u})}^N$, and $\mathcal{J}_{\nu(\bar{u})}^Q = \mathcal{J}_{\nu(\bar{u})}^{Q_i}$

(2) $\pi \restriction \nu_i = \pi_i(\bar{u})$,

and

(3) letting $\mathcal{U} = \mathcal{U}(\pi \circ i^{\mathcal{T}}_{0,i+1}, Q)$ and $p = p(i+1, \pi, \pi \circ i^{\mathcal{T}}_{0,i+1}, Q)$, we have: $\mathcal{U}$ is wellfounded, and there are in order type at least $|p|_{\mathcal{U}}$ cutoff points of $\mathcal{R}$.

Proof. F_b measures the set of such $\bar{u}$, as the quantifiers in its definition range over V^N_κ, and $\mathcal{J}^{Q_i}_\kappa \in N$. Let $\bar{u} \in X$ if and only if $\bar{u} \in [\kappa]^{|b|}$ and the claim fails for $\bar{u}$, and suppose toward a contradiction that $X \in F_b$.

Fix an enumeration $\mathcal{P}_{i+1} = \{x_n \mid n < \omega\}$ of $\mathcal{P}_{i+1}$. For $n < \omega$ let

$$\sigma(x_n) = [c_n, f_n]^{Q_j}_E ,$$

where $c_n \in [\pi_i(\nu_i)]^{<\omega}$ and $f_n \in Q_j$. If $x_n < \nu_i$, so that $\sigma(x_n) = \pi_i(x_n)$, then we choose $c_n = \{\pi_i(x_n)\}$ and $f_n =$ identity function.

Subclaim A. There is a set $Y_n \in F_{c_0 \cup \cdots \cup c_n}$ such that if $t : \bigcup_{i \le n} c_i \to \kappa$ is order preserving, and $t'' \bigcup_{i \le n} c_i \in Y_n$, then

$$(\mathcal{P}_{i+1}, x_0, \ldots, x_n) \equiv (Q_j, f_0(t''c_0) \cdots f_n(t''c_n)) .$$

Proof. Note $(\mathcal{P}_{i+1}, x_0 \cdots x_n) \equiv (Q'_{i+1}, \sigma(x_0), \ldots, \sigma(x_n))$. Now let $Q'_{i+1} \models \varphi[\sigma(x_0), \ldots, \sigma(x_n)]$. By Los' theorem there is a set $Y^\varphi_n \in E_{c_0 \cup \cdots \cup c_n}$ such that if $t : \bigcup_{i \le n} c_i \to \kappa$ is order-preserving and $t'' \bigcup_{i \le n} c_i \in Y^\varphi_n$, then $Q_j \models \varphi[f_0(t''c_0), \ldots, f_n(t''c_n)]$. We can choose $Y^\varphi_n \in \operatorname{ran} \pi_j$ because $\{f_0, \ldots, f_n\} \subseteq \operatorname{ran} \pi_j$. Thus $Y^\varphi_n \in \mathcal{A}$, and hence $Y^\varphi_n \in F_{c_0 \cup \cdots \cup c_n}$. Let $Y_n = \bigcap_\varphi Y^\varphi_n$; this works because $F_{c_0 \cup \cdots \cup c_n}$ is countably complete. □

Subclaim B. If $x_n < \nu_i$, then there is a set $Z_n \in F_{c_n \cup b}$ such that if $t : c_n \cup b \to \kappa$ is order preserving and $t''(c_n \cup b) \in Z_n$, then $\pi_i(t''b)(x_n) = f_n(t''c_n) < \nu(t''b)$.

Proof.

$$\mathrm{Ult}(N, F) \models [b, \lambda\bar{u} \cdot \pi_i(\bar{u})]^N_F(x_n) = [c_n, f_n]^N_F < [b, \lambda\bar{u} \cdot \nu(\bar{u})]^N_F$$

(noting that $[c_n, f_n]^N_F = [\{\pi_i(x_n)\}, \text{identity}]^N_F = \pi_i(x_n)$ since $x_n < \nu_i$). The subclaim now follows from Los' theorem for $\mathrm{Ult}(N, F)$. □

Subclaim C. If $x_n = i^{\mathcal{T}}_{j,i+1}(y)$, then there is a set $W_n \in F_{c_n}$ such that whenever $t''c_n \in W_n$, $f_n(t''c_n) = \pi_j(y)$.

Proof.

$$\begin{aligned} [c_n, f_n]^{Q_j}_E &= \sigma(i^{\mathcal{T}}_{j,i+1}(y)) \\ &= i_E(\pi_j(y)) \\ &= [c_n, \lambda\bar{u} \cdot \pi_j(y)]^{Q_j}_E . \end{aligned}$$

By Los' theorem for $\mathrm{Ult}(Q_j, E)$, there is a set $W_n \in E_{c_n}$ as desired. But we can assume $W_n \in \operatorname{ran} \pi_j$, so that $W_n \in \mathcal{A}$, and thus $W_n \in F_{c_n}$. □

Since F is countably complete, we can find

$$t : \bigcup_{n<\omega} c_n \cup b \to \kappa$$

order preserving such that $t''b \in X$, and $t'' \bigcup_{i \leq n} c_i \in Y_n$, and $t''(c_n \cup b) \in Z_n$, and $t''c_n \in W_n$, for all $n \leq \omega$. Letting

$$\psi(x_n) = f_n(t''c_n)\,,$$

$\psi : \mathcal{P}_{i+1} \to Q_j$ elementarily, and $\psi \restriction \nu_i = \pi_i(t''b)$, and $\pi_j = \psi \circ i^{\mathcal{T}}_{j,i+1}$. Let

$$\mathcal{U} = \mathcal{U}(\pi_j \circ i^{\mathcal{T}}_{0,j}, Q_j)\,,$$

and

$$p = p(j, \pi_j, \pi_j \circ i^{\mathcal{T}}_{0,j}, Q_j)\,,$$

so that by induction $\mathcal{R}_j$ has at least $|p|_{\mathcal{U}}$ many cutoff points. Let

$$q = p(i+1, \psi, \pi_j \circ i^{\mathcal{T}}_{0,j}, Q_j)\,.$$

One can check easily that q is a proper extension of p in $\mathcal{U}$; this is true because $jTi+1$ and $\pi_j = \psi \circ i^{\mathcal{T}}_{j,i+1}$. Thus there is an $\eta \in \mathrm{OR}^{\mathcal{R}_j}$ such that

$$\eta = |q|_{\mathcal{U}}\text{th} \quad \text{cutoff point of} \quad \mathcal{R}_j\,.$$

Set

$$\begin{aligned} \mathcal{R} &= \text{transitive collapse of closure of } V^{\mathcal{R}_j}_{\nu(t''b)} \cup \{Q_j\} \\ &\quad\ \text{under Skolem functions for } V^{\mathcal{R}_j}_{\eta} \text{ and } \omega\text{-sequences} \\ Q &= \text{image of } Q_j \text{ under collapse,} \\ \pi &= \text{image of } \psi \text{ under collapse}\,. \end{aligned}$$

(Since ${}^{\omega}\mathcal{R} \subseteq \mathcal{R}$, $\mathcal{T}$ and ψ belong to the uncollapsed hull. So also do $\mathcal{U}$, p, and q.)

Clearly $(Q, \mathcal{R}, \pi) \in V^{\mathcal{R}_j}_{\kappa} = V^{N}_{\kappa}$. Moreover, $(Q, \mathcal{R}, \pi)$ witnesses the truth of the claim for $\bar{u} = t''b \in X$.

For (2): $\psi \restriction \nu_i = \pi \restriction \nu_i$ because we put all of $\nu(t''b)$ into the hull collapsing to $\mathcal{R}$, and $\psi(x_n) < \nu(t''b)$ for $x_n < \nu_i$ by B. But $\psi \restriction \nu_i = \pi_i(t''b)$ by B, as we observed earlier.

For (3): Since $\pi_j = \psi \circ i^{\mathcal{T}}_{j,i+1}$, $\pi_j \circ i^{\mathcal{T}}_{0,j} = \psi \circ i^{\mathcal{T}}_{0,i+1}$, and

$$\begin{aligned} \mathcal{U} &= \mathcal{U}(\psi \circ i^{\mathcal{T}}_{0,i+1}, Q_j)\,, \\ q &= p(i+1, \psi, \psi \circ i^{\mathcal{T}}_{0,i+1}, Q_j)\,; \end{aligned}$$

moreover $V^{\mathcal{R}_j}_{\eta}$ has at least $|q|_{\mathcal{U}}$ cutoff points. This is first order, so $\mathcal{R}$ has at least $|\bar{q}|_{\bar{\mathcal{U}}}$ cutoff points, where $\bar{q}$ = collapse of q and $\bar{\mathcal{U}}$ = collapse of $\mathcal{U}$. Since $\bar{\mathcal{U}} = \mathcal{U}(\pi \circ i^{\mathcal{T}}_{0,i+1}, Q)$ and $\bar{q} = p(i+1, \pi, \pi \circ i^{\mathcal{T}}_{0,i+1}, Q)$, we are done.

This proves the claim. $\square$

By AC in N, we have a function

$$\bar{u} \mapsto (\mathcal{R}(\bar{u}), Q(\bar{u}), \pi(\bar{u}))$$

defined F_b a.e., said function in N, picking witnesses to the claim. Let

$$\begin{aligned} \mathcal{R}_{i+1} &= [b, \lambda\bar{u} \cdot \mathcal{R}(\bar{u})]_F^N , \\ Q_{i+1} &= [b, \lambda\bar{u} \cdot Q(\bar{u})]_F^N , \end{aligned}$$

and

$$\pi_{i+1} = [b, \lambda\bar{u} \cdot \pi(\bar{u})]_F^N .$$

By (1) of the claim, $V^{\mathcal{R}_{i+1}}_{\pi_i(\nu_i)} = V^{\mathrm{Ult}(N,F)}_{\pi_i(\nu_i)} = V^{\mathcal{R}_i}_{\pi_i(\nu_i)}$. So $\mathcal{R}_{i+1}$ agrees with $\mathcal{R}_k$, $k \leq i$, as desired. By (2), $\pi_{i+1} \restriction \nu_i = \pi_i \restriction \nu_i$. To show that Q_{i+1} agrees with Q_i below $\pi_i(\nu_i)$, we argue as follows. Let $\gamma < \pi_i(\nu_i)$. Since Q_i agrees with $Q(\bar{u})$ below $\nu(\bar{u})$, for F_b a.e. $\bar{u}$, we easily get

$$\mathcal{J}^{Q_{i+1}}_\gamma = [\{\gamma\}, f]_F^N , \text{ where } f(\alpha) = \mathcal{J}^{Q_i}_\alpha \text{ for all } \alpha < \kappa .$$

Now clearly $f \in Q_i$, and the coherence condition on E, which is on the Q_i sequence, gives

$$\mathcal{J}^{Q_i}_\gamma = [\{\gamma\}, f]_E^{\mathcal{J}^{Q_i}_\beta} .$$

But then the agreement between E and F guarantees $\mathcal{J}^{Q_{i+1}}_\gamma = \mathcal{J}^{Q_i}_\gamma$. (It is enough to see that if $\beta_0 < \beta_1 < \omega\gamma$ and $\varphi(v_0, v_1)$ is a formula, then $\mathcal{J}^{Q_{i+1}}_\gamma \models \varphi[\beta_0, \beta_1]$ iff $\mathcal{J}^{Q_i}_\gamma \models \varphi[\beta_0, \beta_1]$, because there is a uniformly definable surjection of $\omega\gamma$ onto $\mathcal{J}^Q_\gamma$. Let us assume $\beta_0 < \beta_1 < \gamma$ and $\mathcal{J}^{Q_{i+1}}_\gamma \models \varphi[\beta_0, \beta_1]$. Then for $F_{\{\beta_0,\beta_1,\gamma\}}$ a.e. $\bar{u}$, $\mathcal{J}^{Q_i}_{u_2} \models \varphi[u_0, u_1]$. The set of such $\bar{u}$ is in $\mathcal{A}$, and so is measured the same way by $E_{\{\beta_0,\beta_1,\gamma\}}$.)

The remaining induction hypotheses for our construction are easy to check. This completes our proof of 2.5 under the simplifying assumptions (1)-(3) above.

We now sketch how to do without our first simplifying assumption, that $lh\ \mathcal{T} = \omega$.

We call a sequence $\langle(\pi_j, Q_j, \mathcal{R}_j) \mid j \leq i\rangle$ satisfying our inductive hypotheses (1)-(4) an *enlargement* of $\mathcal{T}$. Suppose we are given a simple $\mathcal{T}$ of length $\omega + 1$ and want to construct an enlargement $\langle(\pi_j, Q_j, \mathcal{R}_j) \mid j \leq i\rangle$ of $\mathcal{T}$. Let us assume that our simplifying assumptions (2) and (3) hold of $\mathcal{T}$. Let $\mathcal{P}_j$ be the jth model of $\mathcal{T}$. We are given by hypothesis

$$\pi_0 : \mathcal{P}_0 \to Q_0$$

where

$$Q_0 = \mathcal{N}_\theta , \quad \text{the} \quad \theta\text{th model of} \quad \mathbb{C}^V$$

and π_0 is elementary. For any $\tau : \mathcal{P} \to Q$ let

$$\mathcal{U}(\tau, Q) \quad = \quad \text{tree of attempts to build a pair } (c, \sigma), \text{ where } c \text{ is a cofinal branch of } \mathcal{T} \restriction \omega,\ c \neq [0,\omega]_T, \text{ and } \tau = \sigma \circ i^{\mathcal{T}}_{0c}\,.$$

Note because $\mathcal{T}$ is simple, $\mathcal{U}(\tau, Q)$ is wellfounded for all τ, Q. For $j \in \omega$ such that $j \notin [0,\omega]_T$, and $\psi : \mathcal{P}_j \to Q$ such that

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\pi"] \arrow[dr, "i_{0,j}"'] & Q \\
& \mathcal{P}_j \arrow[u, "\psi"']
\end{tikzcd}$$

commutes, set

$$p(j, \psi, \tau, Q) \quad = \quad \text{canonical initial segment of length } |\{k \mid kTj \,\&\, \neg kT\omega\}| \text{ of a branch of } \mathcal{U}(\tau, Q) \text{ given by } (j, \psi)\,.$$

We define by induction on $i \leq \omega$ enlargements $\mathcal{E}^i$ of $\mathcal{T} \restriction i+1$. There are two cases.

Case 1. $i + 1 \notin [0,\omega]_T$.

In this case we proceed exactly as we did in the construction given in the length ω case. Let

$$\mathcal{E}^i = \langle(\pi_k, Q_k, \mathcal{R}_k) \mid k \leq i\rangle\,.$$

Let $j = T\text{-pred}(i{+}1)$. Let $\mathcal{U} = \mathcal{U}(\pi_j \circ i^{\mathcal{T}}_{0,j}, Q_j)$ and $p = p(j, \pi_j, \pi_j \circ i^{\mathcal{T}}_{0j}, Q_j)$. Our inductive hypotheses guarantee that $\mathcal{R}_j$ has $|p|_{\mathcal{U}}$ many cutoff points. Arguing as before, we get a background extender F for $\pi_i(E^{\mathcal{T}}_i)$, and for F_b a.e. $\bar{u}$ an embedding $\psi : \mathcal{P}_{i+1} \to Q_j$ such that $\pi_j = \psi \circ i^{\mathcal{T}}_{j,i+1}$ and $\psi \restriction \nu_i = \pi_i(\bar{u}) \restriction \nu_i$. (Here $[b, \lambda\bar{u} \cdot \pi_i(\bar{u})]^N_F = \pi_i \restriction \nu_i$.) We then set $q = p(i+1, \psi, \pi_j \circ i^{\mathcal{T}}_{0,j}, Q_j)$.)

Since $i + 1 \notin [0,\omega]_T$, q extends p in $\mathcal{U}$. Thus, for F_b a.e. $\bar{u}$, we are given a cutoff point of $\mathcal{R}_j$ at which to take a hull.

As before, we do this and collapse, producing $\mathcal{R}(\bar{u})$, $Q(\bar{u})$, $\pi(\bar{u})$. We then take $\mathcal{R}_{i+1} = [b, \lambda\bar{u}\mathcal{R}(\bar{u})]^N_F$, $Q_{i+1} = [b, \lambda\bar{u}Q(\bar{u})]^N_F$, $\pi_{i+1} = [b, \lambda\bar{u} \cdot \pi_i(\bar{u})]^N_F$. Set $\mathcal{E}^{i+1} = \mathcal{E}^{i\frown}\langle\pi_{i+1}, Q_{i+1}, \mathcal{R}_{i+1}\rangle$.

Case 2. $i + 1 \in [0,\omega]_T$.

Again, let $\mathcal{E}^i = \langle(\pi_j, Q_j, \mathcal{R}_j) \mid j \leq i\rangle$, and let $j = T\text{-pred}(i + 1)$. Arguing as before, we get "measure one many"

$$\psi : \mathcal{P}_{i+1} \to Q_j$$

such that $\pi_j = \psi \circ i^{\mathcal{T}}_{j,i+1}$. In this case, we are not given an ordinal at which to take a hull; we're on the wellfounded branch of $\mathcal{T}$ and so don't expect to get such ordinals. Along this branch, we shall realize the models of $\mathcal{T}$ back in V; that is, we take

$$\begin{aligned} \mathcal{R}_{i+1} &= \mathcal{R}_j\,, \\ Q_{i+1} &= Q_j\,, \\ \pi_{i+1} &= \psi\,, \end{aligned}$$

for a ψ chosen to meet certain "measure one" conditions. (Thus by induction, $\mathcal{R}_{i+1} = (V_\xi, \in, \Omega), Q_{i+1} = \mathcal{N}_\theta$, and $\pi_{i+1} \circ i^{\mathcal{T}}_{0,i+1} = \pi_0$.)

In order to do this, we must redefine $(\pi_k, Q_k, \mathcal{R}_k)$ for $j \leq k \leq i$, as otherwise our inductive hypotheses on agreement will fail. (After all, if $\bar{\kappa} =$ crit $E^{\mathcal{T}}_i$, then $\pi_j(\bar{\kappa}) = \psi \circ i^{\mathcal{T}}_{j,i+1}(\bar{\kappa}) > \psi(\bar{\kappa})$.)

First, we find new $(\pi'_k, Q'_k, \mathcal{R}'_k)$ for $j \leq k \leq i$ such that $(\pi'_k, Q'_k, \mathcal{R}'_k) \in \mathrm{Ult}(N, F)$. (As in case 1, (N, F) is a ran π_i-certificate for $\mathcal{J}^{Q_i}_\beta$ where $\beta = lh(\pi_i(E^{\mathcal{T}}_i))$.) For this, we must suppose that our induction hypothesis on the number of cutoff points gives us, for each k s.t. $j \leq k \leq i$, a cutoff point η_k of $\mathcal{R}_k$ which we can now afford to drop to. Let G be the finite set of relevant parameters, and

$$\begin{aligned} \mathcal{R}'_k &= \text{collapse of Skolem closure of } G \cup V^{\mathcal{R}_k}_{\pi_k(\nu_k)} \text{ inside } V^{\mathcal{R}_k}_{\eta_k}\,, \\ Q'_k &= \text{image of } \ Q_k \ \text{ under collapse}, \\ \pi'_k &= \text{collapse} \circ \pi_k\,, \end{aligned}$$

Now $(\pi'_k, Q'_k, \mathcal{R}'_k)$ is coded by a subset of $V^{\mathcal{R}_k}_{\pi_k(\nu_k)}$ belonging to $\mathcal{R}_k$. Let us add to our inductive agreement hypotheses

$$V^{\mathcal{R}_k}_{\pi_k(\nu_k)+1} \subseteq \mathcal{R}_i \qquad (k \leq i)\,.$$

(As our background extenders are "$\nu + 1$" strong, this is consistent with the construction in Case 1.) It follows that $V^{\mathcal{R}_k}_{\pi_k(\nu_k)+1} \subseteq \mathrm{Ult}(N, F)$, where (N, F) is the background certificate in $\mathcal{R}_i$ for $\pi_i(E^{\mathcal{T}}_i)$. Let for $j \leq k \leq i$,

$$(\pi'_k, Q'_k, \mathcal{R}'_k) = [b, \lambda \bar{u} \cdot (\pi'_k(\bar{u}), Q'_k(\bar{u}), \mathcal{R}'_k(\bar{u}))]^N_F\,.$$

Then using $\langle(\pi'_k(\bar{u}), \mathcal{R}'_k(\bar{u})) \mid j \leq k \leq i\rangle$ in the same way that we used $\pi_i(\bar{u})$ in Case 1, we can define additional measure one sets for F so that by meeting them we guarantee that

$$\mathcal{E}^{i+1} = \ \mathcal{E}^i \restriction j^\frown \langle(\pi'_k(\bar{u}), Q'_k(\bar{u}), \mathcal{R}'_k(\bar{u})) \mid j \leq k \leq i\rangle^\frown(\psi, Q_j, \mathcal{R}_j)$$

is an enlargement with the desired properties. (Notice that if $k < j$, then $\nu_k < \bar{\kappa}$ since $j = T\text{-pred}(i+1)$ and $\bar{\kappa} =$ crit $E^{\mathcal{T}}_i$. But then $i^{\mathcal{T}}_{j,i+1} \restriction \nu_k =$ identity, so $\psi \restriction \nu_k = (\psi \circ i^{\mathcal{T}}_{j,i+1}) \restriction \nu_k = \pi_j \restriction \nu_k = \pi_k \restriction \nu_k$. This is why we do not need to re-define $\mathcal{E}^i \restriction j$.)

The existence of the cutoff point η_k of $\mathcal{R}_k$, for $j \leq k \leq i$, is not a problem because for each $k \in \omega$, only one such cutoff point is used. (It is used at stage $i+1$, where i is least such that $k \leq i$ and $i+1 \in [0,\omega]_T$.)

Let now, for $k \in \omega$

$$\mathcal{E}_k^\omega = \text{eventual value of} \quad \mathcal{E}_k^i \quad \text{as} \quad i \to \omega .$$

The eventual value exists since in fact $\mathcal{E}_k^i$ changes value at most once. Set

$$\begin{aligned} \mathcal{R}_\omega^\omega = \mathcal{R}_0^0 &= \text{common value of} \quad \mathcal{R}_j^i\,, \text{ for } \quad j \in [0,\omega]_T \quad \text{and} \quad i \geq j\,, \\ Q_\omega^\omega = Q_0^0 &= \text{common value of} \quad Q_j^i\,, \text{ for } j \in [0,\omega]_T\,, \\ \pi_\omega^\omega(i_{j,\omega}^{\mathcal{T}}(x)) &= \pi_j^\omega(x), \quad \text{for} \quad j \in [0,\omega]_T\,. \end{aligned}$$

(Here $\mathcal{E}_j^\omega = (\pi_j^\omega, Q_j^\omega, \mathcal{R}_j^\omega)$.) Then $\mathcal{E}^\omega = \langle(\pi_k^\omega, Q_k^\omega, \mathcal{R}_k^\omega) \mid k \leq \omega\rangle$ is the desired enlargement of $\mathcal{T}$.

The extension of this method of enlargement to arbitrary simple countable trees involves only more bookkeeping. The reader can see similar bookkeeping problems handled in [IT]. In one respect, in fact, what we are doing now is simpler than what is done in [IT]. In the present construction, all models at the enlargement level, i.e. all $\mathcal{R}_\beta^\alpha$'s, are ω-closed. This is true even for $\beta \geq \omega$. The construction of [IT] did not have this property, and that led to complications.

The techniques of §12 of [FSIT] allow one to drop our simplifying assumptions (2) and (3). This completes our sketch of the proof of Theorem 2.5. □

Putting together Lemma 2.4 and Theorem 2.5, we get

Corollary 2.7. *Suppose there is no proper class inner model with a Woodin cardinal; then for all $\eta \leq \Omega$ and $k \leq \omega$, $\mathfrak{C}_k(\mathcal{N}_\eta)$ exists and is k-iterable.*

Proof. The reader can find k-iterability defined in 5.1.4 of [FSIT]. Roughly speaking, it means: iterable with respect to simple, k-bounded iteration trees. The existence of $\mathfrak{C}_{k+1}(\mathcal{N}_\eta)$ comes from the k-iterability of $\mathfrak{C}_k(\mathcal{N}_\eta)$ via Theorem 8.1 of [FSIT]. (The Strong Uniqueness theorem, 6.2 of [FSIT], is used here to show that the iteration trees arising in the proof of 8.1 are simple.) So we need only show that if $\mathfrak{C}_k(\mathcal{N}_\eta)$ exists, then it is k-iterable.

Let $\mathcal{T}$ on $\mathfrak{C}_k(\mathcal{N}_\eta)$ be almost normal and k-maximal, and $\mathcal{T} \restriction \lambda$ simple for all $\lambda < lh\ \mathcal{T}$. Suppose $lh\ \mathcal{T}$ is a limit; the case $lh\ \mathcal{T}$ is a successor is similar. We want a cofinal wellfounded branch of $\mathcal{T}$, and from 2.4 and 2.5 we get such a branch in $V^{\mathrm{Col}(\omega,\kappa)}$ for all sufficiently large κ. So if $\mathcal{T}$ is simple, we are done. If not, then letting $\mathcal{M} = \mathfrak{C}_k(\mathcal{N}_\eta)$, we have δ such that $\mathcal{M} \models \delta$ is Woodin, and $\rho_{k+1}(\mathcal{M}) \geq \delta$. Let $\langle\gamma, e\rangle$ be lexicographically least above $\langle\eta, k\rangle$ such that $\mathfrak{C}_e(\mathcal{N}_\gamma) \models \delta$ is not Woodin, or $\rho_{e+1}(\mathcal{N}_\gamma) < \delta$. Such a pair $\langle\gamma, e\rangle$ must exist because otherwise δ is Woodin in a proper class inner model. $\mathcal{M}$ is an initial segment of $\mathfrak{C}_e(\mathcal{N}_\gamma)$, and all extenders from the $\mathcal{M}$-sequence have length $< \delta$, and δ is a cardinal of $\mathfrak{C}_e(\mathcal{N}_\gamma)$. So $\mathcal{T}$ lifts to a j-maximal iteration tree $\mathcal{T}^*$ on $\mathfrak{C}_e(\mathcal{N}_\gamma)$. Since $\mathcal{T}^*$ is simple, it has a cofinal wellfounded branch in V, and thus so does $\mathcal{T}$. □

Corollary 2.8. *If there is no proper class inner model with a Woodin cardinal, then $K^c = \mathcal{N}_\Omega$ exists.*

From 2.7 we get that if there is no proper class model with a Woodin cardinal, then player II wins the full iteration game on any $\mathfrak{C}_k(\mathcal{N}_\eta)$. His winning "iteration strategy" is just to pick the unique cofinal wellfounded branch (after perhaps extending $\mathfrak{C}_k(\mathcal{N}_\eta)$ to some larger $\mathfrak{C}_e(\mathcal{N}_\gamma)$.) The existence of iteration strategies is more important than their nature, and indeed once one gets to premice which are not 1-small, there may be more than one cofinal wellfounded branch from which to choose. In order to state the results of §3 - §5 in their proper generality, we make the following definitions. Since we require only normal, ω-maximal trees in §3 - §5, we restrict ourselves to these.

The full iteration game $\mathcal{G}(\mathcal{M},\theta)$ of §5 of [IT] has an obvious counterpart $\mathcal{G}^*(\mathcal{M},\theta)$ for "fine-structural" premice. In $\mathcal{G}^*(\mathcal{M},\theta)$, I and II build together a normal, ω-maximal tree $\mathcal{T}$ on $\mathcal{M}$. At move $\alpha+1<\theta$, I picks an extender $E_\alpha^\mathcal{T}$ on the $\mathcal{M}_\alpha^\mathcal{T}$ sequence such that $\gamma<\alpha \Rightarrow lh\ E_\gamma^\mathcal{T} < lh\ E_\alpha^\mathcal{T}$. The rules for ω-maximal trees then determine a $\beta\leq\alpha$ such that $\beta = T\text{-pred}(\alpha+1)$, and an initial segment $\mathcal{M}_{\alpha+1}^*$ of $\mathcal{M}_\beta^\mathcal{T}$ and $k\leq\omega$ such that $\mathcal{M}_{\alpha+1}^\mathcal{T} = \text{Ult}_k(\mathcal{M}_{\alpha+1}^*, E_\alpha^\mathcal{T})$. If $\mathcal{M}_{\alpha+1}^\mathcal{T}$ is illfounded, the game is over and I has won. At move $\lambda<\theta$, where λ is a limit, II must pick a cofinal branch b of $\mathcal{T}\restriction\lambda$ such that $D^\mathcal{T}\cap b$ is finite and $\mathcal{M}_b^\mathcal{T}$ is wellfounded; if he fails to do so, then I wins. If he succeeds in doing so, we set $\mathcal{M}_\lambda^\mathcal{T} = \mathcal{M}_b^\mathcal{T}$ and continue play. If I does not win $\mathcal{G}^*(\mathcal{M},\theta)$ at some move $\alpha<\theta$ for one of the reasons just given, then II wins.

We also want to consider a variant of this game which allows almost normal iteration trees to be played. Let $\mathcal{G}^*(\mathcal{M},(\omega,\theta))$ be played as follows. There are ω rounds. Round 1 is just a play of $\mathcal{G}^*(\mathcal{M},\theta)$, except that I must say "exit" at some move $\alpha+1<\theta$. (If he doesn't do so, and II doesn't lose $\mathcal{G}^*(\mathcal{M},\theta)$, then II has already won $\mathcal{G}^*(\mathcal{M},(\omega,\theta))$ after round 1.) If I says "exit" at $\alpha+1$, then play moves to round 2, and in this round I and II play $\mathcal{G}^*(\mathcal{M}_\alpha^{\mathcal{T}_1},\theta)$, where $\mathcal{T}_1$ is the tree produced in round 1. Once again, I must exit at some $\beta+1<\theta$, etc. If no one loses during the ω rounds, then we say that II has won $\mathcal{G}^*(\mathcal{M},(\omega,\theta))$.

Definition 2.9. *A premouse $\mathcal{M}$ is θ-iterable iff II has a winning strategy in $\mathcal{G}^*(\mathcal{M},\theta)$. A winning strategy for II in $\mathcal{G}^*(\mathcal{M},\theta)$ is a θ-iteration strategy for $\mathcal{M}$. Similarly, $\mathcal{M}$ is (ω,θ)-iterable iff II has a winning strategy in $\mathcal{G}^*(\mathcal{M},(\omega,\theta))$ and we call such a strategy an (ω,θ)-iteration strategy.*

An obvious copying construction gives

Lemma 2.10. *If $\mathcal{M}\preceq\mathcal{N}$ and $\mathcal{N}$ is θ-iterable, then $\mathcal{M}$ is θ-iterable. Similarly, if $\mathcal{M}\preceq\mathcal{N}$ and $\mathcal{N}$ is (ω,θ) iterable, then so is $\mathcal{M}$.*

From 2.7 we get

Theorem 2.11. *Suppose there is no proper class model with a Woodin cardinal; then K^c is (ω, θ)-iterable for all θ.*

We shall only make use of the $(\omega, \Omega + 1)$-iterability of K^c. We can prove this without assuming there are no proper class models with a Woodin cardinal, but using instead the measurability of Ω and assuming $K^c \models$ there are no Woodin cardinals. More precisely, we use that $A^\sharp$ exists for all sets $A \in V_\Omega$ in order to see that K^c is well-behaved with respect to trees $\mathcal{T} \in V_\Omega$ (using 2.4 (a)), and then the weak compactness of Ω to see that K^c is well behaved with respect to trees of length Ω. We use that $K^c \models$ there is no Woodin cardinal to show that the appropriate trees are simple, and thus have not just generic branches, but branches in V. In a similar vein, one can omit the hypothesis "there is no proper class model with a Woodin cardinal" in 2.8, by using the measurability of Ω. We have stated 2.8 and 2.11 as we have in order to point out what can be proved without using the measurability of Ω.

Most of the rest of this paper makes heavy use of Theorem 1.4, and we certainly do not know how to avoid the measurability of Ω as a hypothesis in that theorem. So we shall take "$K^c \models$ there is no Woodin cardinal" as our non-large-cardinal hypothesis, when we need one, instead of "there is no proper class model with a Woodin cardinal". We shall use:

Thorem 2.12. *Suppose $K^c \models$ there is no Woodin cardinal; then K^c is $(\omega, \Omega+1)$- iterable.*