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PROJECTING ON POLYNOMIAL DIRAC SPINORS

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Abstract. In this note we adapt Axler and Ramey's method of constructing the harmonic part of a homogeneous polynomial to the Fischer decomposition associated to Dirac operators acting on polynomial spinors. The result yields a constructive solution to a Dirichlet-like problem with polynomial boundary data.

It is well-known [3] that any homogeneous real or complex polynomial p_k of degree $k = 0, 1, 2, \dots$ in $n \geq 2$ real variables $x = (x_1, x_2, \dots, x_n)$ admits an unique decomposition

$$p_k(x) = h_k(x) + |x|^2 p_{k-2}(x) \tag{1}$$

where h_k is a homogeneous harmonic polynomial of degree k, p_{k-2} is a homogeneous polynomial of degree $k-2$, and, as usual, $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

In [1] Axler and Ramey presented an elegant, elementary way of constructing h_k from p_k, which involves only differentiation. In essence, for $k > 0$

$$h_k(x) = \begin{cases} c_k^{-1}|x|^{2k} p_k(D)(\log|x|), & \text{if } n = 2 \\ c_k^{-1}|x|^{n-2+2k} p_k(D)(|x|^{2-n}), & \text{if } n > 2 \end{cases} \tag{2}$$

where

$$c_k = \begin{cases} (-2)^{k-1}(k-1)!, & \text{if } n = 2 \\ \prod_{j=0}^{k-1}(2-n-2j), & \text{if } n > 2 \end{cases} \tag{3}$$

and where $p_k(D)$ is the associated partial differential operator acting on smooth functions defined on open subsets of $\mathbb{R}^n$ obtained by replacing a typical monomial $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, $\alpha_1 + \alpha_2 + \cdots + \alpha_n = k$, of p_k by $\dfrac{\partial^k}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}$.

As a by-product they obtained a speedy solution to the Dirichlet problem on the unit ball of $\mathbb{R}^n$ with polynomial boundary data which eliminates the use of the impractical Poisson integral.

The purpose of this note is to establish similar results when polynomials are replaced by polynomial spinors and harmonic polynomials by polynomial Dirac spinors, i.e., polynomial solutions of Dirac equations.

To this end consider an action of the real Clifford algebra $\mathrm{Cl}_n := \mathrm{Cl}(\mathbb{R}^n)$ on some complex space $\mathbb{C}^N$. Equivalently, one is presented with n skew-Hermitian $N \times N$ complex matrices $E_1, E_2, \ldots, E_n$ such that for every i, $E_i^2 = -\mathrm{Id}$ and $E_iE_j + E_jE_i = 0$, for every $i \neq j$. The Euclidean Dirac operator is then the differential operator

$$\not{D} : C^\infty(U, \mathbb{C}^N) \longrightarrow C^\infty(U, \mathbb{C}^N), \quad U \subseteq \mathbb{R}^n \text{ open}$$

defined for spinors $s \in C^\infty(U, \mathbb{C}^N)$ written in column form by

$$\not{D}s = \sum_{i=1}^{n} E_i \frac{\partial s}{\partial x_i}$$

where $\frac{\partial s}{\partial x_i}$ represents component-wise differentiation of s with respect to x_i. It is easily seen that $\not{D}$ is a self-adjoint first order elliptic differential operator satisfying the following properties:

$$\not{D}(fs) = \operatorname{grad} f \cdot s + f\, \not{D}s, \quad f \in C^\infty(U, \mathbb{C}), \quad \operatorname{grad} f \cdot s := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} E_i s \quad (4)$$

$$\not{D}^2 = -\Delta, \quad \text{where } \Delta \text{ is the component-wise Laplacian on } C^\infty(U, \mathbb{C}^N). \quad (5)$$

Denote now by P_k the subspace of $C^\infty(\mathbb{R}^n, \mathbb{C}^N)$ consisting in spinors with polynomial components, homogeneous of degree k, and by H_k the subspace of P_k consisting in polynomial Dirac spinors, i.e.,

$$H_k := \{p_k \in P_k;\ \not{D}(p_k) = 0\}.$$

Clearly, $\not{D}(P_k) \subseteq P_{k-1}$ ($P_{-1} = 0$). If one denotes by $x\cdot$ the Clifford multiplication in $\mathbb{C}^N$ by $x \in \mathbb{R}^n$, i.e., $x \cdot v = \sum_{i=1}^{n} x_i E_i v$, $v \in \mathbb{C}^N$, then $x \cdot P_k \subseteq P_{k+1}$.

Lemma. *Let $h_k \in H_k$ be a polynomial Dirac spinor of degree k. Then*

$$\not{D}(x \cdot h_k) = -(n + 2k)h_k. \quad (6)$$

Consequently, $x \cdot h_k$ has harmonic components.

Proof: Since $x\cdot = \text{grad}\left(\frac{|x|^2}{2}\right)\cdot$ and $\not{D}h_k = 0$, equations (4) and (5) give

$$\not{D}(x\cdot h_k) = \not{D}\left(\text{grad}\left(\frac{|x|^2}{2}\right)\cdot h_k\right) = \not{D}^2\left(\frac{|x|^2}{2}h_k\right) - \not{D}\left(\frac{|x|^2}{2}\not{D}h_k\right)$$
$$= -\Delta\left(\frac{|x|^2}{2}h_k\right).$$

However,

$$\Delta\left(\frac{|x|^2}{2}h_k\right) = \Delta\left(\frac{|x|^2}{2}\right)h_k + 2\sum_{i=1}^{n}\frac{\partial\left(\frac{|x|^2}{2}\right)}{\partial x_i}\frac{\partial h_k}{\partial x_i} + \frac{|x|^2}{2}\Delta h_k$$
$$= nh_k + 2\sum_{i=0}^{n}x_i\frac{\partial h_k}{\partial x_i} - \frac{|x|^2}{2}\not{D}^2h_k = (n+2k)h_k$$

since for homogeneous polynomials of degree k, $\sum_{i=1}^{n} x_i\frac{\partial h_k}{\partial x_i} = kh_k$. This proves equation (6). By (5) and (6), $\Delta(x\cdot h_k) = -\not{D}^2(x\cdot h_k) = (n+2k)\not{D}h_k = 0$. The proof of the Lemma is complete. □

The following theorem, called sometimes the Fischer decomposition for polynomial spinors [2], holds now true:

Theorem 1. *Any element p_k of P_k can be uniquely decomposed as*

$$p_k(x) = h_k(x) + x\cdot p_{k-1}(x) \tag{7}$$

for suitable $h_k \in H_k$ and $p_{k-1} \in P_{k-1}$.

Although proofs of the Fischer decomposition exist in much more general settings [2] we will reprove it here in a way that is beneficial for what follows.

Proof: Any polynomial spinor $p_k \in P_k$ can be written, by applying equation (1) component-wise, as

$$p_k(x) = \alpha_k(x) + |x|^2\beta_{k-2}(x)$$

where $\alpha_k \in P_k$ has harmonic components and $\beta_{k-2} \in P_{k-2}$. As a result, equation (5) gives $\not{D}^2\alpha_k = 0$. We claim that for a suitable constant $\lambda \in \mathbb{Q}$, to be determined, $\alpha_k + \lambda x\cdot \not{D}\alpha_k$ is a polynomial Dirac spinor of degree k. Indeed, since $\not{D}\alpha_k \in H_{k-1}$, by the Lemma, $\not{D}(\alpha_k + \lambda x\cdot \not{D}\alpha_k) = (1-\lambda(n+2k-2))\not{D}\alpha_k$, and so $\alpha_k + \lambda x\cdot \not{D}\alpha_k \in H_k$ if

$$\lambda = \begin{cases} 0, & \text{if } k = 0 \\ \dfrac{1}{n-2+2k}, & \text{if } k > 0. \end{cases} \tag{8}$$

Setting now $h_k := \alpha_k + \lambda x\cdot \not{D}\alpha_k$ and $p_{k-1} := -\lambda \not{D}\alpha_k - x\cdot\beta_{k-2}$, λ given by (8), proves the existence part of Theorem 1 because $x\cdot x\cdot = -|x|^2$.
The uniqueness part is equivalent to showing that if $0 = h_k + x\cdot p_{k-1}$, $h_k \in H_k$, $p_{k-1} \in P_{k-1}$, then $h_k = 0$ and $p_{k-1} = 0$. It follows that

$$0 = -x\cdot h_k + |x|^2 p_{k-1} \tag{9}$$

and invoking the above Lemma again $-x\cdot h_k$ has harmonic polynomial components of degree $k+1$. $h_k = 0$ and $p_{k-1} = 0$ follow now from the uniqueness of the decomposition (1) in degree $k+1$, applied component-wise to equation (9). □

Theorem 2. *In the Fischer decomposition* (7), $h_0 = p_0$ *and for* $k > 0$ h_k *can be calculated from* p_k *according to the rule*

$$h_k(x) = \begin{cases} -c_{k+1}^{-1}|x|^{2k}x\cdot \not{D}(p_k(D)(\log|x|)), & \textit{if} \quad n=2 \\ -c_{k+1}^{-1}|x|^{n-2+2k}x\cdot \not{D}(p_k(D)(|x|^{2-n})), & \textit{if} \quad n>2 \end{cases}$$

where c_k *is given by* (3) *and* $p_k(D)$ *is the spinor-valued partial differential operator defined according to the recipe following equation* (3).

Proof: By Theorem 1 and equation (2), for $k > 0$ we have

$$h_k = \alpha_k + \frac{1}{n-2+2k}x\cdot \not{D}\alpha_k$$

where

$$\alpha_k(x) = \begin{cases} c_k^{-1}|x|^{2k}p_k(D)(\log|x|), & \text{if } n=2 \\ c_k^{-1}|x|^{n-2+2k}p_k(D)(|x|^{2-n}), & \text{if } n>2. \end{cases}$$

Noticing now that for every $n \geq 2$ we can write $\alpha_k = c_k^{-1}|x|^{n-2+2k}\sigma_k$, where for $x \neq 0$

$$\sigma_k(x) = \begin{cases} p_k(D)(\log|x|), & \text{if } n=2 \\ p_k(D)(|x|^{2-n}), & \text{if } n>2 \end{cases}$$

we have, via equation (4),

$$\begin{aligned} x\cdot \not{D}\alpha_k &= x\cdot \not{D}\left(c_k^{-1}|x|^{n-2+2k}\sigma_k\right) \\ &= c_k^{-1}x\cdot \text{grad}\left(\left(|x|^2\right)^{\frac{n}{2}-1+k}\right)\cdot\sigma_k + c_k^{-1}|x|^{n-2+2k}x\cdot \not{D}\sigma_k \\ &= c_k^{-1}\left(\frac{n}{2}-1+k\right)|x|^{n-4+2k}x\cdot \text{grad}\left(|x|^2\right)\cdot\sigma_k + c_k^{-1}|x|^{n-2+2k}x\cdot \not{D}\sigma_k \\ &= c_k^{-1}(n-2+2k)|x|^{n-4+2k}x\cdot x\cdot\sigma_k + c_k^{-1}|x|^{n-2+2k}x\cdot \not{D}\sigma_k \\ &= c_k^{-1}(n-2+2k)|x|^{n-4+2k}(-|x|^2)\cdot\sigma_k + c_k^{-1}|x|^{n-2+2k}x\cdot \not{D}\sigma_k \\ &= -(n-2+2k)\alpha_k + c_k^{-1}|x|^{n-2+2k}x\cdot \not{D}\sigma_k. \end{aligned}$$

Consequently,

$$h_k = \frac{1}{n-2+2k} c_k^{-1} |x|^{n-2+2k} x \cdot \not{D} \sigma_k = -c_{k+1}^{-1} |x|^{n-2+2k} x \cdot \not{D} \sigma_k$$

which is the Theorem 2 claim. □

By iterating the Fischer decomposition (7) we conclude that every homogeneous polynomial spinor $p_k \in P_k$ can be uniquely represented as

$$p_k = \sum_{j=0}^{k} \underbrace{x \cdot x \cdot \ldots x \cdot}_{j \text{ times}} h_{k-j} \tag{10}$$

for suitable $h_k \in H_k$, $h_{k-1} \in H_{k-1}$, ..., $h_0 \in H_0$. Equation (10) is useful in assessing constructively when the following Dirichlet-like problem for the Dirac operator has solution (see [3, 1] for the harmonic case).

Corollary. *For a given $v \in P_k$ the Dirichlet-like problem on the closed unit ball B_n in $\mathbb{R}^n$ with polynomial boundary data v,*

$$\not{D} u = 0, \qquad u_{|\partial B_n} = v_{|\partial B_n} \tag{11}$$

has a solution $u \in C^\infty(B_n, \mathbb{C}^N)$ if and only if in the decomposition (10) *for v the odd part $\sum_{j \text{ odd}} \underbrace{x \cdot x \cdot \ldots x \cdot}_{j \text{ times}} h_{k-j}$ vanishes. When a solution exists it is unique and it is a polynomial spinor* (*not necessarily homogeneous*) *which can be constructed explicitly by employing the rule given in Theorem 2 to v. More precisely, referring again to the decomposition* (10) *for v,*

$$u = h_k - h_{k-2} + h_{k-4} - \ldots$$

Proof: Assume that $u \in C^\infty(B_n, \mathbb{C}^N)$ is a solution for (11). Since $\not{D}^2 = -\Delta$, u is then also a solution to the usual **Dirichlet problem**

$$\Delta u = 0, \qquad u_{|\partial B_n} = v_{|\partial B_n}. \tag{12}$$

However, (12) has an unique solution [3, 1] which can be obtained in the following way: via (10), express v uniquely as $v = \sum_{j=0}^{k} \underbrace{x \cdot x \cdot \ldots x \cdot}_{j \text{ times}} h_{k-j}$, for suitable $h_k \in H_k$, $h_{k-1} \in H_{k-1}$, ..., $h_0 \in H_0$. Since $x \cdot x \cdot = -|x|^2$, on $\partial B_n = \{|x| = 1\}$ we have

$$\begin{aligned} v(x) &= (h_k(x) + x \cdot h_{k-1}(x)) - |x|^2 (h_{k-2}(x) + x \cdot h_{k-3}(x)) \\ &\quad + |x|^4 (h_{k-4}(x) + x \cdot h_{k-5}(x)) - \cdots = (h_k(x) + x \cdot h_{k-1}(x)) \\ &\quad - \left(h_{k-2(x)} + x \cdot h_{k-3}(x)\right) + (h_{k-4}(x) + x \cdot h_{k-5}(x)) - \ldots \end{aligned}$$

and so the Lemma implies that

$$(h_k + x{\cdot}h_{k-1}) - (h_{k-2} + x{\cdot}h_{k-3}) + (h_{k-4} + x{\cdot}h_{k-5}) + \dots$$

is the solution u of (12). The Lemma also gives

$$\not{D}u = -(n+2k-2)h_{k-1} + (n+2k-6)h_{k-3} - (n+2k-10)h_{k-5} + \dots \quad (13)$$

and since by the original hypothesis $\not{D}u = 0$, the vanishing of the right hand side of equation (13) is easily seen to be equivalent to $\sum_{j \text{ odd}} \underbrace{x{\cdot}x{\cdot}\dots x{\cdot}}_{j \text{ times}} h_{k-j} = 0$.

Conversely, if $\sum_{j \text{ odd}} \underbrace{x{\cdot}x{\cdot}\dots x{\cdot}}_{j \text{ times}} h_{k-j} = 0$ then $v = \sum_{j \text{ even}} \underbrace{x{\cdot}x{\cdot}\dots x{\cdot}}_{j \text{ times}} h_{k-j}$. As before, on ∂B_n,

$$\sum_{j \text{ even}} \underbrace{x{\cdot}x{\cdot}\dots x{\cdot}}_{j \text{ times}} h_{k-j}(x) = h_k(x) - h_{k-2}(x) + h_{k-4}(x) - \dots$$

and so $u := h_k - h_{k-2} + h_{k-4} - \dots$ is a solution to (11). The uniqueness of u follows from the fact that it is also (the unique) solution for (12). Clearly Theorem 2 allows the construction of h_k, then h_{k-2}, then h_{k-4}, etc., therefore, the construction of u. □

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