Free monoid completeness of the Lambek calculus allowing empty premises

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ABSTRACT We prove that the Lambek syntactic calculus allowing empty premises is complete with respect to the class of all free monoid models (i. e., the class of all string models, allowing the empty string).

Introduction

Lambek syntactic calculus (introduced in [7]) is one of the logical calculi used in the paradigm of categorial grammar for deriving reduction laws of syntactic types in natural and formal languages. The intended models for these calculi are free semigroup models (also called language models or string models), where each syntactic category is interpreted as a set of non-empty strings over some alphabet of symbols. Models for Lambek calculus were studied in [2], [3], [4], [5], [6], etc. Completeness of the Lambek calculus with respect to string models was proved in [9], [10], and [11]. Closely related is the result about completeness with respect to relational semantics [8].

There is a natural modification of the original Lambek calculus, which we call the Lambek calculus allowing empty premises (cf. [2, p. 44]). This calculus appears to be a fragment of the noncommutative linear logic. The natural class of string models for the Lambek calculus allowing empty premises is the class of all free monoid models, where also empty string is allowed.

In this paper we prove that the Lambek calculus allowing empty premises is complete with respect to these models.

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1 Lambek calculus allowing empty premises

We consider the *Lambek calculus allowing empty premises* (cf. [2, p. 44]) and denote it by L^*. This calculus is a modification of the syntactic calculus introduced in [7].

The types of L^* are built of primitive types $p_1, p_2, \ldots$, and three binary connectives $\bullet, \backslash, /$. We shall denote the set of all types by Tp. The set of finite sequences of types (resp. finite non-empty sequences of types) is denoted by Tp^* (resp. Tp^+). The symbol Λ will stand for the empty sequence of types.

Sometimes we shall write $A_1\bullet\ldots\bullet A_n$ instead of $(\ldots(A_1\bullet A_2)\bullet\ldots)\bullet A_n$.

Capital letters A, B, ... range over types. Capital Greek letters range over finite (possibly empty) sequences of types.

Sequents of L^* are of the form $\Gamma\to A$. Note that Γ can be the empty sequence.

Axioms: $A\to A$

Rules:

$$\frac{A\Pi\to B}{\Pi\to A\backslash B}\ (\to\backslash) \qquad \frac{\Phi\to A \quad \Gamma B\Delta\to C}{\Gamma\Phi(A\backslash B)\Delta\to C}\ (\backslash\to)$$

$$\frac{\Pi A\to B}{\Pi\to B/A}\ (\to/) \qquad \frac{\Phi\to A \quad \Gamma B\Delta\to C}{\Gamma(B/A)\Phi\Delta\to C}\ (/\to)$$

$$\frac{\Gamma\to A \quad \Delta\to B}{\Gamma\Delta\to A\bullet B}\ (\to\bullet) \qquad \frac{\Gamma AB\Delta\to C}{\Gamma(A\bullet B)\Delta\to C}\ (\bullet\to)$$

$$\frac{\Phi\to B \quad \Gamma B\Delta\to A}{\Gamma\Phi\Delta\to A}\ (CUT)$$

It is known that the cut-elimination theorem holds for this calculus (cf. [2]). We write $L^*\vdash\Gamma\to A$ if the sequent $\Gamma\to A$ is derivable in L^*.

There is an obvious duality phenomenon inherent in L^*.

Definition. The function dual: Tp $\to$ Tp is defined as follows.

$$\begin{aligned}
\mathrm{dual}(p_i) &\rightleftharpoons p_i\\
\mathrm{dual}(A\bullet B) &\rightleftharpoons \mathrm{dual}(B)\bullet\mathrm{dual}(A)\\
\mathrm{dual}(A\backslash B) &\rightleftharpoons \mathrm{dual}(B)/\mathrm{dual}(A)\\
\mathrm{dual}(A/B) &\rightleftharpoons \mathrm{dual}(B)\backslash\mathrm{dual}(A)
\end{aligned}$$

The extension to sequences of types dual: $\mathrm{Tp}^*\to\mathrm{Tp}^*$ is defined as

$$\mathrm{dual}(A_1\ldots A_n)\rightleftharpoons\mathrm{dual}(A_n)\ldots\mathrm{dual}(A_1).$$

Lemma 1.1 *If $L^*\vdash\Gamma\to A$, then $L^*\vdash\mathrm{dual}(\Gamma)\to\mathrm{dual}(A)$.*

Proof. Straightforward induction on the derivation of $\Gamma\to A$. ■

2 Free monoid models

We use the following notation. Let $\mathcal{V}$ be any alphabet, i.e., any set, the elements of which are called symbols. We denote by $\mathcal{V}^+$ the set of all non-empty words over the alphabet $\mathcal{V}$. By $\mathcal{V}^*$ we denote the set of all words over the alphabet $\mathcal{V}$, including the *empty word* ε. Let $\circ$ denote concatenation. Evidently $\mathcal{V}^*$ is a free monoid w.r.t. $\circ$. The unit of the free monoid is ε. Throughout the paper calligraphic letters $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ will denote alphabets.

If α is a word, then $|\alpha|$ (the *length* of α) is the number of symbols in α.

We shall use the following shorthand notation. For any sets $\mathcal{R} \subseteq \mathcal{V}^*$ and $\mathcal{T} \subseteq \mathcal{V}^*$ we write

$$\mathcal{R} \circ \mathcal{T} \rightleftharpoons \{\gamma \in \mathcal{V}^* \mid \text{ there are } \alpha \in \mathcal{R} \text{ and } \beta \in \mathcal{T} \text{ such that } \alpha \circ \beta = \gamma\};$$

$$\mathcal{R} \circ \beta \rightleftharpoons \mathcal{R} \circ \{\beta\}; \qquad \alpha \circ \mathcal{T} \rightleftharpoons \{\alpha\} \circ \mathcal{T}.$$

Since this operation on sets is associative, we omit parentheses in expressions like $\mathcal{R}_1 \circ \mathcal{R}_2 \circ \ldots \circ \mathcal{R}_m$. In the case of $m = 0$ we assume that this expression stands for the set $\{\varepsilon\}$. By $\mathcal{R}^m$ we denote the set $\underbrace{\mathcal{R} \circ \ldots \circ \mathcal{R}}_{m \text{ times}}$.

We shall denote the set of all subsets of a set $\mathcal{S}$ by $\mathbf{P}(\mathcal{S})$.

Definition. A free monoid model $\langle \mathcal{W}^*, w\rangle$ consists of the free monoid $\langle \mathcal{W}^*, \circ, \varepsilon\rangle$ and a valuation $w\colon \mathrm{Tp} \to \mathbf{P}(\mathcal{W}^*)$ associating with each type of L^* a subset of $\mathcal{W}^*$ and satisfying for any types A and B the following conditions.

(1) $w(A \bullet B) = w(A) \circ w(B)$

(2) $w(A \backslash B) = \{\gamma \in \mathcal{W}^* \mid w(A) \circ \gamma \subseteq w(B)\}$

(3) $w(B/A) = \{\gamma \in \mathcal{W}^* \mid \gamma \circ w(A) \subseteq w(B)\}$

For any function $w\colon \mathrm{Tp} \to \mathbf{P}(\mathcal{W}^*)$ and for any types $A_1, \ldots, A_n$, we write $\vec{w}(A_1 \ldots A_n)$ as a shorthand for $w(A_1) \circ \ldots \circ w(A_n)$. Note that $\vec{w}(\Lambda) = \{\varepsilon\}$.

Definition. A sequent $\Gamma \to B$ is *true* in a model $\langle \mathcal{W}^*, w\rangle$ iff $\vec{w}(\Gamma) \subseteq w(B)$. A sequent is *false* in a model iff it is not true in the model.

The following well-known soundness theorem holds.

Theorem 2.1 *If a sequent is derivable in the calculus L^*, then the sequent is true in every free monoid model.*

The rest of the paper is devoted to the proof of the corresponding completeness theorem. In view of the following two lemmas it is sufficient to consider only sequents with empty antecedent.

Lemma 2.2 *For any types A_1, ..., A_n, B, the sequent $A_1 \dots A_n \to B$ is derivable in L^* if and only if the sequent $\Lambda \to (A_1 \bullet \dots \bullet A_n) \backslash B$ is derivable in L^*.*

Lemma 2.3 *For any free monoid model $\langle \mathcal{W}^*, w \rangle$ and for any types A_1, ..., A_n, B, the sequent $A_1 \dots A_n \to B$ is true in $\langle \mathcal{W}^*, w \rangle$ if and only if the sequent $\Lambda \to (A_1 \bullet \dots \bullet A_n) \backslash B$ is true in $\langle \mathcal{W}^*, w \rangle$.*

3 Quasimodels

In this section we introduce the notion of $\mathrm{Tp}(m)$-quasimodels and describe an algorithm of constructing a free monoid model as the limit of an infinite sequence of $\mathrm{Tp}(m)$-quasimodels, which are conservative extensions of each other.

Definition. The *length* of a type is defined as the total number of primitive type occurrences in the type.

$$\|p_i\| \rightleftharpoons 1 \qquad \|A \bullet B\| \rightleftharpoons \|A\| + \|B\|$$

$$\|A \backslash B\| \rightleftharpoons \|A\| + \|B\| \qquad \|A / B\| \rightleftharpoons \|A\| + \|B\|$$

Similarly, for sequences of types we put $\|A_1 \dots A_n\| \rightleftharpoons \|A_1\| + \dots + \|A_n\|$.

Definition. The set of primitive types *occurring* in a type is defined as follows.

$$\mathrm{Var}(p_i) \rightleftharpoons \{p_i\} \qquad \mathrm{Var}(A \bullet B) \rightleftharpoons \mathrm{Var}(A) \cup \mathrm{Var}(B)$$

$$\mathrm{Var}(A \backslash B) \rightleftharpoons \mathrm{Var}(A) \cup \mathrm{Var}(B) \qquad \mathrm{Var}(A / B) \rightleftharpoons \mathrm{Var}(A) \cup \mathrm{Var}(B)$$

Definition. For any integer m, we write $\mathrm{Tp}(m)$ for the finite set of types

$$\mathrm{Tp}(m) \rightleftharpoons \{A \in \mathrm{Tp} \mid \mathrm{Var}(A) \subseteq \{p_1, p_2, \dots, p_m\} \text{ and } \|A\| \leq m\}.$$

By $\mathrm{Tp}(m)^*$ we denote the set of all finite sequences of types from $\mathrm{Tp}(m)$.

Definition. A $\mathrm{Tp}(m)$-*quasimodel* $\langle \mathcal{W}^*, w \rangle$ is a valuation $w \colon \mathrm{Tp} \to \mathbf{P}(\mathcal{W}^*)$ over a free monoid $\langle \mathcal{W}^*, \circ, \varepsilon \rangle$ such that

(1) for any $A \in \mathrm{Tp}$ and $B \in \mathrm{Tp}$, if $A \bullet B \in \mathrm{Tp}(m)$, then $w(A \bullet B) \subseteq w(A) \circ w(B)$;

(2) for any $\Gamma \in \mathrm{Tp}(m)^*$ and $A \in \mathrm{Tp}(m)$, if $L^* \vdash \Gamma \to A$, then $\vec{w}(\Gamma) \subseteq w(A)$;

(3) for any $A \in \mathrm{Tp}(m)$, if $\varepsilon \in w(A)$, then $L^* \vdash \Lambda \to A$.

Lemma 3.1 *Let $\langle \mathcal{W}^*, w \rangle$ be a $\mathrm{Tp}(m)$-quasimodel. Then the following statements hold.*

(i) *If* $A{\bullet}B \in \mathrm{Tp}(m)$, *then* $w(A{\bullet}B) = w(A) \circ w(B)$.

(ii) *If* $A\backslash B \in \mathrm{Tp}(m)$, *then* $w(A\backslash B) \subseteq \{\gamma \in \mathcal{W}^* \mid w(A) \circ \gamma \subseteq w(B)\}$.

(iii) *If* $B/A \in \mathrm{Tp}(m)$, *then* $w(B/A) \subseteq \{\gamma \in \mathcal{W}^* \mid \gamma \circ w(A) \subseteq w(B)\}$.

PROOF. It is sufficient to note that $L^* \vdash AB \to A{\bullet}B$, $L^* \vdash A(A\backslash B) \to B$, $L^* \vdash B(B/A) \to B$, and use (2) from the definition of a $\mathrm{Tp}(m)$-quasimodel. ∎

Definition. A sequent $\Gamma \to A$ is *true* in a $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}^*, w\rangle$ iff $\vec{w}(\Gamma) \subseteq w(A)$.

Definition. A $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}^*, w\rangle$ is a *conservative extension* of another $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{V}^*, v\rangle$ iff

(1) $\mathcal{V} \subseteq \mathcal{W}$;

(2) $w(A) \cap \mathcal{V}^* = v(A)$ for any type A.

Evidently, if $\langle \mathcal{W}^*, w\rangle$ is a conservative extension of $\langle \mathcal{V}^*, v\rangle$, then for any type A we have $v(A) \subseteq w(A)$.

Lemma 3.2 *If* $\langle \mathcal{W}_2^*, w_2\rangle$ *is a conservative extension of* $\langle \mathcal{W}_1^*, w_1\rangle$ *and* $\langle \mathcal{W}_3^*, w_3\rangle$ *is a conservative extension of* $\langle \mathcal{W}_2^*, w_2\rangle$, *then* $\langle \mathcal{W}_3^*, w_3\rangle$ *is a conservative extension of* $\langle \mathcal{W}_1^*, w_1\rangle$.

PROOF. In view of $\mathcal{W}_1^* \subseteq \mathcal{W}_2^*$ we have $w_3(A) \cap \mathcal{W}_1^* = w_3(A) \cap (\mathcal{W}_2^* \cap \mathcal{W}_1^*) = (w_3(A) \cap \mathcal{W}_2^*) \cap \mathcal{W}_1^*$. Further, $(w_3(A) \cap \mathcal{W}_2^*) \cap \mathcal{W}_1^* = w_2(A) \cap \mathcal{W}_1^* = w_1(A)$. ∎

We shall denote by $\mathbf{Z}$ the set of all integers and by $\mathbf{N}$ the set of all natural numbers, including zero.

Definition. We say that a sequence of $\mathrm{Tp}(m)$-quasimodels $\langle \mathcal{W}_i^*, w_i\rangle$ $(i \in \mathbf{N})$ is *conservative* iff, for every $i \in \mathbf{N}$, $\langle \mathcal{W}_{i+1}^*, w_{i+1}\rangle$ is a conservative extension of $\langle \mathcal{W}_i^*, w_i\rangle$. (Here m is constant.)

Definition. The *limit* of a conservative sequence $\langle \mathcal{W}_i^*, w_i\rangle$ $(i \in \mathbf{N})$ is the $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}_\infty^*, w_\infty\rangle$ defined as follows.

(1) $\mathcal{W}_\infty \rightleftharpoons \bigcup_{i \in \mathbf{N}} \mathcal{W}_i$

(2) $w_\infty(A) \rightleftharpoons \bigcup_{i \in \mathbf{N}} w_i(A)$

Lemma 3.3 *The definition of the limit is correct, i.e.,* $\langle \mathcal{W}_\infty^*, w_\infty\rangle$ *is really a* $\mathrm{Tp}(m)$*-quasimodel.*

PROOF.

(1) Let $A{\bullet}B \in \mathrm{Tp}(m)$ and $\gamma \in w_\infty(A{\bullet}B)$. Then for some n we have $\gamma \in w_n(A{\bullet}B) \subseteq w_n(A) \circ w_n(B)$. Thus $\gamma = \alpha \circ \beta$, where $\alpha \in w_n(A)$ and $\beta \in w_n(B)$. Evidently $\alpha \in w_\infty(A)$ and $\beta \in w_\infty(B)$, whence $\gamma = \alpha \circ \beta \in w_\infty(A) \circ w_\infty(B)$.

(2) Let $L^* \vdash A_1 \ldots A_l \to B$, where $A_1 \in \mathrm{Tp}(m)$, ..., $A_l \in \mathrm{Tp}(m)$, and $B \in \mathrm{Tp}(m)$. Assume that $\gamma \in \vec{w}_\infty(A_1 \ldots A_l)$, i.e., $\gamma = \alpha_1 \circ \ldots \circ \alpha_l$, where $\alpha_1 \in w_\infty(A_1)$, ..., $\alpha_l \in w_\infty(A_l)$. Then $\alpha_1 \in w_{i_1}(A_1)$, ..., $\alpha_l \in w_{i_l}(A_l)$ for some $i_1, \ldots, i_l \in \mathbf{N}$. Put $n \rightleftharpoons \max(i_1, \ldots, i_l)$.

According to Lemma 3.2, $\alpha_1 \in w_n(A_1)$, ..., $\alpha_l \in w_n(A_l)$, whence $\gamma = \alpha_1 \circ \ldots \circ \alpha_l \in \vec{w_n}(A_1 \ldots A_l) \subseteq w_n(B) \subseteq w_\infty(B)$.

(3) Obvious. ∎

Lemma 3.4 *The limit of a conservative sequence is a conservative extension of any of the elements of the sequence.*

PROOF. We verify that $w_\infty(A) \cap \mathcal{W}_i^* = w_i(A)$. For any $k \leq i$ we have $w_k(A) \subseteq w_i(A)$. Thus $w_\infty(A) = \bigcup\limits_j w_j(A) = \bigcup\limits_{j \geq i} w_j(A)$, whence $w_\infty(A) \cap \mathcal{W}_i^* = (\bigcup\limits_{j\geq i} w_j(A)) \cap \mathcal{W}_i^* = \bigcup\limits_{j \geq i}(w_j(A) \cap \mathcal{W}_i^*)$. Note that $w_j(A) \cap \mathcal{W}_i^* = w_i(A)$ for any $j \geq i$ (according to Lemma 3.2). Now $\bigcup\limits_{j\geq i}(w_j(A) \cap \mathcal{W}_i^*) = \bigcup\limits_{j \geq i} w_i(A) = w_i(A)$. ∎

4 A simple quasimodel

Definition. We define the *non-negative count* $\bar{\#}$ as the following mapping from types to non-negative integers.

$$\begin{array}{rcl}
\bar{\#}p_i & \rightleftharpoons & 1 \\
\bar{\#}(A{\bullet}B) & \rightleftharpoons & \bar{\#}A + \bar{\#}B \\
\bar{\#}(A\backslash B) & \rightleftharpoons & \begin{cases} \max(0, \bar{\#}B - \bar{\#}A), & \text{if } L^* \vdash \Lambda \to A\backslash B \\ \max(1, \bar{\#}B - \bar{\#}A), & \text{if } L^* \nvdash \Lambda \to A\backslash B \end{cases} \\
\bar{\#}(A/B) & \rightleftharpoons & \begin{cases} \max(0, \bar{\#}A - \bar{\#}B), & \text{if } L^* \vdash \Lambda \to A/B \\ \max(1, \bar{\#}A - \bar{\#}B), & \text{if } L^* \nvdash \Lambda \to A/B \end{cases}
\end{array}$$

The non-negative count of a sequence of types is defined in the natural way.

$$\bar{\#}(A_1 \ldots A_l) \rightleftharpoons \bar{\#}A_1 + \ldots + \bar{\#}A_l$$

By definition, $\bar{\#}(\Lambda) \rightleftharpoons 0$.

Lemma 4.1 *For any type A its non-negative satisfies the inequalities* $0 \leq \bar{\#}A \leq \|A\|$.

PROOF. Induction on $\|A\|$. ■

Lemma 4.2 *If* $\bar{\#}A = 0$, *then* $L^* \vdash \Lambda \to A$.

PROOF. Induction on $\|A\|$. Induction steps for $B\backslash C$ and B/C are easy. In the case of $B{\bullet}C$ we assume that $\bar{\#}(B{\bullet}C) = 0$ and obtain $\bar{\#}B = 0$, $\bar{\#}C = 0$, and

$$\frac{\Lambda \to B \quad \Lambda \to C}{\Lambda \to B{\bullet}C}\ (\to{\bullet}).$$

■

Lemma 4.3 *If* $L^* \vdash \Gamma \to A$ *then* $\bar{\#}\Gamma \geq \bar{\#}A$.

PROOF. Induction on the length of the derivation.
CASE 1: Axiom.
Obvious.
CASE 2: $(\to\backslash)$ Given $\dfrac{A\,\Pi \to B}{\Pi \to A\backslash B}\ (\to\backslash)$.
By the induction hypothesis $\bar{\#}A + \bar{\#}\Pi \geq \bar{\#}B$, whence $\bar{\#}\Pi \geq \bar{\#}B - \bar{\#}A$. Obviously, if $\bar{\#}\Pi \geq 1$, then $\bar{\#}\Pi \geq \max(1, \bar{\#}B - \bar{\#}A) \geq \bar{\#}(A\backslash B)$. Let now $\bar{\#}\Pi = 0$ and $\Pi = C_1 \ldots C_n$. Then $\bar{\#}C_i = 0$ for each $i \leq n$. According to Lemma 4.2 $L^* \vdash \Lambda \to C_i$ for each $i \leq n$. Applying (CUT) n times we derive $L^* \vdash \Lambda \to A\backslash B$, whence $\bar{\#}(A\backslash B) = \max(0, \bar{\#}B - \bar{\#}A)$. From $\bar{\#}B - \bar{\#}A \leq \bar{\#}\Pi = 0$ we obtain $\bar{\#}(A\backslash B) = 0$.
CASE 3: $(\to/)$
Similar.
CASE 4: $(\backslash\to)$ Given $\dfrac{\Phi \to A \quad \Gamma B \Delta \to C}{\Gamma\Phi(A\backslash B)\Delta \to C}\ (\backslash\to)$.
By the induction hypothesis $\bar{\#}\Phi \geq \bar{\#}A$ and $\bar{\#}\Gamma + \bar{\#}B + \bar{\#}\Delta \geq \bar{\#}C$.
Note that $\bar{\#}(A\backslash B) \geq \bar{\#}B - \bar{\#}A$.
Hence $\bar{\#}\Gamma + \bar{\#}\Phi + \bar{\#}(A\backslash B) + \bar{\#}\Delta \geq \bar{\#}\Gamma + \bar{\#}A + (\bar{\#}B - \bar{\#}A) + \bar{\#}\Delta \geq \bar{\#}C$.
CASE 5: $(/\to)$
Similar.
CASE 6: $(\to{\bullet})$ Given $\dfrac{\Gamma \to A \quad \Delta \to B}{\Gamma\Delta \to A{\bullet}B}\ (\to{\bullet})$.
If $\bar{\#}\Gamma \geq \bar{\#}A$ and $\bar{\#}\Delta \geq \bar{\#}B$, then $\bar{\#}\Gamma + \bar{\#}\Delta \geq \bar{\#}A + \bar{\#}B = \bar{\#}(A{\bullet}B)$.
CASE 7: $({\bullet}\to)$ Given $\dfrac{\Gamma A B \Delta \to C}{\Gamma(A{\bullet}B)\Delta \to C}\ ({\bullet}\to)$.
Evidently $\bar{\#}(\Gamma(A{\bullet}B)\Delta) = \bar{\#}(\Gamma A B \Delta)$. ■

Remark. For any type A, we have $\bar{\#}A = 0$ if and only if $L^* \vdash \Lambda \to A$.

Now we define a $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}_0^*, w_0 \rangle$.

$$\mathcal{W}_0 \rightleftharpoons \{a_0\} \qquad w_0(A) \rightleftharpoons \{a_0^k \mid k \geq \bar{\#}A\}$$

Here a_0^k denotes $\underbrace{a_0 \circ \ldots \circ a_0}_{k \text{ times}}$. In particular, $a_0^0 = \varepsilon$.

Lemma 4.4 *$\langle \mathcal{W}_0^*, w_0 \rangle$ is a* Tp(m)*-quasimodel for any natural number m.*

PROOF. (1) We prove that $w_0(A{\bullet}B) \subseteq w_0(A) \circ w_0(B)$.

Let $a_0^k \in w_0(A{\bullet}B)$. We put $k_1 \rightleftharpoons \bar{\#}A$ and $k_2 \rightleftharpoons k - k_1$. In view of $k \geq \bar{\#}(A{\bullet}B) = \bar{\#}A + \bar{\#}B$ we have $k_2 \geq \bar{\#}B$. Evidently $a_0^k = a_0^{k_1} \circ a_0^{k_2}$, $a_0^{k_1} \in w_0(A)$ and $a_0^{k_2} \in w_0(B)$.

(2) We verify that if $L^* \vdash C_1 \ldots C_n \to A$, then $w_0(C_1) \circ \ldots \circ w_0(C_n) \subseteq w_0(A)$.

Let $a_0^{k_i} \in w_0(C_i)$ for every $i \leq n$. Then $\sum k_i \geq \sum \bar{\#}C_i \geq \bar{\#}A$ according to Lemma 4.3. Thus $a_0^{k_1} \circ \ldots \circ a_0^{k_n} \in w_0(A)$.

(3) In view of Lemma 4.2, if $\varepsilon \in w_0(A)$, then $L^* \vdash \Lambda \to A$. ∎

5 Witnesses

Definition. We fix a countable alphabet $\mathcal{U} = \{a_j \mid j \in \mathbf{N}\}$. By $\mathcal{K}^m$ we denote the class of all Tp(m)-quasimodels $\langle \mathcal{V}^*, v \rangle$, such that $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ is finite, and for every $A \in$ Tp(m) there is $\alpha \in v(A)$ satisfying $|\alpha| \leq m$.

Lemma 5.1 *The* Tp(m)*-quasimodel $\langle \mathcal{W}_0^*, w_0 \rangle$ from Lemma 4.4 belongs to the class $\mathcal{K}^m$.*

PROOF. Immediate from Lemma 4.1. ∎

Definition. Let $\langle \mathcal{W}^*, w \rangle$ be a Tp(m)-quasimodel. Let $A, B \in$ Tp, $\alpha \in \mathcal{W}^*$, $\gamma \in \mathcal{W}^*$, and $\gamma \notin w(A\backslash B)$. We say that α is a *witness of* $\gamma \notin w(A\backslash B)$ iff $\alpha \in w(A)$ and $\alpha \circ \gamma \notin w(B)$.

Definition. Let $\langle \mathcal{W}^*, w \rangle$ be a Tp(m)-quasimodel. Let $A, B \in$ Tp, $\alpha \in \mathcal{W}^*$, $\gamma \in \mathcal{W}^*$, and $\gamma \notin w(B/A)$. We say that α is a *witness of* $\gamma \notin w(B/A)$ iff $\alpha \in w(A)$, and $\gamma \circ \alpha \notin w(B)$.

Definition. Let $\mathcal{K}$ be a class of Tp(m)-quasimodels. We say that the class $\mathcal{K}$ is *witnessed* iff

(1) for any $\langle \mathcal{V}^*, v \rangle \in \mathcal{K}$, for any type of the form $A\backslash B$ from Tp(m), and for any $\gamma \in \mathcal{V}^*$, if $\gamma \notin v(A\backslash B)$ then there is a conservative extension $\langle \mathcal{W}^*, w \rangle$ of $\langle \mathcal{V}^*, v \rangle$ in $\mathcal{K}$ and $\langle \mathcal{W}^*, w \rangle$ contains a witness of $\gamma \notin w(A\backslash B)$;

(2) for any $\langle \mathcal{V}^*, v \rangle \in \mathcal{K}$, for any type of the form B/A from Tp(m), and for any $\gamma \in \mathcal{V}^*$, if $\gamma \notin v(B/A)$ then there is a conservative extension $\langle \mathcal{W}^*, w \rangle$ of $\langle \mathcal{V}^*, v \rangle$ in $\mathcal{K}$ and $\langle \mathcal{W}^*, w \rangle$ contains a witness of $\gamma \notin w(B/A)$.

Lemma 5.2 *If the class $\mathcal{K}^m$ is witnessed, then there is a free monoid model $\langle \mathcal{V}^*, v \rangle$ such that*

(i) *for every type* $E \in \mathrm{Tp}(m)$, *if* $L^* \not\vdash \Lambda \to E$, *then the sequent* $\Lambda \to E$ *is false in* $\langle \mathcal{V}^*, v \rangle$;

(ii) $v(E) \neq \emptyset$ *for every type* $E \in \mathrm{Tp}(m)$;

(iii) $\mathcal{V} \subseteq \mathcal{U}$.

At the end of this paper it will be proved that the class $\mathcal{K}^m$ is witnessed. Thus Lemma 5.2 (i) provides a proof of completeness of L^* with respect to free monoid models.

PROOF. Evidently there is a function $\sigma: \mathbf{N} \to \mathrm{Tp}(m) \times \mathcal{U}^*$ such that for any $C \in \mathrm{Tp}(m)$ and for any $\gamma \in \mathcal{U}^*$ there are infinitely many natural numbers i, for which $\sigma(i) = \langle C, \gamma \rangle$.

Starting with the $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}_0^*, w_0 \rangle$ from Lemma 4.4, we define by induction on i a conservative sequence $\langle \mathcal{W}_i^*, w_i \rangle$ $(i \in \mathbf{N})$, consisting of $\mathrm{Tp}(m)$-quasimodels from the class $\mathcal{K}^m$.

Assume that $\langle \mathcal{W}_i^*, w_i \rangle \in \mathcal{K}^m$ has been constructed. We define $\langle \mathcal{W}_i^*, w_i \rangle$ as follows.

CASE 1:
If $\sigma(i) = \langle A \backslash B, \gamma \rangle$, $\gamma \in \mathcal{W}_i^*$, $\gamma \notin w_i(A \backslash B)$, and there are no witnesses of $\gamma \notin w_i(A \backslash B)$ in $\langle \mathcal{W}_i^*, w_i \rangle$, then we take $\langle \mathcal{W}_{i+1}^*, w_{i+1} \rangle$ to be any conservative extension of $\langle \mathcal{W}_i^*, w_i \rangle$ in $\mathcal{K}^m$, containing a witness of $\gamma \notin w_{i+1}(A \backslash B)$. Such a $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}_{i+1}^*, w_{i+1} \rangle$ exists, since $\mathcal{K}^m$ is witnessed.

CASE 2:
If $\sigma(i) = \langle B/A, \gamma \rangle$, $\gamma \in \mathcal{W}_i^*$, $\gamma \notin w_i(B/A)$, and there are no witnesses of $\gamma \notin w_i(B/A)$ in $\langle \mathcal{W}_i^*, w_i \rangle$, then we take $\langle \mathcal{W}_{i+1}^*, w_{i+1} \rangle$ to be any conservative extension of $\langle \mathcal{W}_i^*, w_i \rangle$ in $\mathcal{K}^m$, containing a witness of $\gamma \notin w_{i+1}(B/A)$.

CASE 3:
Otherwise we put $\langle \mathcal{W}_{i+1}^*, w_{i+1} \rangle \rightleftharpoons \langle \mathcal{W}_i^*, w_i \rangle$.

Let $\langle \mathcal{W}_\infty^*, w_\infty \rangle$ be the limit of the conservative sequence $\langle \mathcal{W}_i^*, w_i \rangle$. We put $\mathcal{V} \rightleftharpoons \mathcal{W}_\infty$.

Now we define a valuation $v: \mathrm{Tp} \to \mathbf{P}(\mathcal{W}_\infty^*)$ by induction on the complexity of a type.

$$
\begin{aligned}
v(p_i) &\rightleftharpoons w_\infty(p_i) \\
v(A \bullet B) &\rightleftharpoons v(A) \circ v(B) \\
v(A \backslash B) &\rightleftharpoons \{\gamma \in \mathcal{V}^* \mid v(A) \circ \gamma \subseteq v(B)\} \\
v(B/A) &\rightleftharpoons \{\gamma \in \mathcal{V}^* \mid \gamma \circ v(A) \subseteq v(B)\}
\end{aligned}
$$

Evidently $\langle \mathcal{V}^*, v \rangle$ is a free monoid model. Next we verify by induction on the complexity of C that $w_\infty(C) = v(C)$ for every $C \in \mathrm{Tp}(m)$.

Induction step.

CASE 1: $C = A \bullet B$
Obvious from Lemma 3.1 (i).

CASE 2: $C = A\backslash B$
First we prove that if $\gamma \in w_\infty(A\backslash B)$ then $\gamma \in v(A\backslash B)$. Let $\gamma \in w_\infty(A\backslash B)$. Take any $\alpha \in v(A)$. By the induction hypothesis $\alpha \in w_\infty(A)$. Evidently $\alpha \circ \gamma \in \vec{w_\infty}(A(A\backslash B))$. Hence $\alpha \circ \gamma \in w_\infty(B)$ in view of $L^* \vdash A(A\backslash B) \to B$. By the induction hypothesis $\alpha \circ \gamma \in v(B)$. Thus $\gamma \in v(A\backslash B)$.

Now we prove that if $\gamma \in \mathcal{U}^*$ and $\gamma \notin w_\infty(A\backslash B)$ then $\gamma \notin v(A\backslash B)$. If $\gamma \notin \mathcal{W}_\infty^*$, then this is obvious. Let now $\gamma \in \mathcal{W}_\infty^*$. Recall that $\mathcal{W}_\infty = \bigcup_{j \in \mathbf{N}} \mathcal{W}_j$. Thus $\gamma \in \mathcal{W}_j^*$ for some j. Evidently, there exists an integer $i \geq j$ such that $\sigma(i) = \langle A\backslash B, \gamma \rangle$. According to the construction of $\langle \mathcal{W}_{i+1}^*, w_{i+1} \rangle$ there is a witness $\alpha \in \mathcal{W}_{i+1}^*$ of $\gamma \notin w_\infty(A\backslash B)$. That is, $\alpha \in w_{i+1}(A)$ and $\alpha \circ \gamma \notin w_{i+1}(B)$. Since w_∞ is conservative over w_{i+1}, we have $\alpha \in w_\infty(A)$ and $\alpha \circ \gamma \notin w_\infty(B)$. By the induction hypothesis, $\alpha \in v(A)$ and $\alpha \circ \gamma \notin v(B)$. Thus $\gamma \notin v(A\backslash B)$.
CASE 3: $C = B/A$
Similar to the previous case.

Finally, we prove that the free monoid model $\langle \mathcal{V}^*, v \rangle$ has the desired properties (i)–(iii).

(i) Let $E \in \mathrm{Tp}(m)$ and $L^* \nvdash \Lambda \to E$. We must prove that $\varepsilon \notin v(E)$. According to the definition of a $\mathrm{Tp}(m)$-quasimodel $\varepsilon \notin w_0(E)$. In view of Lemma 3.4 $\varepsilon \notin w_\infty(E) = v(E)$.

(ii) If $E \in \mathrm{Tp}(m)$, then $a_0^m \in w_0(E) \subseteq w_\infty(E) = v(E)$.

(iii) Obvious. ■

6 Noncommutative linear logic

In this paper we consider only the multiplicative fragment of linear logic.

Noncommutative multiplicative linear formulas are defined as follows. We assume that an enumerable set of *variables* $\mathrm{Var} = \{p_1, p_2, \ldots\}$ is given. We introduce the set of formal symbols called *atoms*

$$\mathrm{At} \rightleftharpoons \{p^{\perp n} \mid p \in \mathrm{Var},\ n \in \mathbf{Z}\}.$$

Intuitively, if $n \geq 0$, then $p^{\perp n}$ means 'p with n right negations' and $p^{\perp(-n)}$ means 'p with n left negations'.

Definition. The set of *normal formulas* (or just *formulas* for shortness) is defined to be the smallest set NFm satisfying the following conditions:

1. $\mathrm{At} \subset \mathrm{NFm}$;
2. $\perp \in \mathrm{NFm}$;
3. $\mathbf{1} \in \mathrm{NFm}$;
4. if $A \in \mathrm{NFm}$ and $B \in \mathrm{NFm}$, then $(A \otimes B) \in \mathrm{NFm}$ and $(A \wp B) \in \mathrm{NFm}$.

Here $\otimes$ is the multiplicative conjunction, called 'tensor', and $\wp$ is the multiplicative disjunction, called 'par'. The constants $\bot$ and $\mathbf{1}$ are multiplicative falsity and multiplicative truth respectively.

By NFm* we denote the set of all finite sequences of normal formulas. The empty sequence is denoted by Λ.

Definition. We define by induction the right negation $(\ )^{\bot} : \mathrm{NFm} \to \mathrm{NFm}$ and the left negation ${}^{\bot}(\ ) : \mathrm{NFm} \to \mathrm{NFm}$.

$$\begin{array}{rcl}
(p^{\bot n})^{\bot} & \rightleftharpoons & p^{\bot(n+1)} \\
\bot^{\bot} & \rightleftharpoons & \mathbf{1} \\
\mathbf{1}^{\bot} & \rightleftharpoons & \bot \\
(A \otimes B)^{\bot} & \rightleftharpoons & (B^{\bot}) \wp (A^{\bot}) \\
(A \wp B)^{\bot} & \rightleftharpoons & (B^{\bot}) \otimes (A^{\bot})
\end{array}$$

$$\begin{array}{rcl}
{}^{\bot}(p^{\bot n}) & \rightleftharpoons & p^{\bot(n-1)} \\
{}^{\bot}\bot & \rightleftharpoons & \mathbf{1} \\
{}^{\bot}\mathbf{1} & \rightleftharpoons & \bot \\
{}^{\bot}(A \otimes B) & \rightleftharpoons & ({}^{\bot}B) \wp ({}^{\bot}A) \\
{}^{\bot}(A \wp B) & \rightleftharpoons & ({}^{\bot}B) \otimes ({}^{\bot}A)
\end{array}$$

The two negations are extended to sequences of normal formulas as follows.

$$\begin{array}{rcl}
{}^{\bot}(A_1 \dots A_n) & \rightleftharpoons & {}^{\bot}A_n \dots {}^{\bot}A_1 \\
(A_1 \dots A_n)^{\bot} & \rightleftharpoons & A_n^{\bot} \dots A_1^{\bot}
\end{array}$$

Remark. Several other connectives can be defined in this logic.

The most popular ones are two linear implications, defined as

$$A \multimap B \rightleftharpoons A^{\bot} \wp B \text{ and } B \circ\!\!- A \rightleftharpoons B \wp {}^{\bot}A.$$

Lemma 6.1 *For any $A \in \mathrm{NFm}$ the equalities ${}^{\bot}(A^{\bot}) = A$ and $({}^{\bot}A)^{\bot} = A$ hold true.*

PROOF. Easy induction on the structure of A. ■

In [1] V. M. Abrusci introduced a sequent calculus PNCL for the pure noncommutative classical linear propositional logic. In the same paper two one-sided sequent calculi SPNCL and SPNCL′ were introduced and it was proved that they are equivalent to PNCL.

We shall use a slightly modified (but equivalent) version of the multiplicative fragment of SPNCL′. The sequents of this calculus are of the form $\to\Gamma$, where $\Gamma \in \mathrm{NFm}^*$.

The calculus SPNCL′ has the following axioms and rules.

$$\frac{}{\to(p^{\perp(n+1)})(p^{\perp n})}\ (id) \qquad\qquad \frac{}{\to\mathbf{1}}\ (\mathbf{1})$$

$$\frac{\to\Gamma\Delta}{\to\Gamma\perp\Delta}\ (\perp)$$

$$\frac{\to\Gamma AB\Delta}{\to\Gamma(A\wp B)\Delta}\ (\wp) \qquad\qquad \frac{\to\Gamma A \quad \to B\Delta}{\to\Gamma(A\otimes B)\Delta}\ (\otimes)$$

$$\frac{\to A\Gamma}{\to\Gamma({}^{\perp\perp}A)}\ ({}^{\perp\perp}(\,\cdot\,)) \qquad\qquad \frac{\to\Gamma A}{\to(A^{\perp\perp})\Gamma}\ ((\,\cdot\,)^{\perp\perp})$$

Here capital letters A, B, ... stand for formulas, capital Greek letters denote finite (possibly empty) sequences of formulas, p ranges over Var, and n ranges over $\mathbf{Z}$.

Remark. The rule (id) can be written as $\overline{\to(C^{\perp})C}$ (or equivalently as $\overline{\to C\,({}^{\perp}C)}$), where $C \in$ At. Actually, the restriction $C \in$ At is not essential. It is imposed in this paper only in order to reduce the number of technical details in some proofs.

We define an embedding of L^* into SPNCL′.

Definition. The function $\widehat{(\;)}$: Tp $\to$ NFm is defined as follows.

$$\begin{array}{rcl} \widehat{p_i} & \rightleftharpoons & p_i \\ \widehat{A\bullet B} & \rightleftharpoons & \widehat{A}\otimes\widehat{B} \\ \widehat{A\backslash B} & \rightleftharpoons & \widehat{A}^{\perp}\wp\widehat{B} \\ \widehat{A/B} & \rightleftharpoons & \widehat{A}\wp{}^{\perp}\widehat{B} \end{array}$$

If $\Gamma = A_1 \ldots A_n$, then by $\widehat{\Gamma}$ we denote the sequence $\widehat{A_1}\ldots\widehat{A_n}$.

Lemma 6.2 *For every normal formula $A \in$ NFm there is at most one type $B \in$ Tp such that $\widehat{B} = A$.*

PROOF. We define a function $\natural$: NFm $\to \mathbf{Z}$ by induction as follows.

$$\begin{array}{rcl} \natural(p^{\perp n}) & \rightleftharpoons & \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases} \\ \natural\mathbf{1} & \rightleftharpoons & 0 \\ \natural\perp & \rightleftharpoons & 1 \\ \natural(A\otimes B) & \rightleftharpoons & \natural A + \natural B \\ \natural(A\wp B) & \rightleftharpoons & \natural A + \natural B - 1 \end{array}$$

It is easy to see that $\natural(A^{\perp}) = \natural({}^{\perp}A) = 1 - \natural A$. Now we can verify that if $A \in \mathrm{Tp}$, then $\natural\widehat{A} = 0$. Thus there are no types $A_1, A_2 \in \mathrm{Tp}$ such that $\widehat{A_1}^{\perp} = \widehat{A_2}$ (because $\widehat{A_1}^{\perp} = 1$ and $\widehat{A_2} = 0$).

Given a formula $D = \widehat{C}$ (where $C \in \mathrm{Tp}$), we can automatically decide what is the main connective in C. If $D \in \mathrm{Var}$, then C is primitive. If the main connective of D is $\otimes$, then the main connective of C is $\bullet$. Finally, if $D = D_1 \wp D_2$, then the main connective of C is $\backslash$ or $/$, depending on whether $\natural D_1 = 1$ or $\natural D_1 = 0$. ■

Lemma 6.3 *Let $\Gamma \in \mathrm{Tp}^*$ and $A \in \mathrm{Tp}$. The sequent $\Gamma \to A$ is derivable in L^* if and only if the sequent $\to \widehat{\Gamma}^{\perp}\,\widehat{A}$ is derivable in* SPNCL$'$.

PROOF. Both directions are proved using induction on derivation length. ■

7 Proof nets

We define proof nets for the multiplicative fragment of the noncommutative classical linear propositional logic. The concept of proof net introduced here (an extension of that from [1]) appears to be mathematical folklore.

We prove that a sequent is derivable if and only if there exists a proof net for this sequent.

Definition. For the purposes of this paper it is convenient to measure the length of a normal formula using the function $|\!|\!|\cdot|\!|\!| : \mathrm{NFm} \to \mathbf{N}$ defined in the following way.

$$\begin{aligned}
|\!|\!|p^{\perp n}|\!|\!| &\rightleftharpoons 2\\
|\!|\!|\perp|\!|\!| &\rightleftharpoons 2\\
|\!|\!|\mathbf{1}|\!|\!| &\rightleftharpoons 2\\
|\!|\!|A \otimes B|\!|\!| &\rightleftharpoons |\!|\!|A|\!|\!| + |\!|\!|B|\!|\!|\\
|\!|\!|A \wp B|\!|\!| &\rightleftharpoons |\!|\!|A|\!|\!| + |\!|\!|B|\!|\!|
\end{aligned}$$

Remark. We are going to define formally a total order on the set of all $\mathbf{1}$, $\perp$, $\otimes$, $\wp$ and atom occurrences in a formula (in fact this order coincides with the natural order from left to right). To make the forthcoming definition easier, we have used 2 instead of 1 in the base case in the definition of $|\!|\!|\cdot|\!|\!|$.

The definition of $|\!|\!|\cdot|\!|\!|$ is extended to finite sequences of formulas in the natural way.

$$|\!|\!|A_1 \ldots A_n|\!|\!| \rightleftharpoons |\!|\!|A_1|\!|\!| + \ldots + |\!|\!|A_n|\!|\!|$$

We put $|\!|\!|\Lambda|\!|\!| \rightleftharpoons 0$.

The number of formulas in a finite sequence Γ is denoted by $|\Gamma|$. Thus $|A_1 \ldots A_n| = n$.

To formalize the notion of *occurrences* of subformulas we introduce the set

$$\mathrm{Occ} \rightleftharpoons \mathrm{NFm} \times \mathbf{Z}.$$

A pair $\langle B, k\rangle \in \mathrm{Occ}$ will be intuitively interpreted as a subformula occurrence B. Here k in a way characterizes the position of B in the whole formula.

Definition. We define the function $c\colon \mathrm{NFm} \to \mathbf{N}$ (evaluating the "distance" of the "main connective" of a formula from its left end) formally as follows.

$$\begin{array}{rcl} c(p^{\perp n}) & \rightleftharpoons & 1 \\ c(\perp) & \rightleftharpoons & 1 \\ c(\mathbf{1}) & \rightleftharpoons & 1 \\ c(A \otimes B) & \rightleftharpoons & \|\!|A|\!\| \\ c(A \wp B) & \rightleftharpoons & \|\!|A|\!\| \end{array}$$

Definition. We define the binary relation 'α is a subformula of β' on the set Occ formally as the least transitive binary relation $\prec$ satisfying $\langle A, k - \|\!|A|\!\| + c(A)\rangle \prec \langle (A\lambda B), k\rangle$ and $\langle B, k + c(B)\rangle \prec \langle (A\lambda B), k\rangle$ for every $\lambda \in \{\otimes, \wp\}$, $A \in \mathrm{NFm}$, $B \in \mathrm{NFm}$, and $k \in \mathbf{Z}$.

Definition. The binary relation $\preceq$ on the set Occ is defined in the natural way: $\alpha \preceq \beta$ if and only if $\alpha \prec \beta$ or $\alpha = \beta$.

Given a standalone formula $A \in \mathrm{NFm}$, we usually associate it with the pair $\langle A, c(A)\rangle \in \mathrm{Occ}$. Then each subformula occurrence B is associated with a pair $\langle B, k\rangle \in \mathrm{Occ}$ such that $\langle B, k\rangle \preceq \langle A, c(A)\rangle$ and k is (intuitively) the "$\|\!|\cdot|\!\|$-distance" of the "main connective" of B from the left end of A.

Lemma 7.1 *Let* $A \in \mathrm{NFm}$. *Then*

(i) *the set* $\{\alpha \in \mathrm{Occ} \mid \alpha \preceq \langle A, c(A)\rangle\}$ *contains* $\|\!|A|\!\| - 1$ *elements;*

(ii) *for every* $k \in \mathbf{Z}$ *such that* $0 < k < \|\!|A|\!\|$, *there is a unique formula* $B \in \mathrm{NFm}$ *satisfying* $\langle B, k\rangle \preceq \langle A, c(A)\rangle\}$.

Definition. For any sequence of normal formulas $\Gamma = A_1 \ldots A_n$ we construct a finite set

$$\Omega_\Gamma \subset (\mathrm{NFm} \cup \{\diamond\}) \times \mathbf{N},$$

where $\diamond$ is a new formal symbol which does not belong to NFm.

The set Ω_Γ will act as the domain of all proof structures for the sequent $\to\Gamma$.

$$\begin{array}{rcl} \Omega_\Gamma & \rightleftharpoons & \{\langle B, k + \|\!|A_1 \ldots A_{i-1}|\!\|\rangle \mid 1 \le i \le n \ \text{ and } \ \langle B, k\rangle \preceq \langle A_i, c(A_i)\rangle\} \\ & & \cup\{\langle \diamond, \|\!|A_1 \ldots A_{i-1}|\!\|, \rangle \mid 1 \le i \le n\} \end{array}$$

Example 7.2 Let $\Gamma = ((q^{\perp 3} \otimes p^{\perp 8}) \wp p^{\perp 7})\, q^{\perp 2}$. Then $\Omega_\Gamma = \{\alpha_0, \ldots, \alpha_7\}$, where

$$
\begin{array}{rcl}
\alpha_0 & \rightleftharpoons & \langle \diamond, 0 \rangle, \\
\alpha_1 & \rightleftharpoons & \langle q^{\perp 3}, 1 \rangle, \\
\alpha_2 & \rightleftharpoons & \langle (q^{\perp 3} \otimes p^{\perp 8}), 2 \rangle, \\
\alpha_3 & \rightleftharpoons & \langle p^{\perp 8}, 3 \rangle, \\
\alpha_4 & \rightleftharpoons & \langle ((q^{\perp 3} \otimes p^{\perp 8}) \wp p^{\perp 7}), 4 \rangle, \\
\alpha_5 & \rightleftharpoons & \langle p^{\perp 7}, 5 \rangle, \\
\alpha_6 & \rightleftharpoons & \langle \diamond, 6 \rangle, \\
\alpha_7 & \rightleftharpoons & \langle q^{\perp 2}, 7 \rangle.
\end{array}
$$

The set Ω_Γ can be considered as consisting of six disjoint parts

$$\Omega_\Gamma = \Omega_\Gamma^\diamond \cup \Omega_\Gamma^{\mathrm{At}} \cup \Omega_\Gamma^\perp \cup \Omega_\Gamma^{\mathbf{1}} \cup \Omega_\Gamma^\otimes \cup \Omega_\Gamma^\wp,$$

where

$$
\begin{array}{rcl}
\Omega_\Gamma^\diamond & \rightleftharpoons & \{\langle \diamond, k, \Pi \rangle \in \Omega_\Gamma\}; \\
\Omega_\Gamma^{\mathrm{At}} & \rightleftharpoons & \{\langle p^{\perp n}, k, \Pi \rangle \in \Omega_\Gamma\}; \\
\Omega_\Gamma^\perp & \rightleftharpoons & \{\langle \perp, k, \Pi \rangle \in \Omega_\Gamma\}; \\
\Omega_\Gamma^{\mathbf{1}} & \rightleftharpoons & \{\langle \mathbf{1}, k, \Pi \rangle \in \Omega_\Gamma\}; \\
\Omega_\Gamma^\otimes & \rightleftharpoons & \{\langle A \otimes B, k, \Pi \rangle \in \Omega_\Gamma\}; \\
\Omega_\Gamma^\wp & \rightleftharpoons & \{\langle A \wp B, k, \Pi \rangle \in \Omega_\Gamma\}.
\end{array}
$$

We shall often write $\Omega_\Gamma^{\wp\diamond}$ for $\Omega_\Gamma^\diamond \cup \Omega_\Gamma^\wp$.

Lemma 7.3 $|\Omega_\Gamma| = |\!|\!|\Gamma|\!|\!|$.

Definition. The invariant $\flat$, associating an integer with Ω_Γ, is defined as

$$\flat(\Omega_\Gamma) \rightleftharpoons |\Omega_\Gamma^{\wp\diamond}| - |\Omega_\Gamma^\otimes| - |\Omega_\Gamma^\perp| + |\Omega_\Gamma^{\mathbf{1}}|.$$

Definition. For every subset Θ of Ω_Γ we put

$$\flat(\Theta) \rightleftharpoons |\Omega_\Gamma^{\wp\diamond} \cap \Theta| - |\Omega_\Gamma^\otimes \cap \Theta| - |\Omega_\Gamma^\perp \cap \Theta| + |\Omega_\Gamma^{\mathbf{1}} \cap \Theta|.$$

Remark. $\flat(\Omega_{C^\perp}) = \flat(\Omega_{{}^\perp C}) = 2 - \flat(\Omega_C)$.

Lemma 7.4 *For all* Γ

(i) $|\Omega_\Gamma^{\mathrm{At}}| + |\Omega_\Gamma^\perp| + |\Omega_\Gamma^{\mathbf{1}}| = |\Omega_\Gamma^\otimes| + |\Omega_\Gamma^\wp| + |\Omega_\Gamma^\diamond|$;

(ii) *if* $\rightarrow\Gamma$ *is derivable in* SPNCL$'$, *then* $\flat(\Omega_\Gamma) = 2$.

PROOF. (i) Easy induction on $|\!|\!|\Gamma|\!|\!|$.
(ii) Straightforward induction on the length of the derivation in SPNCL$'$.
■

For each sequent $\rightarrow\Gamma$ we define two binary relations on Ω_Γ.

Definition. Let $\alpha \in \Omega_\Gamma$ and $\beta \in \Omega_\Gamma$. Then $\alpha \prec_\Gamma \beta$ if and only if $\alpha \notin \Omega_\Gamma^\circ$, $\beta \notin \Omega_\Gamma^\circ$, and $\alpha \prec \beta$.

Remark. The relation $\prec_\Gamma$ is a strict partial order on Ω_Γ.

Definition. Let $\langle A, k, \Delta\rangle \in \Omega_\Gamma$ and $\langle B, l, \Pi\rangle \in \Omega_\Gamma$.
Then $\langle A, k, \Delta\rangle <_\Gamma \langle B, l, \Pi\rangle$ if and only if $k < l$.

Remark. The relation $<_\Gamma$ is an irreflexive linear order on Ω_Γ.

Definition. For any sequent $\rightarrow\Gamma$ we denote by $\mathbf{\Omega}_\Gamma$ the triple $\langle \Omega_\Gamma, \prec_\Gamma, <_\Gamma\rangle$.

Lemma 7.5 *Let $\alpha, \beta, \gamma \in \Omega_\Gamma$ and $\alpha <_\Gamma \beta <_\Gamma \gamma$.*

(i) *If $\alpha \prec_\Gamma \gamma$, then $\beta \prec_\Gamma \gamma$.*

(ii) *If $\gamma \prec_\Gamma \alpha$, then $\beta \prec_\Gamma \alpha$.*

Definition. Let $\langle \Omega, \mathcal{C}\rangle$ be an undirected graph, where Ω is the set of vertices and $\mathcal{C}$ is the set of edges. Let $<$ be a strict linear order on the set Ω. We say that the graph $\langle \Omega, \mathcal{C}\rangle$ is *$<$-planar* iff for every edge $\{\alpha, \beta\} \in \mathcal{C}$ and every edge $\{\gamma, \delta\} \in \mathcal{C}$, if $\alpha < \gamma < \beta$, then $\alpha < \delta < \beta$ or $\delta = \alpha$ or $\delta = \beta$.

Remark. Intuitively, a graph is $<$-planar if and only if its edges can be drawn without intersections on a semiplane while the vertices of the graph are ordered according to $<$ on the border of the semiplane.

Lemma 7.6 *If $\langle \Omega, \mathcal{C}_1\rangle$ is $<$-planar and $\mathcal{C}_2 \subseteq \mathcal{C}_1$, then $\langle \Omega, \mathcal{C}_2\rangle$ is $<$-planar.*

Lemma 7.7 *Let $\langle \Omega, \mathcal{C}\rangle$ be an undirected graph, where $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Let $<$ and $<'$ be two linear orders on Ω such that*

$$\begin{array}{l}
(\forall\alpha \in \Omega_1)(\forall\beta \in \Omega_2)\ \alpha < \beta;\\
(\forall\alpha \in \Omega_1)(\forall\beta \in \Omega_2)\ \beta <' \alpha;\\
(\forall\alpha \in \Omega_1)(\forall\beta \in \Omega_1)\ \alpha < \beta \text{ iff } \alpha <' \beta;\\
(\forall\alpha \in \Omega_2)(\forall\beta \in \Omega_2)\ \alpha < \beta \text{ iff } \alpha <' \beta.
\end{array}$$

Then $\langle \Omega, \mathcal{C}\rangle$ is $<$-planar if and only if $\langle \Omega, \mathcal{C}\rangle$ is $<'$-planar.

Definition. If $\mathcal{C}$ is a set of directed edges, then by $\mathcal{C}^\#$ we denote the associated set of undirected edges.

$$\mathcal{C}^\# \rightleftharpoons \{\{\alpha, \beta\} \mid \langle \alpha, \beta\rangle \in \mathcal{C}\}$$

Definition. A *proof structure* is a quadruple $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$, where

(A1) $\mathcal{A} \subseteq \Omega_\Gamma^\otimes \times \Omega_\Gamma^{\wp\circ}$;

(A2) $\mathcal{B} \subseteq \Omega_\Gamma^\perp \times (\Omega_\Gamma^{\mathrm{At}} \cup \Omega_\Gamma^{\mathbf{1}})$;

(A3) $\mathcal{E} \subseteq \Omega_\Gamma^{\mathrm{At}} \times \Omega_\Gamma^{\mathrm{At}}$;

(A4) the relations $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$ are total functions on domains $\Omega_\Gamma^\otimes$, $\Omega_\Gamma^\perp$, and $\Omega_\Gamma^{\mathrm{At}}$ respectively.

(A5) if $\langle \alpha, \beta \rangle \in \mathcal{E}$, then $\langle \beta, \alpha \rangle \in \mathcal{E}$;

(A6) if $\langle \alpha, \beta \rangle \in \mathcal{E}$ and $\alpha <_\Gamma \beta$, then there are $p \in \mathrm{Var}$ and $n \in \mathbf{Z}$ such that $\alpha = p^{\perp(n+1)}$ and $\beta = p^{\perp n}$;

(A7) the graph $\langle \Omega_\Gamma, (\mathcal{A} \cup \mathcal{B} \cup \mathcal{E})^\# \rangle$ is $<_\Gamma$-planar.

If $\alpha \in \Omega_\Gamma^\otimes$, then we denote by $\mathcal{A}\alpha$ the only element $\beta \in \Omega_\Gamma$ such that $\langle \alpha, \beta \rangle \in \mathcal{A}$. Similarly for $\mathcal{B}$ and $\mathcal{E}$.

Definition. A *proof net* is a proof structure $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E} \rangle$ such that

(A8) $\flat(\Omega_\Gamma) = 2$;

(A9) the graph $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A} \rangle$ is acyclic (i. e., the transitive closure of $\prec_\Gamma \cup \mathcal{A}$ is irreflexive).

Example 7.8 We continue Example 7.2, where

$$\Gamma = ((q^{\perp 3} \otimes p^{\perp 8}) \wp\, p^{\perp 7}) q^{\perp 2}.$$

Let $\mathcal{A} = \{\langle \alpha_2, \alpha_6 \rangle\}$, $\mathcal{B} = \emptyset$, and $\mathcal{E} = \{\langle \alpha_1, \alpha_7 \rangle, \langle \alpha_3, \alpha_5 \rangle, \langle \alpha_5, \alpha_3 \rangle, \langle \alpha_7, \alpha_1 \rangle\}$. Then $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E} \rangle$ is a proof net.

Remark. In the definition of a proof structure one may in addition require that, if $\langle \alpha, \beta \rangle \in \mathcal{B}$ and $\langle \beta, \gamma \rangle \in \mathcal{E}$, then $\beta <_\Gamma \gamma$.

Before establishing that a sequent is derivable if and only if it has a proof net we prove some auxiliary lemmas.

Definition. Let $\Gamma \in \mathrm{NFm}^*$, $\alpha, \beta \in \Omega_\Gamma$, and $\alpha <_\Gamma \beta$. Then by $\Theta_\Gamma^{\alpha,\beta}$ we denote the set $\{\gamma \in \Omega_\Gamma \mid \alpha <_\Gamma \gamma <_\Gamma \beta\}$ and by $\Xi_\Gamma^{\alpha,\beta}$ we denote the set $\{\gamma <_\Gamma \alpha \text{ or } \beta <_\Gamma \gamma\}$.

Lemma 7.9 *Let $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E} \rangle$ be a proof structure, $\{\alpha, \beta\} \in \mathcal{A}^\#$, and $\alpha <_\Gamma \beta$. Then $\flat(\Theta_\Gamma^{\alpha,\beta}) \geq 1$ and $\flat(\Xi_\Gamma^{\alpha,\beta}) \geq 1$.*

PROOF. For shortness we denote $\Theta \rightleftharpoons \Theta_\Gamma^{\alpha,\beta}$ and $\Xi \rightleftharpoons \Xi_\Gamma^{\alpha,\beta}$. We shall verify only $\flat(\Theta) \geq 1$. The proof of $\flat(\Xi) \geq 1$ is analogous.

According to Lemma 7.6 the graph $\langle \Omega_\Gamma, \mathcal{A}^\# \rangle$ is $<_\Gamma$-planar. Thus the set $\mathcal{A}$ is divided into three disjoint subsets

$$\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}^\Theta \cup \mathcal{A}^\Xi,$$

where $\mathcal{A}_0^{\#} = \{\{\alpha, \beta\}\}$, $\mathcal{A}^{\Theta} \subseteq \Theta \times (\Theta \cup \{\alpha, \beta\})$ and $\mathcal{A}^{\Xi} \subseteq \Xi \times (\Xi \cup \{\alpha, \beta\})$.

Similarly, the graph $\langle \Omega_\Gamma, \{\{\alpha, \beta\}\} \cup \mathcal{B}^{\#} \rangle$ is $<_\Gamma$-planar and thus $\mathcal{B}$ is divided into two disjoint subsets

$$\mathcal{B} = \mathcal{B}^{\Theta} \cup \mathcal{B}^{\Xi},$$

where $\mathcal{B}^{\Theta} \subseteq \Theta \times \Theta$ and $\mathcal{B}^{\Xi} \subseteq \Xi \times \Xi$.

Once again, the graph $\langle \Omega_\Gamma, \{\{\alpha, \beta\}\} \cup \mathcal{E}^{\#} \rangle$ is $<_\Gamma$-planar and thus $\mathcal{E}$ is divided into two disjoint subsets

$$\mathcal{E} = \mathcal{E}^{\Theta} \cup \mathcal{E}^{\Xi},$$

where $\mathcal{E}^{\Theta} \subseteq \Theta \times \Theta$ and $\mathcal{E}^{\Xi} \subseteq \Xi \times \Xi$.

Note that $\langle \Theta \cup \{\alpha, \beta\}, (\mathcal{A}_0 \cup \mathcal{A}^{\Theta} \cup \mathcal{B}^{\Theta} \cup \mathcal{E}^{\Theta})^{\#} \rangle$ is an undirected graph. Furthermore, this graph is $<_\Theta$-planar, where $<_\Theta$ is the restriction of $<_\Gamma$ on the set $\Theta \cup \{\alpha, \beta\}$.

Let us draw this $<_\Theta$-planar graph on a semiplane as described after the definition of a $<$-planar graph. We denote the segment of the semiplane border between α and β by $[\alpha, \beta]$. The border segment $[\alpha, \beta]$ and the edge $\{\alpha, \beta\}$ surround a closed area, which contains all edges from the set $(\mathcal{A}^{\Theta} \cup \mathcal{B}^{\Theta} \cup \mathcal{E}^{\Theta})^{\#}$. The edges from $(\mathcal{B}^{\Theta} \cup \mathcal{E}^{\Theta})^{\#}$ divide this area into $|(\mathcal{B}^{\Theta} \cup \mathcal{E}^{\Theta})^{\#}| + 1$ regions. We are interested in all these regions except the one adjacent to the edge $\{\alpha, \beta\}$.

Consider any of these regions. We claim that it has at least one nontrivial segment of $[\alpha, \beta]$ at its border. (Otherwise every vertex from $\Omega_\Gamma^{\mathrm{At}} \cup \Omega_\Gamma^{\mathbf{1}} \cup \Omega_\Gamma^{\perp}$ adjacent to the region considered would belong to two edges from $(\mathcal{B}^{\Theta} \cup \mathcal{E}^{\Theta})^{\#}$, but this is impossible.)

Any such segment of $[\alpha, \beta]$ contains at least one element of $\Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp\diamond}$. Thus some elements of $\Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp\diamond}$ are adjacent to the region considered. It is impossible that all of these would belong to $\Omega_\Gamma^{\otimes}$ (because $\mathcal{A}$ is a total function).

Thus the number of regions considered does not exceed the cardinality of the set $\Omega_\Gamma^{\wp\diamond} \cap \Theta$.

$$|\Omega_\Gamma^{\wp\diamond} \cap \Theta| \geq |(\mathcal{B}^{\Theta} \cup \mathcal{E}^{\Theta})^{\#}|$$

Taking into account that

$$|(\mathcal{B}^{\Theta})^{\#}| = |\mathcal{B}^{\Theta}| = |\Omega_\Gamma^{\perp} \cap \Theta|$$

and

$$|(\mathcal{E}^{\Theta})^{\#}| = \frac{1}{2}|\mathcal{E}^{\Theta}| = \frac{1}{2}|\Omega_\Gamma^{\mathrm{At}} \cap \Theta|$$

we obtain that

$$|\Omega_\Gamma^{\wp\diamond} \cap \Theta| \geq |\Omega_\Gamma^{\perp} \cap \Theta| + \frac{1}{2}|\Omega_\Gamma^{\mathrm{At}} \cap \Theta|.$$

Analogously to Lemma 7.4 (i) we notice that

$$|(\Omega_\Gamma^{\wp\diamond} \cup \Omega_\Gamma^{\otimes}) \cap \Theta| = |(\Omega_\Gamma^{\perp} \cup \Omega_\Gamma^{\mathbf{1}} \cup \Omega_\Gamma^{\mathrm{At}}) \cap \Theta| - 1.$$

Subtracting

$$|\Omega_\Gamma^{\wp\diamond} \cap \Theta| + |\Omega_\Gamma^{\otimes} \cap \Theta| = |\Omega_\Gamma^{\perp} \cap \Theta| + |\Omega_\Gamma^{1} \cap \Theta| + |\Omega_\Gamma^{\mathrm{At}} \cap \Theta| - 1$$

from

$$2|\Omega_\Gamma^{\wp\diamond} \cap \Theta| \geq 2|\Omega_\Gamma^{\perp} \cap \Theta| + |\Omega_\Gamma^{\mathrm{At}} \cap \Theta|$$

we obtain the desired inequality

$$|\Omega_\Gamma^{\wp\diamond} \cap \Theta| - |\Omega_\Gamma^{\otimes} \cap \Theta| \geq |\Omega_\Gamma^{\perp} \cap \Theta| - |\Omega_\Gamma^{1} \cap \Theta| + 1.$$

■

Lemma 7.10 *Let* $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ *be a proof net,* $\{\alpha, \beta\} \in \mathcal{A}^{\#}$, *and* $\alpha <_\Gamma \beta$. *Then* $\flat(\Theta_\Gamma^{\alpha,\beta}) = 1$ *and* $\flat(\Xi_\Gamma^{\alpha,\beta}) = 1$.

PROOF. Note that $2 = \flat(\Omega_\Gamma) = \flat(\Theta_\Gamma^{\alpha,\beta}) + \flat(\Xi_\Gamma^{\alpha,\beta}) + \flat(\alpha, \beta) = \flat(\Theta_\Gamma^{\alpha,\beta}) + \flat(\Xi_\Gamma^{\alpha,\beta})$, since $\flat(\alpha, \beta) = 0$. It remains to use the previous lemma. ■

Proposition 7.11 *Let* $\langle \mathbf{\Omega}_{\Gamma\Delta(A\otimes B)\Pi}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ *be a proof net and* $\mathcal{A}\langle A\otimes B, \|\Gamma\| + \|\Delta\| + \|A\|\rangle = \langle \diamond, \|\Gamma\|\rangle$. *Then the sequents* $\rightarrow\Delta A$ *and* $\rightarrow\Gamma B\Pi$ *are derivable in* SPNCL$'$.

PROOF. Proof structures for $\rightarrow\Delta A$ and $\rightarrow\Gamma B\Pi$ are easily constructed from the given proof net. To verify that they are proof nets we use Lemma 7.10. ■

Theorem 7.12 *A sequent* $\rightarrow\Gamma$ *is derivable in* SPNCL$'$ *if and only if there exists a proof net* $\langle \mathbf{\Omega}_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ *for the sequent* $\rightarrow\Gamma$.

PROOF. Sketch. Proving the 'only if' part is easy. To prove the 'if' part we proceed by induction on the cardinality of the set $\Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp}$.

Induction base. Let $\Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp} = \emptyset$. From $\flat(\Omega_\Gamma) = 2$ we conclude that either $\Gamma = \perp \ldots \perp \mathbf{1} \perp \ldots \perp$ or $\Gamma = \perp \ldots \perp q^{\perp k} \perp \ldots \perp p^{\perp n} \perp \ldots \perp$. In the latter case $q = p$ and $k = n + 1$ in view of (A6). Evidently all sequents $\rightarrow\perp \ldots \perp \mathbf{1} \perp \ldots \perp$ and $\rightarrow\perp \ldots \perp p^{\perp n+1} \perp \ldots \perp p^{\perp n} \perp \ldots \perp$ are derivable in SPNCL$'$.

Induction step. Assume now that $\Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp}$ is not empty. We introduce on $\Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp}$ a binary relation $\ll$ stipulating that $\alpha \ll \beta$ if and only if $\alpha \prec_\Gamma \beta$ or $\langle \alpha, \beta\rangle \in \mathcal{A}$. In other words, $\ll$ is the restriction of $\prec_\Gamma \cup \mathcal{A}$ on the set $\Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp}$.

According to (A9) there is an element $\delta_0 \in \Omega_\Gamma^{\otimes} \cup \Omega_\Gamma^{\wp}$ maximal with respect to $\ll$. We consider two cases.

CASE 1: $\delta_0 \in \Omega_\Gamma^{\wp}$

We can use the induction hypothesis and apply the rule $(\wp)$.

CASE 2: $\delta_0 \in \Omega_\Gamma^{\otimes}$

In view of $\mathcal{A}$ being a function there exists $\beta \in \Omega_\Gamma^{\wp\diamond}$ such that $\langle \delta_0, \beta\rangle \in \mathcal{A}$. Since δ_0 is maximal with respect to $\ll$, we have $\beta \in \Omega_\Gamma^{\diamond}$.

We consider two subcases.
CASE 2a: $\beta = \langle \diamond, 0\rangle$ (i.e., β is the least element of Ω_Γ w. r. t. $<_\Gamma$)
In view of Proposition 7.11 we can use the induction hypothesis and apply the rule $(\otimes)$.
CASE 2b: $\beta \neq \langle \diamond, 0\rangle$
We use Lemma 7.7 and the rules $({}^{\perp\perp}(\,\cdot\,))$, $((\,\cdot\,)^{\perp\perp})$ to reduce this case to the previous one. ■

Remark. Analogous result can be easily established also for the multiplicative fragment of cyclic linear logic defined in [12].

8 Properties of proof nets

Lemma 8.1 *Let $\langle \Omega_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ be a proof structure. If the graph $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A}\rangle$ contains a cycle, then there exists a cycle*

$$(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n)$$

such that

(i) *$\alpha_i \in \Omega_\Gamma^{\wp}$ and $\beta_i \in \Omega_\Gamma^{\otimes}$ for each $i \leq n$;*

(ii) *$\alpha_i \prec_\Gamma \beta_i$ for each $i \leq n$;*

(iii) *$\langle \beta_i, \alpha_{i+1}\rangle \in \mathcal{A}$ for each $i < n$;*

(iv) *$\langle \beta_n, \alpha_1\rangle \in \mathcal{A}$;*

(v) *either $\alpha_1 <_\Gamma \beta_1 <_\Gamma \alpha_2 <_\Gamma \cdots <_\Gamma \beta_n$ or $\beta_n <_\Gamma \alpha_n <_\Gamma \beta_{n-1} <_\Gamma \cdots <_\Gamma \alpha_1$.*

Definition. Let $g\colon \Omega_1 \to \Omega_2$ be a bijection and $\mathcal{R}$ be a binary relation on Ω_1. Then by $\mathcal{R}^g$ we denote the binary relation $\{\langle g(\alpha), g(\beta)\rangle \mid \langle \alpha, \beta\rangle \in \mathcal{R}\}$ on Ω_2.

Proposition 8.2 *Let $\langle \Omega_\Gamma, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ be a proof net. Let Γ' be obtained from Γ by replacing an occurrence of a subformula $(A\otimes(B\otimes C))$ by $((A\otimes B)\otimes C)$ or vice versa. Let g denote the unique isomorphism of $\langle \Omega_\Gamma, <_\Gamma\rangle$ and $\langle \Omega_{\Gamma'}, <_{\Gamma'}\rangle$. Then $\langle \Omega_{\Gamma'}, \mathcal{A}^g, \mathcal{B}^g, \mathcal{E}^g\rangle$ is a proof net.*

PROOF. Sketch. Let Γ' be obtained from Γ by replacing an occurrence of a subformula $(A \otimes (B \otimes C))$ by $((A \otimes B) \otimes C)$. Assume that the graph $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A}\rangle$ is acyclic, whereas the graph $\langle \Omega_{\Gamma'}, <_{\Gamma'} \cup \mathcal{A}^g\rangle$ is not. Applying Lemma 8.1 we find in $\langle \Omega_{\Gamma'}, <_{\Gamma'} \cup \mathcal{A}^g\rangle$ a cycle of special form $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_n, \beta_n)$.

Evidently there is $m \leq n$ such that $\beta_m = \langle (A \otimes B) \otimes C, k + \|A\| + \|B\|\rangle$ and $\alpha_m \preceq_{\Gamma'} \langle A, k + c(A)\rangle$ for some k.

We denote $\gamma \rightleftharpoons \langle A \otimes B, k + \|A\| \rangle$. In view of (A7) and the special form of the cycle there must be $l \leq n$ such that $\alpha_l <_{\Gamma'} \mathcal{A}^g\gamma <_{\Gamma'} \beta_l$ or $\alpha_l = \mathcal{A}^g\gamma$. But then there is another cycle in $\langle \Omega_{\Gamma'}, <_{\Gamma'} \cup \mathcal{A}^g \rangle$ containing the edge $\langle \gamma, \mathcal{A}^g\gamma$ and not involving the vertex β_m (see Lemma 7.5). This cycle is mapped by g^{-1} to a cycle in $\langle \Omega_\Gamma, \prec_\Gamma \cup \mathcal{A} \rangle$. Contradiction. ■

Proposition 8.3 *Let* $\langle \mathbf{\Omega}_{\Gamma(A\otimes(B\otimes C))\Pi}, \mathcal{A}, \mathcal{B}, \mathcal{E} \rangle$ *be a proof net and*

$$\mathcal{A}\langle A \otimes (B \otimes C), \|\Gamma\| + \|A\| \rangle = \mathcal{A}\langle B \otimes C, \|\Gamma\| + \|A\| + \|B\| \rangle.$$

Then the sequent $\rightarrow B$ *is derivable in* SPNCL$'$.

PROOF. Sketch. The proof structure for $\rightarrow B$ is copied from the relevant part of the given proof net. To prove (A8) we apply Lemma 7.10 twice. ■

Proposition 8.4 *Let* $\langle \mathbf{\Omega}_{\Gamma\Delta_1(A_1\otimes B_1)\Pi_1}, \mathcal{A}_1, \mathcal{B}_1, \mathcal{E}_1 \rangle$ *be a proof net and* $\langle \mathbf{\Omega}_{\Gamma\Delta_2(A_2\otimes B_2)\Pi_2}, \mathcal{A}_2, \mathcal{B}_2, \mathcal{E}_2 \rangle$ *be another proof net. If* $\mathcal{A}_1\langle A_1 \otimes B_1, \|\Gamma\| + \|\Delta_1\| + \|A_1\| \rangle = \mathcal{A}_2\langle A_2 \otimes B_2, \|\Gamma\| + \|\Delta_2\| + \|A_2\| \rangle \in \Omega_\Gamma \cup \{\langle \diamond, \|\Gamma\| \rangle\}$, *then the sequents* $\rightarrow\Gamma\Delta_1(A_1 \otimes B_2)\Pi_2$ *and* $\rightarrow\Gamma\Delta_2(A_2 \otimes B_1)\Pi_1$ *are derivable in* SPNCL$'$.

PROOF. Sketch. Let $\beta_1 \rightleftharpoons \langle A_1 \otimes B_1, \|\Gamma\| + \|\Delta_1\| + \|A_1\| \rangle$, $\beta_2 \rightleftharpoons \langle A_2 \otimes B_2, \|\Gamma\| + \|\Delta_2\| + \|A_2\| \rangle$, and $\alpha \rightleftharpoons \mathcal{A}_1\beta_1 = \mathcal{A}_2\beta_2$.

To obtain a proof structure for $\rightarrow\Gamma\Delta_1(A_1 \otimes B_2)\Pi_2$ we combine the parts of the given proof nets corresponding to $\Theta^{\alpha\beta_1}_{\Gamma\Delta_1(A_1\otimes B_1)\Pi_1}$ and $\Xi^{\alpha\beta_2}_{\Gamma\Delta_2(A_2\otimes B_2)\Pi_2}$.

Using Lemma 8.1 one can verify that the proof structure is a proof net.

The claim $\rightarrow\Gamma\Delta_2(A_2 \otimes B_1)\Pi_1$ follows from the other one due to the symmetry of the conditions of the theorem. ■

Proposition 8.5 *Let* SPNCL$' \vdash \rightarrow B$ *and* $\alpha \in \Omega^{\wp\diamond}_\Gamma$. *Then there are* $C, D \in$ NFm *and there is a proof net*

$$\langle \mathbf{\Omega}_{\Gamma C(\perp\otimes(B\otimes\perp))D}, \mathcal{A}, \mathcal{B}, \mathcal{E} \rangle$$

such that $\mathcal{A}\langle \perp \otimes (B \otimes \perp), \|\Gamma\| + \|C\| + \|\perp\| \rangle = \mathcal{A}\langle B \otimes \perp, \|\Gamma\| + \|C\| + \|\perp\| + \|B\| \rangle = \alpha$.

PROOF. Sketch. Given a sequence $\Gamma \in$ NFm* and a vertex $\alpha \in \Omega^{\wp\diamond}_\Gamma$ it is easy to construct two formulas C, D and a proof net $\langle \mathbf{\Omega}_{\Gamma C\otimes D}, \mathcal{A}_0, \mathcal{B}_0, \mathcal{E}_0 \rangle$ such that $\mathcal{A}_0\langle C \otimes D, \|\Gamma\| + \|C\| \rangle = \alpha$ and each edge $\langle \langle E, k \rangle, \langle F, m \rangle \rangle \in \mathcal{A}_0 \cup \mathcal{B}_0 \cup \mathcal{E}_0$ satisfies $k + m = 2\|\Gamma\|$.

On the other hand, there is a proof net for the sequent $\rightarrow B$. It remains to combine these two proof nets. Again, Lemma 8.1 is useful for checking (A9). ■

Proposition 8.6 *Let* $\langle \mathbf{\Omega}_{\Gamma(B\otimes(C\otimes D))\Pi}, \mathcal{A}_1, \mathcal{B}_1, \mathcal{E}_1 \rangle$ *be a proof net. Let the sequent* $\rightarrow C^\perp\, E$ *be derivable in* SPNCL$'$. *Then there exists a proof net*

$$\langle \mathbf{\Omega}_{\Gamma(B\otimes(E\otimes D))\Pi}, \mathcal{A}, \mathcal{B}, \mathcal{E} \rangle$$

such that $\mathcal{A}\langle B\otimes(E\otimes D), |||\Gamma||| + |||B|||\rangle = \mathcal{A}_1\langle B\otimes(C\otimes D), |||\Gamma||| + |||B|||\rangle$ *and* $\mathcal{A}\langle E\otimes D, |||\Gamma||| + |||B||| + |||E|||\rangle = \mathcal{A}_1\langle C\otimes D, |||\Gamma||| + |||B||| + |||C|||\rangle$.

PROOF. Sketch. According to Theorem t-complete there is a proof net

$$\langle \mathbf{\Omega}_{C^\perp\, E}, \mathcal{A}_2, \mathcal{B}_2, \mathcal{E}_2\rangle.$$

There is a natural one-to-one mapping (an anti-isomorphism of linear orders) between the part of $\Omega_{\Gamma(B\otimes(C\otimes D))\Pi}$ corresponding to C and the part of $\Omega_{C^\perp\, E}$ corresponding to $C^\perp$. We denote the graph of this mapping by $\mathcal{G}$ and the graph of its inverse by $\mathcal{G}^{-1}$.

We define $\mathcal{H}$ as the transitive closure of $\mathcal{A}_1\cup\mathcal{B}_1\cup\mathcal{E}_1\cup\mathcal{A}_2\cup\mathcal{B}_2\cup\mathcal{E}_2\cup\mathcal{G}\cup\mathcal{G}^{-1}$ on the disjoint union of $\Omega_{\Gamma(B\otimes(C\otimes D))\Pi}$ and $\Omega_{C^\perp\, E}$.

Finally, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{E}$ are chosen so that $\mathcal{A}\cup\mathcal{B}\cup\mathcal{E}$ coincides with the restriction of $\mathcal{H}$ to the domain excluding C and $C^\perp$. ∎

9 Tp(m)-maps

The aim of this section is to introduce Tp(m)-maps $\langle \mathbf{V}_n, v_n\rangle$ and $\langle \mathbf{V}'_n, v'_n\rangle$, which will later be used in the proof of Lemma 10.20, where we construct a Tp(m)-quasimodel containing a witness for a given word $\delta\notin v(E\backslash F)$ (resp. $\delta\notin v(F/E)$).

We need some notation. If $\mathcal{R}$ and $\mathcal{T}$ are two binary relations on a set $\mathbf{D}$, then we define

$$\mathcal{R}\odot\mathcal{T} \rightleftharpoons \{\langle r,t\rangle\in\mathbf{D}\times\mathbf{D} \mid (\exists s\in\mathbf{D})\ \langle r,s\rangle\in\mathcal{R} \text{ and } \langle s,t\rangle\in\mathcal{T}\}.$$

Evidently, $\odot$ is associative.

Given a set $\mathbf{D}$ and a function $w\colon \mathrm{Tp}\to\mathbf{P}(\mathbf{D}\times\mathbf{D})$ we denote by $\vec{w}$ the function from Tp^* to $\mathbf{P}(\mathbf{D}\times\mathbf{D})$ defined as follows:

$$\begin{aligned}\vec{w}(\Lambda) &\rightleftharpoons \{\langle s,s\rangle \mid s\in\mathbf{D}\};\\ \vec{w}(\Gamma A) &\rightleftharpoons \vec{w}(\Gamma)\odot w(A).\end{aligned}$$

Remark. $\vec{w}(\Gamma\Pi) = \vec{w}(\Gamma)\odot\vec{w}(\Pi)$.

Definition. A Tp(m)-*map* $\langle\mathbf{D},\mathbf{W},w\rangle$ consists of a finite set $\mathbf{D}$, a reflexive linear order $\mathbf{W}\subseteq\mathbf{D}\times\mathbf{D}$, and a valuation $w\colon\mathrm{Tp}\to\mathbf{P}(\mathbf{W})$ such that

(1) for any $A\in\mathrm{Tp}$, $B\in\mathrm{Tp}$, if $A{\bullet}B\in\mathrm{Tp}(m)$, then $w(A{\bullet}B)\subseteq w(A)\odot w(B)$;

(2) for any $\Gamma\in\mathrm{Tp}(m)^*$, $B\in\mathrm{Tp}(m)$, if $L^*\vdash\Gamma\to B$, then $\vec{w}(\Gamma)\subseteq w(B)$;

(3) for any $B\in\mathrm{Tp}$, if $\langle s,s\rangle\in w(B)$ for some $s\in\mathbf{D}$, then $L^*\vdash\Lambda\to B$.

Lemma 9.1 *For any given number* $m\in\mathbf{N}$ *there is a family of* Tp(m)*-maps* $\langle\mathbf{D}_\Gamma,\mathbf{V}_\Gamma,v_\Gamma\rangle$ *indexed by sequences of types* $\Gamma\in\mathrm{Tp}^*$ *and there are elements* $\chi_\Gamma\in\mathbf{D}_\Gamma$ *such that*

(i) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall\Pi \in \mathrm{Tp}^*)\ \mathbf{D}_\Gamma \subseteq \mathbf{D}_{\Pi\Gamma}$;

(ii) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall\Pi \in \mathrm{Tp}^*)\ \mathbf{V}_\Gamma \subseteq \mathbf{V}_{\Pi\Gamma}$;

(iii) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall\Pi \in \mathrm{Tp}^*)\ (\forall\langle s,t\rangle \in \mathbf{V}_{\Pi\Gamma})$ *if* $s \in \mathbf{D}_\Gamma$ *then* $\langle s,t\rangle \in \mathbf{V}_\Gamma$;

(iv) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall\Pi \in \mathrm{Tp}^*)\ (\forall A \in \mathrm{Tp})\ v_\Gamma(A) \subseteq v_{\Pi\Gamma}(A)$;

(v) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall\Pi \in \mathrm{Tp}^*)\ (\forall A \in \mathrm{Tp})\ (\forall\langle s,t\rangle \in v_{\Pi\Gamma}(A))$ *if* $s \in \mathbf{D}_\Gamma$ *then*
$\langle s,t\rangle \in v_\Gamma(A)$;

(vi) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall\Delta \in \mathrm{Tp}^*)$ *if* $\chi_\Gamma \in \mathbf{D}_\Delta$ *then* $(\exists\Pi \in \mathrm{Tp}^*)\ \Delta = \Pi\Gamma$;

(vii) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall C \in \mathrm{Tp})\ \langle\chi_\Gamma, \chi_\Lambda\rangle \in v_\Gamma(C) \Leftrightarrow L^* \vdash \Gamma \to C$;

(viii) $(\forall\Gamma \in \mathrm{Tp}^*)\ (\forall B \in \mathrm{Tp})\ \langle\chi_{B\Gamma}, \chi_\Gamma\rangle \in v_{B\Gamma}(B)$.

PROOF. The construction of $\mathrm{Tp}(m)$-maps is based on the proof nets introduced in section 7. The domain $\mathbf{D}_\Gamma$ of the $\mathrm{Tp}(m)$-map corresponding to Γ will be a finite subset of $\mathrm{NFm} \times \mathrm{Occ} \times \mathrm{NFm}^*$.

Let $\widehat{\Gamma}^\perp = A_1 \dots A_n$. We define $\mathbf{D}_\Gamma^-$ as the set of 3-tuples $\langle B, k, A_1 \dots A_j\rangle$ such that $\langle B, k\rangle \in \Omega^{\wp\diamond}_{\widehat{\Gamma}^\perp}$ and $j \in \mathbf{N}$ is the smallest natural number satisfying $\|A_1 \dots A_j\| \geq k$.

We put $\chi_\Gamma \rightleftharpoons \langle\diamond, \|\widehat{\Gamma}^\perp\|, \widehat{\Gamma}^\perp\rangle$ and $\mathbf{D}_\Gamma \rightleftharpoons \mathbf{D}_\Gamma^- \cup \{\chi_\Gamma\}$.

For any 3-tuple $s = \langle B, k, \Phi\rangle$ we denote by $\widetilde{s}$ the 2-tuple $\langle B, k\rangle$. Evidently the mapping $s \mapsto \widetilde{s}$ establishes a one-to-one correspondence between $\mathbf{D}_\Gamma^-$ and $\Omega^{\wp\diamond}_{\widehat{\Gamma}^\perp}$.

The linear order $\mathbf{V}_\Gamma \subseteq \mathbf{D}_\Gamma \times \mathbf{D}_\Gamma$ is defined by stipulating that a pair $\langle\langle B_1, k_1, \Phi_1\rangle, \langle B_2, k_2, \Phi_2\rangle\rangle$ belongs to $\mathbf{V}_\Gamma$ if and only if $k_1 \geq k_2$.

Finally, the function $v_\Gamma\colon \mathrm{Tp} \to \mathbf{P}(\mathbf{V}_\Gamma)$ is defined by stating that $\langle s,t\rangle \in v_\Gamma(C)$ if and only if there are $E \in \mathrm{NFm}$, $F \in \mathrm{NFm}$, $\Delta \in \mathrm{NFm}^*$, $\Pi \in \mathrm{NFm}^*$, and there is a proof net $\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,\Delta(E\otimes(\widehat{C}\otimes F))\Pi}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ such that $\mathcal{A}\langle E\otimes(\widehat{C}\otimes F), \|\widehat{\Gamma}^\perp\| + \|\Delta\| + \|E\|\rangle = \widetilde{s}$ and $\mathcal{A}\langle \widehat{C}\otimes F, \|\widehat{\Gamma}^\perp\| + \|\Delta\| + \|E\| + \|\widehat{C}\|\rangle = \widetilde{t}$.

First we verify that for each $\Gamma \in \mathrm{Tp}^*$ the triple $\langle\mathbf{D}_\Gamma, \mathbf{V}_\Gamma, v_\Gamma\rangle$ is a $\mathrm{Tp}(m)$-map for every $m \in \mathbf{N}$.

(1) Let $\langle s,t\rangle \in v_\Gamma(A \bullet B)$.

This means that

- there is a proof net $\langle\mathbf{\Omega}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ for a derivable sequent of the form

$$\to \widehat{\Gamma}^\perp\, \Delta(E \otimes ((\widehat{A} \otimes \widehat{B}) \otimes F))\Pi;$$

- $\mathcal{A}\langle E \otimes ((\widehat{A} \otimes \widehat{B}) \otimes F), \|\widehat{\Gamma}^\perp\| + \|\Delta\| + \|E\|\rangle = \widetilde{s}$;
- $\mathcal{A}\langle(\widehat{A} \otimes \widehat{B}) \otimes F, \|\widehat{\Gamma}^\perp\| + \|\Delta\| + \|E\| + \|\widehat{A}\| + \|\widehat{B}\|\rangle = \widetilde{t}$.

Evidently there is $u \in \mathbf{D}_\Gamma$ such that $\mathcal{A}\langle \widehat{A}\otimes\widehat{B}, |||\widehat{\Gamma}^\perp|||+|||\Delta|||+|||E|||+|||\widehat{A}|||\rangle = \widetilde{u}$. Using Proposition 8.2 it is easy to establish that $\langle s,u\rangle \in v_\Gamma(A)$ and $\langle u,t\rangle \in v_\Gamma(B)$, whence $\langle s,t\rangle \in v_\Gamma(A) \odot v_\Gamma(B)$. Thus we have established that $v_\Gamma(A{\bullet}B) \subseteq v_\Gamma(A) \odot v_\Gamma(B)$.

(2) Let $L^* \vdash A_1 \dots A_n \to B$. We must verify that $v_\Gamma(A_1) \odot \dots \odot v_\Gamma(A_n) \subseteq v_\Gamma(B)$. If $n = 0$, then we use Proposition 8.5.

Assume now that $n > 0$. Let $\langle s,u\rangle \in v_\Gamma(A_1)$ and $\langle u,t\rangle \in v_\Gamma(A_2)$. According to the definition of v_Γ there are proof nets

$$\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,\Delta_1(E_1\otimes(\widehat{A_1}\otimes F_1))\Pi_1}, \mathcal{A}_1, \mathcal{B}_1, \mathcal{E}_1\rangle$$

and

$$\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,\Delta_2(E_2\otimes(\widehat{A_2}\otimes F_2))\Pi_2}, \mathcal{A}_2, \mathcal{B}_2, \mathcal{E}_2\rangle$$

such that $\mathcal{A}_1\langle E_1\otimes(\widehat{A_1}\otimes F_1), |||\widehat{\Gamma}^\perp|||+|||\Delta_1|||+|||E_1|||\rangle = \widetilde{s}$, $\mathcal{A}_1\langle \widehat{A_1}\otimes F_1, |||\widehat{\Gamma}^\perp|||+|||\Delta_1|||+|||E_1|||+|||\widehat{A_1}|||\rangle = \widetilde{u}$, $\mathcal{A}_2\langle E_2\otimes(\widehat{A_2}\otimes F_2), |||\widehat{\Gamma}^\perp|||+|||\Delta_2|||+|||E_2|||\rangle = \widetilde{u}$, $\mathcal{A}_2\langle \widehat{A_2}\otimes F_2, |||\widehat{\Gamma}^\perp|||+|||\Delta_2|||+|||E_2|||+|||\widehat{A_2}|||\rangle = \widetilde{t}$. From Proposition 8.2 and Proposition 8.4 we obtain a proof net

$$\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,\Delta_1(E_1\otimes((\widehat{A_1}\otimes\widehat{A_2})\otimes F_2))\Pi_2}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$$

such that

$$\mathcal{A}\langle E_1\otimes((\widehat{A_1}\otimes\widehat{A_2})\otimes F_2), |||\widehat{\Gamma}^\perp|||+|||\Delta_1|||+|||E_1|||\rangle = \widetilde{s}, \mathcal{A}\langle (\widehat{A_1}\otimes\widehat{A_2})\otimes F_2, |||\widehat{\Gamma}^\perp|||+|||\Delta_1|||+|||E_1|||+|||\widehat{A_1}|||+|||\widehat{A_2}|||\rangle = \widetilde{t}.$$

Thus $\langle s,t\rangle \in v_\Gamma(A_1{\bullet}A_2)$. We have established that $v_\Gamma(A_1) \odot v_\Gamma(A_2) \subseteq v_\Gamma(A_1{\bullet}A_2)$. By induction on n we obtain $\vec{v_\Gamma}(A_1 \dots A_n) \subseteq v_\Gamma(A_1{\bullet}\dots{\bullet}A_n)$.

It remains to apply Proposition 8.6.

(3) Let $\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,\Delta(E\otimes(\widehat{B}\otimes F))\Pi}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ be a proof net such that

$$\mathcal{A}\langle E\otimes(\widehat{B}\otimes F), |||\widehat{\Gamma}^\perp|||+|||\Delta|||+|||E|||\rangle = \mathcal{A}\langle \widehat{B}\otimes F, |||\widehat{\Gamma}^\perp|||+|||\Delta|||+|||E|||+|||\widehat{B}|||\rangle.$$

According to Proposition 8.3 the sequent $\to\widehat{B}$ is derivable in SPNCL$'$.

Now we verify that the elements χ_Γ and Tp(m)-maps $\langle \mathbf{V}_\Gamma, v_\Gamma\rangle$ satisfy (i)–(viii).

(i)

Evident from $\widehat{(\Pi\Gamma)}^\perp = \widehat{\Gamma}^\perp\,\widehat{\Pi}^\perp$.

(ii)

Similar.

(iii)

Obvious from the fact that if $\langle B_1, k_1, \Phi_1\rangle \in \mathbf{D}_\Gamma$, $\langle B_2, k_2, \Phi_2\rangle \in \mathbf{D}_\Gamma$, and $k_1 \geq k_2$, then there is $\Pi \in \mathrm{NFm}^*$ such that $\Phi_1 = \Pi\Phi_2$.

(iv)

We verify that $v_\Gamma(A) \subseteq v_{\Pi\Gamma}(A)$. Let $\langle s,t\rangle \in v_\Gamma(A)$. This means that there is a proof net $\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,\Delta(E\otimes(\widehat{A}\otimes F))\Pi}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ such that $\mathcal{A}\langle E\otimes(\widehat{A}\otimes F), |||\widehat{\Gamma}^\perp||| + |||\Delta||| + |||E|||\rangle = \widetilde{s}$ and $\mathcal{A}\langle \widehat{A}\otimes F, |||\widehat{\Gamma}^\perp||| + |||\Delta||| + |||E||| + |||\widehat{A}|||\rangle = \widetilde{t}$. Let $\Pi = B_1 \ldots B_n$. Then one can easily construct another proof net $\langle \mathbf{\Omega}', \mathcal{A}', \mathcal{B}', \mathcal{E}'\rangle$ for the sequent

$$\to\widehat{\Gamma}^\perp\,\widehat{B_n}^\perp\ldots\widehat{B_1}^\perp\,(\widehat{B_1}\otimes\ldots\otimes\widehat{B_n}\otimes\perp)\Delta(E\otimes(\widehat{A}\otimes F))$$

such that $\mathcal{A}'\langle E\otimes(\widehat{A}\otimes F), |||\widehat{\Gamma}^\perp||| + |||\widehat{\Pi}^\perp||| + |||\widehat{\Pi}||| + |||\perp||| + |||\Delta||| + |||E|||\rangle = \widetilde{s}$ and $\mathcal{A}'\langle \widehat{A}\otimes F, |||\widehat{\Gamma}^\perp||| + |||\widehat{\Pi}^\perp||| + |||\widehat{\Pi}||| + |||\perp||| + |||\Delta||| + |||E||| + |||\widehat{A}|||\rangle = \widetilde{t}$. Thus $\langle s, t\rangle \in v_{\Pi\Gamma}(A)$.

(v)
Similar to (iii).

(vi)
Follows from the definition of $\mathbf{D}_\Gamma$.

(vii)
Let $L^* \vdash \Gamma\to C$. According to Lemma 6.3 and Theorem 7.12 there is a proof net for the sequent $\widehat{\Gamma}^\perp\,\widehat{C}$. By an easy modification we obtain a proof net $\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,(\mathbf{1}\otimes(\widehat{C}\otimes\mathbf{1}))}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ such that $\mathcal{A}\langle \mathbf{1}\otimes(\widehat{C}\otimes\mathbf{1}), |||\widehat{\Gamma}^\perp||| + |||\mathbf{1}|||\rangle = \chi_\Gamma$ and $\mathcal{A}\langle \widehat{C}\otimes\mathbf{1}, |||\widehat{\Gamma}^\perp||| + |||\mathbf{1}||| + |||\widehat{C}|||\rangle = \chi_\Lambda$.

For the converse assume that $\langle \mathbf{\Omega}_{\widehat{\Gamma}^\perp\,\Delta(E\otimes(\widehat{C}\otimes F))\Pi}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$ is a proof net such that $\mathcal{A}\langle E\otimes(\widehat{C}\otimes F), |||\widehat{\Gamma}^\perp||| + |||\Delta||| + |||E|||\rangle = \chi_\Gamma$ and $\mathcal{A}\langle \widehat{C}\otimes F, |||\widehat{\Gamma}^\perp||| + |||\Delta||| + |||E||| + |||\widehat{C}|||\rangle = \chi_\Lambda$. Using Proposition 7.11 twice we can obtain a proof net for $\to\widehat{\Gamma}^\perp\,\widehat{C}$.

(viii)
Let $\Gamma = A_1 \ldots A_n$. It is easy to construct a proof net

$$\langle \mathbf{\Omega}_{\widehat{A_n}^\perp\ldots\widehat{A_1}^\perp\,\widehat{B}^\perp\,(\mathbf{1}\otimes(\widehat{B}\otimes(\widehat{A_1}\otimes\ldots\otimes\widehat{A_n})))}, \mathcal{A}, \mathcal{B}, \mathcal{E}\rangle$$

such that $\mathcal{A}\langle \mathbf{1}\otimes(\widehat{B}\otimes(\widehat{A_1}\otimes\ldots\otimes\widehat{A_n})), |||\widehat{\Gamma}^\perp||| + |||\widehat{B}^\perp||| + |||\mathbf{1}|||\rangle = \chi_{BA_1\ldots A_n}$ and $\mathcal{A}\langle \widehat{B}\otimes(\widehat{A_1}\otimes\ldots\otimes\widehat{A_n}), |||\widehat{\Gamma}^\perp||| + |||\widehat{B}^\perp||| + |||\mathbf{1}||| + |||\widehat{B}|||\rangle = \chi_{A_1\ldots A_n}$. ■

Definition. For any two integers m and n, we write $\mathrm{LST}_{m,n}$ for the following finite subset of $\mathrm{Tp}(m)^*$.

$$\mathrm{LST}_{m,n} \rightleftharpoons \{A_1 \ldots A_l \mid 1 \le l \le n,\ A_1 \in \mathrm{Tp}(m),\ \ldots,\ A_l \in \mathrm{Tp}(m)\}$$

Lemma 9.2 *For any given number $m \in \mathbf{N}$ there is a family of $\mathrm{Tp}(m)$-maps $\langle \mathbf{D}_n, \mathbf{V}_n, v_n\rangle$ indexed by $n \in \mathbf{N}$, there is an element g, and there is a family of elements h_Γ indexed by $\Gamma \in \mathrm{Tp}(m)^*$, such that*

(i) $(\forall n)\ g \in \mathbf{D}_n$;

(ii) $(\forall n)\ (\forall\Gamma \in \mathrm{LST}_{m,n})\ h_\Gamma \in \mathbf{D}_n$;

(iii) $(\forall n)\ (\forall\Gamma \in \mathrm{LST}_{m,n})\ (\forall C \in \mathrm{Tp}(m))\ \langle h_\Gamma, g\rangle \in v_n(C) \Leftrightarrow L^* \vdash \Gamma\to C$;

(iv) $(\forall n)\ (\forall \Gamma \in \mathrm{LST}_{m,n-1})\ (\forall B \in \mathrm{Tp}(m))\ \langle h_{B\Gamma}, h_\Gamma\rangle \in v_n(B)$.

PROOF. Take arbitrary $m, n \in \mathbf{N}$. We construct the $\mathrm{Tp}(m)$-map $\langle \mathbf{D}_n, \mathbf{V}_n, v_n\rangle$, using the $\mathrm{Tp}(m)$-maps $\langle \mathbf{D}_\Gamma, \mathbf{V}_\Gamma, v_\Gamma\rangle$ from the previous lemma.

We put $\mathbf{D}_n \rightleftharpoons \bigcup\limits_{\Gamma\in\mathrm{LST}_{m,n}} \mathbf{D}_\Gamma$. Let $\mathbf{V}_n$ be any linear order containing the binary relation $\bigcup\limits_{\Gamma\in\mathrm{LST}_{m,n}} \mathbf{V}_\Gamma$. The valuation v_n is defined by $v_n(C) \rightleftharpoons \bigcup\limits_{\Gamma\in\mathrm{LST}_{m,n}} v_\Gamma(C)$.

We put $g \rightleftharpoons \chi_\Lambda$ and $h_\Gamma \rightleftharpoons \chi_\Gamma$.

It remains to check that $\langle \mathbf{D}_n, \mathbf{V}_n, v_n\rangle$ is a $\mathrm{Tp}(m)$-map.

(1) Obvious.

(2) Let $L^* \vdash A_1 \dots A_k \to B$, $A_i \in \mathrm{Tp}(m)$, and $B \in \mathrm{Tp}(m)$. Assume that $\langle s_0, s_1\rangle \in v_n(A_1), \dots, \langle s_{k-1}, s_k\rangle \in v_n(A_k)$. Then there is $\Delta \in \mathrm{LST}_{m,n}$ such that $\langle s_0, s_1\rangle \in v_\Delta(A_1)$.

By induction on $i < k$ it can be proved that $\langle s_i, s_{i+1}\rangle \in v_\Delta(A_{i+1})$ for the same Δ. Thus $\langle s_0, s_k\rangle \in v_\Delta(B) \subseteq v_n(B)$.

(3) Obvious. ■

We shall also need the dual of Lemma 9.2.

Lemma 9.3 *For any given number $m \in \mathbf{N}$ there is a family of $\mathrm{Tp}(m)$-maps $\langle \mathbf{D}'_n, \mathbf{V}'_n, v'_n\rangle$ indexed by $n \in \mathbf{N}$, there is an element g', and there is a family of elements h'_Γ indexed by $\Gamma \in \mathrm{Tp}(m)^*$, such that*

(i) $(\forall n)\ g' \in \mathbf{D}'_n$;

(ii) $(\forall n)\ (\forall \Gamma \in \mathrm{LST}_{m,n})\ h'_\Gamma \in \mathbf{D}'_n$;

(iii) $(\forall n)\ (\forall \Gamma \in \mathrm{LST}_{m,n})\ (\forall C \in \mathrm{Tp}(m))\ \langle g', h'_\Gamma\rangle \in v'_n(C) \Leftrightarrow L^* \vdash \Gamma \to C$;

(iv) $(\forall n)\ (\forall \Gamma \in \mathrm{LST}_{m,n-1})\ (\forall B \in \mathrm{Tp}(m))\ \langle h'_\Gamma, h'_{\Gamma B}\rangle \in v'_n(B)$.

10 Construction of witnesses

In this section we prove that the class $\mathcal{K}^m$ is witnessed.

We assume being given a number $m \in \mathbf{N}$, a $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{V}^*, v\rangle \in \mathcal{K}^m$, two types E and F such that $E\backslash F \in \mathrm{Tp}(m)$, and a word $\delta \in \mathcal{V}^*$ such that $\delta \notin v(E\backslash F)$. We fix m, $\mathcal{V}$, v, E, F, and δ until the end of this section. Our aim is to find a $\mathrm{Tp}(m)$-quasimodel $\langle \mathcal{W}^*, w\rangle \in \mathcal{K}^m$ and a word $\zeta \in \mathcal{W}^*$ such that $\zeta \in w(E)$, $\zeta \circ \delta \notin w(F)$, and $\langle \mathcal{W}^*, w\rangle$ is a conservative extension of $\langle \mathcal{V}^*, v\rangle$.

First, we put $n \rightleftharpoons |\delta| + 1$. For the given m and n we take the $\mathrm{Tp}(m)$-map $\langle \mathbf{D}'_n, \mathbf{V}'_n, v'_n\rangle$ from Lemma 9.3. Let $k \rightleftharpoons |\mathbf{D}'_n|$. The reflexive linear order $\mathbf{V}'_n$ is isomorphic to the natural order $\leq$ of the set $[0, k-1] \rightleftharpoons \{i \in \mathbf{Z} \mid 0 \leq i \leq k-1\}$.

Throughout this section we shall identify $\mathbf{D}_n$ with $[0, k-1]$ and the linear order $\mathbf{V}'_n$ with $\leq$.

Let x, z, $y_1, y_2, \ldots y_k$ be any $k+2$ distinct elements of $\mathcal{U} = \{a_j \mid j \in \mathbf{N}\}$, which do not occur in $\mathcal{V}$. We denote $\mathcal{Y} \rightleftharpoons \{x, z, y_1, y_2, \ldots, y_k\}$ and put $\mathcal{W} \rightleftharpoons \mathcal{V} \cup \mathcal{Y}$.

We shall work with subwords of

$$(x^m \circ y_1 \circ z^m) \circ (x^m \circ y_2 \circ z^m) \circ \ldots \circ (x^m \circ y_k \circ z^m).$$

Here $x^m \rightleftharpoons \underbrace{x \circ \ldots \circ x}_{m \text{ times}}$.

To define the mapping w we need several auxiliary words and sets. For any integers s and t such that $0 \leq s \leq t \leq k$ we define the word $\pi\langle s, t\rangle \in \mathcal{Y}^*$ as follows:

$$\pi\langle s, t\rangle \rightleftharpoons (x^m \circ y_{s+1} \circ z^m) \circ (x^m \circ y_{s+2} \circ z^m) \circ \ldots \circ (x^m \circ y_t \circ z^m).$$

By definition, $\pi\langle s, s\rangle = \varepsilon$ for every s.

Note that if $0 \leq r \leq s \leq t \leq k$, then $\pi\langle r, s\rangle \circ \pi\langle s, t\rangle = \pi\langle r, t\rangle$.

We shall denote by Subword(β) the set of all subwords of β. Formally,

$$\text{Subword}(\beta) \rightleftharpoons \{\alpha \in \mathcal{W}^* \mid \beta = \gamma_1 \circ \alpha \circ \gamma_2 \ \text{ for some } \ \gamma_1, \gamma_2 \in \mathcal{W}^*\}.$$

We define the finite set $\mathcal{R}$ as follows.

$$\mathcal{R} \ \rightleftharpoons \ \{\rho \in \mathcal{V}^* \mid \rho \circ \alpha = \delta \ \text{ for some } \ \alpha \in \mathcal{V}^*\}$$

We define several functions associating subsets of $\mathcal{W}^*$ with sequences of types from Tp(m). For any $\Theta \in \text{Tp}(m)^*$ we put

$$\begin{aligned}
u_0(\Theta) \ &\rightleftharpoons \ \{\pi\langle s, t\rangle \mid 0 \leq s < t < k \ \text{ and } \ \langle s, t\rangle \in \vec{v'_n}(\Theta)\}; \\
u_2(\Theta) \ &\rightleftharpoons \ \{\pi\langle s, k\rangle \circ \rho \mid 0 \leq s < k, \ \rho \in \mathcal{R}, \ \text{ and } \ \exists \Delta \in \text{Tp}(m)^*, \\
&\qquad\qquad |\Delta| \leq |\rho|, \ \rho \in \vec{v}(\Delta), \ \langle s, h'_{E\Delta}\rangle \in \vec{v'_n}(\Theta)\}; \\
u(\Theta) \ &\rightleftharpoons \ u_0(\Theta) \cup u_2(\Theta) \cup \vec{v}(\Theta).
\end{aligned}$$

We define some subsets of $\mathcal{W}^*$.

$$\begin{aligned}
\mathcal{M}_1 \ &\rightleftharpoons \ \{\alpha \in \mathcal{W}^* \mid \alpha \notin \mathcal{V}^* \ \text{ and } \ \alpha \notin \text{Subword}(\pi\langle 0, k\rangle \circ \delta)\} \\
\mathcal{M}_2 \ &\rightleftharpoons \ z \circ \mathcal{W}^* \\
\mathcal{M}_3 \ &\rightleftharpoons \ \mathcal{W}^* \circ x \\
\mathcal{M} \ &\rightleftharpoons \ \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3
\end{aligned}$$

We define a function $\text{Subst}_{\mathcal{M}}\colon \mathcal{W}^* \to \mathbf{P}(\mathcal{W}^*)$ and two valuations $\tilde{v}\colon \text{Tp}(m) \to \mathbf{P}(\mathcal{W}^*)$ and $w\colon \text{Tp}(m) \to \mathbf{P}(\mathcal{W}^*)$.

$$\begin{aligned}\mathrm{Subst}_{\mathcal{M}}(\varepsilon) &\rightleftharpoons \{\varepsilon\}\\ \mathrm{Subst}_{\mathcal{M}}(\alpha \circ q) &\rightleftharpoons \mathrm{Subst}_{\mathcal{M}}(\alpha) \circ (\{q\} \cup \mathcal{M}) \text{ if } \alpha \in \mathcal{W}^* \text{ and } q \in \mathcal{W}\end{aligned}$$

Informally, for every word $\beta \in \mathcal{W}^*$, the set $\mathrm{Subst}_{\mathcal{M}}(\beta)$ consists of all words that are obtained replacing some (may be none) of symbol occurrences in β by words from the set $\mathcal{M}$.

$$\begin{aligned}\tilde{v}(A) &\rightleftharpoons \bigcup_{\alpha \in v(A)} \mathrm{Subst}_{\mathcal{M}}(\alpha)\\ w(A) &\rightleftharpoons u(A) \cup \tilde{v}(A)\end{aligned}$$

Finally, we put $\zeta \rightleftharpoons \pi\langle g', k\rangle$.

Lemma 10.1 $\zeta \in u(E)$.

PROOF. From $L^* \vdash E \to E$ and Lemma 9.3 (iii) we obtain $\langle g', h'_E\rangle \in v'_n(E)$. Thus $\pi\langle g', k\rangle \in u_2(E)$. ■

Lemma 10.2 $\zeta \circ \delta \notin u(F)$.

PROOF. Assume, for the contrary, that $\pi\langle g', k\rangle \circ \delta \in u(F)$. Evidently $\pi\langle g', k\rangle \in \mathcal{Y}^+$ and $\delta \in \mathcal{V}^+$. Thus $\pi\langle g', k\rangle \circ \delta \in u_2(F)$. According to the definition of u_2 there is $\Delta \in \mathrm{Tp}(m)^*$ such that $|\Delta| \leq |\delta|$, $\delta \in \vec{v}(\Delta)$ and $\langle g', h'_{E\Delta}\rangle \in v'_n(F)$. From $n = |\delta| + 1$ we get $|\Delta| \leq n - 1$, whence $E\Delta \in \mathrm{LST}_{m,n}$. From Lemma 9.3 (iii) we obtain $L^* \vdash E\Delta \to F$. Applying the rule $(\to\backslash)$ we derive $L^* \vdash \Delta \to E\backslash F$, whence $\vec{v}(\Delta) \subseteq v(E\backslash F)$. Thus $\delta \in v(E\backslash F)$. Contradiction. ■

Lemma 10.3 *If* $0 \leq r \leq s < k$, $A \in \mathrm{Tp}(m)$, *and* $\langle r, s\rangle \in v'_n(A)$, *then* $\pi\langle r, s\rangle \in u(A)$.

PROOF. Let $\langle r, s\rangle \in v'_n(A)$. If $r < s$, then $\pi\langle r, s\rangle \in u_0(A)$. If $r = s$, then $L^* \vdash \Lambda \to A$ and thus $\varepsilon \in v(A)$. ■

Lemma 10.4 *If* $A{\bullet}B \in \mathrm{Tp}(m)$, *then* $u(A{\bullet}B) \subseteq u(A) \circ u(B)$.

PROOF. Let $A{\bullet}B \in \mathrm{Tp}(m)$ and $\gamma \in u(A{\bullet}B)$.
CASE 1: $\gamma \in u_0(A{\bullet}B)$
By definition $\gamma = \pi\langle r, t\rangle$, $0 \leq r < t < k$, $\langle r, t\rangle \in \vec{v'_n}(A{\bullet}B) = v'_n(A{\bullet}B)$ for some r and t.

Since $\langle \mathbf{V}'_n, v'_n\rangle$ is a $\mathrm{Tp}(m)$-map, there is $s \in [0, k-1]$ such that $\langle r, s\rangle \in v'_n(A)$ and $\langle s, t\rangle \in v'_n(B)$.

Now $\gamma = \pi\langle r, t\rangle = \pi\langle r, s\rangle \circ \pi\langle s, t\rangle \in u_0(A) \circ u_0(B)$ according to Lemma 10.3.

CASE 2: $\gamma \in u_2(A \bullet B)$
By definition $\gamma = \pi\langle r, k\rangle \circ \rho$, $\rho \in \mathcal{R}$, $\Delta \in \mathrm{Tp}(m)^*$, $|\Delta| \leq |\rho|$, $\rho \in \vec{v}(\Delta)$, $\langle r, h'_{E\Delta}\rangle \in v'_n(A \bullet B)$, $0 \leq r < k$.

Again, there is s such that $\langle r, s\rangle \in v'_n(A)$ and $\langle s, h'_{E\Delta}\rangle \in v'_n(B)$. According to Lemma 10.3, $\pi\langle r, s\rangle \in u_0(A)$. On the other hand $\pi\langle s, k\rangle \circ \rho \in u_2(B)$, whence $\gamma \in u_0(A) \circ u_2(B)$.
CASE 3: $\gamma \in v(A \bullet B)$
Obvious from $v(A \bullet B) \subseteq v(A) \circ v(B)$. ■

We define

$$\mathcal{T} \rightleftharpoons \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}}$$

and establish several properties of $\mathcal{M}$ and $\mathcal{T}$.

Lemma 10.5

(i) *If* $\beta \in \mathcal{W}^*$, $\alpha \in \mathrm{Subword}(\beta)$, *and* $\alpha \in \mathcal{M}_1$, *then* $\beta \in \mathcal{M}_1$.

(ii) $\mathcal{M}_2 \circ \mathcal{W}^* \subseteq \mathcal{M}_2$

(iii) $\mathcal{W}^* \circ \mathcal{M}_3 \subseteq \mathcal{M}_3$

PROOF. (i) According to the definition of $\mathcal{M}_1$, if $\beta \in \mathcal{W}^*$ and $\beta \notin \mathcal{M}_1$, then $\beta \in \mathcal{V}^*$ or $\beta \in \mathrm{Subword}(\pi\langle 0, k\rangle \circ \delta)$. But then one has also $\alpha \in \mathcal{V}^*$ or $\alpha \in \mathrm{Subword}(\pi\langle 0, k\rangle \circ \delta)$, whence $\alpha \notin \mathcal{M}_1$. ■

Lemma 10.6

(i) $\mathcal{M} \circ \mathcal{M} \subseteq \mathcal{M}$

(ii) $\mathcal{T} \subseteq \mathcal{M}$

PROOF.

(i)
Let $\alpha \in \mathcal{M}$ and $\beta \in \mathcal{M}$. We verify that $\alpha \circ \beta \in \mathcal{M}$. If $\alpha \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\alpha \in \mathcal{M}_2$ then $\alpha \circ \beta \in \mathcal{M}_2$. If $\beta \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\beta \in \mathcal{M}_3$ then $\alpha \circ \beta \in \mathcal{M}_3$.

The only complicated case is $\alpha \in \mathcal{M}_3$ and $\beta \in \mathcal{M}_2$, i.e., $\alpha = \alpha' \circ x$ and $\beta = z \circ \beta'$. Note that then $x \circ z \in \mathrm{Subword}(\alpha \circ \beta)$ and $x \circ z \in \mathcal{M}_1$. Thus $\alpha \circ \beta \in \mathcal{M}_1$.

(ii)
Follows from (i). ■

We introduce some subsets of $\mathcal{W}^*$.

$$\begin{aligned}
\mathcal{P}_0 &\rightleftharpoons \{\pi\langle s, t\rangle \mid 0 \leq s < t < k\} \\
\mathcal{P}_1 &\rightleftharpoons \{\pi\langle s, k\rangle \mid 0 \leq s < k\} \\
\mathcal{P}_2 &\rightleftharpoons \mathcal{P}_1 \circ \mathcal{R} \\
\mathcal{P} &\rightleftharpoons \mathcal{P}_0 \cup \mathcal{P}_2
\end{aligned}$$

Note that $\varepsilon \in \mathcal{R}$ and thus $\mathcal{P}_1 \subseteq \mathcal{P}_2$. Note that $u_0(\Theta) \subseteq \mathcal{P}_0$, $u_2(\Theta) \subseteq \mathcal{P}_2$, and $u(\Theta) \subseteq \mathcal{P} \cup \mathcal{V}^*$.

Lemma 10.7 *If $A \in \mathrm{Tp}(m)$, then $\mathcal{T} \subseteq w(A)$.*

PROOF. Since $\langle \mathcal{V}^*, v\rangle \in \mathcal{K}^m$, we can choose a word $\alpha \in v(A)$ such that $|\alpha| \leq m$. Evidently $\underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{|\alpha| \text{ times}} \subseteq \mathrm{Subst}_{\mathcal{M}}(\alpha) \subseteq \tilde{v}(A) \subseteq w(A)$. In view of $\mathcal{M} \circ \mathcal{M} \subseteq \mathcal{M}$ and taking into account that $|\alpha| \leq m$, we have $\underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}} \subseteq \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{|\alpha| \text{ times}}$. Thus $\mathcal{T} = \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}} \subseteq w(A)$. ■

Lemma 10.8

(i) $\mathcal{P} \cap \mathcal{V}^* = \emptyset$

(ii) $\mathcal{P} \cap \mathcal{M} = \emptyset$

(iii) $\mathcal{V}^* \cap \mathcal{M} = \emptyset$

PROOF.
(i)
Evident.
(ii)
Let $\alpha \in \mathcal{P}$. Then the leftmost symbol of α is x and the rightmost symbol of α belongs to $\mathcal{V} \cup \{z\}$. Thus $\alpha \notin \mathcal{M}_2$ and $\alpha \notin \mathcal{M}_3$. Note that $\mathcal{P} \subseteq \mathrm{Subword}(\pi\langle 0, k\rangle \circ \delta)$. Thus $\alpha \notin \mathcal{M}_1$.
(iii)
Obvious. ■

Lemma 10.9

(i) $v(A) \subseteq u(A)$

(ii) $u(A) \subseteq v(A) \cup \mathcal{P}$

Lemma 10.10

(a) $\mathcal{V}^* \circ \mathcal{M} \subseteq \mathcal{M}$

(b) $\mathcal{M} \circ \mathcal{V}^* \subseteq \mathcal{M}$

(c) $\mathcal{P} \circ \mathcal{M} \subseteq \mathcal{T}$

(d) $\mathcal{M} \circ \mathcal{P} \subseteq \mathcal{T}$

(e) $\mathcal{V}^+ \circ \mathcal{P} \subseteq \mathcal{T}$

(f) $\mathcal{P}_0 \circ \mathcal{V}^+ \subseteq \mathcal{T}$

(g) $\mathcal{P}_1 \circ \{\beta \in \mathcal{W}^+ \mid \beta \notin \mathcal{R}\} \subseteq \mathcal{T}$

(h) *If* $0 \leq r < s < k$, $0 \leq s' < t < k$, *and* $s \neq s'$, *then* $\pi\langle r, s\rangle \circ \pi\langle s', t\rangle \in \mathcal{T}$.

PROOF.

(a)

Let $\alpha \in \mathcal{V}^+$ and $\beta \in \mathcal{M}$. We verify that $\alpha \circ \beta \in \mathcal{M}$. If $\beta \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\beta \in \mathcal{M}_3$ then $\alpha \circ \beta \in \mathcal{M}_3$. The only complicated case is $\beta = z \circ \beta'$. Note that in that case $\alpha \circ z \in \text{Subword}(\alpha \circ \beta)$ and $\alpha \circ z \in \mathcal{M}_1$.

(b)

Let $\alpha \in \mathcal{M}$ and $\beta \in \mathcal{V}^+$. We verify that $\alpha \circ \beta \in \mathcal{M}$. If $\alpha \in \mathcal{M}_1$ then $\alpha \circ \beta \in \mathcal{M}_1$. If $\alpha \in \mathcal{M}_2$ then $\alpha \circ \beta \in \mathcal{M}_2$. The only complicated case is $\alpha = \alpha' \circ x$. In this case $x \circ \beta \in \text{Subword}(\alpha \circ \beta)$ and $x \circ \beta \in \mathcal{M}_1$.

(c)

Let $\gamma = \alpha \circ \beta$, where $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{M}$.

CASE 1: $\alpha \in \mathcal{P}_0 \cup \mathcal{P}_1$

By definition $\gamma = \pi\langle s, t\rangle \circ \beta$, where $0 \leq s < t < k$.

Evidently $\gamma = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where $\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1, t\rangle \circ \beta$. We must verify that $\phi \in \mathcal{M}$.

If $\beta \in \mathcal{M}_1$ then $\phi \in \mathcal{M}_1$. If $\beta \in \mathcal{M}_3$ then $\phi \in \mathcal{M}_3$. Let now $\beta \in \mathcal{M}_2$, i.e. $\beta = z \circ \beta'$. Evidently $z^{m+1} \in \text{Subword}(y_{s+1} \circ z^m \circ \pi\langle s+1, t\rangle \circ z \circ \beta'$ and $z^{m+1} \in \mathcal{M}_1$.

CASE 2: $\alpha \in \mathcal{P}_2$ and $\alpha \notin \mathcal{P}_1$

By definition $\gamma = \pi\langle s, k\rangle \circ \rho \circ \beta$, $\rho \in \mathcal{R}$, $\rho \neq \varepsilon$.

Evidently $\gamma = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where $\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1, k\rangle \circ \rho \circ \beta$.

The only complicated case is $\beta \in \mathcal{M}_2$, i.e., $\beta = z \circ \beta'$. Note that $\rho \circ z \in \text{Subword}(\phi)$ and $\rho \circ z \in \mathcal{M}_1$.

(d) and (e)

Let $\alpha \in \mathcal{M} \cup \mathcal{V}^+$ and $\beta \in \mathcal{P}$. We must prove that $\alpha \circ \beta \in \underbrace{\mathcal{M} \circ \ldots \circ \mathcal{M}}_{m \text{ times}}$.

CASE 1: $\beta = \pi\langle s, t\rangle \in \mathcal{P}_0$

Evidently $\alpha \circ \beta = \phi \circ \underbrace{z \circ \ldots \circ z}_{m-1 \text{ times}}$, where $\phi = (\alpha \circ \pi\langle s, t-1\rangle \circ x^m \circ y_t \circ z)$.

Obviously $z \in \mathcal{M}_2$. It remains to verify that $\phi \in \mathcal{M}$.

CASE 1a: $\alpha \in \mathcal{M}_1$

Obvious from Lemma 10.5 (i).

CASE 1b: $\alpha \in \mathcal{M}_2$

Obvious from Lemma 10.5 (ii).

CASE 1c: $\alpha \in \mathcal{M}_3$

Note that the rightmost symbol of α is x and the first m symbols of the word $\pi\langle s, t-1\rangle \circ x^m \circ y_t \circ z$ are x^m. Thus $x^{m+1} \in \text{Subword}(\phi)$. In view of $x^{m+1} \in \mathcal{M}_1$ we have $\phi \in \mathcal{M}_1$.

CASE 1d: $\alpha \in \mathcal{V}^+$
Evidently $\alpha \circ x \in \mathrm{Subword}(\phi)$. On the other hand, $\alpha \circ x \in \mathcal{V}^+ \circ \mathcal{Y}^+$ and $\mathcal{V}^+ \circ \mathcal{Y}^+ \subseteq \mathcal{M}_1$. According to Lemma 10.5 (i'), $\phi \in \mathcal{M}_1$.
CASE 2: $\beta \in \mathcal{P}_2$
By definition $\beta = \pi\langle s,k\rangle \circ \rho$, $\rho \in \mathcal{R}$.

Now $\alpha \circ \beta = \phi \circ \underbrace{z \circ \ldots \circ z}_{m-2 \text{ times}} \circ (z \circ \rho)$, where ϕ is the same as in the previous case. We have already verified that $z \in \mathcal{M}$ and $\phi \in \mathcal{M}$. Evidently $z \circ \rho \in \mathcal{M}_2$.

(f)
Let $\alpha \in \mathcal{P}_0$ and $\beta \in \mathcal{V}^+$. By definition $\alpha = \pi\langle s,t\rangle$, $0 \le s < t < k$.

Evidently $\alpha \circ \beta = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where $\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1,t\rangle \circ \beta$.
Note that $y_t \circ z^m \circ \beta \in \mathrm{Subword}(\phi)$. On the other hand, $y_t \circ z^m \circ \beta \in \mathcal{M}_1$, since $t \neq k$. Thus $\phi \in \mathcal{M}_1$.

(g)
Let $\alpha \in \mathcal{P}_1$, $\beta \in \mathcal{W}^*$, and $\beta \notin \mathcal{R}$. By definition $\alpha = \pi\langle s,k\rangle$, where $0 \le s < k$.

Evidently $\alpha \circ \beta = \underbrace{x \circ \ldots \circ x}_{m-1 \text{ times}} \circ \phi$, where $\phi = x \circ y_{s+1} \circ z^m \circ \pi\langle s+1,k\rangle \circ \beta$.
Note that $z \circ \beta \in \mathrm{Subword}(\phi)$. On the other hand, $z \circ \beta \in \mathcal{M}_1$, since β is not a left subword of δ (see the definition of $\mathcal{R}$). Thus $\phi \in \mathcal{M}_1$.

(h)
Evidently $\pi\langle r,s\rangle \circ \pi\langle s',t\rangle = \phi \circ \underbrace{z \circ \ldots \circ z}_{m-1 \text{ times}}$, where

$$\phi = (\pi\langle r,s\rangle \circ \pi\langle s',t-1\rangle \circ x^m \circ y_t \circ z).$$

We only need to prove that $\phi \in \mathcal{M}$. Note that $y_s \circ z^m \circ x^m \circ y_{s'+1} \in \mathrm{Subword}(\phi)$. On the other hand $y_s \circ z^m \circ x^m \circ y_{s'+1} \in \mathcal{M}_1$, since $s \neq s'$. According to Lemma 10.5 (i'), $\phi \in \mathcal{M}_1$. ■

To make the formulation of the following lemmas more readable we introduce two subsets of $\mathcal{W}^*$.

$$\begin{aligned} \mathcal{Q} &\rightleftharpoons \mathcal{P} \cup \mathcal{V}^* \cup \mathcal{M} \\ \mathcal{Q}_\infty &\rightleftharpoons \{\varepsilon\} \cup \mathcal{Q} \cup (\mathcal{Q} \circ \mathcal{Q}) \cup (\mathcal{Q} \circ \mathcal{Q} \circ \mathcal{Q}) \cup \ldots \end{aligned}$$

Lemma 10.11

(i) $\mathcal{T} \circ \mathcal{Q} \subseteq \mathcal{T}$

(ii) $\mathcal{Q} \circ \mathcal{T} \subseteq \mathcal{T}$

(iii) $\mathcal{P} \circ \mathcal{V}^* \subseteq \mathcal{P} \cup \mathcal{T}$

(iv) $\mathcal{P} \circ \mathcal{P} \subseteq \mathcal{P} \cup \mathcal{T}$

(v) $(\mathcal{P} \cup \mathcal{T}) \circ \mathcal{Q} \subseteq (\mathcal{P} \cup \mathcal{T})$

(vi) $\mathcal{Q} \circ (\mathcal{P} \cup \mathcal{T}) \subseteq (\mathcal{P} \cup \mathcal{T})$

PROOF.
(i)
Evident from Lemma 10.10 (d), (b), and Lemma 10.6.
(ii)
Evident from Lemma 10.10 (c), (a), and Lemma 10.6.
(iii)
Let $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{V}^*$. We must prove that $\alpha \circ \beta \in \mathcal{P} \cup \mathcal{T}$. If $\beta = \varepsilon$, then $\alpha \circ \beta = \alpha \in \mathcal{P}$. Let $\beta \in \mathcal{V}^+$.
CASE 1: $\alpha \in \mathcal{P}_0$
According to Lemma 10.10 (f), $\alpha \circ \beta \in \mathcal{T}$.
CASE 2: $\alpha = \pi\langle s, k\rangle \circ \rho \in \mathcal{P}_2$
If $\rho \circ \beta \in \mathcal{R}$, then $\alpha \circ \beta \in \mathcal{P}_2$, else $\alpha \circ \beta \in \mathcal{T}$ in view of Lemma 10.10 (g).
(iv)
Let $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{P}$. We must prove that $\alpha \circ \beta \in \mathcal{P} \cup \mathcal{T}$.
CASE 1: $\alpha \in \mathcal{P}_0 \cup \mathcal{P}_1$
By definition $\alpha = \pi\langle r, s\rangle$, where $0 \leq r < s \leq k$.
CASE 1a: $\beta \in \mathcal{P}_0 \cup \mathcal{P}_1$
By definition $\beta = \pi\langle s', t\rangle$, where $0 \leq s' < t \leq k$.

If $s = s'$, then $\alpha \circ \beta = \pi\langle r, s\rangle \circ \pi\langle s, t\rangle = \pi\langle r, t\rangle \in \mathcal{P}$ according to the definition of the function π. If $s \neq s'$, then $\alpha \circ \beta \in \mathcal{T}$ according to Lemma 10.10 (h).
CASE 1b: $\beta \in \mathcal{P}_2$
Evidently $\alpha \circ \beta \in \alpha \circ \mathcal{P}_1 \circ \mathcal{R}$. According to case 1a, $\alpha \circ \beta \in (\mathcal{P} \cup \mathcal{T}) \circ \mathcal{R} \subseteq (\mathcal{P} \cup \mathcal{T}) \circ \mathcal{V}^*$. From (iii) and (i) we obtain $(\mathcal{P} \cup \mathcal{T}) \circ \mathcal{V}^* \subseteq \mathcal{P} \cup \mathcal{T}$.
CASE 2: $\alpha \in \mathcal{P}_2$, $\alpha \notin \mathcal{P}_1$
By definition $\alpha = \pi\langle s, k\rangle \circ \rho$, $\rho \in \mathcal{R}$, $\rho \neq \varepsilon$.

Evidently, $\mathcal{P}_2 \circ \mathcal{P} = \mathcal{P}_1 \circ \mathcal{R} \circ \mathcal{P} \subseteq \mathcal{P}_1 \circ (\mathcal{V}^* \circ \mathcal{P})$.

From (ii) and Lemma 10.10 (e) we get $\mathcal{P}_1 \circ (\mathcal{V}^+ \circ \mathcal{P}) \subseteq \mathcal{P}_1 \circ \mathcal{T} \subseteq \mathcal{T}$.
(v)
Immediate from (i), (iv), (iii), and Lemma 10.10 (c).
(vi)
Immediate from (ii), (iv), and Lemma 10.10 (e), (d). ■

Lemma 10.12

(i) $\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty \subseteq \mathcal{P} \cup \mathcal{T}$

(ii) $\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty \circ \mathcal{M} \circ \mathcal{Q}_\infty \subseteq \mathcal{T}$

(iii) $\mathcal{Q}_\infty \circ \mathcal{M} \circ \mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty \subseteq \mathcal{T}$

PROOF.
(i)
From Lemma 10.11 (vi) we obtain

$$\mathcal{Q}_\infty \circ (\mathcal{P} \cup \mathcal{T}) \subseteq \mathcal{P} \cup \mathcal{T}.$$

From Lemma 10.11 (v) we obtain

$$(\mathcal{P} \cup \mathcal{T}) \circ \mathcal{Q}_\infty \subseteq \mathcal{P} \cup \mathcal{T}.$$

Thus $(\mathcal{Q}_\infty \circ \mathcal{P}) \circ \mathcal{Q}_\infty \subseteq (\mathcal{P} \cup \mathcal{T}) \circ \mathcal{Q}_\infty \subseteq \mathcal{P} \cup \mathcal{T}$.
(ii)
According to (i), Lemma 10.10 (c), and Lemma 10.6 $(\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty) \circ \mathcal{M} \subseteq \mathcal{T}$. It remains to apply Lemma 10.11 (i).
(iii)
According to (i), Lemma 10.10 (d), and Lemma 10.6 $\mathcal{M} \circ (\mathcal{Q}_\infty \circ \mathcal{P} \circ \mathcal{Q}_\infty) \subseteq \mathcal{T}$. It remains to apply Lemma 10.11 (ii). ■

Lemma 10.13 *Let* $\Delta \in \mathrm{Tp}(m)^*$, $\Pi \in \mathrm{Tp}(m)^*$, *and* $\varepsilon \in u(\Delta)$. *Then* $v_n(\vec{\Delta\Pi}) = v_n(\vec{\Pi})$ *and* $v_n(\vec{\Pi\Delta}) = v_n(\vec{\Pi})$.

PROOF. Evidently $\varepsilon \in \vec{v}(\Delta)$.

Let $\Delta = A_1 \dots A_l$. Take arbitrary $i \leq l$. Evidently, $\varepsilon \in A_i$ and thus $L^* \vdash \Lambda \to A_i$. We obtain $\langle s, s \rangle \in v_n(A_i)$ for every $s \in [0, k-1]$. Hence $\langle s, s \rangle \in \vec{v_n}(\Delta)$ for every $s \in [0, k-1]$. ■

Lemma 10.14 *Let* $\Theta \in \mathrm{Tp}(m)^*$ *and* $B \in \mathrm{Tp}(m)$. *Then* $u(\Theta) \circ u(B) \subseteq u(\Theta B) \cup \mathcal{T}$.

PROOF. Let $\gamma \in u(\Theta) \circ u(B)$. Then $\gamma = \alpha \circ \beta$ for some $\alpha \in u(\Theta) = u_0(\Theta) \cup u_2(\Theta) \cup \vec{v}(\Theta)$ and $\beta \in u(B) = u_0(B) \cup u_2(B) \cup v(B)$. We consider the corresponding nine cases and prove that $\alpha \circ \beta \in u_0(\Theta B) \cup u_2(\Theta B) \cup \vec{v}(\Theta B) \cup \mathcal{T}$.

CASE 1: $\alpha \in u_0(\Theta)$
By definition $\alpha = \pi\langle r, s \rangle$, $0 \leq r < s < k$, $\langle r, s \rangle \in \vec{v_n'}(\Theta)$.
CASE 1a: $\beta \in u_0(B)$
By definition $\beta = \pi\langle s', t \rangle$, $0 \leq s' < t < k$, $\langle s', t \rangle \in v_n'(B)$.

If $s \neq s'$, then $\alpha \circ \beta \in \mathcal{T}$ in view of Lemma 10.10 (h).

If $s = s'$, then $\alpha \circ \beta = \pi\langle r, s \rangle \circ \pi\langle s, t \rangle = \pi\langle r, t \rangle \in u_0(\Theta B)$, since $\langle r, t \rangle \in \vec{v_n'}(\Theta) \odot v_n'(B) = \vec{v_n'}(\Theta B)$.
CASE 1b: $\beta \in u_2(B)$
By definition $\beta = \pi\langle s', k \rangle \circ \rho$, $\rho \in \mathcal{R}$, $\Delta \in \mathrm{Tp}(m)^*$, $|\Delta| \leq |\rho|$, $\rho \in \vec{v}(\Delta)$, $\langle s', h'_{E\Delta} \rangle \in v_n'(B)$, $0 \leq s' < k$.

If $s \neq s'$, then $\alpha \circ \beta = \pi\langle r, s \rangle \circ \pi\langle s', k \rangle \circ \rho \in \mathcal{T} \circ \rho \subseteq \mathcal{T} \circ \mathcal{V}^* \subseteq \mathcal{T}$ in view of Lemma 10.10 (h) and (i).

If $s = s'$, then $\alpha \circ \beta = \pi\langle r, k \rangle \circ \rho \in u_2(\Theta B)$, since $\langle r, h'_{E\Delta} \rangle \in \vec{v_n'}(\Theta) \odot v_n'(B) = \vec{v_n'}(\Theta B)$.

CASE 1c: $\beta \in v(B)$
If $\beta \neq \varepsilon$, then $\alpha \circ \beta \in \mathcal{P}_0 \circ \mathcal{V}^+ \subseteq \mathcal{T}$ in view of Lemma 10.10 (f).
If $\beta = \varepsilon$, then $\alpha \circ \beta = \alpha \in u_0(\Theta) \subseteq u_0(\Theta B)$ in view of Lemma 10.13.

CASE 2: $\alpha \in u_2(\Theta)$
$\alpha = \pi\langle r, k\rangle \circ \rho$, $\rho \in \mathcal{R}$, $\Delta \in \mathrm{Tp}(m)^*$, $|\Delta| \leq |\rho|$, $\rho \in \vec{v}(\Delta)$, $\langle r, h'_{E\Delta}\rangle \in \vec{v_n'}(\Theta)$, $0 \leq r < k$
Note that $\alpha \in \mathcal{P}_1 \circ \mathcal{V}^*$.
CASE 2ab: $\beta \in u_0(B) \cup u_2(B) \subseteq \mathcal{P}$
In view of Lemma 10.10 (g) we have $\alpha \circ \beta \in \mathcal{P}_1 \circ (\mathcal{V}^* \circ \mathcal{P}) \subseteq \mathcal{T}$.
CASE 2c: $\beta \in v(B)$
If $\rho \circ \beta \notin \mathcal{R}$, then $\alpha \circ \beta = \pi\langle r, k\rangle \circ (\rho \circ \beta) \in \mathcal{T}$ in view of Lemma 10.10 (g).
Now we prove that if $\rho \circ \beta \in \mathcal{R}$ then $\alpha \circ \beta \in u_2(\Theta B)$. We take $\Delta' = \Delta B$ and $\rho' = \rho \circ \beta$. Evidently $\rho \circ \beta \in \vec{v}(\Delta) \circ v(B) = \vec{v}(\Delta B)$.
If $\beta \neq \varepsilon$, then $|\Delta'| = |\Delta| + 1 \leq |\rho| + 1 \leq |\rho| + |\beta| = |\rho'|$. Further, $|E\Delta| = |\Delta| + 1 \leq |\rho'| \leq |delta| = n - 1$. By Lemma 9.3 (iv) we have $\langle h'_{E\Delta}, h'_{E\Delta B}\rangle \in v_n'(B)$.
Thus $\langle r, h'_{E\Delta B}\rangle \in \vec{v_n'}(\Theta) \odot v_n'(B) = \vec{v_n'}(\Theta B)$.
If $\beta = \varepsilon$, then $\alpha \circ \beta = \alpha \in u_2(\Theta) \subseteq u_2(\Theta B)$ in view of Lemma 10.13.

CASE 3: $\alpha \in \vec{v}(\Theta)$
CASE 3ab: $\beta \in u_0(B) \cup u_2(B) \subseteq \mathcal{P}$
If $\alpha \neq \varepsilon$, then in view of Lemma 10.10 (e) we have $\alpha \circ \beta \in \mathcal{V}^* \circ \mathcal{P} \subseteq \mathcal{T}$.
If $\alpha = \varepsilon$, then $\alpha \circ \beta = \beta \in u(B) \subseteq u(\Theta B)$ in view of Lemma 10.13.
CASE 3c: $\beta \in v(B)$
Evidently $\alpha \circ \beta \in \vec{v}(\Theta) \circ v(B) = \vec{v}(\Theta B)$. ∎

Lemma 10.15 *Let $l \geq 0$, $B_1 \in \mathrm{Tp}(m)$, ..., $B_l \in \mathrm{Tp}(m)$. Then*

$$u(B_1) \circ \ldots \circ u(B_l) \subseteq u(B_1 \ldots B_l) \cup \mathcal{T}.$$

PROOF. Induction on l.
Induction base. For $l = 0$ we have to prove that $\{\varepsilon\} \subseteq u(\Lambda) \cup \mathcal{T}$. Indeed, $\varepsilon \in \vec{v}(\Lambda) \subseteq u(\Lambda)$.
Induction step. We must prove that if $u(B_1) \circ \ldots \circ u(B_l) \subseteq u(B_1 \ldots B_l) \cup \mathcal{T}$ then $u(B_1) \circ \ldots \circ u(B_l) \circ u(B_{l+1}) \subseteq u(B_1 \ldots B_l B_{l+1}) \cup \mathcal{T}$. In view of the induction hypothesis, it is sufficient to verify that $(u(B_1 \ldots B_l) \cup \mathcal{T}) \circ u(B_{l+1}) \subseteq u(B_1 \ldots B_l B_{l+1}) \cup \mathcal{T}$.
From Lemma 10.14 we obtain $u(B_1 \ldots B_l) \circ u(B_{l+1}) \subseteq u(B_1 \ldots B_l B_{l+1}) \cup \mathcal{T}$. According to Lemma 10.11 (i), $\mathcal{T} \circ u(B_{l+1}) \subseteq \mathcal{T}$. ∎

Lemma 10.16 *If $B_1, \ldots, B_l, C \in \mathrm{Tp}(m)$ and $L^* \vdash B_1 \ldots B_l \rightarrow C$, then*

$$u(B_1) \circ \ldots \circ u(B_l) \subseteq u(C) \cup \mathcal{T}.$$

PROOF. Let $B_1, \ldots, B_l, C \in \mathrm{Tp}(m)$ and $L^* \vdash B_1 \ldots B_l \rightarrow C$. According to Lemma 10.15, $u(B_1) \circ \ldots \circ u(B_l) \subseteq u(B_1 \ldots B_l) \cup \mathcal{T}$. It remains to prove that $u(B_1 \dots B_l) \subseteq u(C)$. This follows from $\vec{v}(B_1 \dots B_l) \subseteq v(C)$ and $\vec{v'_n}(B_1 \dots B_l) \subseteq v'_n(C)$ (cf. the definition of u at page 197). ■

Lemma 10.17 *Let $A \in \mathrm{Tp}(m)$. Then*

(i) $v(A) \subseteq \tilde{v}(A)$;

(ii) $\tilde{v}(A) \subseteq v(A) \cup \mathcal{M}$.

PROOF. (i) Let $\alpha \in v(A)$. Evidently $\alpha \in \mathrm{Subst}_{\mathcal{M}}(\alpha) \subseteq \tilde{v}(A)$.

(ii) By induction on $|\alpha|$ we prove that $\mathrm{Subst}_{\mathcal{M}}(\alpha) \subseteq \{\alpha\} \cup \mathcal{M}$ for any $\alpha \in \mathcal{V}^*$. For the induction step it is sufficient to verify that $(\{\alpha\} \cup \mathcal{M}) \circ (\{q\} \cup \mathcal{M}) \subseteq \{\alpha \circ q\} \cup \mathcal{M}$ whenever $\alpha \in \mathcal{V}^*$ and $q \in \mathcal{V}$. This follows from Lemma 10.10 (a), (b) and Lemma 10.6 (i). ■

Lemma 10.18 *If $A \in \mathrm{Tp}(m)$, then $w(A) \cap \mathcal{V}^* = v(A)$.*

PROOF. Let $A \in \mathrm{Tp}(m)$. According to Lemma 10.9 (ii) and Lemma 10.8 (i) $u(A) \cap \mathcal{V}^* = v(A)$.

According to Lemma 10.17 (ii) and Lemma 10.8 (iii) $\tilde{v}(A) \cap \mathcal{V}^* = v(A)$. ■

Lemma 10.19 *$\langle \mathcal{W}^*, w\rangle$ is a $\mathrm{Tp}(m)$-quasimodel.*

PROOF. We verify the conditions (1), (2) and (3) from the definition of a $\mathrm{Tp}(m)$-quasimodel at page 174.

(1)

Let $A{\bullet}B \in \mathrm{Tp}(m)$ and $\gamma \in w(A{\bullet}B)$. We must prove that $\gamma \in w(A) \circ w(B)$.

CASE 1: $\gamma \in u(A{\bullet}B)$

Obvious from Lemma 10.4.

CASE 2: $\gamma \in \tilde{v}(A{\bullet}B)$

By definition, $\gamma \in \mathrm{Subst}_{\mathcal{M}}(\gamma')$ for some $\gamma' \in v(A{\bullet}B) \subseteq v(A) \circ v(B)$. Thus $\gamma' = \alpha' \circ \beta'$, where $\alpha' \in v(A)$ and $\beta' \in v(B)$. Evidently, $\mathrm{Subst}_{\mathcal{M}}(\gamma') = \mathrm{Subst}_{\mathcal{M}}(\alpha') \circ \mathrm{Subst}_{\mathcal{M}}(\beta') \subseteq \tilde{v}(A) \circ \tilde{v}(B)$.

(2)

Let $A_1, \dots, A_l, B \in \mathrm{Tp}(m)$, $L^* \vdash A_1 \dots A_l \to B$, $\alpha_1 \in w(A_1), \dots \alpha_l \in w(A_l)$. We must prove that $\alpha_1 \circ \dots \circ \alpha_l \in w(B)$.

CASE 1: $(\forall i \leq l)\, \alpha_i \in u(A_i)$

According to Lemma 10.16, $\alpha_1 \circ \dots \circ \alpha_l \in u(B) \cup \mathcal{T}$.

In view of Lemma 10.7, $\alpha_1 \circ \dots \circ \alpha_l \in w(B)$.

CASE 2: $(\forall j \leq l)\, \alpha_j \in \tilde{v}(A_j)$

This means that for every index $j \leq l$ there is a word $\beta_j \in v(A_j)$ such that $\alpha_j \in \mathrm{Subst}_{\mathcal{M}}(\beta_j)$. Evidently, $\alpha_1 \circ \dots \circ \alpha_l \in \mathrm{Subst}_{\mathcal{M}}(\beta_1 \circ \dots \circ \beta_l)$. Note that $\beta_1 \circ \dots \circ \beta_l \in v(A_1) \circ \dots \circ v(A_l) \subseteq v(B)$, since $\langle \mathcal{V}^*, v\rangle$ is a $\mathrm{Tp}(m)$-quasimodel. Thus $\alpha_1 \circ \dots \circ \alpha_l \in \tilde{v}(B)$.

CASE 3: $(\exists i \leq l)\, \alpha_i \notin u(A_i)$ and $(\exists j \leq l)\, \alpha_j \notin \tilde{v}(A_j)$

Evidently $\alpha_i \in \tilde{v}(A_i)$. From Lemma 10.17 (ii) we obtain $\alpha_i \in v(A_i) \cup \mathcal{M}$. In view of $v(A_i) \subseteq u(A_i)$ we have $\alpha_i \notin v(A_i)$. Thus $\alpha_i \in \mathcal{M}$.

Evidently $\alpha_j \in u(A_j) \subseteq v(A_j) \cup \mathcal{P}$. On the other hand, from Lemma 10.17 (i) we obtain $\alpha_j \notin v(A_j)$. Thus $\alpha_j \in \mathcal{P}$.

Note that $\alpha_h \in \mathcal{Q}$ for every $h \leq l$. According to Lemma 10.12 (i) and (ii), $\alpha_1 \circ \ldots \circ \alpha_l \in \mathcal{T}$. It remains to apply Lemma 10.7.

(3)

Immediate from Lemma 10.18. ■

Lemma 10.20

(i) $\langle \mathcal{W}^*, w \rangle \in \mathcal{K}^m$;

(ii) $\langle \mathcal{W}^*, w \rangle$ *is a conservative extension of* $\langle \mathcal{V}^*, v \rangle$;

(iii) $\zeta \in w(E)$;

(iv) $\zeta \circ \delta \notin w(F)$.

PROOF. (i) Obvious.

(ii) Immediate from Lemma 10.18.

(iii) Obvious from Lemma 10.1.

(iv) Follows from Lemma 10.2, Lemma 10.17 (ii), and Lemma 10.8 (i), (ii).
■

Lemma 10.21 *The class* $\mathcal{K}^m$ *is witnessed.*

PROOF. Immediate from Lemma 10.20 and its dual for F/E. ■

11 Main result

Theorem 11.1 *Let* $\Gamma \in \mathrm{Tp}^*$ *and* $A \in \mathrm{Tp}$. *Then* $L^* \vdash \Gamma \rightarrow A$ *if and only if the sequent* $\Gamma \rightarrow A$ *is true in every free monoid model over a countable alphabet.*

PROOF. The 'only if' part coincides with Theorem 2.1. The 'if' part is immediate from Lemma 10.21, Lemma 5.2, Lemma 2.2, and Lemma 2.3. ■

Theorem 11.2 *Let* $\Gamma \in \mathrm{Tp}^*$ *and* $A \in \mathrm{Tp}$. *Then* $L^* \vdash \Gamma \rightarrow A$ *if and only if the sequent* $\Gamma \rightarrow A$ *is true in every free monoid model over a two symbol alphabet.*

PROOF. Following the proof of Theorem 11.1 we reduce the proof to the case of a sequent $\Lambda \rightarrow F$ and we find a free monoid model $\langle \mathcal{V}^*, v \rangle$, where $\mathcal{V} \subseteq \{a_j \mid j \in \mathbf{N}\}$ such that $\varepsilon \notin v(F)$ and $v(A) \neq \emptyset$ for every $A \in \mathrm{Tp}(m)$. Here $m = \|F\|$.

We take $\mathcal{W} \rightleftharpoons \{b, c\}$ and define a function $g \colon \mathcal{V}^* \rightarrow \mathcal{W}^*$ as follows.

$$g(a_j) \rightleftharpoons b \circ \underbrace{c \circ \ldots \circ c}_{j \text{ times}} \circ b \qquad g(\alpha \circ \beta) \rightleftharpoons g(\alpha) \circ g(\beta)$$

Note that g is injective.

Now we put $w(p_i) \rightleftharpoons \{g(\gamma) \mid \gamma \in v(p_i)\}$ for every primitive type p_i and define $w(A)$ for complex types by induction according to the definition of a free monoid model (cf. p. 173).

By induction on the complexity of A it is easy to prove that $w(A) = \{g(\gamma) \mid \gamma \in v(A)\}$ for every $A \in \mathrm{Tp}(m)$. In the proof of $\{\gamma \in \mathcal{W}^* \mid w(A) \circ \gamma \subseteq w(B)\} \subseteq w(A\backslash B)$ we use Lemma 5.2 (ii) and the fact that if $\alpha' \in \mathcal{V}^*$, $\beta' \in \mathcal{V}^*$, $\gamma \in \mathcal{W}^*$, and $g(\alpha') \circ \gamma = g(\beta')$ then there is $\gamma' \in \mathcal{V}^*$ such that $\gamma = g(\gamma')$.

Similarly for the dual case $\{\gamma \in \mathcal{W}^* \mid \gamma \circ w(A) \subseteq w(B)\} = w(B/A)$. Other cases of the induction step are trivial. ∎

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12 References

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