Simple groups definable in O-minimal structures

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1 Introduction

In this survey article we wish to describe our classification, or identification, of those (definably) simple groups G which are definable in O-minimal structures. Our basic result says that any such group is a linear semialgebraic group over some real closed field R. An even sharper result says that the structure $(G, .)$ is *bi-interpretable* with either the field R or with the field $R(i)$, where i is the square root of -1. These results will appear in [8] and [9]. (It should be stated here that for technical reasons our hypothesis on the group G is that it is actually definable in M rather than interpretable. We, however, expect all the results to go through under only the interpretability hypothesis.)

Our results yield an O-minimal analogue of the well-known Cherlin conjecture. Recall that the (as yet unproved) Cherlin conjecture states any (noncommutative) simple group of finite Morley rank is a linear algebraic group over an algebraically closed field. It can be shown that if G is a simple group of finite Morley rank, then there is some strongly minimal set D such that G can be defined in D. Thus the finite Morley rank hypothesis in the Cherlin conjecture can be replaced by : G is definable in some strongly minimal structure. So it is quite natural to ask the O-minimal analogue: what are the simple groups definable in O-minimal structures? And this is what we have answered. In fact our second result (the bi-interpretability result mentioned above) clearly yields the full Cherlin conjecture for simple groups of finite Morley rank which happen to be definable in O-minimal structures. Any such group is a linear algebraic group over an algebraically closed field of characteristic 0. Moreover this yields even a model-theoretic *characterisation* of simple algebraic groups over algebraically closed fields of characteristic 0.

For the remainder of this introduction we shall define our terms and give

[1]Received September 10, 96; revised version November 21, 1996.
[2]Partially supported by an NSF grant.
[3]Partially supported by an NSF grant

clear statements of the results.

As a rule when we speak of definable sets (in a structure) we allow parameters in the defining formulas.

By an O-minimal structure we mean a structure $(M, <, ...)$ where $<$ is a dense linear ordering, and every definable (with parameters) subset of M is a finite union of points and open intervals. We should emphasize that the hypothesis is on subsets of M not M^n for $n > 1$. Also by an interval we mean something of the form (a, b) where $a, b \in M \cup \{\pm\infty\}$.

By a group definable in the structure M we mean a group G such that both the universe of $(G, \cdot)$ and the graph of the group operation $\cdot$ are subsets of M^n and M^{3n} respectively (some $n > 1$) which are definable in the structure M. We will say that G is definably simple in the sense of M if G is noncommutative and has no proper nontrivial normal subgroup definable in M. If we replace the latter condition by: G has no proper nontrivial normal subgroups definable in the structure $(G, \cdot)$ we say that G is definably simple in the sense of $(G, \cdot)$.

Now suppose that R is a real closed field. The term "semialgebraic" (with respect to R) means definable in the structure $(R, +, .)$. Of course the ordering on R is semialgebraic, and in fact all semialgebraic sets will be quantifier-free definable after adding a symbol for the ordering. By a semialgebraic linear group with respect to R we mean a subgroup G of $GL(n, R)$ (some n) which is semialgebraic, namely definable in $(R, +, .)$. Such a group is said to be semialgebraically connected, if G has no proper semialgebraic subgroups of finite index. If R happens to be the field of real numbers $\mathbf{R}$, then G is a semialgebraic linear Lie group, and semialgebraic connectedness of G is equivalent to topological connectedness (in the Euclidean topology). Returning to the general situation where R is real closed and G is a semialgebraic linear group with respect to R, G will have a Zariski closure G_1 in $GL(n, R)$, G_1 being the smallest subset of $GL(n, R)$ defined by polynominal equations, which contains G. G_1 will be the universe of a subgroup of $GL(n, R)$ (which we still call G_1) and G will be a finite index subgroup of G_1. (Explanation:the Zariski closure of a subgroup G of $GL(n, R)$ will always also be a subgroup. On the other hand, if G is also semialgebraic, then it comes equipped with a dimension, which turns out to be the same as the dimension of its Zariski closure. Basic properties of this notion of dimension imply that if G and G_1 are semialgebraic groups with the same dimension, and G is a subgroup of G_1, then G is an open subgroup of finite index in G_1.) Now $R(i) = K$ is an algebraically closed field, and the subset of $GL(n, K)$ defined by the zero set of all polynomials over R vanishing on G_1 will be a subgroup of $GL(n, K)$, H say. H is a linear algebraic group, defined over R, and $H \cap GL(n, R) = G_1$. In general, by a linear algebraic group H we mean a subgroup of $GL(n, K)$ defined by polynomial equations over K, where K is an algebraically closed field. If these polynomial equations can be chosen over a subfield L of K, we say (at least in the characteristic 0 case) that H is defined over L. By

$H(L)$ we will mean $H \cap GL(n, L)$, which we call the group of L-rational points of H. H is said to be L-simple, if H has no proper nontrivial normal *algebraic* subgroup defined over L. To be K-simple (K algebraically closed) is the same as being abstractly simple. However, the L-simplicity of H does not even guarantee the abstract simplicity of $H(L)$. An example is given by taking $H = SO(3)$, the group of 3 by 3 orthogonal matrices of determinant 1, identified with $SO(3, K)$ for some big algebraically closed field (of characteristic 0 say). This group is abstractly simple. However, if R is a nonarchimedean real closed field, then $H(R)$ ($= SO(3, R)$) is not abstractly simple, as its elements which are infinitely close to the identity form a normal subgroup. On the other hand, $SO(3, R)$ is semialgebraically simple. This explains why we make the assumption "definably simple" in place of "simple" in the following result.

Theorem 1.1 *Suppose M is an* 0*-minimal structure, and G is a group definable in M which is definably simple in the sense of M. Then there is a real closed field R definable in M, and an R-simple linear algebraic group H defined over R such that G is definably isomorphic (in M) to the semialgebraically connected component of $H(R)$. Otherwise said, G is definably isomorphic to a semialgebraic, semialgebraically connected linear group with respect to some real closed field R definable in M.*

Remark. There is a similar version if G is assumed only to be definably simple in the sense of $(G, \cdot)$. The conclusion is for now a bit weaker: H need not be connected, or R-simple, although its connected component is semisimple. G will be definably isomorphic to a finite index semialgebraic subgroup of $H(R)$.

A key concern of current model theory is the question of bi-interpretability. A structure M is said to be interpretable in a structure N if there is an isomorphic copy $f(M)$ of M say whose universe and basic relations are all definable sets in N^{eq}. Now if M and N are each interpretable in the other, witnessed by $f(M)$ and $g(N)$, then $f.g$ yields an isomorphism of N with $f(g(N))$, the latter being also definable in N. Similarly $g.f$ is an isomorphism of M with $g(f(M))$. M and N are said to be bi-interpretable if one can choose both f and g such that the isomorphism $f.g$ is definable in N and the isomorphism $g.f$ is definable in M. Bi-interpretability means that the structures are essentially the same. An important observation of Poizat ([12]) is that if G is a simple algebraic group (with respect to a given algebraically closed field K), then the structures $(G, .)$ and $(K, +, .)$ are bi-interpretable. This is a model-theoretic version (and easily implies) a simple case of Borel-Tits theory: any abstract group automorphism of a simple algebraic group over an algebraically closed field is a composition of a field automorphism of K with a quasi-isomorphism of algebraic groups (i.e. an isomorphism definable in $(K, +, .)$). Much less trivial is the question of abstract automorphisms of groups of the form $G(k)$ where G is a k-simple algebraic group defined over an arbitrary field k. This is the concern of general Borel-Tits theory ([2]). Our second result bears on this question in the case where k is a real closed field. Note that if R is real closed then the algebraic closure of R is obtained by adjoining i, the square root of -1. Moreover clearly $R(i)$ is definable in R. We will have two fundamental cases for (definably) simple groups definable in the field R. An example of the first case is $PSL(2, R)$, which will be bi-interpretable with R. An example of the second case is $PSL(2, R(i))$ which will be bi-interpretable with $R(i)$ (and not with R). Actually in the proof the first case has 2 subcases: on the one hand, groups like $PSL(2, R)$, which for $R = \mathbf{R}$ are noncompact, and and the other hand groups like $SO(3, R)$ which for $R = \mathbf{R}$ *are* compact. The first (isotropic) subcase fits into Borel-Tits theory, whereas the second (anisotropic) case was studied by Weisfeiler [14]. It should be said, however, that our treatment is independent of either [2] or [14]

One would have liked to prove the next theorem by Theorem 1.1 and inspection. However we were not sure what to inspect and where to find it.

Theorem 1.2 *Let G be an infinite group which is definably simple in the sense of $(G, \cdot)$, and which is definable in some O-minimal structure. Then there is a real closed field R such that $(G, \cdot)$ is bi-interpretable either with $(R, +, .)$ or with $(R(i), +, .)$. In particular any simple Lie group is bi-interpretable with the field of reals or the field of complexes.*

With Theorem 1.2, one sees that definable simplicity of G in the sense of $(G, \cdot)$ is equivalent to definable simplicity in the ambient O-minimal structure. Theorem 1.2 also has bearing on Cherlin's original conjecture for it implies that bi-interpretability with R corresponds to the group being unstable.

Corollary 1.3 *Suppose G is an infinite simple group. Then the following are equivalent:*
(i) $Th(G, \cdot)$ is stable and $(G, \cdot)$ is definable in some O-minimal structure,
(ii) G is an algebraic group over some algebraically closed field of characteristic 0.

It should be said that the problem of proving O-minimal analogues of Cherlin's conjecture (i.e. for Lie groups) was originally raised by the second author, and the solution for groups of small dimension was given in [6].

In the next section we discuss the general ideology of the proof, and mention some aspects which may be of independent interest.

2 Outline of the proof.

Let us first discuss Poizat's original strategy for proving Cherlin's conjecture. Let G be an infinite simple group of finite Morley rank. His idea was to find an infinite solvable nonnilpotent definable subgroup B of G, and then use a result of Zilber which gives an infinite definable field K. K, having finite Morley rank too, has to be algebraically closed. The idea was then to show that the structure induced on K from $(G, \cdot)$ is just the field structure, then to use some model theory to show that $(G, \cdot)$ is definable back in $(K, +, .)$, then to conclude, by a version of Weil's Theorem, that G is an algebraic group over K. The existence of solvable nonnilpotent subgroups of a simple group of finite Morley rank, is the "no bad groups" hypothesis. The truth of this is still unknown, although it is not unlikely that there is a counterexample. The second thing one needs to know is that if K is an algebraically closed field definable in a structure M of finite Morley rank, then no structure on K is induced from M other than the field structure. This is a strong version of the "no bads fields" hypothesis. This turns out to be false: Hrushovski constructed counteramples. (It should be said that Cherlin and others have now a program to prove Cherlin's conjecture under the "no bad groups" hypothesis and a weaker version of the "no bad fields" hypothesis.) So the Poizat strategy falls through. But note that one of the main points was to define a field and show the field to be well-behaved.

Let us now pass to the O-minimal situation. Let M be an O-minimal structure. Sometimes it is convenient to assume M to be saturated. There is a general theory of definable sets and functions in O-minimal structures ([11],[3],[13]), yielding a notion of dimension and independence. Essentially algebraic closure on M is a pregeometry, and the associated dimension of definable sets has a topological significance: if $X \subset M^n$ then $dim(X)$ is the greatest r such that some projection of X on M^r contains an open set (in the product topology). Also definable functions are piecewise continuous.

Now let G be an infinite definably simple (in the sense of M) group definable in M. It turns out that the Poizat strategy is much more successful here. However, this is made possible by a theorem of the first and last authors ([7]) which shows the existence of a definable real closed field in an O-minimal structure M under some reasonable assumption on its complexity. In fact their result yields a definable real closed field in a neigbourhood U of a point $a \in M$, whenever there is a family F of germs of definable functions at a with $dim(F) > 1$. The noncommutativity of G (together with quite a bit of work) enables one to find a point $a \in M$ closely related to G satisying the above complexity hypothesis, yielding a definable real closed field R. Some additional work, using definable simplicity of G, is required to prove that some definable subset X of G of maximal dimension is contained in $dcl(R)$. Now R may have additional structure (other than semialgebraic structure) induced on it from M. However, what O-minimality yields is that definable (in M) functions from R^n to R are piecewise continuously differentiable (in fact piecewise C^k for abitrarily large k). Note that R is a densely ordered field so differentiability makes sense. Methods from [10] allow us to equip G with a "definable group manifold" structure over M such that moreover, some open neighbouhood U of the identity is identified with an open subset of R^n for some n, and that the group operation on this neighbourhood is continuously differentiable. At this point, a version of the classical adjoint representation comes into play: for any $g \in G$, the conjugation by g map $Inn_g : G \to G$ is continuously differentiable on a neighbourhood of the identity of G. Its differential is an element of $GL(n, R)$. G being centreless, this yields a definable embedding of G in $GL(n, R)$. We now have G as a *definable* subgroup of $GL(n, R)$, but we would like it to be a *semialgebraic* subgroup. For this, one has to develop some classical Lie theory (the relation between Lie groups and Lie algebras) in the "O-minimal expansions of real closed fields" context. This works quite successfully. We define a Lie algebra structure on the tangent space at the identity of G in a natural fashion, to obtain $L(G)$ the Lie algebra of G. This is an object definable in $(R, +, .)$. From semisimplicity of G (no definable normal abelian subgroups) we conclude semisimplicity of $L(G)$ (no proper abelian ideals). Properties of semisimple Lie algebras over *arbitrary* fields of characteristic 0 now imply that the (semialgebraic) group of automorphisms of $L(G)$ (linear transformations preserving the Lie brackets), has the same dimension as $L(G)$ and thus as G. The adjoint representation now yields an isomorphism of G with a finite index subgroup of $Aut(L(G))$ which must be semialgebraic. This completes the sketch of the proof of Theorem 1.1.

We now say a few words about the proof of Theorem 1.2. As already remarked, Poizat had already proved that if G is a simple algebraic group over an algebraically closed field (i.e. $G = G(K)$) then $(G, \cdot)$ and $(K, +, .)$ are bi-interpretable. The main point was to find an infinite field K' definable in $(G, \cdot)$, and then to show that (in the structure $(G, \cdot)$, G is contained in $dcl(K')$. Both these steps require some work when transferred to the real closed field setting. (We will here suppress the problem of working with definable simplicity in the sense of $(G, \cdot)$ rather than in the sense of M (or R).) So the situation now will be: R is a real closed field, H is an R-simple linear algebraic group defined over R, and G is the semialgebraic connected component of $H(R)$. The first problem is to find an infinite field definable in $(G, \cdot)$. At the outset we have a division into two cases: (i) H is anisotropic over R and (ii) H is isotropic over R. Condition (i) means that H has *no* (infinite) R-definable algebraic subgroup which is R-isomorphic to some product of copies of the multiplicative group. A typical example is when $G = S0(3, \mathbf{R})$, and corresponds in this classical case to G being *compact.* In this case, with a little work, the results on definability in compact Lie groups from [5] transfer to R, to give a 1-dimensional field definable (actually interpretable) in $(G, \cdot)$. In case (ii), where H is R-isotropic, we can fabricate, using the theory of parabolic subgroups from [1], a connected solvable centreless R-subgroup B of H. $B(R)$ will be enveloped in a solvable nonnilpotent subgroup of G definable in $(G, \cdot)$, and various methods enable one to define in $(G, \cdot)$ an infinite field K. (An O-minimal version of Zilber's Indecomposability Theorem is used here, as well as at other points of the proof.) Now the field K is definable in the real closed field $(R, +, .)$, and is known to be semialgebraically isomorphic to R or to $R(i)$. In the first case $dim(K) = 1$ and there are methods, using simplicity of G, for showing that G is (in $(G, \cdot)$) contained in $dcl(K)$, so we finish. In the second case, we can consider the structure N consisting of $(K, +, .)$ equipped with all structure induced from $(G, \cdot)$, as a semialgebraic expansion of $(R(i), +, .)$. Results of Marker [4] imply that either a 1-dimensional field is definable in N, or N is simply an algebraically closed field with some constants. In the first subcase, we have a 1-dimensional real closed field definable in $(G, \cdot)$ and can proceed as before. In the second subcase, K with all its induced structure is strongly minimal. The original Zilber Indecomposability Theorem (or rather its proof), together with simplicity of G implies that $(G, \cdot)$ is a structure of finite Morley rank, and it is then easy to show that this structure is the definable closure of K.

This completes the outline, as well as the paper.

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