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ON THE INVERSE PROBLEM OF THE SCATTERING THEORY FOR A BOUNDARY-VALUE PROBLEM

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Abstract. In the present work the inverse problem of the scattering theory for Sturm-Liouville differential equation with a spectral parameter in the boundary condition is investigated. The Gelfand–Marchenko–Levitan fundamental equation is obtained, the uniqueness of the solution of the inverse problem is proved and some properties of the scattering data are given.

1. Introduction

We consider the boundary problem generated by the differential equation

$$-y'' + q(x)y = \lambda^2 y \quad (0 < x < \infty) \tag{1}$$

and the boundary condition

$$(\alpha_2 + \mathrm{i}\beta_2\lambda)y'(0) - (\alpha_1 + \mathrm{i}\beta_1\lambda)y(0) = 0 \tag{2}$$

where $q(x)$ is a real-valued function satisfying the condition

$$\int_0^{+\infty} (1 + x)|q(x)|\,\mathrm{d}x < \infty \tag{3}$$

and α_i, β_i $(i = 1, 2)$ are real numbers such that

$$\delta = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} > 0.$$

In the present work the **inverse problem of scattering theory** (IPST) for the boundary problem of (1)–(3) is investigated. For the equation (1) IPST was completely solved in [6], [9], [10] when the boundary condition (2) was not including any spectral parameter. When the boundary condition (2) was including a spectral parameter, the similar problem was discussed in [7], [8] and the inverse problem with respect to the spectral function was investigated in [12], also with respect to the Weyl function was investigated in [13]. In finite intervals, the inverse problem with respect to distinct characterizations for the spectral parameter dependent boundary conditions has been considered by many authors (see [1], [2], [3], [11]). The physical applications of such problems were given [4].

According to [10], for any λ from closed upper-half plane, the equation (1) has a solution $e(\lambda, x)$ described by

$$e(\lambda, x) = \mathrm{e}^{\mathrm{i}\lambda x} + \int_x^{+\infty} K(x,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \tag{4}$$

and for the **kernel function** $K(x,t)$ the inequality

$$|K(x,t)| \leq \frac{1}{2}\sigma(\frac{x+t}{2})\exp\left\{\sigma_1(x) - \sigma_1\left(\frac{x+t}{2}\right)\right\}$$

holds where

$$\sigma(x) \equiv \int_x^{+\infty} |q(t)|\,\mathrm{d}t, \qquad \sigma_1(x) \equiv \int_x^{+\infty} \sigma(t)\,\mathrm{d}t.$$

Moreover,

$$K(x,x) = \frac{1}{2}\int_x^{+\infty} q(t)\,\mathrm{d}t.$$

The solution $e(\lambda, x)$ is an analytic function of λ in the upper half plane $\operatorname{Im}\lambda \geq 0$ and is continuous on the real line. The following estimates hold through the half plane $\operatorname{Im}\lambda \geq 0$

$$\begin{gathered} |e(\lambda,x)| \leq \exp\{-\operatorname{Im}\lambda x + \sigma_1(x)\} \\ |e(\lambda,x) - \mathrm{e}^{\mathrm{i}\lambda x}| \leq \left\{\sigma_1(x) - \sigma_1\left(x + \frac{1}{|\lambda|}\right)\right\}\exp\{-\operatorname{Im}\lambda x + \sigma_1(x)\} \end{gathered} \tag{5}$$

and

$$|e'(\lambda,x) - \mathrm{i}\lambda\mathrm{e}^{\mathrm{i}\lambda x}| \leq \sigma(x)\exp\{-\operatorname{Im}\lambda x + \sigma_1(x)\}. \tag{6}$$

For real $\lambda \neq 0$, the functions $e(\lambda, x)$ and $e(-\lambda, x)$ form a fundamental system of solutions of the equation (1) and their Wronskian is equal to $2\mathrm{i}\lambda$

$$W\{e(\lambda,x), e(-\lambda,x)\} = e'(\lambda,x)e(-\lambda,x) - e(\lambda,x)e'(-\lambda,x) = 2\mathrm{i}\lambda.$$

2. Main Results

Let $\omega(\lambda, x)$ be a solution of the equation (1) satisfying the initial-value conditions

$$\omega(\lambda, 0) = \alpha_2 + \mathrm{i}\beta_2\lambda, \qquad \omega'(\lambda,0) = \alpha_1 + \mathrm{i}\beta_1\lambda.$$

Lemma 1. *The identity*

$$\frac{2\mathrm{i}\lambda\omega(\lambda,x)}{(\alpha_2 + \mathrm{i}\beta_2\lambda)e'(\lambda,0) - (\alpha_1 + \mathrm{i}\beta_1\lambda)e(0,\lambda)} = \overline{e(\lambda,x)} - S(\lambda)e(\lambda,x) \tag{7}$$

holds for all real $\lambda \neq 0$ *where*

$$S(\lambda) = \frac{(\alpha_2 + \mathrm{i}\beta_2\lambda)\overline{e'(\lambda,0)} - (\alpha_1 + \mathrm{i}\beta_1\lambda)\overline{e(\lambda,0)}}{(\alpha_2 + \mathrm{i}\beta_2\lambda)e'(\lambda,0) - (\alpha_1 + \mathrm{i}\beta_1\lambda)e(0,\lambda)}$$

and

$$\overline{S(\lambda)} = S(-\lambda).$$

Proof: It can be easily proved in similar way the proof of the Theorem 1 in [8]. □

Let

$$E(\lambda) = (\alpha_2 + \mathrm{i}\beta_2\lambda)e'(\lambda,0) - (\alpha_1 + \mathrm{i}\beta_1\lambda)e(0,\lambda).$$

Lemma 2. *The function* $E(\lambda)$ *may have only a finite number of zeros on the half plane* $\operatorname{Im}\lambda > 0$*, they are all simple and lie on the imaginary.*

Proof: Since $E(\lambda) \neq 0$ for all real $\lambda \neq 0$, the point $\lambda = 0$ is the possible real zero of the function $E(\lambda)$. Since the function $E(\lambda, 0)$ is analytic in the upper half plane $\operatorname{Im}\lambda > 0$ we have that the zeros of $E(\lambda)$ are at most countable. Now to show that the set of the zeros of $E(\lambda)$ is bounded we assume, by way of contradiction, that this set is bounded, so that there exist the numbers λ_k such that these numbers satisfy the relation $E(\lambda_k) = 0$ for $\operatorname{Im}\lambda_k > 0$ and $|\lambda_k| \to \infty$ or

$$e'(\lambda_k, 0) = \frac{\alpha_1 + \mathrm{i}\beta_1\lambda_k}{\alpha_2 + \mathrm{i}\beta_2\lambda_k} e(\lambda_k, 0).$$

On the other hand, taking $x = 0$, $\lambda = \lambda_k$ in the inequality (6) we have

$$|e'(\lambda_k, x) - \mathrm{i}\lambda_k| \leq \sigma(0)\exp\{\sigma_1(0)\}$$

or

$$\left| \frac{\alpha_1 + \mathrm{i}\beta_1\lambda_k}{\alpha_2 + \mathrm{i}\beta_2\lambda_k} e(\lambda_k, 0) - \mathrm{i}\lambda_k \right| \leq \sigma(0)\exp\{\sigma_1(0)\}.$$

Thus, we can write

$$|\lambda_k| \leq \left| \frac{\alpha_1 + \mathrm{i}\beta_1\lambda_k}{\alpha_2 + \mathrm{i}\beta_2\lambda_k} e(\lambda_k, 0) \right| + \sigma(0)\exp\{\sigma_1(0)\}.$$

According to (5) since $\lim_{k\to\infty} e(\lambda_k, 0) = 1$, the right side of the last inequality has a finite limit. This contradiction shows that the set $\{\lambda_k\}$ is bounded. Hence, the set of zeros of the function $E(\lambda)$ is bounded and form at most countable set having 0 the only possible limit point.

Now, we shall show that all the zeros of the function $E(\lambda)$ lie in the imaginary axis. Suppose that λ_1 and λ_2 are two arbitrary zeros of the function $E(\lambda)$. Multiplying the first of the relations

$$-e''(\mu_1, x) + q(x)e(\mu_1, x) = \mu_1^2 e(\mu_1, x)$$
$$-\overline{e''(\mu_2, x)} + q(x)\overline{e(\mu_2, x)} = \overline{\mu}_2^2 \overline{e(\mu_2, x)}$$

by $\overline{e(\mu_2,x)}$ and the second relation by $e(\mu_1,x)$, subtracting the second resulting relation from the first, and integrating the resulting difference from zero to infinity, we obtain

$$(\mu_1^2-\overline{\mu}_2^2)\int_0^{+\infty} e(\mu_1,x)\overline{e(\mu_2,x)}\,\mathrm{d}x - W[e(\mu_1,x),\overline{e(\mu_2,x)}]\Big|_{x=0}=0. \tag{8}$$

On the other hand, since

$$e(\mu_j,x)=\frac{1}{\delta}[\beta_2 e'(\mu_j,0)-\beta_1 e(\mu_j,0)]\omega(\mu_j,x) \tag{9}$$

then

$$\begin{aligned}&W[e(\mu_1,x),\overline{e(\mu_2,x)}]\Big|_{x=0}\\&\quad=\frac{1}{\delta}[\beta_2 e'(\mu_1,0)-\beta_1 e(\mu_1,0)]\cdot[\beta_2 e'(\mu_2,0)-\beta_1 e(\mu_2,0)](\overline{\mu}_2^2-\mu_1^2).\end{aligned}$$

Thus, the equality (8) takes of the form

$$\begin{aligned}(\mu_1^2-\overline{\mu}_2^2)\bigg\{&\int_0^{+\infty} e(\mu_1,x)\overline{e(\mu_2,x)}\,\mathrm{d}x\\&+\frac{1}{\delta}[\beta_2 e'(\mu_1,0)-\beta_1 e(\mu_1,0)]\cdot\overline{[\beta_2 e'(\mu_2,0)-\beta_1 e(\mu_2,0)]}\bigg\}=0.\end{aligned} \tag{10}$$

Specially, taking $\mu_2=\overline{\mu}_1$ in (10) we have $\mu_1^2-\overline{\mu}_2^2=0$ and obtain $\mu_2=\mathrm{i}\lambda_1$ where $\lambda_1>0$. That is, the zeros of the function $E(\lambda)$ lie in the imaginary axis.

Now we shall prove that the function $E(\lambda)$ has finitely many zeros in the half plane $\operatorname{Im}\lambda>0$. This is obvious if $E(0)\neq 0$ because under this assumption the set of zeros cannot have any limit point. To verify that the number of zeros of $E(\lambda)$ is finite in the general case too, we show that the distance between neighboring zeros is bounded away from zero.

We let δ denote the infimum of the distances between two neighboring zeros of $E(\lambda)$ and show next that $\delta>0$. Using the same way in [7], [8] it can be easily shown that $\delta>0$. Thus, the function $E(\lambda)$ has finitely many zeros.

From the equality

$$\begin{aligned}m_k^{-2}&=\int_0^{+\infty}[e(\mathrm{i}\lambda_k,x)]^2\,\mathrm{d}x+\frac{[\beta_2 e'(\mathrm{i}\lambda_k,0)-\beta_1 e(\mathrm{i}\lambda_k,0)]^2}{\delta}\\&=\frac{1}{2\mathrm{i}\mu_k\delta}[\beta_2 e'(\mathrm{i}\lambda_k,0)-\beta_1 e(\mathrm{i}\lambda_k,0)]\dot{E}(\mathrm{i}\lambda_k).\end{aligned} \tag{11}$$

it follows that the zeros of the function $E(\lambda)$ are simple. The lemma is proved. □

We need also the following lemma.

Lemma 3. *The function $S(\infty)-S(\lambda)$ is the Fourier transform of a function $F_S(x)$ in the form*

$$F_S(x) = F_S^{(1)}(x) + F_S^{(2)}(x)$$

where

$$F_S^{(1)}(x) \in L_1(-\infty,+\infty),\ F_S^{(2)}(x) \in L_2(-\infty,+\infty),\quad \sup_{-\infty<x<\infty} |F_S^{(2)}(x)| < \infty$$

and

$$S(\infty) = \begin{cases} -1, & \text{if } \beta_2 \neq 0 \\ 1, & \text{if } \beta_2 = 0. \end{cases}$$

Proof: From the formula (4) it follows that

$$e(\lambda,0) = 1 + \int_0^{+\infty} K(0,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t$$
$$e'(\lambda,0) = \mathrm{i}\lambda - K(0,0) + \int_0^{+\infty} K_x(0,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t.$$

We shall use the following notations

$$q_0 = K(0,0), \qquad \varphi_0(\lambda) = (\alpha_2 + \mathrm{i}\beta_2\lambda)\mathrm{i}\lambda - (\alpha_1 + \mathrm{i}\beta_1\lambda)$$
$$K_1(t) = \alpha_2 K_x(0,t) - \alpha_1 K(0,t), \qquad K_2(t) = \beta_2 K_x(0,t) - \beta_1 K(0,t)$$

and

$$\widetilde{K_j}(-\lambda) = \int_0^{+\infty} K_j(t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t, \qquad j = 1,2.$$

Let $\beta_2 \neq 0$. Then we have

$$\begin{aligned} S(\infty) - S(\lambda) &= -[1 + S(\lambda)] \\ &= \frac{T(\lambda)}{\varphi_0(\lambda) - q_0(\alpha_2 + \mathrm{i}\beta_2\lambda) + \widetilde{K_1}(-\lambda) + \mathrm{i}\lambda\widetilde{K_2}(-\lambda)} \end{aligned} \tag{12}$$

where

$$\begin{aligned} T(\lambda) = {} & 2(\alpha_1 + \mathrm{i}\beta_1\lambda) + 2q_0(\alpha_2 + \mathrm{i}\beta_2\lambda) - [\widetilde{K_1}(-\lambda) + \mathrm{i}\lambda\widetilde{K_2}(-\lambda)] \\ & - \widetilde{K_1}(\lambda) - \mathrm{i}\lambda\widetilde{K_2}(\lambda). \end{aligned}$$

Every one of the functions

$$\widetilde{f}_j(\lambda) = \frac{\alpha_j + \mathrm{i}\beta_j\lambda}{\varphi_0(\lambda)} q_0^{j-1}, \qquad j = 1,2, \qquad \widetilde{f}^{\pm}(\lambda) = \frac{\widetilde{K_1}(\pm\lambda) + \mathrm{i}\lambda\widetilde{K_2}(\pm\lambda)}{\varphi_0(\lambda)}$$

is the Fourier transformation of a summable function. Hence we have

$$S(\infty) - S(\lambda) = \frac{\widetilde{f}(\lambda)}{1 + \widetilde{K}(-\lambda)}$$

where

$$\widetilde{f}(\lambda) = 2\widetilde{f}_1(\lambda) + 2\widetilde{f}_2(\lambda) + \widetilde{f}^-(\lambda) + \widetilde{f}^+(\lambda)$$
$$\widetilde{K}(-\lambda) = -\widetilde{f}_2(\lambda) + \widetilde{f}(\lambda).$$

We can rewrite the formula (12) in the form

$$\begin{aligned} S(\infty) - S(\lambda) = \widetilde{f}(\lambda)\left[\left\{1 + \left(1 - \widetilde{h}(\lambda N^{-1})\right)\widetilde{K}(-\lambda)\right\}^{-1} - 1\right] + \widetilde{f}(\lambda) \\ - \widetilde{f}(\lambda)\left\{\frac{1}{1 + \left\{1 - \widetilde{h}(\lambda N^{-1})\widetilde{K}(-\lambda)\right\}} - \frac{1}{1 + \widetilde{K}(-\lambda)}\right\} \end{aligned} \tag{13}$$

where

$$\widetilde{h}(\lambda) = \begin{cases} 1 & \text{if } |\lambda| \le 1 \\ 2 - |\lambda| & \text{if } 1 \le \lambda \le 2 \\ 0 & \text{if } |\lambda| > 2 \end{cases}$$

is the Fourier transform of a function $h(x) \in L_1(-\infty, +\infty)$, also, $\widetilde{h}(\lambda N^{-1})$ is the Fourier transform of the function $h_N(x) = Nh(xN)$, and

$$\lim N \to \infty \|f(x) - h_N * f(x)\|_{L_1} = 0$$

for all $f(x) \in L_1(-\infty, +\infty)$, where $h_N * f(x)$ is the convolution of functions $h_N(x)$ and $f(x)$ from $L_1(-\infty, \infty)$: $h_N * f(x) = \int_{-\infty}^{+\infty} h_N(x-t) f(t)\, \mathrm{d}t$.

The function $\left\{1 + \left(1 - \widetilde{h}(\lambda N^{-1})\right)\widetilde{K}(-\lambda)\right\}^{-1}$ is the Fourier transform of a summable function for sufficiently large numbers N. In this case the summation of the first two terms of the formula (13) is the Fourier transform of a function $F_S^{(1)}(x) \in L_1(-\infty, +\infty)$. For $|\lambda| > 2$ the third term equals to zero and bounded. As such, it is the Fourier transform of a bounded function $F_S^{(2)}(x) \in L_2(-\infty, +\infty)$. For $\beta_2 = 0$ the statement can be proved similarly and in this way the lemma is proved as well. □

Now we shall obtain a linear integral equation for the kernel function $K(x,t)$ of the special solution (4). For this we use the equality (7) proved in Lemma 1

$$\frac{2\mathrm{i}\lambda\omega(\lambda, x)}{E(\lambda)} = e(-\lambda, x) - S(\lambda)e(\lambda, x).$$

Using (4) in this relation we obtain

$$\begin{aligned} \frac{2\mathrm{i}\lambda\omega(\lambda, x)}{E(\lambda)} &= \mathrm{e}^{-\mathrm{i}\lambda x} - S(\infty)\mathrm{e}^{\mathrm{i}\lambda x} + \int_x^{+\infty} K(x,t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t + [S(\infty) - S(\lambda)]\mathrm{e}^{\mathrm{i}\lambda x} \\ &+ \int_x^{+\infty} K(x,t)[S(\infty) - S(\lambda)]\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}t - S(\infty)\int_x^{+\infty} K(x,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \end{aligned}$$

where

$$S(\infty)=\begin{cases}-1 & \text{if } \beta_2\neq 0\\ \ \ 1 & \text{if } \beta_2=0.\end{cases}$$

If $\beta_2\neq 0$ the last equality takes the form

$$\begin{aligned}\frac{2\mathrm{i}\lambda\omega(\lambda,x)}{E(\lambda)}-2\cos\lambda x &= \int_x^{+\infty}K(x,t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t+[S(\infty)-S(\lambda)]\mathrm{e}^{\mathrm{i}\lambda x}\\ &+\int_x^{+\infty}K(x,t)[S(\infty)-S(\lambda)]\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t-S(\infty)\int_x^{+\infty}K(x,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t\end{aligned}\tag{14}$$

and if $\beta_2=0$ it takes the form

$$\begin{aligned}\frac{2\mathrm{i}\lambda\omega(\lambda,x)}{E(\lambda)}+2\sin\lambda x &= \int_x^{+\infty}K(x,t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t+[S(\infty)-S(\lambda)]\mathrm{e}^{\mathrm{i}\lambda x}\\ &+\int_x^{+\infty}K(x,t)[S(\infty)-S(\lambda)]\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t-S(\infty)\int_x^{+\infty}K(x,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t.\end{aligned}\tag{15}$$

We let multiple both sides of the equalities (14) and (15) by $\frac{1}{2\pi}\mathrm{e}^{\mathrm{i}\lambda y}$ and integrate from $-\infty$ to $+\infty$ with respect to λ. Taking $y>x$, by Lemma 3 on the right side we have

$$K(x,y)+F_S(x+y)+\int_x^{+\infty}K(x,t)F_S(t+y)\,\mathrm{d}t$$

and on left side using Jordan's lemma and the residue theorem we have

$$\mathrm{i}\sum_{k=1}^{n}\frac{\mathrm{i}2\mathrm{i}\lambda_k\omega(\mathrm{i}\lambda_k,x)}{\dot{E}(\mathrm{i}\lambda_k)}\mathrm{e}^{-\lambda_k y}.$$

Using (9) and (11) the last statement can be rewritten as

$$\begin{aligned}-\sum_{k=1}^{n}\frac{2\mathrm{i}\lambda_k\omega(\mathrm{i}\lambda_k,x)}{\dot{E}(\mathrm{i}\lambda_k)}\mathrm{e}^{-\lambda_k y} &= -\sum_{k=1}^{n}\frac{2\mathrm{i}\lambda_k\delta e(\mathrm{i}\lambda_k,x)\mathrm{e}^{-\lambda_k y}}{[\beta_2 e'(\mathrm{i}\lambda_k,0)-\beta_1 e(\mathrm{i}\lambda_k,0)]\dot{\varphi}(\mathrm{i}\lambda)}\\ &= -\sum_{k=1}^{n}m_k^2 e(\mathrm{i}\lambda_k,x)e^{-\lambda_k y}\\ &= -\sum_{k=1}^{n}m_k^2\left\{\mathrm{e}^{-\lambda_k(x+y)}+\int_x^{+\infty}K(x,t)\mathrm{e}^{-\lambda_k(t+y)}\,\mathrm{d}t\right\}.\end{aligned}$$

Thus, for $y>x$ we obtain

$$\begin{aligned}&-\sum_{k=1}^{n}m_k^2\left\{\mathrm{e}^{-\lambda_k(x+y)}+\int_x^{+\infty}K(x,t)\mathrm{e}^{-\lambda_k(t+y)}\,\mathrm{d}t\right\}\\ &\qquad = F_S(x+y)+K(x,y)+\int_x^{+\infty}K(x,t)F_S(t+y)\,\mathrm{d}t\end{aligned}$$

or

$$F(x+y) + K(x,y) + \int_x^{+\infty} K(x,t)F(t+y)\,\mathrm{d}t = 0 \tag{16}$$

where

$$F(x) = \sum_{k=1}^{n} m_k^2 \mathrm{e}^{-\lambda_k x} + \frac{1}{2\pi}\int_x^{+\infty} [S(\infty) - S(\lambda)]\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x. \tag{17}$$

The equation (16) is called the fundamental equation for the boundary problem (1)–(3).

Hence, we have proved the following theorem.

Theorem 1. *For all $x \geq 0$ the kernel function $K(x,y)$ of the solution* (4) *satisfies the fundamental equation* (16).

As it is seen that, to construct the fundamental equation we have to know the function $F(x)$. The function $F(x)$ itself is determined by the set $\{S(\lambda)\ (-\infty < \lambda < \infty);\ \lambda_k, m_k\ (k = 1, \dots, n)\}$.

Definition. *The set $\{S(\lambda)\ (-\infty < \lambda < \infty);\ \lambda_k, m_k\ (k = 1, \dots, n)\}$ is called the* ***scattering data*** *for the boundary problem* (1), (2).

In fact, if the scattering data is known, the function $F(x)$ is constructed by the formula (17) and the fundamental equation (16) is constructed with respect to the unknown function $K(x,y)$. Solving this equation the kernel function $K(x,y)$ of the solution (4) is found, and using the kernel function $K(x,y)$, the coefficient $q(x) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}K(x,x)$ is obtained.

Theorem 2. *For each fixed $x > 0$ the fundamental equation* (16) *has a unique solution $K(x,y) \in L_1[x,\infty)$.*

Proof: The **transition function** $F_S(x)$ possesses properties similar to those of the transition function for the problem without spectral parameter in the boundary conditions and, therefore, the proof of Theorem 2 follows ([10], Theorem 3.3.1). □

Theorem 3 (see [10], p. 210). *The function $F_S(x)$ is differentiable on $(0,+\infty)$ and its derivative satisfies the condition*

$$\int_0^{+\infty} (1+x)|F_S'(x)|\,\mathrm{d}x < \infty.$$

Theorem 4. *The scattering function $S(\lambda)$ is continuous on the whole real axis.*

Proof: For all real $\lambda \neq 0$ the continuity of the function $S(\lambda)$ can be obtained from Lemma 1. In the case $E(0) \neq 0$ the continuity of the function $S(\lambda)$ at $\lambda = 0$ is clear and $S(0)=1$. Now we shall prove the continuity of the function $S(\lambda)$ in the case

$$\begin{aligned} E(0) &= \alpha_2 e'(0,0) - \alpha_1 e(0,0) \\ &= \alpha_2 \left[-K(0,0) + \int_0^{+\infty} K_x(0,t)\,\mathrm{d}t \right] - \alpha_1 \left[1 + \int_0^{+\infty} K(0,t)\,\mathrm{d}t \right] = 0. \end{aligned} \tag{18}$$

Substituting $x=0$ into the fundamental equation (16) and integrating from z to infinity with respect to y we have

$$\begin{aligned} &\left\{ 1 + \int_0^{+\infty} K(0,t)\,\mathrm{d}t \right\} \int_z^{+\infty} F(y)\,\mathrm{d}y + \int_z^{+\infty} K(0,y)\,\mathrm{d}y \\ &\qquad - \int_0^{+\infty} \left\{ \int_t^{+\infty} K(0,\xi)\,\mathrm{d}\xi \right\} F(t+z)\,\mathrm{d}t = 0. \end{aligned} \tag{19}$$

Deriving the fundamental equation (16) with respect to x we have

$$\begin{aligned} &\left[-K(0,0) + \int_0^{+\infty} K_x(0,t)\,\mathrm{d}t \right] \int_z^{+\infty} F(y)\,\mathrm{d}y - F(z) \\ &+ \int_z^{+\infty} K_x(0,y)\,\mathrm{d}y - \int_0^{+\infty} \left\{ \int_t^{+\infty} K_x(0,\xi)\,\mathrm{d}\xi \right\} F(t+z)\,\mathrm{d}t = 0. \end{aligned} \tag{20}$$

If the equality (18) satisfies, from (19) and (20) we obtain that the function

$$K_1(z) = \int_z^{+\infty} [\alpha_2 K_x(0,t) - \alpha_1 K(0,t)]\,\mathrm{d}t$$

is a solution of the equation

$$K_1(z) - \int_0^{+\infty} K_1(t) F(t+z)\,\mathrm{d}t = \alpha_2 F(z).$$

Every bounded solution of this equation is summable on semi-axis, that is, $K_1(z) \in L_1[0,\infty)$. Hence, in the considered case we have

$$\begin{aligned} E(\lambda) &= (\alpha_2 + \mathrm{i}\beta_2\lambda) e'(\lambda,0) - (\alpha_1 + \mathrm{i}\beta_1\lambda) e(\lambda,0) \\ &= -\alpha_2 K(0,0) + \alpha_2 \int_0^{+\infty} K_x(0,t)\,\mathrm{d}t - \alpha_1 - \alpha_1 \int_0^{+\infty} K(0,t)\,\mathrm{d}t \\ &\quad + \mathrm{i}\lambda \left\{ \alpha_2 + \mathrm{i}\beta_2\lambda - \beta_2 K(0,0) + \beta_2 \int_0^{+\infty} K_x(0,t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t - \beta_1 \right. \\ &\quad \left. - \beta_1 \int_0^{+\infty} K(0,t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t + \int_0^{+\infty} K_1(t) \mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t \right\} = \mathrm{i}\lambda \widetilde{K}(\lambda). \end{aligned}$$

In the similar manner we can obtain

$$\begin{aligned}E_1(\lambda) &= (\alpha_2 + \mathrm{i}\beta_2\lambda)e'(-\lambda,0) - (\alpha_1 + \mathrm{i}\beta_1\lambda)e(-\lambda,0) \\ &= -\alpha_2 K(0,0) + \alpha_2 \int_0^{+\infty} K_x(0,t)\,\mathrm{d}t - \alpha_1 - \alpha_1 \int_0^{+\infty} K(0,t)\,\mathrm{d}t \\ &\quad -\mathrm{i}\lambda\left\{\alpha_2 + \mathrm{i}\beta_2\lambda + \beta_2 K(0,0) - \beta_2 \int_0^{+\infty} K_x(0,t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t + \beta_1\right. \\ &\quad \left. +\beta_1 \int_0^{+\infty} K(0,t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t + \int_0^{+\infty} K_1(t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t\right\} = \mathrm{i}\lambda\widetilde{K_1}(\lambda).\end{aligned}$$

Hence we obtain the result

$$S(\lambda) = -\frac{\widetilde{K_1}(\lambda)}{\widetilde{K}(\lambda)}.$$

According to equality (7) and the formula (21) it follows that

$$2\omega(x,\lambda) = \widetilde{K}(\lambda)\left[e(-\lambda,x) - S(\lambda)e(\lambda,x)\right].$$

So, we have that $\widetilde{K}(\lambda) \neq 0$. This results show that the scattering function $S(\lambda)$ is continuous at $\lambda = 0$, and $S(0) = -\frac{\widetilde{K_1}(0)}{\widetilde{K}(0)}$. This completes the proof of the theorem. □

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