AN EXTENSION OF THE LEBESGUE MEASURE OF LINEAR SETS

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1. Introduction

The present paper develops the following idea of which I told Lebesgue almost forty years ago. It is possible to define on the circle of length unity a set Γ such that the circle is the union of a countable infinity of sets Γ_k which are disjoint and superposable on Γ by rotation. If K_h is the union of those Γ_k for which $k \equiv h \bmod p$, where $p > 1$ is an integer, the circle is also the union of p disjoint sets K_h which are superposable by rotation. For an extension of the Lebesgue measure invariant under rotation the Γ_k cannot be measurable. However, the K_h can be measurable, their common measure being $1/p$.

After a brief preliminary section (section 2), these results will be established in section 3.

Lebesgue, who was not interested in arguments depending on Zermelo's axiom of choice, led me away from a further study of these sets. Besides, I was soon to learn that the sets Γ had already been defined by Vitali. I do not know whether the sets K had also attracted his attention.

The reason for my present return to these problems is the remark that my old results can be substantially improved by using G. Hamel's theorem according to which the real numbers possess a basis $\{\omega_\nu\}$. Every real number x is then uniquely defined by a noncountable family of *rational components* a_ν. Restricting one of the a_ν to be an element of the interval $[0, 1)$, one obtains a set of the Vitali type. From this it is possible to construct sets K_ν^h which are easily made measurable by a suitable extension of the Lebesgue measure. The application of the same procedure to any finite set of components a_ν yields a new extension.

An extension of the Lebesgue measure which reaches considerably further can be obtained through the use of two Hamel bases $\{\omega_\nu\}$ and $\{\omega'_\nu\}$. Consider then the numbers x and $y = f(x)$ which correspond for these two bases to the same rational components a_ν. If $f(1) = 1$, as will be assumed here, the fractional part $g(x)$ of $f(x)$, defined for $x \in [0, 1)$, is a one-to-one function of x. Suppose that, in the square $[0, 1) \times [0, 1)$, the complement of the graph $\mathcal{G}$ of the function g has interior measure zero. If E is a measurable subset of the square having Lebesgue measure $m(E)$ and if $\mathcal{E}$ denotes the set of values of x for which $[x, g(x)] \in E$, then $\mu^*(\mathcal{E}) = m(E)$ defines an extension of the Lebesgue measure.

This extension is countably additive and invariant under shifts provided that x and y be defined modulo unity.

It is a consequence of a theorem due to H. Cartan and G. Choquet and communicated to me by one of these authors, that the condition imposed on $\mathcal{G}$ is satisfied under very general circumstances. It even appears that the cases where this condition is not satisfied may be considered very special. Indeed, the belief that this was in fact the case is the reason why I undertook the present work, whose interest would have been considerably lessened had this not been so.

Since the completion of this work, H. Cartan has brought to my attention a paper by K. Kodaira and S. Kakutani [1]. These authors have established the existence of invariant extensions of the Lebesgue measure. Their main theorems go further than mine in the sense that they point out the vastness of the σ-fields on which these measures can be defined. In a subsequent article [2], S. Kakutani and J. C. Oxtoby construct a measure which is invariant under all transformations preserving Lebesgue measure.

The following exposition, at least when the notion of the Hamel basis is accepted, is considerably more concrete, and so may not be valueless.

2. The notion of a randomly chosen integer

When an integer-valued random variable can effectively take any one of an infinite set of values, the probability cannot possibly be uniformly distributed over these values. Indeed, these probabilities are the terms of a convergent series having sum unity, so that the terms converge to zero without being equal to zero.

When speaking of a randomly chosen integer N, one attempts to salvage the uniformity of the distribution of probability, giving up in the process the requirement that the probability of belonging to a sequence S be a countably additive function of S. To this end consider an integer-valued random variable uniformly distributed over the interval $[1, n]$ if the positive integers are under discussion or over the interval $[-n, +n]$ if all real integers are being considered. The probability α_n of belonging to S is then the relative frequency of the numbers belonging to S among those of the interval. By definition, the probability of S is the limit of α_n if this limit exists.

This is a limit frequency and not a true probability. The limit does not always exist. Furthermore, on the family of sequences for which it is defined, it is finitely but not countably additive.

In the sequel we shall restrict our considerations to sequences S_p which are arithmetic progressions of step p, indefinitely extended in both directions. For every p, there are p sequences of type S_p, containing respectively the numbers $1, 2, \cdots, p$. The probability of each S_p is then $1/p$. A finite family of such sequences is contained in a finite Borel field presenting no difficulty, since the question of countable additivity does not arise there. However one can inquire whether it is possible to define a countably additive probability $P(E)$ such that $P(S_p) = 1/p$ for every sequence S_p.

The answer is in the negative, even if one deals only with the sequences S'_q obtained by taking $p = 2^q$. The sequence S'_0 is the same as S_1, that is, S'_0 is the set of all real integers. The result is a consequence of the following theorem.

THEOREM. *The smallest Borel field B containing all the sequences S'_q contains every singleton and, consequently, every subset of the set of real integers.*

PROOF. Let $\{a\}$ be the set containing only the element a. Every integer $k \neq a$ can be written in a unique manner in the form $k = a + (2h + 1)2^{q-1}$, where h and q are integers and $q > 0$. Take for S'_q the set of integers $a + (2h + 1)2^{q-1}$ obtained when q is a fixed positive integer and h varies from $-\infty$ to $+\infty$. Every $k \neq a$ belongs to one and only one S'_q having a positive subscript. Therefore $S'_0 = \bigcup[\{a\}, S'_1, S'_2, \cdots]$, and since $S'_q \in B$ we have $\{a\} \in B$ also.

The requirement of countable additivity for $P(E)$ is satisfied here for the union of disjoint sets S'_q, for $q > 0$, but not for the union of sets of the type $\{a\}$.

Another example can be constructed as follows. Let us restrict ourselves to the set of positive integers so that each S_p will begin with its first positive element. Let S''_q, for $q > 0$ be the sequence S_p having first element q with $p = p_q > q$ and such that $\sum (1/p_q) < \epsilon$ for an arbitrarily small positive ϵ. This implies $\sum P(S''_q) < \epsilon$. However, the union of the sets S''_q contains every q, hence all the positive integers, and therefore must have measure unity.

This example shows that countable additivity can fail for a union of sequences S_p. In the previous example the set $\{a\}$ was the complement of such a union.

However, it is possible to find infinite families of sets of the S_p type leading to a Borel field for which the requirement of countable additivity is satisfied. This will be the case, for instance, if one considers only those S_p which contain the element zero.

3. Vitali's construction of nonmeasurable sets

3.1. We shall consider here subsets E of the circle of unit length. With each point x is associated the family of its circular abscissas so that the linear image of x is the set of points $\{x + k\}$ where k is an integer taking values from $-\infty$ to $+\infty$. The linear image $\mathcal{E}$ of a set E is invariant under the shift T_1 which changes x into $x + 1$. In order to define E and $\mathcal{E}$ uniquely it is sufficient to give the part of $\mathcal{E}$ which belongs to an arbitrary semiopen interval of length unity. We shall call a *shift* the operation T_l which changes x into $x + l$ even though, on the circle, this operation is a rotation. We shall deal with E instead of $\mathcal{E}$. However, it is easy to translate all our statements so that they apply to the line by considering that x is defined only mod 1.

Let ω be an irrational number. Let C_0 be the set of points $\{h\omega\}$, where h is an integer ranging from $-\infty$ to $+\infty$. Let C_l be the set obtained from C_0 by a shift of length l. Let C and C' be two of the sets obtained in this fashion. The two sets C and C' are either disjoint or identical. In other words the sets C_l are equivalence classes.

In each equivalence class let us select (by a Zermelian choice) one point x_0, called the *central point* of the class and let Γ_0 be the set of points so selected. If h denotes a real integer varying from $-\infty$ to $+\infty$, the shifts $T_{h\omega}$ form a group. The set of transforms of x_0 by the elements of this group is the class C from which x_0 was selected, each $x \in C$ being obtained once and only once. It follows that the sets $\Gamma_h = T_{h\omega}\Gamma_0$ are all disjoint and that their union is the entire circle.

The sets C are countable everywhere dense sets, while the sets Γ_h are uncountable. One cannot assert that the sets Γ_h will be dense; in fact one can assume that all the central points x_0 are chosen in an interval of arbitrarily small length, and the set Γ_0 will then be a subset of this interval. On the other hand it is possible to make Γ_0 everywhere dense, and all the sets Γ_h will then have the same property, since they are obtained from Γ_0 by shifts. However, under no circumstances can the sets Γ_h be measurable with regard to invariant Lebesgue measure, for, if they were measurable, they would all possess the same measure. Their measure, however, could be neither zero nor positive since the circle which is their union has a nonnull finite measure.

3.2. Let $p > 1$ be an integer. Let K_h be the union of the sets Γ_k with $k \equiv h \pmod{p}$. The circle is then the union of p classes K_h pairwise disjoint and permuted cyclically by the shift operation T_ω. These classes will play the same role as the arithmetic progressions considered in section 2. There is no objection to our attributing to them a measure.

These classes are everywhere dense, since each of the intersections $C \cap K_h$ is countable and everywhere dense. In fact $C \cap K_h$ is the set of points admitting an expression $x_0 + (h + np)\omega$, where x_0 is the central point of the class C and where the integer n varies from $-\infty$ to $+\infty$. Since each one of the $C \cap K_h$ is uniformly distributed over the circle, the class K_h which is their union will possess the same property. This observation will be made precise below.

Let i be an interval of length l. Then for all $\epsilon > 0$ there exists a $\lambda \in (0, \epsilon)$ having the form $\lambda = (h + np)\omega + n'$ for suitable integer n and n'. Since the classes K_h are invariant under the operation $T_{np\omega+n'}$ one can write $T_\lambda K_0 = T_{h\omega}K_0 = K_h$. Let i' be the interval $T_\lambda i$. Then $T_\lambda(i \cap K_0) = i' \cap K_h$. Moreover, $i \cap K_h$ and $i' \cap K_h$ can be obtained by adding to their intersection $i \cap i' \cap K_h$ two sets which are each contained in intervals of length $\lambda < \epsilon$, so that these sets are almost identical. For an extension of the Lebesgue measure, their measures differ by at most $\lambda < \epsilon$. If in addition we wish the extended measure to be invariant under T_ω, then the measures of $i \cap K_0$ and $i' \cap K_h$ are equal. Consequently the measure of $i \cap K_0$ and $i \cap K_h$ differ by at most $\lambda < \epsilon$, and so they are equal, their common value being necessarily l/p.

These remarks lead to an extension of the Lebesgue measure. Let B be the family of Lebesgue measurable sets. Since each $E \in B$ is, up to a set of arbitrarily small measure, a finite union of intervals, every $E \cap K_h$ has measure $\mu(E)/p$. These sets generate a Borel field B^* whose elements have the form

$$\mathcal{E} = \bigcup_h [E_h \cap K_h], \qquad E_h \in B; h = 1, 2, \cdots, p. \tag{1}$$

For every such $\mathcal{E}$ one can define, without any risk of contradiction, an extension $\mu^*(\mathcal{E})$ of $\mu(E)$ by the formula

$$\mu^*(\mathcal{E}) = \frac{1}{p} \sum_1^p \mu(E_h). \tag{2}$$

3.3. The preceding considerations can easily be generalized, replacing C_0 by the set of points $h_1\omega_1 + \cdots + h_n\omega_n$, where n is fixed, the ω_ν are linearly independent and where each h_ν varies from $-\infty$ to $+\infty$. With each subscript ν is associated an integer $p_\nu > 1$, then the number of distinct classes K is $p = p_1p_2 \cdots p_n$. However, the extension to the case where n is infinite presents some difficulties which can easily be overcome by a different approach, as will now be shown.

4. A new extension of μ

4.1. THEOREM (G. Hamel). *There exist uncountable sets of numbers ω_ν having the following property: every real number x can be represented in a unique manner as a finite sum of terms $a_\nu\omega_\nu$ where the coefficients a_ν are rational numbers.*

Let us recall the proof, which uses transfinite induction. The ω_ν must, of course, be linearly independent. That is, there does not exist among the ω_ν any linear homogeneous relation with integer coefficients. Consequently the same is true of rational coefficients and finite number of terms. This amounts to saying that a given number x has only at most one representation of the described form.

Suppose that a certain linearly independent set S' of numbers ω_ν has been constructed. If there be numbers x which are not representable in the required form, we can select one and adjoin it to the set S'. If all the x have been previously ordered in a transfinite sequence one can take the first number of the sequence which is not a finite sum $\sum a_\nu\omega_\nu$. The process is then continued, indefinitely and transfinitely, of adding new ω_ν to the set S. When this becomes impossible the desired goal is attained.

The set S obtained in this fashion will be called a *complete Hamel basis* and every $S' \subset S$ will be an *incomplete basis*. If R' is the set of x of the form $\sum a_\nu\omega_\nu$ where the ω_ν belong to S', we shall say that S' is the base of R'.

Once the Hamel basis $\{\omega_\nu\}$ has been selected, there will be no need for us to use transfinite induction again. Since this set has necessarily the power of the continuum, ν can be assumed to be a continuous parameter. The a_ν are well defined functions of x and they will be called the *rational components* of x. These components are always almost all equal to zero, only a finite number $N(x)$ of them being nonzero rational numbers. Suppose the set $\{\omega_\nu\}$ is divided in a completely arbitrary way into two complementary subsets S' and S''. The conditions imposed on the components a_ν of the $\omega_\nu \in S'$ are entirely independent of the conditions concerning the coefficients of the $\omega_\nu \in S''$.

Let us consider these two classes of numbers, those x', formed with the terms of the first set and those x'', formed with the terms of the second set. Every real number can be represented in one and only one way as a sum $x' + x''$. In other words the set R of real numbers is the *direct sum* of the sets R' and R'' having for bases S' and S'' respectively.

We shall assume that ω_0 is equal to unity. Under these circumstances, if x is the abscissa of a point on the circle of length unity, the fractional part of a_0 and all the other a_ν are well-determined functions of x.

4.2. We are now going to define two kinds of equivalence classes, denoted by letters C and Γ respectively. These classes will not be the same as in section 3 but they will behave in a similar way, in the sense that the points x will again be ordered in a two-way classification with a countable infinity of rows corresponding to the classes Γ and an uncountable infinity of columns, corresponding to the classes C.

For every rational r let Γ_ν^r be the set of x for which $a_\nu = r$. The classes C_ν on the other hand, are obtained by fixing all the rational components of x except a_ν and letting a_ν vary.

In each class of type C, the point x_0 selected as central point will be defined by the condition $a_\nu = 0$. Thus the theorem proved in (4.1) with the help of Zermelo's axiom of choice renders unnecessary a new use of this axiom. Furthermore, the choice made in the present fashion is rather special, since the set of central points x_0 is everywhere dense, although it would obviously have been possible to choose all these points in an interval of arbitrarily small length.

Obviously

$$T_{r\omega_\nu}\Gamma_\nu^0 = \Gamma_\nu^r, \tag{3}$$

so that, for each value of ν, all the classes Γ_ν^r are superposable by shift. The circle is the union of a countable infinity of disjoint superposable classes Γ_ν^r. As was the case for the Γ_h of section 3, these classes will not be measurable. However, in a certain sense, each one of them is only a negligible part of the circle. This is true in particular for Γ_ν^0.

This statement is valid for every value of ν. On the other hand for each x all the a_ν except a finite number are equal to zero. Hence, each x belongs to almost all the Γ_ν^0.

The combination of these two statements constitute a paradox reminiscent of Hausdorff's paradox, but much less surprising than the latter. Indeed, it is closely akin to the following, too simple to be surprising. Let N and N' be two positive integers "selected at random" independently of each other according to the prescriptions of section 2. If the value n of N is known, there is zero probability that $N' \leqq N$, or, in other words it is almost sure that $N' > N$. However, if N' is chosen first, exactly the opposite conclusion is reached. Similarly in the problem under consideration, if x is known, its nonzero rational components are known, and for a ν chosen at random it is almost sure that $a_\nu = 0$. If on the contrary ν is given and x is chosen at random, it is almost sure that $a_\nu \neq 0$. This paradox, like the preceding one, is only an example of two well-known facts. First, in infinite sets the meaning of the expressions "in general" or "almost always" depends in an essential manner on the ordering of the elements. Second, one easily obtains paradoxical statements by using probabilistic language in connection with measures which are not countably additive.

4.3. To define the classes K playing the same role as those considered in section 3, we shall first introduce the sets

$$U_\nu^s = \bigcup_{s \leq r < s+1} \Gamma_\nu^r, \tag{4}$$

where s represents an arbitrary integer and where ν is assumed to be different from zero, so that ω_ν is irrational. The set U_ν^s is a Vitali set to which the results of section 3 can be applied. The transforms of U_ν^s by the shift operations $T_{k\omega_\nu}$, for integers k in $-\infty$ to $+\infty$, are pairwise disjoint and cover the circle. (For $\nu = 0$ these transforms are all identical; each one of them covers the circle.)

For an integer $p_\nu > 1$ let h be defined mod p_ν and let

$$K_\nu^h = \bigcup \{U_\nu^s;\ s \equiv h \ (\text{mod}\ p_\nu)\}. \tag{5}$$

In this fashion the Γ_ν^r are classified into p_ν disjoint classes whose union is the circle and which are obtainable from each other by the shift T_{ω_ν}. It is natural to assign to each of them the measure $\mu^*(K_\nu^h) = 1/p_\nu$. The notation μ^* is used to indicate that this measure is not the Lebesgue measure.

This procedure can be applied further. Let $\mathcal{K}_n^h$ be the intersection of n sets of the $K_\nu^{h_\nu}$ type for n distinct values of $\nu \neq 0$. Let p be the product of the p_ν and let h and ω be the vectors whose components are the h_ν and the ω_ν respectively. Letting i be an interval of length l as in section 3, it is natural to define $\mu^*[i \cap \mathcal{K}_n^h] = l/p$. We shall show that this can be done without contradiction, and, further, that it is the only possibility if the measure μ^* is required to be invariant under the T_{ω_ν} shifts.

Each K_ν^h is invariant under all T_{ω_ρ} for $\rho \neq \nu$ as well as under $T_{p_\nu\omega_\nu}$. In addition $K_\nu^h = T_{h\omega_\nu}K_\nu^0$. If $\mathcal{K}_n^0$ is the set obtained when all h_ν are taken equal to zero in the definition of $\mathcal{K}_n = \mathcal{K}_n^h$ one can write

$$\mathcal{K}_n = T_{h\omega}\mathcal{K}_n^0, \tag{6}$$

where $h\omega$ denotes the sum $\sum h_\nu\omega_\nu$ taken over all the indices ν which occur in the definition of $\mathcal{K}_n^h$.

The other indices being denoted by ρ, the shifts $T_{p_\nu\omega_\nu}$ and T_{ω_ρ} do not alter anything. Consequently the same is true for all the shifts belonging to the group generated by the $T_{p_\nu\omega_\nu}$ and T_{ω_ρ}. It follows that, for every arbitrarily small ϵ, it is possible to find a number $\lambda \in (0, \epsilon)$ such that $\mathcal{K}_n^h = T_\lambda\mathcal{K}_n^0$, where T_λ is one of the elements of the group G generated by the T_{ω_ν} and the T_{ω_ρ}.

The consequences are the same as in section 3.2. The interval $i' = T_\lambda i$ differs very little from i. For any extension μ^* of the Lebesgue measure which is left invariant by the operations of the group G, the measure of $i' \cap \mathcal{K}_n^h = T_\lambda(i \cap \mathcal{K}_n^0)$

is equal to the measure of $i \cap \mathcal{K}_n^0$ and differs by at most $\lambda < \epsilon$ from the measure of $i \cap \mathcal{K}_n$. Hence

$$|\mu^*(i' \cap \mathcal{K}_n^h) - \mu^*(i \cap \mathcal{K}_n^h)| < \epsilon, \tag{7}$$

and, consequently,

$$\mu^*(i \cap \mathcal{K}_n^h) = \mu^*(i \cap \mathcal{K}_n^0) = \frac{l}{p}. \tag{8}$$

The p sets $\mathcal{K}_n^h = T_{h\omega}\mathcal{K}_n^0$ and the Lebesgue sets then define a Borel field B^* whose elements have the general form

$$\mathcal{E} = \bigcup_h (E_h \cap \mathcal{K}_n^h), \tag{9}$$

where the E_h are Lebesgue measurable. Furthermore,

$$\mu^*(\mathcal{E}) = \frac{1}{p} \sum_h \mu(E_h). \tag{10}$$

4.4. Suppose now that the integer n and the subscripts ν which occur in the definition of the $\mathcal{K}_n^h$ are left variable, while, for every fixed ν, the ω_ν and p_ν are fixed. A natural idea would be to seek an extension of μ^* to a Borel field which would contain all the K_ν^h, hence all the $\mathcal{K}_n^h$. We shall show that *such an extension is impossible. The function μ^* cannot be well defined and countably additive for the sets $\mathcal{E}$ of such a family.*

To prove this it is sufficient to consider a countable infinity of values of ν. Identify the μ^*-probability that a point X chosen at random on the circle belongs to K_ν^0 with $\mu^*(K_\nu^0) = 1/p_\nu$. The complementary probability is $1 - 1/p_\nu \geqq 1/2$. It follows from (8) that, for every finite n, the n events of the form $X \in K_\nu^0$ are independent. Then the countable additivity of μ^* would imply, according to Borel's lemma, that the number of nonzero components a_ν is almost surely infinite. However, the number of such components is always finite.

It is therefore necessary to restrict ourselves to the study of finite unions $\mathcal{K}'$ of sets of the $\mathcal{K}_n^h$ type. This, however, does not prevent the consideration of infinite sequences of sets $\mathcal{K}_n'$ corresponding to disjoint measurable sets E_n. For each of these sequences the union

$$\mathcal{E} = \bigcup_n E_n \cap \mathcal{K}_n' \tag{11}$$

will be given the measure

$$\mu^*(\mathcal{E}) = \sum_1^\infty \mu(E_n)\mu^*(\mathcal{K}_n'). \tag{12}$$

Let $\mathfrak{F}$ be the set of all $\mathcal{E}$ having the representation (11). It is easily verified that if $\mathcal{E}_1$ and $\mathcal{E}_2$ belong to $\mathfrak{F}$ the same is true of $\mathcal{E}_1 \cup \mathcal{E}_2$ and $\mathcal{E}_1 \cap \mathcal{E}_2$. In addition the measure defined by (12) is countably additive.

4.5. Returning to the effect of shifts on the $\mathcal{E} \in \mathfrak{F}$ and on μ^*, the group G

defined by the T_{ω_ν} leaves $\mathfrak{F}$ and μ^* invariant. To study the effect of other shifts it is sufficient to investigate the effect on each a_ν of the shifts $T_{c\omega_\nu}$ with c rational, $c \in (0, 1)$. The $T_{c\omega_\rho}$ with $\rho \neq \nu$ do not alter a_ν. The operation $T_{c\omega_\nu}$ changes a_ν to $a_\nu + c$ and consequently, on the axis of components a_ν, changes $[h, h + 1)$ to $[h + c, h + 1 + c)$ and transforms the class K_ν^h into a set which overlaps K_ν^h and K_ν^{h+1}. This set does not belong to the family $\mathfrak{F}$.

This does not prevent our regarding μ^* as invariant under $T_{c\omega_\nu}$ and consequently under all shifts. The measure will then be defined for every $\mathcal{E}$ obtained by shift from the elements of $\mathfrak{F}$. Let $\mathfrak{F}'$ be the family of such sets. The extension of μ^* to $\mathfrak{F}'$ does not give rise to any contradiction. One can, in particular, assign to each set of values of a_ν having the form $\{a + kp_\nu\}$, where k is an integer ranging from $-\infty$ to $+\infty$, any probability one pleases with the sole restriction that this probability should not change if a is replaced by $a + 1$. On the axis of the component a_ν an interval of length 1 has then always a probability $1/p_\nu$ of covering one of the $a + kp_\nu$. Consequently the addition to a of an arbitrary rational number does not alter this probability.

On the other hand the relations $\mathcal{E}_1 \in \mathfrak{F}'$ and $\mathcal{E}_2 \in \mathfrak{F}'$ do not imply $\mathcal{E}_1 \cap \mathcal{E}_2 \in \mathfrak{F}'$. For instance, the intersection of $[h + c, h + 1 + c)$ and $[h + 1, h + 2)$ is the interval $[h + 1, h + 1 + c)$ defining a set $\mathcal{E}$ which does not belong to $\mathfrak{F}'$. One cannot always assign to this set a probability equal to its length since the probability distribution is discrete. It is therefore possible to consider a distribution on the rational numbers of the interval $[0, p_\nu)$ only if one gives up the requirement of homogeneity on the axis of components a_ν and the requirement of invariance under the shifts $T_{c\omega_\nu}$ for rational nonintegral c. In any case such a distribution would be absolutely arbitrary, whereas so far we have introduced only very rational assumptions. It must not be forgotten that the choice of the values ω_ν and p_ν were already arbitrary choices.

Bearing this in mind, and considering each a_ν as defined mod p_ν, nothing prevents us from renouncing invariance under the shifts T_{ω_ν} and distributing the probability in a completely arbitrary manner on the rationals of the interval $[0, p_\nu)$. It should be noted that under such circumstances p_ν and ω_ν enter only through their product.

The same operation can be performed for each subscript ν. Then every possible system of numbers a_ν defined mod p_ν, or rather, of numbers $a_\nu\omega_\nu$ defined mod $p_\nu\omega_\nu$, corresponding to a finite set of values of ν has a well-defined probability α. Further, for each $E \in B$ the μ^*-measure of the set of values of x for which such conditions are satisfied is equal to $\alpha\mu(E)$.

One can even generalize further by relaxing the condition that the a_ν be stochastically independent and independent of the intervals i from which x is chosen. Thus the Lebesgue measure can be extended very far and in very many ways. However, the most interesting extensions seem to be the measure which is invariant under the elements of G and the measure defined in $\mathfrak{F}'$ and invariant under all these shifts.

5. Hamel functions

5.1. Consider two systems of numbers $\{\omega_\nu\}$ and $\{\omega'_\nu\}$ satisfying the conditions given in the statement of the theorem of section 4. Between two such systems there must exist relations of the type

$$\omega'_\nu = \sum_\rho c_{\nu,\rho}\omega_\rho, \qquad \omega_\nu = \sum_\rho c'_{\nu,\rho}\omega'_\rho, \tag{13}$$

where the coefficients $c_{\nu,\rho}$ and $c'_{\nu,\rho}$ are rational and every such sum has only a finite number of nonzero terms. A natural problem presents itself in the comparison of the two representations

$$x = \sum a_\nu\omega_\nu = \sum a'_\nu\omega'_\nu \tag{14}$$

of the same number x. Formally,

$$a_\nu = \sum_\rho c_{\rho,\nu}a'_\nu. \tag{15}$$

Since there can be, for each ν, an uncountable infinity of nonzero coefficients $c_{\rho,\nu}$, this formula does not make sense in general. However, the a'_ν are different from zero only for a finite set of values of ν, so that, for the systems of values to which the formula will be used, no difficulty will be encountered. The same applies to the inverse formula

$$a'_\nu = \sum_\rho c'_{\rho,\nu}a_\rho. \tag{16}$$

One remarkable case occurs when the coefficients $c_{\nu,\rho}$ are zero whenever $\rho \neq \nu$. Then, for each ν, one can write $a_\nu\omega_\nu = a'_\nu\omega'_\nu$ and, consequently, $\omega'_\nu = c_\nu\omega_\nu$, where c_ν is rational.

5.2. A second problem arises when, for the same rational coefficients a_ν, one writes

$$x = \sum a_\nu\omega_\nu, \qquad y = \sum a_\nu\omega'_\nu. \tag{17}$$

Under the conditions given above these formulas define a one-to-one correspondence between x and $y = f(x)$. However, in the sequel we shall not use the assumption that the ω'_ν form a complete Hamel basis. In this case y remains a well-defined function of x but the correspondence is not one-to-one.

This function f will be called a Hamel function in the sequel. It possesses the very simple and characteristic property already noted by Hamel, that $f(u + v) = f(u) + f(v)$. More generally, if λ and μ are rational numbers, then $f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$. However, except for the very special case where $\omega'_\nu = c\omega_\nu$ with c independent of ν, the function f is very irregular. When the condition $\omega'_\nu = c\omega_\nu$ is satisfied for all ν except $\nu = 1$ but when $\omega'_1 - c\omega_1 = (c_1 - c)\omega_1 \neq 0$ one can write

$$f(x) = cx + a_1(c_1 - c)\omega_1. \tag{18}$$

Even in this case, since a_1 is not μ^*-measurable, the function f is not μ^*-measurable. A μ^*-measurable function is obviously a measurable function of μ and of a finite number of coefficients h_ν. It seems interesting to seek new extensions of the Lebesgue measure, independently of those given by (12), for which f would be measurable.

5.3. We shall assume that $f(1) = 1$ and that $f(x)$ is not of the form cx. This condition can be ensured, in particular, by taking $\omega_0 = \omega_0' = 1$, as was already assumed in section 4 for ω_0. The fractional part of a_0 and of all the other a_ν then depends only on the fractional part of n. The fractional parts of x and $f(x)$ are then in a one-to-one correspondence. That is, if $x \in [0, 1)$, the function $g(x) = f(x) \pmod 1$ is a one-to-one map of the interval $[0, 1)$ into itself.

The image by g of an interval i of length l and the inverse image i^* of an interval j of length l' are both everywhere dense in $[0, 1)$. In a sense to be elucidated, they can each be regarded as uniformly distributed in this interval. More precisely, consider two intervals j_1 and j_2 both interior to $[0, 1)$ and each having length l'. Whatever the number $\epsilon > 0$ one can find a $\lambda \in (0, \epsilon)$ such that $f(\lambda)$ differs by less than ϵ from any previously given number. Since $f(x + \lambda) = f(x) + f(\lambda)$, the λ can be chosen in such a manner that in the (x, y) plane the transformation $x \to x + \lambda$, $y \to y + f(\lambda)$, which leaves the graphs of $y = f(x)$ and $y = g(x)$ invariant, will bring the rectangle $i \times j_2$ onto a rectangle $i' \times j_1'$ differing very little from $i \times j_1$, corresponding sides being at distances less than ϵ. For any extension of the Lebesgue measure the difference between the measures of $i' \times j_1'$ and $i \times j_1$ tends to zero as $\epsilon \to 0$. If in addition the measure is required to be invariant under the transformation $x \to x + \lambda$, then the sets defined by $x \in i$, $g(x) \in j_2$, and $x \in i'$, $g(x) \in j_1'$ must have the same measure. Therefore, the subsets of i defined respectively by $g(x) \in j_1$ and by $g(x) \in j_2$ have the same measure equal to ll'. In other words

$$\mu^*(i \cap i^*) = m(i \times j), \tag{19}$$

where m denotes the Lebesgue measure in the plane. Of course the measure μ^* so defined does not coincide with the one described in sections 4.3 to 4.5.

Consider now a measurable plane set E inside the square $Q = [0, 1) \times [0, 1)$. The set E is constructed from rectangles $i \times j$ by means of disjoint unions, by subtraction and by the addition of a subset of a null set obtained by the first two operations. The same operations performed on the linear sets $i \cap i^*$ lead to a set $\mathcal{E}$. Since each set $i \times i^*$ has a μ^*-measure equal to the measure $m(i \times j)$ of the corresponding rectangle, one is led to define $\mu^*(\mathcal{E})$ by the formula

$$\mu^*(\mathcal{E}) = m(E). \tag{20}$$

Such is the formula which appears to give the most interesting extension of the Lebesgue measure. (It must be remembered that this extension depends on the choice of the ω_ν and the ω_ν'. No particular extension has a pre-eminent claim to be chosen; indeed there are an infinity of possible extensions, none of which is more natural than the others, and we have no reason to believe that they may be compatible.) However, it should be noticed that formula (20) can be valid only if $m(E)$ depends on $\mathcal{E}$ alone, or, equivalently, if the formula cannot ascribe a positive measure to an empty set. Therefore, whether or not f is a Hamel function it is necessary that there be no set of positive measure included in the complement in Q of the graph $\mathcal{G}$ of $g(x)$. Without introducing this complementary set, let us say that $\mathcal{G}$ has maximum outer measure to indicate that for every set E the outer measure of $E \cap \mathcal{G}$ is equal to its maximum possible value, namely $m(E)$.

When this condition is satisfied it becomes obvious that the measure μ^* is well defined and countably additive in a Borel field. If in addition $f(x)$ is a Hamel function, this measure is invariant under shifts.

6. Properties of the graphs of Hamel functions

6.1. It remains then to find out whether the graph $\mathcal{G}$ [that is, the set of points $(x, y) \in Q$ such that $y = g(x)$] can have maximum outer measure. A very simple argument shows that if f is a Hamel function *either this graph has maximum outer measure, or it has measure zero.* The argument is the same as the one leading to formula (18). Taking x and y to be defined modulo unity, the graph $\mathcal{G}$ is invariant under all the shifts of an everywhere dense set. One concludes from this that the outer measure of $\mathcal{G} \cap (i \times j)$ is invariant under all shifts, and hence of the form kll' for some constant k. Let $\overline{\mathcal{G}}$ be a measurable set such that $\mathcal{G} \subset \overline{\mathcal{G}}$ and such that $m(\overline{\mathcal{G}})$ be equal to the outer measure $\bar{m}(\mathcal{G})$ of $\mathcal{G}$. As is well known, a measurable set has almost everywhere a well-defined density equal to unity in the set and to zero outside it. In the present case the density of $\overline{\mathcal{G}}$ is equal to k. Hence $k = 0$ or $k = 1$ and finally $\bar{m}(\mathcal{G}) = 0$ or $\bar{m}(\mathcal{G}) = 1$.

The same result is also valid for the graph $\mathcal{F}$ of the function f itself. From now on we shall be concerned mostly with $\mathcal{F}$; however, it is to be noted that formula (20) acquires a meaning only through the function $g(x)$. Furthermore, it is easily seen that $\mathcal{F}$ and $\mathcal{G}$ have simultaneously measure zero or maximum outer measure.

6.2. Let us show first that $\mathcal{F}$ can have measure zero. According to a remark of H. Cartan, such is the case in the preceding situation, when $y - cx$ reduces to a single term $(\omega'_\nu - c\omega_\nu)a_\nu$, admitting only a countable infinity of distinct values. The same conclusion still holds whenever all the differences $\omega'_\nu - c\omega_\nu$ are algebraic numbers. Indeed, $y - cx$ is then the sum of a finite number of terms of the type $(\omega'_\nu - c\omega_\nu)a_\nu$, hence algebraic. Much more generally, suppose that there exists a (zero or nonzero) constant c such that the numbers $(\omega'_\nu - c\omega_\nu)a_\nu$ constitute the Hamel base of a set of measure zero. Then, the graph $\mathcal{F}$ is entirely contained in a family of straight lines $y - cx = c'$ intersecting the y-axis in a null set. Thus $\mathcal{F}$ has measure zero.

We shall see that the other possibility, $k = 1$, occurs under conditions of a considerably less restrictive character. While at present it is not known, in the most general case, whether $k = 0$ or 1, the case $k = 1$ seems to be the most general. It is natural to ask whether the sufficient condition we have just indicated is not also necessary for k to be zero. This problem is as yet unsolved.

6.3. That k can be equal to unity is a consequence of the following theorem due to H. Cartan and G. Choquet and communicated to me by H. Cartan in a letter of December 22, 1959.

THEOREM. *The additive group of real numbers is the direct sum of a countable family of subgroups R_n which all have maximum outer measure.*

To prove this let $\{P_\nu\}$ be a transfinite ordering of the perfect subsets of R. By perfect subset we mean here a subset which is identical with its set of accumulation points. Since the family of perfect sets has the cardinality of the continuum, this ordering can be performed using only the ordinals ν which precede the first ordinal whose cardinality is the continuum. In each P_ν we shall choose a countable infinity of numbers $\omega_{\nu,n}$. Let us agree that (ρ, p) precedes (ν, n) if either ρ precedes ν or if $\rho = \nu$ and $p < n$. Select each $\omega_{\nu,n}$ in such a way that $\omega_{\nu,n}$ does not belong to the additive group $R_{\nu,n}$ whose Hamel base is constructed by the previously chosen $\omega_{\rho,p}$. Such a selection is possible. Indeed, the cardinality of the set of $\omega_{\rho,p}$ is strictly less than that of the continuum and so is the cardinality of $R_{\nu,n}$. Therefore, there is in the set P_ν whose cardinality is the continuum, at least one point which does not belong to $R_{\nu,n}$.

Once the selection has been accomplished, the set of $\omega_{\nu,n}$ is the Hamel basis of the set $R' \subset R$ which is the union of the $R_{\nu,n}$. If R' is a proper subset of R, a complete Hamel basis can be constituted by adjunction of new numbers ω'_ν. Define S_n as the union of the set of $\omega_{\nu,n}$ whose second subscript is n and of a subset of the ω'_ν. These latter may be distributed among the S_n in an arbitrary fashion. R_n is the group of real numbers having base S_n. Since the union S of the disjoint sets S_n is a Hamel basis of R the group R is the direct sum of the R_n.

On the other hand every closed uncountable subset of the line includes one of the perfect sets P_ν. Thus this set contains all the $\omega_{\nu,n}$ whose first subscript is ν and therefore intersects all the S_n and a fortiori all the R_n. Each R_n having a nonempty intersection with every uncountable closed subset of R has maximum outer measure, since otherwise it could be covered by a family of open intervals whose complement would be an uncountable closed set. This completes the proof of the theorem.

6.4. Let us return now to the graph $\mathfrak{F}$. It follows from the preceding theorem that two sets R' and R'' can be found, each having maximum outer measure, and such that R is the direct sum of R' and R''. One can for instance put $S' = S_1$ and $S'' = \bigcup\{S_n, n = 2, 3, \cdots\}$ and then take for R' and R'' the sets having respectively S' and S'' for bases. Then, as was shown in section 4 every real x possesses one and only one representation of the form $x = x' + x''$ with $x' \in R'$ and $x'' \in R''$. Let us now assume that $\omega'_\nu = c'\omega_\nu$ if $\omega_\nu \in S'$ and $\omega'_\nu = c''\omega_\nu$ with $c'' \neq c'$ if $\omega_\nu \in S''$. Then $y = f(x) = c'x' + c''x''$. Consider the intersection of $\mathfrak{F}$ by the straight line D defined by $y - c'x = (c'' - c')x''$, with $x'' \in R''$ fixed. Since x' and x'' vary independently, the projection of this intersection on the x-axis is the set $x = x' + x''$ with x'' fixed and $x' \in R'$. Since R' has maximum outer measure, so also does this projection. If $\mathfrak{F}$ had measure zero such a situation would occur only for values x'' forming a set of measure zero. However, since the property holds for every $x'' \in R''$ the graph $\mathfrak{F}$ cannot have measure zero. It follows from this that $k > 0$, hence $k = 1$, so that the graph $\mathfrak{F}$ has maximum outer measure.

6.5. The preceding argument, due to H. Cartan, does not rely at all on the fact that every x can be written $x = x' + x''$. We can therefore use the representation $x = \sum x_n$, where $x_n \in R_n$, obtainable from the theorem of section 6.3 and apply the preceding result for instance with $x' = x_1$ and $x'' = x_2$. The set of points $x = x_1 + x_2$, $y = c'x_1 - c''x_2$ has maximum outer measure. However, this set is only a proper subset of the graph of $f(x)$, namely the subset obtained by equating to zero all the x_n for $n > 2$. A fortiori, the graph of f has maximum outer measure.

This extension of H. Cartan's result shows that, in order to ensure that k be equal to unity, it is sufficient to impose on the ratios ω'_ν/ω_ν restrictions which involve only a negligible part of the set of indices ν. Indeed, there exists an infinity of disjoint sets S_n, any two of which can play the role of S_1 and S_2. For the indices ν which do not belong to the chosen sets one can take for the ω'_ν arbitrary values, without requiring that they form a complete Hamel basis, or even that they be linearly independent.

Sufficient conditions for k to be equal to unity seem therefore to be relatively lax. This explains the conjecture made in section 6.2 although, for the values $x_1 + x_2$ of x, the condition $f(x) = c'x_1 + c''x_2$ is very restrictive.

The same remark would also apply to functions $f(x)$ other than Hamel functions. If $f(x) = c'x_1 + c''x_2$ for $x = x_1 + x_2$ the graph of f has maximum outer measure and formula (16) defines a completely additive set function.

7. Probability theory

An application of these results to probability theory is immediate but leads to rather paradoxical results. In all the preceding arguments one can replace the Lebesgue measure by the joint distribution of two random variables X and Y. Provided the joint distribution be absolutely continuous nothing is changed. If X and Y are linked by the relation $Y = f(x)$ one can define a μ^*-probability by the formula

$$P^*\{\mathcal{E}\} = P\{E\} \tag{21}$$

corresponding to formula (20). One can in particular assume that the laws of X and Y are independent provided they be absolutely continuous. *Thus the deterministic relation $Y = f(x)$ does not preclude the stochastic independence of X and Y.*

Clearly this remark cannot extend to discontinuous distributions. If there exists a value ξ such that $P\{X = \xi\} = \alpha > 0$ then $Y = f(x)$ implies $P\{Y = f(\xi)\} \geqq \alpha$ with equality if $f(x) = f(\xi)$ implies $x = \xi$. In the completely discontinuous (discrete) case the law of X determines that of Y.

We have not studied the case where the law of X is merely continuous. It would be interesting to know what distributions for Y can be associated with such laws.

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