EXISTENCE AND PROPERTIES OF CERTAIN OPTIMAL STOPPING RULES

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1. Introduction

The main purpose of this note is to prove the existence of optimal stopping rules for certain problems involving sums of independent, identically distributed random variables. A special case was treated by Y. S. Chow and H. E. Robbins [2]. Their problem is very easily stated: let s_n be the excess of the number of heads over the number of tails in the first n tosses of an infinite sequence of independent tosses of a fair coin. Does there exist a stopping variable τ for which the expected average gain is maximal? In other words, does there exist a τ for which the expectation of s_τ/τ is at least as great as the expectation of s_t/t for any other stopping variable t? It turns out that this simple problem is not reducible to any of the available standard results on the existence of optimal stopping rules. Chow and Robbins do prove the existence of an optimal τ by an ingenious method which is, at least in part, suited only to the special case which they consider. Here, following in part the method of [2] and substituting general considerations for the specific ones used there, we establish the existence of an optimal stopping variable, maximizing the expected average gain under the sole assumption that the random variables involved have finite variance.

In section 2 we prove the above result. Our method also yields interesting information on the structure of the optimal τ which we present in section 3. For the sake of clarity, we confined the main exposition to the problem of maximizing the expected average gain; however, the methods developed here can deal with more general situations, and one generalization is presented in section 4. The last section contains various remarks.

Throughout we denote by $(\Omega, \mathcal{B}, P)$ the underlying probability space. Also, E denotes expectation, and we write $\{\cdots\}$ to denote $\{\omega\colon \cdots\}$, the set of ω having the indicated properties. All random variables are, of course, defined only almost surely but, in the interest of brevity, this qualification is usually omitted.

Throughout the paper, $x_1, x_2, \cdots, x_n, \cdots$, is a sequence of *independent, iden-*

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tically distributed random variables with zero mean and positive finite variance σ^2. We put

$$s_n = x_1 + \cdots + x_n, \qquad (n = 1, 2, \cdots) \tag{1}$$

and denote by $\mathcal{B}_n$ the σ-field generated by $x_1, \cdots, x_n$. By a *stopping variable* t^- (relative to the sequence $x_1, x_2, \cdots$) we understand a random variable whose range is the set of positive integers and such that

$$\{t = n\} \in \mathcal{B}_n, \qquad (n = 1, 2, \cdots). \tag{2}$$

This definition implies

$$P(t < \infty) = 1. \tag{3}$$

For technical reasons we find it convenient to consider also *generalized stopping variables.* These are defined as random variables t whose range is the set consisting of the positive integers and $+\infty$, and which satisfy (2).

We shall denote by T the set of all generalized stopping variables, that is, those $t \in T_\infty$ which satisfy (3). If $y_1, y_2, \cdots$ is a sequence of random variables and $t \in T_\infty$, then we define

$$Ey_t = \int_{\{t<\infty\}} y_t \, dP, \tag{4}$$

provided the right side is defined. For $t \in T$, this reduces to the usual definition of the expectation of y_t.

2. Existence of an optimal stopping variable

In this section we shall prove the following result.

THEOREM 1. *There exists a stopping variable* $\tau \in T$ *such that*

$$E \frac{s_\tau}{\tau} = \sup_{t \in T_\infty} E \frac{s_\tau}{\tau}. \tag{5}$$

Moreover, we have

$$0 < E \frac{s_\tau}{\tau} < \frac{\pi}{\sqrt{6}} \sigma. \tag{6}$$

Since the proof is somewhat long, we shall break it into several auxiliary assertions.

LEMMA 1. *Let* $t \in T_\infty$, *and let* $t(m)$, $(m = 0, 1, 2, \cdots)$ *be defined by*

$$t(m) = \begin{cases} t & \text{if } t \le m, \\ \infty & \text{otherwise.} \end{cases} \tag{7}$$

Then $t(m) \in T_\infty$ *and*

$$Es^2_{t(m)} \le m\sigma^2. \tag{8}$$

PROOF. That $t(m)$ is a generalized stopping variable follows immediately from the definition. To establish (8) we proceed as follows:

$$(9)\qquad Es_{t(m)}^2 = \sum_{i=1}^{m} \int_{\{t=i\}} s_i^2\, dP \le \sum_{i=1}^{m} \int_{\{t=i\}} (s_i^2 + (s_m - s_i)^2)\, dP$$
$$= \sum_{i=1}^{m} \int_{\{t=i\}} (s_i + (s_m - s_i))^2\, dP = \int_{\{t\le m\}} s_m^2\, dP \le Es_m^2 = m\sigma^2.$$

In the passage from the first to the second line we used the facts that $\{t = i\} \in \mathcal{B}_i$ and $E(x_j|\mathcal{B}_i) = 0$, for $j > i$.

LEMMA 2. *For all $t \in T_\infty$ and all $a > -1$ we have*

$$(10)\qquad E\left(\frac{s_t}{a+t}\right)^2 < \sum_{i=1}^{\infty} \frac{\sigma^2}{(a+1)^2}.$$

PROOF. Defining $t(m)$ by (8) we have

$$(11)\qquad E\left(\frac{s_t}{a+t}\right)^2 = \sum_{i=1}^{\infty} \frac{1}{(a+i)^2} \int_{\{t=i\}} s_i^2\, dP$$
$$= \sum_{i=1}^{\infty} \frac{1}{(a+i)^2} (Es_{t(i)}^2 - Es_{t(i-1)}^2).$$

Putting, for $i = 1, 2, \cdots$,

$$(12)\qquad v_i = Es_{t(i)}^2 - Es_{t(i-1)}^2,$$

the right side of (11) becomes

$$(13)\qquad \sum_{i=1}^{\infty} \frac{v_i}{(a+i)^2}$$

with v_i satisfying, by (12) and (8),

$$(14)\qquad v_i \ge 0, \qquad \sum_{i=1}^{m} v_i \le m\sigma^2, \qquad (m = 1, 2, \cdots).$$

Since $(a+i)^2 > 0$ and is strictly increasing with i, (13) is increased if some v_i is increased and a v_j, with $j > i$, is decreased by the same amount. Hence, the maximum of (13), for v_i satisfying (14), is obtained for, and only for, $v_1 = v_2 = \cdots = \sigma^2$.

This proves (10) with $\le$ instead of the sharp inequality. Though this is quite enough for our purposes, we add the short argument which yields (10). By the preceding, equality of the two sides of (10) would imply $v_1 = v_2 = \cdots = \sigma^2$. But $v_1 = \int_{\{t=1\}} x_1^2\, dP = \sigma^2$ implies that $x_1 \ne 0 \Rightarrow t = 1$ a.s., and thus, by independence, $v_2 = \int_{\{t=2\}} (x_1 + x_2)^2\, dP \le \int_{\{t>1\}} x_2^2\, dP = \sigma^2 P(t > 1) < \sigma^2$.

LEMMA 3. *For all $t \in T_\infty$ and $a > -1$ we have*

$$(15)\qquad E\frac{s_t}{a+t} < \sigma\left(\sum_{i=1}^{\infty} \frac{1}{(a+i)^2}\right)^{1/2}.$$

In particular we have for $a > 0$,

$$(16)\qquad E\frac{s_t}{a+t} < \frac{\sigma}{\sqrt{a}}.$$

PROOF. Inequality (15) is an immediate consequence of (10) and implies (16), since, for $a > 0$,

$$\sum_{i=1}^{\infty} \frac{1}{(a+i)^2} < \sum_{i=1}^{\infty} \int_{i-1}^{i} \frac{du}{(a+u)^2} = \int_0^{\infty} \frac{du}{(a+u)^2} = \frac{1}{a}. \tag{17}$$

The next lemmas study the expected value of $b + s_t/a + t$ for suitable generalized stopping variables t.

LEMMA 4. *If for some $a > 0$, real b and $t \in T_\infty$ we have*

$$E\frac{b + s_t}{a + t} \geq \frac{b}{a}, \tag{18}$$

then there exists $t' \in T_\infty$ satisfying

$$t' < \infty \Rightarrow s_{t'} > -b \tag{19}$$

and

$$E\frac{b + s_{t'}}{a + t'} \geq \frac{b}{a}. \tag{20}$$

PROOF. Put $t' = t$ if $t < \infty$ and $s_t > -b$; and $t' = \infty$ otherwise.

LEMMA 5. *Let $a > 0$, b real, and $t' \in T_\infty$ satisfy* (19) *and* (20). *Then we have*

$$E\frac{b' + s_{t'}}{a' + t'} \geq \frac{b'}{a'} \tag{21}$$

for all $a' \geq a$ and $b' \leq b$. (Moreover, for $b > 0$ the equality sign occurs in (21) *only for $a' = a$, $b' = b$.)*

PROOF. The derivative, with respect to b of the right side of (18), or (20), is $1/a$, while that of the left side is bounded by $1/(a + 1)$. Hence, (18) implies the same relation (even with strict inequality, unless the left side vanishes identically, that is $P(t = \infty) = 1$, and $b \leq 0$) when b is replaced by $b' < b$.

It remains to prove that (20) implies the same relation (even with strict inequality unless $P(t = \infty) = 1$) when a is replaced by $a' > a$. But (20) is equivalent to $E(a(b + s_{t'})/(a + t')) \geq b$ and, by (19), the left side is increasing in a (strictly, unless $t' = \infty$ almost surely).

LEMMA 6. *Let $a' \geq a > 0$, $b' \leq b$, $t \in T_\infty$, and* (18) *hold. Then there exists, for every $m = 0, 1, 2, \cdots$, a generalized stopping variable $t_m \in T_\infty$ satisfying*

$$t_m \leq m \Rightarrow s_{t_m} > b - b' \tag{22}$$

and

$$E\frac{b' + s_{t_m}}{a' + t_m} \geq \frac{b'}{a'}. \tag{23}$$

PROOF. For $m = 0$ the requirement (22) is vacuously satisfied, and by lemmas 4 and 5 there exists $t_0 = t'$ (the same for all $a' \geq a$, $b' \leq b$) satisfying (23). Having proved the existence of t_i for $i < m$, we put

$$t_0^{(m)}(x_1, x_2, \cdots, x_m, x_{m+1}, x_{m+2}, \cdots) = t_0(x_{m+1}, x_{m+2}, \cdots) \tag{24}$$

and define

$$(25) \qquad t_m = \begin{cases} t_{m-1} + t_0^{(m)} & \text{if} \quad t_{m-1} = m \quad \text{and} \quad s_m \le b - b', \\ \infty & \text{otherwise.} \end{cases}$$

Since t_0 is a fixed generalized stopping variable, it follows that $t_m \in T_\infty$ and, by definition, satisfies (22). Moreover, it follows from (23) with $m = 0$ that

$$(26) \qquad E\,\frac{b' + s_{t_m}}{a' + t_m} \ge E\,\frac{b' + s_{t_{m-1}}}{a' + t_{m-1}}, \qquad (m = 1, 2, \cdots),$$

and thus (23) is valid for all m.

(We used a somewhat abbreviated notation in (24). The precise meaning is the following: if ω and ω' on Ω are such that $x_i(\omega') = x_{m+i}(\omega)$ for all $i = 1, 2, \cdots$, then $t_0^{(m)}(\omega) = t_0(\omega')$. The function $t_0^{(m)}$ itself is not a generalized stopping variable, since the set $\{t = i\}$ need not belong to $\mathcal{B}_i$, but it belongs to $\mathcal{B}_{m+i}$, and therefore, $t_m \in T_\infty$). It may be remarked that lemmas 4 and 5 hold for arbitrary sequences of random variables, but lemma 6 utilizes the stationarity of the sequence $x_1, x_2, \cdots$. In the following lemma, even more properties of this sequence are used.

LEMMA 7. *Let* (18) *hold for some* $a > 0$, $b > 0$, *and* $t \in T_\infty$. *Then there exists* $t^* \in T_\infty$ *satisfying*

$$(27) \qquad E\,\frac{\frac{1}{2}b + s_{t^*}}{a + t^*} > \frac{b}{2a}$$

and

$$(28) \qquad E\,\frac{1}{a + t^*} < \frac{1}{2a} + \frac{2\sigma^2}{b^2}.$$

PROOF. Let t^* be the t_m, whose existence is assured by the previous lemma, corresponding to $a' = a$, $b' = b/2$, and $m = [a]$ (namely, the greatest integer $\le a$). Then (27) holds and, by (22),

$$(29) \qquad t^* \le a \Rightarrow \max\,(s_1, s_2, \cdots, s_{[a]}) > b/2.$$

Therefore, by Kolmogorov's inequality,

$$(30) \qquad P(t^* \le a) \le \frac{\sigma^2[a]}{(b/2)^2} \le \frac{4\sigma^2 a}{b^2}.$$

But

$$(31) \qquad E\,\frac{1}{a + t^*} \le \frac{P(t^* \le a)}{a + 1} + \frac{P(t^* > a)}{a + [a] + 1} < \frac{P(t^* \le a)}{a} + \frac{P(t^* > a)}{2a} = \frac{1}{2a} + \frac{P(t^* \le a)}{2a}$$

and (28) follows from (30).

LEMMA 8. *Let* $a > 0$ *and*

$$(32) \qquad b \ge 5\sigma\sqrt{a}.$$

Then

$$(33) \qquad E\,\frac{b + s_t}{a + t} < \frac{b}{a}$$

for all $t \in T_\infty$.

PROOF. If (33) were false, we would have, by (27), (28), and lemma 3, for the t^* of the preceding lemma,

$$\frac{b}{2a} < \frac{b}{2} E \frac{1}{a + t^*} + E \frac{s_{t^*}}{a + t^*} < \frac{b}{4a} + \frac{\sigma^2}{b} + \frac{\sigma}{\sqrt{a}}. \tag{34}$$

Hence, $b^2 - 4\sigma\sqrt{a}b - 4\sigma^2 a < 0$, thus

$$b < 2\sigma\sqrt{a} + \sqrt{4\sigma^2 a + 4\sigma^2 a} = 2(1 + \sqrt{2})\sigma\sqrt{a}, \tag{35}$$

contradicting (32).

This lemma asserts that the conditional expectation of s_t/t, given that $t \geq n$ and $s_n \geq 5\sigma\sqrt{n}$, is maximized by putting $t = n$.

LEMMA 9. *We have*

$$E\left(\sup_{n=1,2,\cdots} \frac{s_n^+}{n}\right) < \infty. \tag{36}$$

PROOF. Denoting the sup in (36) by s, we have for every $u > 0$,

$$\begin{aligned} P(s \geq u) &\leq \sum_{i=1}^{\infty} P\left(\max_{2^{i-1} \leq n < 2^i} \frac{s_n}{n} \geq u\right) \\ &\leq \sum_{i=1}^{\infty} P(\max_{1 \leq n < 2^i} s_n \geq 2^{i-1}u). \end{aligned} \tag{37}$$

Therefore, by Kolmogorov's inequality,

$$P(s \geq u) < \sigma^2 \sum_{i=1}^{\infty} \frac{2^i}{2^{2(i-1)}u^2} = \frac{4\sigma^2}{u^2}, \tag{38}$$

whence

$$Es = -\int_0^{\infty} u\, dP(s \geq u) = \int_0^{\infty} P(s \geq u)\, du < \infty. \tag{39}$$

To complete the proof of the theorem we need the following result. It may be found in [1] (see lemma 2 there) however, for the sake of self-containedness we reproduce its proof here.

LEMMA 10. *Let $\bar{t} \in T$, let $y_n (n = 1, 2, \cdots)$ be a sequence of random variables satisfying*

$$E(\sup_{n=1,2,\cdots} y_n^+) < \infty, \tag{40}$$

and let $\overline{T}$ be the family of all stopping variables $t \leq \bar{t}$ (that is, those $t \in T$ for which $t(\omega) \leq \bar{t}(\omega)$ almost surely). Then there exists $\tau \in \overline{T}$ satisfying

$$Ey_\tau = \sup_{t \in \overline{T}} Ey_t. \tag{41}$$

PROOF. The stopping time $t \in T$ is called regular if $t > j \Rightarrow E(y_t|\mathcal{B}_j) > y_j$ for all $j = 1, 2, \cdots$. If t and t' are regular and $t \leq t'$, then we have on $\{t = j\}$ the inequality $E(y_{t'}|\mathcal{B}_j) \geq y_j = y_t$, hence $t \leq t' \Rightarrow Ey_{t'} \geq Ey_t$. Now, if the right side of (41) is $-\infty$, there is nothing to prove. We may therefore assume that it is finite, M say. Then there exists for every $n = 1, 2, \cdots$, a stopping variable $t_n \in \overline{T}$ with $Et_n \geq M - 1/n$. Let t_n' = smallest integer $i \geq 1$ for which

$E(y_{\bar{t}_n}|\mathcal{B}_i) \leq y_i$. Then $t_n' \in \bar{T}$, $Ey_{t'_n} \geq Ey_{t_n}$, and t_n' is regular. Putting $\bar{t}_n = \max(t_1', \cdots, t_n')$, it is clear that $t_n' \leq \bar{t}_n \leq \bar{t}_{n+1}$, and that $\bar{t}_n \in \bar{T}$ is regular. Finally let $\tau = \lim_{n=\infty} \bar{t}_n$. Because of (40) we have, by Fatou's lemma, $E(y_\tau|\mathcal{B}_j) \geq \limsup_{n=\infty} E(y_{\bar{t}_n}|\mathcal{B}_j)$ and, since $\tau > j \Rightarrow \bar{t}_n > j$ for some n, it follows that $\tau > j \Rightarrow E(y_\tau|\mathcal{B}_j) > y_j$. Hence, τ is regular. Since $\tau \geq \bar{t}_n \geq t_n'$ we have $Ey_\tau > M - 1/n$ for all n, that is, $Ey_\tau = M$.

PROOF OF (5). Let $\bar{t} \in T_\infty$ be defined as follows: $\bar{t} = j$ when j is the smallest positive integer for which $s_j \geq 5\sigma\sqrt{j}$. Then, by the law of the iterated logarithm, $\bar{t} \in T$. If $t \in T_\infty$ is arbitrary and we put $t' = \min(t, \bar{t})$, then, by lemma 8, $E(s_{t'}/t') \geq E(s_t/t)$. Thus the sup in (5) may be taken only over the class $\bar{T} \subset T$ of all stopping variables $\leq \bar{t}$. Since, by lemma 9, the sequence $y_n = s_n/n$ $(n = 1, 2, \cdots)$ satisfies (40), we can apply lemma 10 and deduce the existence of an optimal $\tau \in \bar{T}$ satisfying (5).

PROOF OF (6). For the generalized stopping variable t defined by $t = 1$ if $x_1 > 0$ and $t = \infty$ otherwise, we have $E(s_t/t) = Ex_1^+ > 0$. This gives the first inequality of (6). The other inequality follows from (15) with $a = 0$.

For reference in the sequel we state the following slight extension of theorem 1, which is proved in exactly the same way.

THEOREM 1'. *Let a and b be real, then there exists a stopping variable $\tau \in T$ for which*

$$E\,\frac{b + s_\tau}{a + \tau} = \sup_{t \in T_\infty} \frac{b + s_t}{a + t}. \tag{42}$$

Since we do not need the analogue of (6), we do not state it here.

3. Structure of the optimal rule

By Theorem 1', there exists for every $\beta > 0$ and $n = 1, 2, \cdots$ a stopping variable $\tau(\beta, n)$ for which

$$V_n(\beta) = E\,\frac{\beta + s_{\tau(\beta,n)}}{n + \tau(\beta, n)} = \sup_{t \in T_\infty} E\,\frac{\beta + s_t}{n + t}. \tag{43}$$

Since we have for all real β, β', and every $t \in T_\infty$,

$$\left| E\,\frac{\beta' + s_t}{n + t} - E\,\frac{\beta + s_t}{n + t} \right| \leq \frac{\beta' - \beta}{n + 1}, \tag{44}$$

it follows that $V_n(\beta)$ is continuous in β. Consider now the equation

$$V_n(\beta) = \frac{\beta}{n}. \tag{45}$$

Since $V_n(\beta) > 0$ for all β (consider the rule: stop when $s_i > -\beta$ for the first time), and since, by lemma 8, $V_n(\beta) < \beta/n$ for large β, it follows from the continuity of $V_n(\beta)$ that equation (45) is solvable for every n. If β_n is a solution of (45), then it follows from the last sentence of lemma 5 that $V_n(\beta) < \beta/n$ for $\beta > \beta_n$.

The equation

$$\frac{\beta_n}{n} = \sup_{t \in T_\infty} \frac{\beta_n + s_t}{n + t} \tag{46}$$

defines β_n *uniquely.*

By lemma 5, the sequence

$$\beta_1, \beta_2, \cdots, \beta_n, \cdots \tag{47}$$

is a strictly increasing sequence of positive numbers, and by the considerations of section 2, an optimal stopping variable τ_0 satisfying (5) may be defined by

$$\tau_0 = n \Rightarrow s_n \geq \beta_n \quad \text{and} \quad s_i < \beta_i \qquad \text{for} \quad i = 1, \cdots, n-1, \tag{48}$$

that is, by the rule: stop whenever $s_n \geq \beta_n$ for the first time.

The analysis of section 2 also shows that if τ is any stopping variable optimal in the sense of (5), then $P(\tau \geq \tau_0) = 1$, and that there exists an optimal τ with $P(\tau \neq \tau_0) > 0$ if and only if $P(s_n = \beta_n, \tau_0 = n) > 0$ for at least one $n = 1, 2, \cdots$.

Thus, if we identify two stopping variables which are equal almost surely, τ_0 given by (48) is the unique minimal optimal stopping variable. It is interesting to study the rate of growth of the sequence (47), and the following result contains information about this rate.

THEOREM 2. *The numbers* β_n $(n = 1, 2, \cdots)$ *defined by* (46) *satisfy*

$$\limsup_{n=\infty} \frac{\beta_n}{\sqrt{n}} \leq c_1 \sigma \tag{49}$$

and

$$\liminf_{n=\infty} \frac{\beta_n}{\sqrt{n}} \geq c_2 \sigma, \tag{50}$$

where $c_1 = 4.06 \cdots$ *is the infimum for* $0 < \nu < 1$ *of the positive root* c *of the equation*

$$c - \nu c^2 + \frac{\nu}{(1-\nu)^2} \log\left(1 + (1-\nu)^2 c^2\right) = 0 \tag{51}$$

and $c_2 = 0.32 \cdots$ *is the supremum for* $\nu > 0$ *of the positive root* c *of the equation*

$$c - \frac{\sqrt{\nu}}{(1+\nu)\sqrt{2\pi}} \left(e^{-\frac{c^2}{2\nu}} + \frac{c}{\sqrt{\nu}} \int_{-c/\sqrt{\nu}}^{\infty} e^{-u^2/2}\, du \right) = 0. \tag{52}$$

The assertion (49) with c_1 replaced by 5 was proved in lemma 8 (or with c_1 replaced by $4.828 \cdots$ in (35)). Since c_1 is not the best possible constant, our main reason for stating (49) is to introduce the following strengthening of lemma 7, which is of certain independent interest.

LEMMA 11. *Let* (18) *hold for some* $a > 0$, $b > 0$ *and* $t \in T_\infty$, *and let* ν *satisfy* $0 < \nu < 1$. *Then there exists* $t^* \in T_\infty$ *satisfying*

$$E \frac{\nu b + s_{t^*}}{a + t^*} > \frac{\nu b}{a} \tag{53}$$

and

$$E \frac{1}{a + t^*} < \frac{\sigma^2}{(1-\nu)^2 b^2} \log\left(1 + \frac{1 + (1-\nu)^2 b^2/\sigma^2}{a}\right). \tag{54}$$

PROOF. Let $m = [(1-\nu)^2 b^2/\sigma^2]$ and define t^* as the t_m given by lemma 6 for this m and $a' = a$, $b' = \nu b$. Then (53) holds and we have

$$E \frac{1}{a+t^*} = \sum_{i=1}^{\infty} \frac{P(t^* = i)}{a+i}. \tag{55}$$

But, by (22), $t^* \leq i$ implies, for $i \leq m$, the inequality $\max(s_1, \cdots, s_i) > (1-\nu)b$, and therefore, by Kolmogorov's inequality,

$$P(t^* \leq i) < \frac{\sigma^2 i}{(1-\nu)^2 b^2}, \qquad (i = 1, \cdots, m). \tag{56}$$

The right side of (56) is >1 for $i \geq m+1$, and hence, by a reasoning similar to that which leads from (13) to (10), the infinite sum in (55) is smaller than

$$\frac{\sigma^2}{(1-\nu)^2 b^2} \sum_{i=1}^{m+1} \frac{1}{a+i}. \tag{57}$$

Estimating the sum in (57) by an integral, we obtain (54).

PROOF OF (49). We first remark that the left side of (51) vanishes for $c = 0$, is a convex function of c, and tends to $-\infty$ as $c \to \infty$; therefore, the equation (51) has indeed a unique positive root.

From (46), (43), lemma 11, and (16), we have

$$\frac{\nu\beta_n}{n} < \frac{\nu\sigma^2}{(1-\nu)^2\beta_n} \log\left(1 + \frac{1 + (1-\nu)^2\beta_n^2/\sigma^2}{n}\right) + \frac{\sigma}{\sqrt{n}} \tag{58}$$

or, putting $c = \beta_n/(\sigma\sqrt{n})$,

$$\nu c^2 < \frac{\nu}{(1-\nu)^2} \log\left(1 + \frac{1}{n} + (1-\nu)^2 c^2\right) + c. \tag{59}$$

Letting $n \to \infty$ and remarking that (59) holds for all $0 < \nu < 1$, we obtain (49).

PROOF OF (50). We first remark that the left side of (52) is negative for $c = 0$ and that it has, relative to c, a positive derivative bounded away from zero; therefore, the equation (52) has indeed a unique positive root.

Let now $\nu > 0$ and $c > 0$ be given. For $a > 0$, define b and $t \in T_\infty$ by

$$b = c\sigma\sqrt{a}, \qquad t = \begin{cases} [\nu a] & \text{if } s_{[\nu a]} > -b \\ \infty & \text{otherwise.} \end{cases} \tag{60}$$

Then

$$E \frac{b + s_t}{a+t} = \frac{1}{a + [\nu a]} \int_{-b}^{\infty} (b+u)\, dP\,(s_{[\nu a]} \leq u). \tag{61}$$

Using the fact that $s_n/(\sigma\sqrt{n})$ is asymptotically normal with zero mean and unit variance and that its first absolute moment converges to the same moment of the limiting distribution, we see from (60) and (61) that

$$\lim_{a=\infty} \frac{a}{b} E \frac{b+s_t}{a+t} = \frac{\sqrt{\nu}}{(1+\nu)c\sqrt{2\pi}} \int_{-c/\sqrt{\nu}}^{\infty} \left(\frac{c}{\sqrt{\nu}} + u\right) e^{-u^2/2}\, du. \tag{62}$$

But, by the definition of c_2, the right side of (62) is smaller than 1 for $c < c_2$ and an appropriate ν. Hence, we have for $c < c_2$ and large n,

$$\frac{c\sigma\sqrt{n}}{n} < V_n(c\sigma\sqrt{n}), \tag{63}$$

or $c\sigma\sqrt{n} < \beta_n$, which completes the proof.

4. Generalization

In order not to lengthen the paper, we state only one generalization whose proof follows very closely those given above.

THEOREM 3. *Let $\alpha > \frac{1}{2}$; then there exists a stopping variable $\tau \in T$ for which*

$$E\frac{s_\tau}{\tau^\alpha} = \sup_{t\in T_\infty} E\frac{s_t}{t^\alpha}. \tag{64}$$

For such τ we have

$$0 < E\frac{s_\tau}{\tau^\alpha} < \sigma\left(\sum_{i=1}^{\infty}\frac{1}{i^{2\alpha}}\right)^{1/2}. \tag{65}$$

Such a τ may be defined through a suitable increasing sequence of positive numbers $\beta_1(\alpha)$, $\beta_2(\alpha)$, $\cdots$ as follows: $\tau = n$ if n is the smallest positive integer for which $s_n \geq \beta_n(\alpha)$. The $\beta_n(\alpha)$ satisfy

$$c_2(\alpha) \leq \liminf_{n=\infty}\frac{\beta_n(\alpha)}{\sigma\sqrt{n}} \leq \limsup_{n=\infty}\frac{\beta_n(\alpha)}{\sigma\sqrt{n}} \leq c_1(\alpha) \tag{66}$$

with positive finite $c_1(\alpha)$ and $c_2(\alpha)$.

PROOF. As above for $\alpha = 1$ we obtain, similarly to (10),

$$E\frac{s_t^2}{(a+t)^{2\alpha}} < \sum_{i=1}^{\infty}\frac{\sigma^2}{(a+i)^{2\alpha}}, \tag{67}$$

from which we have in lieu of (16),

$$E\frac{s_t}{(a+t)^\alpha} < \frac{\sigma}{(2\alpha-1)^{1/2}a^{\alpha-1/2}}. \tag{68}$$

Lemmas 4, 5, and 6 hold with the same proof if we replace everywhere in the denominators a, $a + t$, $\cdots$, by a^α, $(a + t)^\alpha$, and so on. Similarly, lemma 7 remains valid when $a + t^*$ and $2a$ in (27) are replaced by $(a + t^*)^\alpha$ and $(2a)^\alpha$ respectively, and we have, instead of (28),

$$E\frac{1}{(a+t^*)} < \frac{1}{(2a)^\alpha} + \frac{2^\alpha-1}{(2a)^\alpha}\cdot\frac{4\sigma^2 a}{b^2}. \tag{69}$$

As in the proof of lemma 8, we obtain from (68) and (69) an inequality similar to (34), which may be rewritten as $Q_\alpha\,(b/\sigma\sqrt{a}) < 0$ where Q_α is a polynomial of the second degree with leading coefficient 1, and the other coefficients negative (and depending on α). This shows, similarly to lemma 8, that the conditional expectation of s_t/t^α, given that $t \geq n$ and $s_n \geq c_1(\alpha)\sigma\sqrt{n}$, is maximized by putting $t = n$. Since lemma 9 remains valid, with the same proof, when s_n/n is replaced by s_n/n^α, we can apply lemma 10 and deduce the existence of $\tau \in T$ satisfying (64) as well as the last inequality of (66). The second inequality of

(65) follows from (68), and the first is proved exactly as the first inequality of (6). Finally, the first inequality of (66) follows again from considerations of asymptotic normality, similarly to the proof of (50).

5. Remarks

1. Various other generalizations and extensions are possible. Thus we can study in a similar fashion $E((b + s_t)^+)^\beta/(a + t)^\alpha$ provided $2\alpha > \max(1, \beta)$. Also more complicated, and less explicit, functions of t and s_t may be considered.

2. The left inequality of (6) cannot be improved (without considering other features of the distribution of the x_i besides the variance). This is seen trivially by remarking that a random variable with zero mean and given variance can have an arbitrary small positive bound. On the other hand, $\pi/\sqrt{6}$ is not the best possible constant in (6).

3. The inequality (10) of lemma 2 cannot be improved. Indeed, let $0 < p < 1$, $q = 1 - p$, and each x_i assume the values $\sigma\sqrt{q/p}$ and $-\sigma\sqrt{q/p}$ with probabilities p and q respectively. Then for $t \in T_\infty$ defined by: $t = i$ if i is the smallest integer for which $x_i > 0$ and $i < q/p$, and $t = \infty$ otherwise, we have

$$E\left(\frac{s_t}{a + t}\right)^2 = \sigma^2 \sum_{i=1}^{[q/p]} pq^{i-1} \frac{((q/p)^{1/2} - (i - 1)(p/q)^{1/2})^2}{(a + i)^2}. \tag{70}$$

But as $p \to 0$ the right side of (70) approaches the limit $\sigma^2 \sum_{i=1}^\infty (a + i)^{-2}$. However, (15) in lemma 3 can be improved as is seen by examining the conditions under which little is lost by the passage from (10) to (15), as well as those under which (10) gives a close to best estimate, and observing that it is impossible to satisfy both simultaneously. This establishes, in particular, the assertion about (6) in the preceding remark.

A similar remark applies to (68) and (67).

4. It is not difficult to improve the constants c_1 and c_2 in (49) and (50) in theorem 2. But we do not know how to obtain the best constants. It is very likely that the lim sup and lim inf in (49) and (50) must coincide; that is, that $\lim_{n=\infty} \beta_n/\sqrt{n}$ must exist (though it may depend on aspects of the distribution of the x_i other than the variance). But we cannot prove this in general. The existence of this limit seems to be a most interesting problem connected with the structure of the optimal stopping rule.

5. Since the sequence $s_1, s_2, \cdots$ is Markovian, it is possible to describe every optimal stopping variable τ by a sequence of "absorbing" sets B_n of real numbers as follows: $\tau = n$ if n is the smallest positive integer for which $s_n \in B_n$.

It follows from lemma 5 that it is natural to take the sets B_n as half-lines: $B_n = \{u: u \geq b_n\}$. Theorem 2 gives such a description of the regular stopping variable τ_0, but more is true. If $\tau = n$ when n is the smallest integer for which $s_n \geq b_n$ describes an optimal stopping variable (in the sense of (5)), then the b_n necessarily satisfy (49) and (50) (with b_n in place of β_n). Indeed, it follows from the considerations of the beginning of section 3 that τ can be optimal only if

$P(\tau_0 = n, s_n \in I_n) = 0$ for all n, where I_n is the open interval with endpoints β_n and b_n. It can be easily seen that τ_0 is not bounded. If the random variables x_i are such that $P(x_i \in I) > 0$ for every interval I whose length exceeds some given number, it can be easily inferred that the sequence $b_n - \beta_n (n = 1, 2, \cdots)$ is bounded, and thus the b_n satisfy (49) and (50). The general case requires more elaborate arguments.

6. The assumption that the random variables x_i have zero mean is, of course, unimportant and can be dropped. We did, however, make essential use of the fact that they have finite variance. It seems possible to replace this condition by a weaker one, such as assuming the finiteness of an absolute moment of order greater than one, but this has not yet been done. It would also be interesting to relax the conditions of independence and identical distribution of the x_i.

7. We assumed that $\mathcal{B}_n$ is the σ-field generated by $x_1, \cdots, x_n$. All our considerations remain valid if we assume instead that $\mathcal{B}_n$ is the σ-field generated by $x_1, \cdots, x_n$ and $\mathcal{C}$, where $\mathcal{C}$ is a subfield of $\mathcal{B}$ independent of those generated by the x_n. This device makes it possible to treat certain randomized stopping variables.

8. Our method can easily be adapted to treat stochastic processes with a continuous time parameter. This may, however, necessitate a slight reformulation of the problem. Consider, for example, the standard Brownian motion process $s(h)$, $0 \leq h < \infty$. Denoting by $\mathcal{B}(h)$ the σ-field generated by $s(h')$, $0 \leq h' \leq h$, a nonnegative random variable t is called a stopping variable if $\{t \leq h\} \in \mathcal{B}(h)$ for all $0 \leq h < \infty$. Now, sup $Es(t)/t$ taking over all stopping variables is ∞, and indeed if we define τ to be the smallest $h > 0$ for which $s(h) \geq \sqrt{h \log^+ 1/h}$, then we find $Es(\tau)/\tau = \infty$ (the almost sure continuity of $s(h)$ and the law of the iterated logarithm imply that τ is a stopping variable). If we wish to consider problems for which sup $Es(t)/t$ is finite we must modify the original problem somewhat. We might, for example, confine our attention to stopping variables t satisfying $P(t \geq \delta) = 1$ for a given $\delta > 0$, or consider $s(t)/(a + t)$ instead of $s(t)/t$. Then the supremum of the expectations would be finite, and there would exist a stopping variable τ for which this supremum is achieved and, moreover, a result similar to theorem 2 would hold.

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