THE MATHEMATICAL FORMULATION OF STRATEGIC PROBLEMS

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THIS PAPER does not record the results of exhaustive research, leading to the development of promising new techniques, which a group of scholars such as this has a right to expect. Nor is it a mathematical paper. It does, however, touch upon a need which is all too evident in various human activities today, namely, the application of analytical methods to the broad major problems which confront our world, and illustrates this need by discussion of a particular problem which was uppermost in the minds of many for some time—how to win a war. The paper, furthermore, is addressed to mathematicians, since they seem best equipped to take the necessary next step of specifying, for any given problem, what data are relevant, and of showing how to manipulate those data in such a way that the probable consequences of our actions may be predicted. It is to be hoped that your interest, both in a discussion of war and in the broader implications of that discussion, may outweigh the shortcomings and inadequacies of the following remarks.

Total war, in which all the people of a nation, and to a lesser degree all nations, are involved, is not alone a conflict between armed forces; it is a struggle between complex organizations of men. Rates of expenditure of all kinds are likely to reach such proportions that reliance cannot be placed in stocks of finished goods. Within the limits of man's ingenuity, production must be so organized that maximum expenditures can be maintained, the only restrictive factor being the availability of manpower and raw materials. A nation thus resembles a living, growing organism that survives by healing its wounds. Under these circumstances war no longer is a question of fixed destruction, of defeating an army, but becomes a race between destruction and production. In this sense it has some of the elements of the problem of emptying a bathtub into which a tap is pouring water. How much effort should be expended on turning off the tap and how much on opening the drain? In order to bring out this aspect of the problem, particular attention must be given to long-range bombardment, attacking the national productive system and thus shutting off the source of supply so that the problem of attrition in combat may be undertaken with some assurance of finality.

In this discussion the central theme is strategy, or how to use available resources to maximum advantage in any given situation. Important as the details of the situation are, their study is relevant to the selected central theme only in supplying the basic data for arriving at strategic decisions. What kinds of data do we need, and what kinds of answers are we likely to get in response to questions about the component parts of the problem?

For a complex synthesis such as we are interested in, it would seem advisable to reduce the component parts to their barest essentials, adding complicating factors only after some insight has been gained into the methods of handling drastically oversimplified analogous problems. It seems desirable, however, to review these component parts in some detail in order to indicate the nature of the complexities that will have to be introduced at a later stage.

The simplest of these component problems is that of physical vulnerability, or the specification of the amount of physical damage produced by a weapon, or a group of weapons in some known geometrical arrangement, employed in attack on material objects whose physical properties can be determined by direct measurement. The units in which such damage may be specified are various: square feet of floor space destroyed, degree of damage to equipment, number of people killed and wounded, and so forth. Similarly, the means for specifying geometrical arrangements and distributions are various, and none are completely satisfactory for all purposes. But these are problems of complication, and, as will be seen in what follows, the errors introduced by oversimplification in the description of damage are negligible compared to uncertainties resulting from inadequate treatment of other factors.

One of these factors is the estimation of the effect of physical damage on production. The obvious first approximation is to assume that, apart from reconstruction, production loss is proportional to total damage inflicted. This very crude assumption is used in what follows, but it is obviously inadequate for other than illustrative purposes. The problem is one of great importance to economists, and methods developed by Dr. W. W. Leontief at Harvard, together with data which are being assembled by the Bureau of Labor Statistics, give hope of providing a more satisfactory solution. This consists of dividing the productive system into a number of parts, and specifying the requirements of each part per unit of production. Since requirements of one part of industry are the end products of others, this scheme results in a set of simultaneous equations. These equations turn out to be linear, and it is therefore possible to specify the physically practicable methods of operating an economic system for which the necessary constants are known. In time of war these physical limits are especially important, since the desires of individual producers and consumers play a relatively unimportant part. There is, then, some hope that we may be able to specify the changes in these physical limits of production due to any specified distribution of physical damage among the components of industry. It must be remembered that the operation of an economic system is flexible, and that no unique definition of production in terms of size and constitution of the productive establishment is possible. But once the limits on possible modes of operation are given, we have a good foundation for discussing the strategic question of how it can be operated most advantageously from the point of view of winning a war.

Once these two factors of physical and economic vulnerability are established, the next problem is to relate them to each other. The solution of parts of this problem is in sight, and requires only the interest of competent men. Two kinds of questions need to be answered.

a) Given targets of specified value, physical vulnerability, and geographical distribution, and given the physical properties and aiming characteristics of weapons to be used in their destruction, how should effort be distributed among these targets in order to maximize the damage produced?

b) A productive system is being progressively destroyed. What fraction of its activities should be assigned to recuperation in order to maximize its total useful output at some later date? The same kind of question is of interest to the attacker. Given the effectiveness of attack on factories and end products, what division of destructive effort will minimize the total production at some future date?

The two last questions taken separately can be answered precisely. When, however, both attacker and defender are free to choose how they will use their destructive and productive efforts, a new element is introduced into the problem, since the best course for the one depends on what the other does. This introduces the elements of a game, as discussed at length by von Neumann and Morgenstern in their recent book *The Theory of Games and Economic Behavior*. Since the object of this game is to win a war rather than to keep production up or down, further discussion is postponed to permit prior consideration of the really vital element, conflict between armed forces.

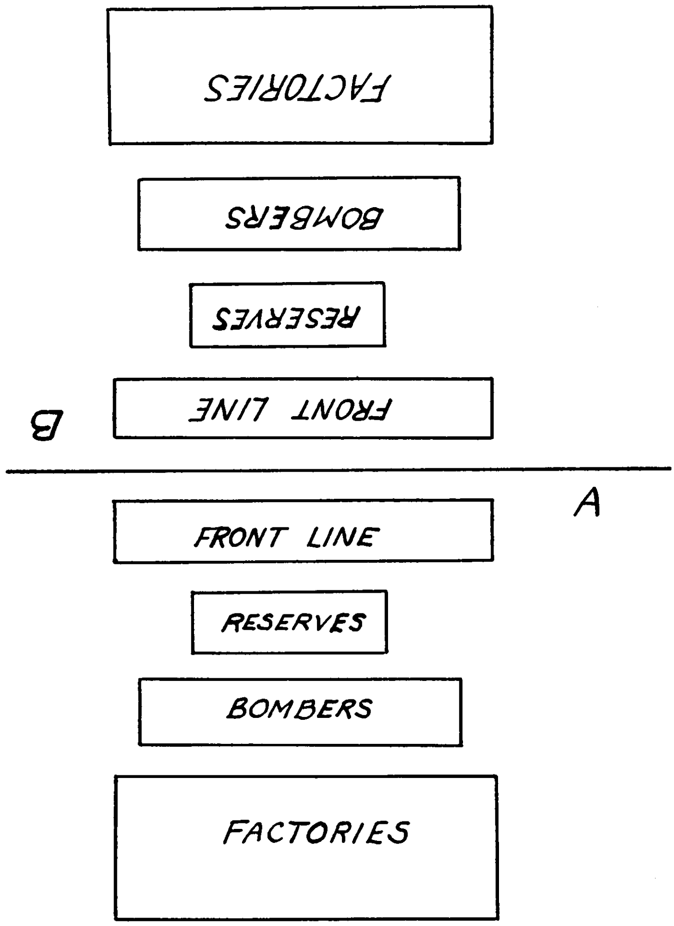
From the point of view of this analysis, conflict between armed forces determines, first, rates of attrition, or the demands put upon the economic system, and finally, which side wins. The simplest assumption seemingly compatible with the above is that, as long as neither side is strongly superior, rates of expenditure of both sides are approximately equal and are determined by that side desiring the greater attrition. When one side has a sufficient superiority it decisively defeats the other.

While this simplified version of armed combat may suffice for considerations concerned primarily with attacks on production, it misses the important problem of how to use reserves. In fact, there is always the possibility of defeat in any engagement, particularly under conditions of approximate equality. It is therefore undesirable to commit all available strength to combat until conditions are sufficiently favorable to make the risk of defeat negligible. And this factor is important in considerations involving long-range weapons, because strength in reserve, though not exposed to loss by combat, is exposed to direct and indirect attack from the air (and from the sea, as we had cause to know). The problem of the proper use of reserves is of great interest in its own right. It is essentially, again, a game, and analysis of a simple related game may prove useful in establishing principles, or doctrine. For example, a generalization of matching pennies might be appropriate, in which each player could risk any part of his holdings on any throw, the probability of losing being somehow adjusted to his "front-line superiority," or the relative amounts at stake.

The multiplicity of problems and complications presented may make a combination of the separate factors into a single problem appear hopeless. Whether or not the difficulties are at present insurmountable remains to be seen. It is, however, possible to present the over-all problem in fairly simple form, and to show how solutions may be of practical value.

The essential elements of the problem are: (1) the role of the armed forces in producing attrition and in deciding victory, (2) attack on production and stocks by long-range bombardment, and (3) the role of the national economy in supplying the armed forces, and in national survival in the face of bombardment.

This simplification omits coalitions, disregards limitations on the supply of raw materials and manpower, and discounts the importance of ingenuity, skill,



and doubtless many other vital factors. It retains, however, enough of the vital ingredients to be of considerable interest. In this form, the problem may be represented as a game, of which two versions are presented. The first, called "Total War," includes all of the elements that are considered essential. The second, called "Target Selection," is a simplified version in which the use of reserves is greatly simplified, and risk of defeat, except against overwhelming front-line superiority, is avoided. This version emphasizes the economic aspects of war. For these games a board like that illustrated in the accompanying figure is used.

"*Total War.*"—Each player starts out with 4 counters representing factories, 4 counters representing bombers, and 4 counters representing armies deployed in defense of his cities, partly in the front line and partly in reserves.

The game starts by having each player fly bomber sorties in attacks on factories or on reserves, as he may elect. Not over one-half of the available bomber counters may be used in one turn. Counters representing bomber losses are given by each player to his opponent in return for an equal number of counters taken from the opponent's factories or reserve.

The next step consists in having each player take counters from a central pool equal in number to the number of his factories. This "production" can be distributed in any desired manner among factories, bombers, and armies, but, once the assignment has been made, it cannot be changed. The decision with respect to the allocation of production is made in ignorance of the opponent's decision in this regard, for instance, by having each player indicate his choice on a slip of paper.

RULES FOR DETERMINING THE OUTCOME OF BATTLE, USING ONE DIE

Outcome of battle	Relative front-line strengths		
	1/1 to 5/4	Greater than 5/4	
		Not greater than 2/1	Greater than 2/1
Side A (the stronger side) loses all; opponent loses nothing	1 or 2	1	......
Both sides lose at least one-half	3 or 4	2 or 3	1 or 2
Side B (the weaker side) loses all; opponent loses nothing	5 or 6	4, 5, or 6	3, 4, 5, or 6

After production counters have been placed on the board by both sides, either player may force one or more battles between the ground forces in the front lines. The outcome of any one battle is to be determined by weighted probabilities according to some scheme such as that shown in the table above.

Upon the conclusion of the battle each side may move up reserves or withdraw troops from the front line at will until both sides are satisfied. The front line may not be left empty. If desired, a further battle may then take place.

The previous steps of bombardment, production, and battle (if either side desires it) are repeated until one or the other army is eliminated.

The object of the game is not only to win by destroying the enemy's armies, but to minimize the cost of victory.

"*Target Selection.*"—In order to bring out more clearly the factors involved in target selection, the game "Total War" may be simplified by eliminating the elements of risk and tactical front-line maneuver, thereby producing a situation in which the role of front-line strength is primarily to produce attrition of reserve stocks.

In this simplified form of the game, each side starts with 4 counters representing factories, 4 counters representing bombers, 4 counters representing reserve equipment for ground forces, and 10 counters representing fully equipped combat divisions in the front line (this number is assumed to be the maximum number which each side has the manpower to maintain.)

The game begins, as in "Total War," by having each player use not more than one-half of his available bomber sorties in attacks on factories or reserve stocks. Chips representing bomber losses are exchanged on a one-for-one basis with chips representing targets attacked.

Each player then supplies himself secretly with a number of chips representing factories, bombers, or reserve stocks, and equal in number to the total number of factories at his disposal.

Thereupon either player may call for front-line attrition up to one-half the number of equipped divisions he has in the front line. Front-line chips are then exchanged on a one-for-one basis equal in number to the larger of the above demands. Following this, front-line troops may be re-equipped by moving chips out of the reserve stock pool into the front line. The total number of chips in the front line may not exceed the original ten.

The sequence of events is then repeated until one side or the other has no troops in the front line.

It is clear that games of this kind are capable of almost endless variation. The significance of a detailed analysis of any one game is primarily that it may throw light on how to tackle a whole group of games as a class—how to make general statements defining sound procedure even when complete knowledge of all relevant conditions is lacking. Only general rules would be of value in practice, since the conditions of warfare, the effectiveness of effort expended and the probabilities involved, may in reality be expected to change at any time and are never precisely known. It is further interesting to observe that doctrine is at present expressed in statements having very general applicabilty, such as, for instance, "Attack an enemy's most menacing strength," or "Establish progressively more favorable conditions for combat." Is it possible to give a sounder basis for any particular item of doctrine? In the analysis of games there appears to be promise of answering this question.

Further, studies along the lines suggested could at least provide us with rules for games that are more carefully thought out than the examples given. Even if a complete analysis is not possible, familiarity with a variety of games can undoubtedly produce a facility which military-staff colleges may wish to impart to students. And finally, experience on the part of mathematicians in setting up and solving problems of this kind will probably prove of value at some future time in connection with other matters of interest to us all.