

INVARIANTS OF SMOOTH FOUR-MANIFOLDS: TOPOLOGY, GEOMETRY, PHYSICS

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Abstract. The profound and beautiful interaction between smooth four-manifold topology and the quantum theory of fields often seems as impenetrable as it is impressive. The objective of this series of lectures is to provide a very modest introduction to this interaction by describing, in terms as elementary as possible, how Atiyah and Jeffrey [1] came to view the partition function of Witten’s first topological quantum field theory [21], which coincides with the zero-dimensional Donaldson invariant, as an “Euler characteristic” for an infinite-dimensional vector bundle.

1. Motivation: Donaldson–Witten Theory

From 1982 to 1994 the study of smooth four-manifolds was dominated by the ideas of Simon Donaldson who showed how to construct remarkably sensitive differential topological invariants for such a manifold B from moduli spaces of anti-self-dual connections on principal $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$ bundles over B (we will describe the simplest of these in Section 4). In 1988, Edward Witten [21], prompted by Atiyah, constructed a quantum field theory in which the Donaldson invariants appeared as expectation values of certain observables. This came to be known as *Donaldson–Witten theory* and the action underlying it was of the form

$$S_{\mathrm{DW}} \propto \int_B \mathrm{Tr} \left\{ -\frac{1}{4} F_\omega \wedge *F_\omega - \frac{1}{4} F_\omega \wedge F_\omega - \frac{1}{2} \psi \wedge [\phi, \psi] \right. \\ \left. + i d^\omega \chi \wedge \psi + 2i[\chi, * \chi] \lambda - i * (\phi \Delta_0^\omega \lambda) + \chi \wedge * d^\omega \eta \right\}, \quad (1.1)$$

where λ , ϕ and η are 0-forms, ω and χ are 1-forms and ψ is a self-dual 2-form, all in the adjoint representation (we will explain all of this in more detail in Section 6). Witten introduced these fields and choose this action to ensure certain symmetries that guaranteed expectation values that would be independent of the Riemannian metric on B and the coupling constant of the theory, i. e., that would be “topological” in the sense that the term is used by physicists.

This construction of Witten’s was a remarkable achievement and provided the most direct sort of link between topology and physics. Topologists were (and still are) eager to understand and exploit the insights that gave rise to such a radically new view of smooth four-manifold invariants. These insights, however, sprang from the deepest regions of contemporary theoretical physics and relatively few mathematicians have gained access to these regions. It is in some measure comforting then that Atiyah and Jeffrey [1] have shown how to arrive at the very same action by purely geometrical means, utilizing the work of Mathai and Quillen [15] on equivariant cohomology. Subsequently, their procedure has been turned around to produce analogous topological quantum field theories for other invariants, e. g., the Seiberg–Witten invariants, and it is this procedure that we intend to describe here.

2. Euler Numbers for Oriented Real Vector Bundles

We consider an oriented, real vector bundle $\xi = (\pi_E: E \rightarrow X)$ of even rank (fiber dimension) $r = 2k$ over a compact smooth manifold X of dimension r , e. g., the tangent bundle of an even-dimensional, oriented manifold. Generally, the typical fiber of ξ will be denoted V and will come equipped with a positive definite inner product (e. g., from a fiber metric on ξ). The exterior algebra of V will be denoted $\wedge V$ and $\{\psi^1, \dots, \psi^r\}$ will denote an oriented, orthonormal basis for V (regarded as odd generators for $\wedge V$). The corresponding volume form for V will be written $\text{vol} = \psi^1 \cdots \psi^r \in \wedge^r V$ (we suppress the customary wedge \wedge and write the product in $\wedge V$ by juxtaposition). The **Euler number** of ξ is defined by

$$\chi(\xi) = \int_X e(\xi), \quad (2.1)$$

where $e(\xi) \in H^r(X; \mathbb{R})$ is the “Euler class” of ξ , for which we offer two (equivalent) definitions.

The Euler class can be defined by the Chern–Weil procedure in a manner entirely analogous to the familiar definitions of Chern and Pontryagin classes. We briefly recall the ideas behind this procedure. Let $G \hookrightarrow P \xrightarrow{\pi_P} X$ be

a principal G -bundle with connection ω and curvature Ω . Let $\{\xi_1, \dots, \xi_n\}$ be a basis for the Lie algebra \mathcal{G} of G and write $\omega = \omega^a \xi_a$ and $\Omega = \Omega^a \xi_a$, where $\omega^a \in \Omega^1(X)$ and $\Omega^a \in \Omega^2(X)$ for $a = 1, \dots, n$. Let $\mathbb{C}[\mathcal{G}]^G$ be the algebra of complex-valued polynomial functions on \mathcal{G} that are $\text{ad } G$ -invariant ($\mathcal{P}(g\xi g^{-1}) = \mathcal{P}(\xi)$ for all $\xi \in \mathcal{G}$ and $g \in G$). Denote by $\Omega^*(P)_{BAS}$ the graded algebra of forms on P that are **basic**, i. e. **G -invariant** ($\sigma_g^* \varphi = \varphi$ for all $g \in G$, where $\sigma_g(p) = p \cdot g$) and **horizontal** ($\iota_W \varphi = 0$ for all vertical vector fields W , where ι_W denotes interior multiplication by W). These are precisely the forms on P which descend to forms on X , i. e., for which there is a $\bar{\varphi} \in \Omega^*(X)$ such that $\pi_P^* \bar{\varphi} = \varphi$. Then there is a map

$$CW_\omega: \mathbb{C}[\mathcal{G}]^G \rightarrow \Omega^*(P)_{BAS},$$

called the **Chern–Weil homomorphism**, defined by “evaluating the polynomial on the curvature of ω ”. More precisely, if $\mathcal{P} \in \mathbb{C}[\mathcal{G}]^G$ has degree k and if we denote by \mathcal{P} also the corresponding symmetric k -linear map (obtained by polarization), then

$$\begin{aligned} CW_\omega(\mathcal{P}) &= \mathcal{P}(\Omega) = \mathcal{P}(\Omega^{a_1} \xi_{a_1}, \dots, \Omega^{a_k} \xi_{a_k}) \\ &= \mathcal{P}(\xi_{a_1}, \dots, \xi_{a_k}) \Omega^{a_1} \wedge \dots \wedge \Omega^{a_k}. \end{aligned} \quad (2.2)$$

Being basic, $\mathcal{P}(\Omega)$ is the pullback by π_P of a form $\bar{\mathcal{P}}(\Omega)$ on X which can be shown to be closed and whose cohomology class does not depend on the choice of ω . Making specific choices for \mathcal{P} gives rise to various characteristic classes of the bundle (see Chapter XII, Vol. II, of [10] for more details).

Now, to define the Euler class of $\xi = (\pi_E: E \rightarrow X)$ we require a principal bundle and an invariant polynomial. For the former we select a fiber metric on ξ and consider the corresponding oriented, orthonormal frame bundle

$$\mathbb{S}\mathbb{O}(2k) \hookrightarrow F_{\mathbb{S}\mathbb{O}}(\xi) \xrightarrow{\pi_{\mathbb{S}\mathbb{O}}} X. \quad (2.3)$$

For the $\mathbb{S}\mathbb{O}(2k)$ -invariant polynomial we select the Pfaffian

$$\text{Pf}: \mathfrak{so}(2k) \rightarrow \mathbb{R}$$

defined as follows: to each skew-symmetric matrix $Q = (q_{ij}) \in \mathfrak{so}(2k)$ we associate an element

$$\sum_{i < j} q_{ij} \psi^i \psi^j = \frac{1}{2} \psi^\top Q \psi \quad (2.4)$$

of $\wedge^2 V$. Then $(\frac{1}{2} \psi^\top Q \psi)^k$ is in $\wedge^{2k} V$ and so is a multiple of the volume form $\text{vol} = \psi^1 \dots \psi^{2k}$. We define $\text{Pf}(Q)$ by

$$\frac{1}{k!} \left(\frac{1}{2} \psi^\top Q \psi \right)^k = \text{Pf}(Q) \text{vol}. \quad (2.5)$$

One can show that Pf is $\text{ad}(\mathbb{S}\mathbb{O}(2k))$ -invariant and satisfies $(\text{Pf}(Q))^2 = \det Q$. Now, choose a connection ω on the frame bundle (2.3), denote its curvature by Ω and define the **Euler class** $e(\xi) \in H^{2k}(X; \mathbb{R})$ of $\xi = (\pi_E: E \rightarrow X)$ by

$$e(\xi) = (-2\pi)^{-k} \left[\overline{\text{Pf}(\Omega)} \right]. \quad (2.6)$$

For any section s of the frame bundle this is given locally by

$$e(\xi) = (-2\pi)^{-k} \left[\text{Pf}(s^*\Omega) \right]. \quad (2.7)$$

Remark. *We will show in the next section that the form $\text{Pf}(s^*\Omega)$ representing the Euler class can be written as a “Fermionic integral” and this is the key to the Mathai–Quillen generalization.*

There is an alternative definition of the Euler class of ξ which proceeds as follows. Denote by $H_{CV}^*(E; \mathbb{R})$ the compact-vertical cohomology of ξ (generated by the differential forms on E whose restriction to each fiber has compact support). It can be shown that there is a unique element $U(\xi)$ of $H_{CV}^r(E; \mathbb{R})$ whose integral over each fiber is 1. This is called the **Thom class** and it has the property that, if s is any section of ξ (e. g., the 0-section), then

$$e(\xi) = s^*U(\xi). \quad (2.8)$$

Having defined the Euler class, and therefore the Euler number of ξ we recall the so-called **Poincaré–Hopf theorem** which asserts that $\chi(\xi)$ can also be computed as the intersection number of any generic section s of ξ with the 0-section s_0 of ξ (generic means that $s(X)$ intersects $s_0(X)$ transversally).

3. Fermionic Integration and Equivariant Cohomology

We regard the elements of the exterior algebra $\wedge V$ as polynomials with real coefficients in the odd (anti-commuting) variables ψ^1, \dots, ψ^r and provide $\wedge V$ with its usual \mathbb{Z}_2 -grading

$$\wedge V = \bigoplus_{i=0}^{\infty} \wedge^i V = \left(\bigoplus_{i=0}^{\infty} \wedge^{2i} V \right) \oplus \left(\bigoplus_{i=0}^{\infty} \wedge^{2i+1} V \right) = (\wedge V)_0 \oplus (\wedge V)_1.$$

$\wedge V$ is therefore a super-commutative superalgebra. One can define the exponential map on $\wedge V$ by the usual power series, noting that the series eventually terminates for any element of $\wedge V$ due to the anti-commutativity of the multiplication.

The **fermionic** (or **Berezin**) **integral** of an element f of $\wedge V$ is the (real) coefficient of $\psi^1 \cdots \psi^r = \text{vol}$ in the polynomial f and we will write this as

$$\int f \mathcal{D}\psi = f_{\text{vol}}. \quad (3.1)$$

For example, our definition (2.5) of the Pfaffian of $Q \in \mathfrak{so}(2k)$ can be written

$$\int e^{\frac{1}{2} \psi^T Q \psi} \mathcal{D}\psi = \text{Pf}(Q) \quad (3.2)$$

and so the local description (2.7) of the Euler class becomes

$$e(\xi) = (-2\pi)^{-k} \int e^{\frac{1}{2} \psi^T (s^* \Omega) \psi} \mathcal{D}\psi. \quad (3.3)$$

It will also be useful to extend this notion of fermionic integration as follows. Let A be any other supercommutative superalgebra and $A \otimes \wedge V$ the (super) tensor product. Regard the elements of $A \otimes \wedge V$ as polynomials in the odd variables ψ^1, \dots, ψ^r with coefficients in A and define the fermionic integral of such an $F \in A \otimes \wedge V$ to be the coefficient (in A) of $\psi^1 \cdots \psi^r = \text{vol}$.

$$\int F \mathcal{D}\psi = F_{\text{vol}}. \quad (3.4)$$

As an example we introduce coordinates u_1, \dots, u_r corresponding to the basis ψ^1, \dots, ψ^r and let $A = \Omega^*(V)$ be the algebra of complex-valued differential forms on V . We show that

$$\int e^{i\psi^j du_j} \mathcal{D}\psi = du_1 \cdots du_r. \quad (3.5)$$

Indeed,

$$\begin{aligned} \int e^{i\psi^j du_j} \mathcal{D}\psi &= \int e^{i(\psi^1 du_1 + \cdots + \psi^r du_r)} \mathcal{D}\psi = \int e^{i\psi^1 du_1} \cdots e^{i\psi^r du_r} \mathcal{D}\psi \\ &\quad \text{(the elements } \psi^j du_j \text{ are even in } \Omega^*(V) \otimes \wedge V \\ &\quad \text{and therefore commute)} \\ &= \int (1 + i\psi^1 du_1) \cdots (1 + i\psi^r du_r) \mathcal{D}\psi \\ &= \int (i\psi^1 du_1) \cdots (i\psi^r du_r) \mathcal{D}\psi \\ &\quad \text{(only this product contributes to the coefficient of } \psi^1 \cdots \psi^r) \\ &= i^r \int (-1)^{\frac{1}{2} r(r+1)} du_1 \cdots du_r \psi^1 \cdots \psi^r \mathcal{D}\psi \\ &= (-1)^k (-1)^{\frac{1}{2} (2k)(2k+1)} du_1 \cdots du_r = du_1 \cdots du_r. \end{aligned}$$

Notice that, if we write $\|u\|^2 = u_1^2 + \cdots + u_r^2$, then

$$(2\pi)^{-k} \int e^{-\frac{1}{2}\|u\|^2 + i\psi^j du_j} \mathcal{D}\psi = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} du_1 \cdots du_r \quad (3.6)$$

which is a form on V that integrates to 1 over V . Shortly we will introduce the so-called “universal Thom form” of Mathai and Quillen which adds one more term to the exponent to produce what is called an “equivariant differential form”. For this we require a brief digression.

The idea behind equivariant cohomology is as follows. Suppose M is a smooth manifold and G is a compact, connected Lie group which acts smoothly on M on the left (we will write the action as $\sigma(g, m) = g \cdot m = \sigma_g(m) = \sigma_m(g)$ for all $g \in G$ and $m \in M$). If the action is free, then the orbit space M/G admits a natural manifold structure and so has an ordinary (de Rham) cohomology. One would like to compute this “equivariantly”, i. e., from a chain co-complex constructed on M (which is presumably simpler). Should this be possible one could then attach a meaning to the notion of the “de Rham” cohomology of M/G even when the action is not free and M/G is not a smooth manifold. There are, in fact, many ways to accomplish this (see [9]) and we will briefly describe one (the so-called “Cartan model” of equivariant cohomology).

Let \mathcal{G} denote the Lie algebra of G and $\mathbb{C}[\mathcal{G}]$ the algebra of complex-valued polynomials on \mathcal{G} . $\Omega^*(M)$ will denote the algebra of complex-valued forms on M and we will consider the tensor product $\mathbb{C}[\mathcal{G}] \otimes \Omega^*(M)$ of $\Omega^*(M)$ -valued polynomials on \mathcal{G} . Rather than the usual product grading on $\mathbb{C}[\mathcal{G}] \otimes \Omega^*(M)$, however, we will “double the degrees” in $\mathbb{C}[\mathcal{G}]$. More precisely, if $\alpha = \mathcal{P} \otimes \varphi$ is a homogeneous element of $\mathbb{C}[\mathcal{G}] \otimes \Omega^*(M)$ we define

$$\deg \alpha = \deg(\mathcal{P} \otimes \varphi) = 2 \deg \mathcal{P} + \deg \varphi, \quad (3.7)$$

where $\deg \mathcal{P}$ is the algebraic degree of the polynomial \mathcal{P} and $\deg \varphi$ is the cohomological degree of the form φ . G acts on the elements of $\mathbb{C}[\mathcal{G}] \otimes \Omega^*(M)$ as follows: For $g \in G$ and $\xi \in \mathcal{G}$,

$$(g \cdot \alpha)(\xi) = (g \cdot (\mathcal{P} \otimes \varphi))(\xi) = \mathcal{P}(g^{-1}\xi g) \sigma_{g^{-1}}^* \varphi. \quad (3.8)$$

An element α of $\mathbb{C}[\mathcal{G}] \otimes \Omega^*(M)$ is said to be G -invariant if $g \cdot \alpha = \alpha$ and the set of all such will be denoted

$$\Omega_G^*(M) = [\mathbb{C}[\mathcal{G}] \otimes \Omega^*(M)]^G.$$

It is easy to see that α is G -invariant if and only if $\alpha(g\xi g^{-1}) = \sigma_{g^{-1}}^*(\alpha(\xi))$. The elements of $\Omega_G^*(M)$ are called **G -equivariant differential forms** on M . Next

we define the **G -equivariant exterior derivative** d_G on $\Omega_G^*(M)$ as follows. For $\alpha \in \Omega_G^*(M)$ and $\xi \in \mathcal{G}$,

$$(d_G \alpha)(\xi) = d(\alpha(\xi)) - \iota_{\xi^\#}(\alpha(\xi)), \quad (3.9)$$

where $\xi^\#$ is the vector field on M defined, at each $m \in M$, by

$$\xi^\#(m) = \left. \frac{d}{dt} (\exp(-t\xi) \cdot m) \right|_{t=0} \quad (3.10)$$

and $\iota_{\xi^\#}$ denotes interior multiplication by $\xi^\#$. Alternatively, if $\{\xi_1, \dots, \xi_n\}$ is a basis for \mathcal{G} and if we write $\iota_a = \iota_{\xi_a^\#}$, then

$$d_G = 1 \otimes d - x^a \otimes \iota_a, \quad (3.11)$$

where $\{x^1, \dots, x^n\}$ is the basis for \mathcal{G}^* dual to $\{\xi_1, \dots, \xi_n\}$ and we regard each x^a as an element of $\mathbb{C}[\mathcal{G}]$. Because of the grading (3.7) we have introduced on $\mathbb{C}[\mathcal{G}] \otimes \Omega^*(M)$, d_G increases the degree of homogeneous elements by one and one can show that, on G -invariant elements, $d_G \circ d_G = 0$ so

$$(\Omega_G^*(M), d_G)$$

is a cochain complex. The cohomology of this complex is called the **Cartan model** of the **G -equivariant cohomology** of M and is denoted $H_G^*(M)$. Cartan has shown that, if the action of G on M is free, then this cohomology is isomorphic to the (complex) de Rham cohomology of M/G .

We describe a specific example of an equivariant cohomology class called the Mathai-Quillen “universal Thom class”. In this case, M is our vector space V and the group G is $\mathbb{S}\mathbb{O}(V)$ with its defining action on V . Thus, we seek an element of

$$\Omega_{\mathbb{S}\mathbb{O}(V)}^*(V) = [\mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V)]^{\mathbb{S}\mathbb{O}(V)}$$

and it will be defined as the fermionic integral of an element of $A \otimes \wedge V$, where $A = \mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V)$. In addition to the notation established for (3.6) and (3.11) we define, for each $\xi \in \mathcal{G}$, the linear transformation

$$M_\xi: V \rightarrow V$$

by

$$M_\xi(\psi) = \left. \frac{d}{dt} (\exp(t\xi) \cdot \psi) \right|_{t=0} \quad (3.12)$$

and write M_a for M_{ξ_a} . Notice that, for each $\xi = x^a(\xi)\xi_a \in \mathfrak{so}(V)$,

$$M_\xi = x^a(\xi)M_a \quad (3.13)$$

and, if (M_ξ) denotes the matrix of M_ξ relative to $\{\psi^1, \dots, \psi^r\}$, then

$$-\frac{1}{2} \sum_l \psi^l x^a(\xi) M_a \psi^l = \frac{1}{2} \psi^\top(M_\xi) \psi. \quad (3.14)$$

Thus, (3.2) can be written

$$\int e^{-\frac{1}{2} \sum_l \psi^l x^a(\xi) M_a \psi^l} \mathcal{D}\psi = \text{Pf}(M_\xi). \quad (3.15)$$

The Mathai–Quillen **universal Thom form** ν for V is defined by

$$\begin{aligned} V &= (2\pi)^{-k} \int e^{-\frac{1}{2} \|u\|^2 + i\psi^j du_j - \frac{1}{2} \sum_l \psi^l x^a M_a \psi^l} \mathcal{D}\psi \\ &= (2\pi)^{-k} e^{-\frac{1}{2} \|u\|^2} \int \exp\left(i\psi^j du_j - \frac{1}{2} \sum_l \psi^l x^a M_a \psi^l\right) \mathcal{D}\psi \end{aligned} \quad (3.16)$$

where products such as $\psi^j du_j$ and $\psi^l x^a (M_a \psi^l)$ take place in the algebra $\mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V) \otimes \wedge V$.

Remark. Note that the normalizing factor here is $(2\pi)^{-k}$ rather than $(-2\pi)^{-k}$. This is to guarantee that (3.16) will reduce to (3.6), a form that integrates to 1 over V , when the term $-\frac{1}{2} \sum_l \psi^l (x^a M_a) \psi^l$ is not present. However, we would also like ν to eventually give rise to representatives of the Euler class analogous to (3.3) (with the substitution (3.14)). This would require $(-2\pi)^{-k}$ rather than $(2\pi)^{-k}$. We have chosen to adopt the normalization in (3.16) and simply keep in mind that the ‘‘Euler class’’ we will get from (3.16) is actually $(-1)^k e(\xi)$. In particular, the factor $(-1)^k$ will have to be included when we use these representatives to compute Euler characteristics.

For example, if one carries out the fermionic integration in (3.16) for $V = \mathbb{R}^2$ (usual orientation and inner product) and $\mathbb{S}\mathbb{O}(V) = \mathbb{S}\mathbb{O}(2)$, the result is

$$\nu_{\mathbb{R}^2} = (2\pi)^{-1} e^{-\frac{1}{2}(u_1^2 + u_2^2)} du_1 du_2 + (2\pi)^{-1} x^1 e^{-\frac{1}{2}(u_1^2 + u_2^2)}. \quad (3.17)$$

Note that each term in (3.17) has degree 2 and the first (in $\mathbb{C}^0[\mathfrak{so}(2)] \otimes \Omega^2(\mathbb{R}^2)$) integrates to 1 over \mathbb{R}^2 . In general, one verifies the following properties of the form ν given by (3.16):

1. ν is an $\mathbb{S}\mathbb{O}(V)$ -invariant element of $\mathbb{C}[\mathfrak{so}(V)] \otimes \Omega^*(V)$ of degree $r = 2k$.
2. ν is $\mathbb{S}\mathbb{O}(V)$ -equivariantly closed, i. e., $d_{\mathbb{S}\mathbb{O}(V)} \nu = 0$, and so determines an equivariant cohomology class

$$[\nu] \in H_{\mathbb{S}\mathbb{O}(V)}^{2k}(V).$$

3. The integral of (the $\mathbb{C}^0[\mathfrak{so}(V)] \otimes \Omega^{2k}(V)$ -part of) ν over V is 1.

To understand why ν is referred to as a universal Thom form we first make the following observation. Suppose G is a Lie group and $\rho: G \rightarrow \mathbb{S}\mathbb{O}(V)$ is a representation of G on V . Then ρ induces a homomorphism

$$\rho_*: \mathcal{G} \rightarrow \mathfrak{so}(V)$$

of Lie algebras (just the derivative of ρ at the identity element of G). It is then easy to show that

$$\nu_G = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^j du_j - \frac{1}{2} \sum_l \psi^l (x^a \circ \rho_*) M_a \psi^l\right) \mathcal{D}\psi \quad (3.18)$$

is a G -equivariantly closed form on V . Now, suppose $G \hookrightarrow P \xrightarrow{\pi_P} X$ is a principal G -bundle over a compact, smooth manifold X of dimension $r = 2k = \dim V$. Then ρ determines an associated vector bundle $P \times_\rho V \xrightarrow{\pi_\rho} X$ over X with typical fiber V . We show now that there is a G -equivariant generalization of the Chern–Weil map which associates with each G -equivariantly closed form on V an ordinary form on $P \times_\rho V$ and that, when applied to ν_G , the result is a representative of the Thom class of $P \times_\rho V$.

Begin by considering the commutative diagram

$$\begin{array}{ccc} P \times V & \xrightarrow{\text{Proj}} & P \\ q \downarrow & & \downarrow \pi_P \\ P \times_\rho V & \xrightarrow{\pi_\rho} & X \end{array}$$

where Proj is the projection of $P \times V$ onto the first factor and q is the map $q(p, v) = (p \cdot g, \rho(g^{-1})(v)) = [p, v]$ defining the vector bundle space $P \times_\rho V$. Since the action of G on P is free, so is the action of G on $P \times V$ ($(p, v) \cdot g = (p \cdot g, \rho(g^{-1})(v))$) so we may consider $q: P \times V \rightarrow P \times_\rho V$ a principal G -bundle. In particular, we have an isomorphism

$$\Omega^*(P \times_\rho V) \cong \Omega^*(P \times V)_{\text{BAS}} \quad (3.19)$$

between the spaces of ordinary forms on $P \times_\rho V$ and the forms on $P \times V$ that are basic with respect to the action $(p, v) \cdot g = (p \cdot g, \rho(g^{-1})(v))$ of G on $P \times V$. Thus, to specify a form on $P \times_\rho V$ (e. g., a Thom form) it is enough to specify a basic form on $P \times V$.

Now choose a connection ω on P with curvature Ω . Then the pullback ω' of ω to $P \times V$ by Proj is a connection on $G \hookrightarrow P \times V \xrightarrow{q} P \times_\rho V$. Given an $\alpha \in \Omega_G^*(V)$ we evaluate the polynomial parts of α at the curvature as in the ordinary Chern–Weil homomorphism to obtain a form $\alpha(\Omega)$ on $P \times V$ (e. g., if $\alpha = \mathcal{P} \otimes \varphi$ is homogeneous, then $\alpha(\Omega) = \mathcal{P}(\Omega) \wedge \varphi$). Then $\alpha(\Omega)$ is always

G -invariant, but need not be horizontal. We remedy this by evaluating $\alpha(\Omega)$ only on ω' -horizontal parts. Thus, we denote by

$$\text{Hor}_{\omega'}(\alpha(\Omega))$$

the form on $P \times V$ which evaluates $\alpha(\Omega)$ on the projections of tangent vectors into the horizontal subspaces determined by ω' . This form is both G -invariant and horizontal, i. e., it is basic and so corresponds to a form on $P \times_{\rho} V$. Define the **G -equivariant Chern–Weil homomorphism**

$$CW_{\omega}^G : \Omega_G^*(V) = [\mathbb{C}[\mathcal{G}] \otimes \Omega_G^*(V)]^G \rightarrow \Omega^*(P \times V)_{\text{BAS}} \cong \Omega^*(P \times_{\rho} V)$$

by

$$CW_{\omega}^G(\alpha) = \text{Hor}_{\omega'}(\alpha(\Omega)).$$

It is not difficult to show that

$$d \circ CW_{\omega}^G = CW_{\omega}^G \circ d_G \quad (3.20)$$

so that CW_{ω}^G descends to a map on cohomology for which we use the same symbol.

$$CW_{\omega}^G : H_G^*(V) \rightarrow H^*(P \times_{\rho} V).$$

Thus, we have a machine for producing closed forms on the vector bundle $P \times_{\rho} V$ from G -equivariantly closed forms on V . Essentially, the machine evaluates the polynomial part on the curvature Ω of the connection ω and then evaluates the resulting form on $P \times V$ on horizontal parts for the connection on $P \times V$ obtained by pulling ω back by the projection $P \times V \rightarrow P$. This gives a closed, basic form on $P \times V$ and therefore a closed form on $P \times_{\rho} V$. Applying this procedure to the form ν_G of (3.18) gives the closed form

$$U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp \left(i\psi^j du_j - \frac{1}{2} \sum_l \psi^l (x^a(\rho_*(\Omega))) M_a \psi^l \right) \mathcal{D}\psi \quad (3.21)$$

(evaluated on horizontal parts)

which we will write more simply as

$$U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp \left(i\psi^{\top} du + \frac{1}{2} \psi^{\top} (\rho_*(\Omega)) \psi \right) \mathcal{D}\psi \quad (3.22)$$

(evaluated on horizontal parts)

by thinking of ψ as a column matrix of basis vectors, du as the column matrix $(du_1 \cdots du_r)^{\top}$ and $\rho_*(\Omega)$ as an $\mathfrak{so}(2k)$ matrix of 2-forms. As an example of what the result of such a calculation might look like we record the following example. As in (3.17) we take $V = \mathbb{R}^2$ with its usual orientation and inner product and $\mathbb{S}\mathbb{O}(V) = \mathbb{S}\mathbb{O}(2)$. For the principal bundle we select the oriented

orthonormal frame bundle of the 2-sphere S^2 , which has structure group $G = \mathbb{S}\mathbb{O}(2)$.

$$\mathbb{S}\mathbb{O}(2) \hookrightarrow F_{\mathbb{S}\mathbb{O}}(S^2) \rightarrow S^2$$

Taking the representation ρ to be the identity $\rho = \text{Id}_{\mathbb{S}\mathbb{O}(2)}$ with $\mathbb{S}\mathbb{O}(2)$ acting on \mathbb{R}^2 by matrix multiplication, the associated vector bundle is the tangent bundle of S^2 .

$$F_{\mathbb{S}\mathbb{O}}(S^2) \times_{\text{Id}_{\mathbb{S}\mathbb{O}(2)}} \mathbb{R}^2 \cong TS^2$$

For any connection $\omega = \begin{pmatrix} 0 & \omega^1 \\ -\omega^1 & 0 \end{pmatrix}$ on $F_{\mathbb{S}\mathbb{O}}(S^2)$ one finds that the image of $\nu_{\mathbb{S}\mathbb{O}(S^2)}$ under $CW_{\omega}^{\mathbb{S}\mathbb{O}(2)}$ is

$$\frac{1}{2\pi} e^{-\frac{1}{2}(u_1^2+u_2^2)} \left(du_1 du_2 + \Omega^1 - \left(\text{Proj}_{F_{\mathbb{S}\mathbb{O}}(S^2)} \right)^* \omega^1 \wedge (u_1 du_1 + u_2 du_2) \right), \quad (3.23)$$

where $\text{Proj}_{F_{\mathbb{S}\mathbb{O}}(S^2)}: F_{\mathbb{S}\mathbb{O}}(S^2) \times \mathbb{R}^2 \rightarrow F_{\mathbb{S}\mathbb{O}}(S^2)$ is the projection onto the first factor. Note that each term has degree 2 and that the $\mathbb{C}^0[\mathfrak{so}(2)] \otimes \Omega^2(\mathbb{R}^2)$ -part

$$(2\pi)^{-1} e^{-\frac{1}{2}(u_1^2+u_2^2)} du_1 du_2,$$

integrates to 1 over each fiber (it is not of compact support on the fiber, but it is Gaussian and this is enough).

The properties described for this last example can be shown to persist in general. Thus, each term in the form U given by (3.22) has degree $2k$ and the $\mathbb{C}^0[\mathcal{G}] \otimes \Omega^{2k}(V)$ -part is Gaussian and integrates to 1 over each fiber of $P \times_{\rho} V$. In this sense, U is a *Gaussian representative* of the Thom class of $P \times_{\rho} V$. Pulling U back to X by some section of $P \times_{\rho} V$ gives a representative of the Euler class of the vector bundle. Now, every section of $P \times_{\rho} V$ is given locally by $x \rightarrow [s(x), S(s(x))]$, where s is a section of the principal bundle $G \hookrightarrow P \rightarrow X$ and $S: P \rightarrow V$ is an equivariant map ($S(p \cdot g) = \rho(g^{-1})(S(p))$) and the result of pulling back U by such a section is

$$(2\pi)^{-k} e^{-\frac{1}{2}\|s\|^2} \int \exp \left(i\psi^{\top} dS + \frac{1}{2} \psi^{\top} (\rho_*(S^*\Omega)) \psi \right) \mathcal{D}\psi. \quad (3.24)$$

Integrating over X then gives the Euler number of the vector bundle (indeed, one can refrain from pulling back to X by s and integrate the resulting form on P over P to get the Euler number). Continuing the example above, if we choose the Levi-Civita connection on the frame bundle $\mathbb{S}\mathbb{O}(2) \hookrightarrow F_{\mathbb{S}\mathbb{O}}(S^2) \rightarrow S^2$ and pull back by the vector field (section of TS^2) given, relative to the orthonormal frame field $\{E_1, E_2\} = \left\{ \frac{\partial}{\partial \phi}, \frac{1}{\sin \phi} \frac{\partial}{\partial \theta} \right\}$ on S^2 , by $\gamma \sin \theta E_1 + \gamma \cos \theta \cos \phi E_2$,

where $\gamma \in \mathbb{R}$ is an arbitrary parameter, then we obtain the following representation of the Euler characteristic of S^2 .

$$\chi(S^2) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\pi e^{-\frac{1}{2}\gamma^2(\sin^2\theta + \cos^2\theta \cos^2\phi)} \sin\phi \left(1 + \gamma^2 \cos^2\theta \sin^2\phi\right) d\phi d\theta$$

Of course, the value of the integral on the right-hand side is 2 for any $\gamma \in \mathbb{R}$ (since $\chi(S^2) = 2$), but it is not altogether clear how one might evaluate it directly to show this (unless $\gamma = 0$).

We will return to these expressions for the Thom and Euler classes in Section 5, but first we must pause to discuss something (apparently) quite different.

4. The 0-Dimensional Donaldson Invariant

We now describe (very briefly) the standard construction of the simplest of the Donaldson invariants and then present an alternative view of the invariant which suggests an analogy with the Euler characteristic. To ease the exposition we will consider only the case in which the structure group G is $\mathrm{SU}(2)$, but the somewhat more general case of $\mathrm{SO}(3)$ is similar. Throughout this section B will denote a compact, simply connected, oriented, smooth 4-manifold (additional assumptions, on $b_2^+(B)$, will be made as the need arises). The principal $\mathrm{SU}(2)$ -bundle over B with second Chern class $k \in \mathbb{Z}$ is denoted

$$SU(2) \hookrightarrow P_k \xrightarrow{\pi_k} B$$

and (for reasons to be discussed shortly) we will restrict our attention to those for which $k > 0$. $\mathcal{A}(P_k)$ will denote the set of all connection 1-forms on P_k and $\mathcal{G}(P_k)$ is the gauge group of all automorphisms of P_k (diffeomorphisms of P_k onto itself which preserve the group action and cover the identity map on B). $\mathcal{G}(P_k)$ is isomorphic to the group of sections of the nonlinear adjoint bundle $P_k \times_{\mathrm{Ad}} \mathrm{SU}(2)$ (the fiber bundle associated to P_k by the adjoint action of $\mathrm{SU}(2)$ on itself), where the group operation is pointwise multiplication. $\mathcal{G}(P_k)$ acts on $\mathcal{A}(P_k)$ on the right by pullback ($\omega \circ f = f^*\omega$) and we are interested in the moduli space of gauge equivalence classes of connections, denoted

$$\mathcal{B}(P_k) = \mathcal{A}(P_k)/\mathcal{G}(P_k).$$

Remark. *To do the required analysis these C^∞ objects must be replaced by “appropriate Sobolev completions”. Our discussion will be sufficiently cursory that we will simply consider this done and not concern ourselves with the technical details.*

$\mathcal{A}(P_k)$ is an affine space modelled on the vector space $\Omega^1(B, \text{ad } P_k)$ of 1-forms on B with values in the adjoint bundle $\text{ad } P_k$ (the vector bundle associated to P_k by the adjoint action of $\text{SU}(2)$ on its Lie algebra $\mathfrak{su}(2)$). Thus, $\mathcal{A}(P_k)$ is an infinite-dimensional manifold whose tangent space at every point $\omega \in \mathcal{A}(P_k)$ is

$$T_\omega(\mathcal{A}(P_k)) \cong \Omega^1(B, \text{ad } P_k). \quad (4.1)$$

We wish to find a decomposition of this tangent space into a piece tangent to the orbit $\omega \cdot \mathcal{G}(P_k)$ of ω under the action of $\mathcal{G}(P_k)$ and an orthogonal complement which, under favorable circumstances, projects injectively into the moduli space $\mathcal{B}(P_k)$ ($\mathcal{A}(P_k)$ is an affine space so we can canonically identify $T_\omega(\mathcal{A}(P_k))$ with a subset of $\mathcal{A}(P_k)$). To this end we consider the covariant exterior derivative

$$d^\omega : \Omega^0(B, \text{ad } P_k) \rightarrow \Omega^1(B, \text{ad } P_k) \quad (4.2)$$

associated with $\omega \in \mathcal{A}(P_k)$. The spaces $\Omega^i(B, \text{ad } P_k)$ of forms on B with values in $\text{ad } P_k$ all have natural inner products arising from a choice of Riemannian metric on B and ad-invariant inner product on $\mathfrak{su}(2)$ and so d^ω has a formal adjoint

$$\delta^\omega : \Omega^1(B, \text{ad } P_k) \rightarrow \Omega^0(B, \text{ad } P_k). \quad (4.3)$$

One can show that $\delta^\omega \circ d^\omega$ is an elliptic, formally self-adjoint operator and from this it follows that $\Omega^1(B, \text{ad } P_k)$ has an orthogonal Hodge decomposition

$$T_\omega(\mathcal{A}(P_k)) \cong \Omega^1(B, \text{ad } P_k) \cong \text{im}(d^\omega) \oplus \ker(\delta^\omega). \quad (4.4)$$

Now, $\text{im}(d^\omega)$ turns out to be the tangent space to the orbit $\omega \cdot \mathcal{G}(P_k)$ at ω and we now describe the ‘‘favorable circumstances’’ under which a neighborhood of ω in $\ker(\delta^\omega)$ projects injectively into $\mathcal{B}(P_k)$. The subgroup of $\mathcal{G}(P_k)$ which leaves ω invariant (i. e., the stabilizer of ω) is always at least \mathbb{Z}_2 (the sections of $P_k \times_{\text{Ad}} \text{SU}(2)$ that send every point to $\pm \text{Id}$). If the stabilizer of ω is precisely \mathbb{Z}_2 , then ω is said to be *irreducible* (this is the case if and only if $\ker(d^\omega)$ is trivial). Otherwise, ω is *reducible*. When ω is irreducible one can show that a sufficiently small open ball $\mathcal{O}_{\omega, \epsilon}$ about ω in $\ker(\delta^\omega)$ (thought of as a subset of $\mathcal{A}(P_k)$) intersects an orbit no more than once and so projects injectively into the moduli space and so provides a local model for the moduli space $\mathcal{B}(P_k)$ near $[\omega]$. In particular, if $\hat{\mathcal{A}}(P_k)$ denotes the (open) set of irreducible elements of $\mathcal{A}(P_k)$ and $\hat{\mathcal{B}}(P_k) = \hat{\mathcal{A}}(P_k)/\mathcal{G}(P_k)$ is the moduli space of irreducible connections on P_k , then each $[\omega] \in \hat{\mathcal{B}}(P_k)$ has a local manifold structure with $T_{[\omega]}(\hat{\mathcal{B}}(P_k)) \cong \ker(\delta^\omega)$. For ω reducible, however, $\mathcal{B}(P_k)$ generally is not smooth near $[\omega]$.

Donaldson theory is concerned primarily with certain subspaces of $\mathcal{B}(P_k)$ and $\widehat{\mathcal{B}}(P_k)$ to which we now turn our attention. To do so we must choose a Riemannian metric \mathfrak{g} on B (since we wish to construct a differential topological invariant of B we must eventually address the issue of showing that this construction does not depend on the choice of \mathfrak{g}). This, together with the given orientation of B , gives a Hodge star operator $*$ on B and, since the Hodge dual of a 2-form on a 4-manifold is another 2-form it is meaningful to say that a connection $\omega \in \mathcal{A}(P_k)$ is **self-dual** (respectively, **anti-self-dual**) if its curvature $F_\omega \in \Omega^2(B, \text{ad } P_k)$ satisfies $*F_\omega = F_\omega$ (respectively, $*F_\omega = -F_\omega$). We define

$$\begin{aligned}\text{Asd}(P_k, \mathfrak{g}) &= \{\omega \in \mathcal{A}(P_k); *F_\omega = -F_\omega\}, \\ \widehat{\text{Asd}}(P_k, \mathfrak{g}) &= \{\omega \in \text{Asd}(P_k, \mathfrak{g}); \omega \text{ irreducible}\}, \\ \mathcal{M}(P_k, \mathfrak{g}) &= \text{Asd}(P_k, \mathfrak{g})/\mathcal{G}(P_k),\end{aligned}$$

and

$$\widehat{\mathcal{M}}(P_k, \mathfrak{g}) = \widehat{\text{Asd}}(P_k, \mathfrak{g})/\mathcal{G}(P_k).$$

Remark. Given a connection ω on $\text{SU}(2) \hookrightarrow P_k \rightarrow B$ the Chern class can be written

$$k = \frac{1}{8\pi^2} \int_B (|F_\omega^-|^2 - |F_\omega^+|^2) \text{vol}_\mathfrak{g}$$

so anti-self-dual connections ($F_\omega^+ = 0$) can exist only if $k \geq 0$. Moreover, if $k = 0$ any such connection is flat. This accounts for our restriction to bundles with $k > 0$.

It is the moduli spaces $\mathcal{M}(P_k, \mathfrak{g})$ and $\widehat{\mathcal{M}}(P_k, \mathfrak{g})$ of \mathfrak{g} -anti-self-dual connections on P_k that are of most interest to us. Their structure is most conveniently unraveled by studying what is called the *fundamental elliptic complex* $\mathcal{E}(\omega)$ associated with any $\omega \in \text{Asd}(P_k, \mathfrak{g})$:

$$0 \rightarrow \Omega^0(B, \text{ad } P_k) \xrightleftharpoons[\delta^\omega]{d^\omega} \Omega^1(B, \text{ad } P_k) \xrightleftharpoons[\delta_+^\omega]{d_+^\omega} \Omega_+^2(B, \text{ad } P_k) \rightarrow 0.$$

Here d^ω and δ^ω have already been introduced, $\Omega_+^2(B, \text{ad } P_k)$ is the space of self-dual 2-forms on B with values in $\text{ad } P_k$, d_+^ω is the covariant exterior derivative $d^\omega : \Omega^1(B, \text{ad } P_k) \rightarrow \Omega^2(B, \text{ad } P_k)$ followed by the projection onto the self-dual part (any 2-form can be written uniquely as the sum of a self-dual and an anti-self-dual 2-form), and δ_+^ω is the formal adjoint of d_+^ω with respect to the natural inner products on the spaces of forms.

Notice that $\mathcal{E}(\omega)$ really is a complex, i. e., $d_+^\omega \circ d^\omega = 0$ because, in general, $d^\omega \circ d^\omega = [F_\omega, \cdot]$ so $d_+^\omega \circ d^\omega = [F_\omega^+, \cdot]$, where F_ω^+ is the self-dual part of the

curvature ($F_\omega^+ = \frac{1}{2}(F_\omega + F_\omega)$) and, since ω is anti-self-dual, $F_\omega^+ = 0$. One can compute the symbol sequence of $\mathcal{E}(\omega)$ explicitly and show that it is exact for each nonzero cotangent vector so that $\mathcal{E}(\omega)$ is an elliptic complex. Elliptic theory (generalized Hodge decomposition) then guarantees that the cohomology of the complex $\mathcal{E}(\omega)$ is *finite-dimensional* and given by

$$H^0(\omega) = \ker(d^\omega) \quad (4.5)$$

$$H^1(\omega) = \ker(d_+^\omega) / \text{im}(d^\omega) \cong \ker(d_+^\omega | \ker(\delta^\omega)) \quad (4.6)$$

$$H^2(\omega) = \Omega_+^2(B, \text{ad } P_k) / \text{im}(d_+^\omega) \cong \text{im}(d_+^\omega | \ker(\delta^\omega))^\perp. \quad (4.7)$$

Here is the significance of this. We have already pointed out that a connection ω is irreducible if and only if $\ker(d^\omega)$ is trivial so

$$\omega \text{ is irreducible} \iff H^0(\omega) = 0.$$

Keep in mind also that the irreducible connections are those that project nicely into the moduli space. To understand the relevance of $H^1(\omega)$ and $H^2(\omega)$ we point out that $d_+^\omega | \ker(\delta^\omega)$ turns out to be the derivative at ω of the “self-dual curvature map”

$$\mathcal{O}_{\omega, \epsilon} \rightarrow \Omega_+^2(B, \text{ad } P_k)$$

on some ball $\mathcal{O}_{\omega, \epsilon}$ about ω in $\ker(\delta^\omega)$ which projects injectively into the moduli space. Consequently, if $H^2(\omega) = 0$, then this derivative is surjective so $0 \in \Omega_+^2(B, \text{ad } P_k)$, which is the value of the self-dual curvature map at the anti-self-dual connection ω , is a regular value. The Implicit Function Theorem then implies

$$H^2(\omega) = 0 \implies \mathcal{A}\text{sd}(P_k, \mathfrak{g})$$

has a manifold structure near ω of dimension $\dim(\ker(d_+^\omega | \ker(\delta^\omega))) = \dim H^1(\omega)$.

Now suppose that *both* $H^0(\omega)$ and $H^2(\omega)$ are trivial. Then $\mathcal{A}\text{sd}(P_k, \mathfrak{g})$ admits a local smooth structure near ω of dimension $\dim H^1(\omega)$ and this projects injectively into the moduli space $\mathcal{M}(P_k, \mathfrak{g})$ to give a local manifold structure of the same dimension near $[\omega]$. Note also that, in this case, the dimension can be written

$$\dim H^1(\omega) = -\dim H^0(\omega) + \dim H^1(\omega) - \dim H^2(\omega)$$

and this is minus the index of the elliptic complex $\mathcal{E}(\omega)$. Now, there is a machine (called the Atiyah-Singer Index Theorem) that computes indices of elliptic complexes in terms of topological data. An application of this machine to $\mathcal{E}(\omega)$ gives for the index the expression $-8k + 3(1 + b_2^+(B))$, where k is

the Chern class and $b_2^+(B)$ can be thought of as the dimension of the space $\Omega_+^2(B; \mathbb{R})$ of self-dual 2-forms on B . Thus,

$$H^0(\omega) = 0 \text{ and } H^2(\omega) = 0 \implies \dim H^1(\omega) = 8k - 3(1 + b_2^+(B)) \quad (4.8)$$

so, near $[\omega]$, $\mathcal{M}(P_k, \mathfrak{g})$ has the structure of a manifold of dimension $8k - 3(1 + b_2^+(B))$ (notice that this is independent of ω).

The remarkable part of the story is that “generically” $H^0(\omega)$ and $H^2(\omega)$ are always trivial. Somewhat more precisely, one can prove that there is a dense G_δ -subset \mathfrak{Gen} in the space of all Riemannian metrics on B such that, for every $\mathfrak{g} \in \mathfrak{Gen}$, every \mathfrak{g} -anti-self-dual connection ω satisfies $H^0(\omega) = 0$ and $H^2(\omega) = 0$. We conclude that if \mathfrak{g} is generic (i. e., $\mathfrak{g} \in \mathfrak{Gen}$), then $\mathcal{M}(P_k, \mathfrak{g}) = \hat{\mathcal{M}}(P_k, \mathfrak{g})$ is either empty or a smooth manifold of dimension

$$8k - 3(1 + b_2^+(B)) . \quad (4.9)$$

One can show further that this manifold is always orientable and, in fact, that an orientation for $\mathcal{M}(P_k, \mathfrak{g})$ is uniquely determined by the choice of an orientation for the vector space $H_+^2(B; \mathbb{R})$.

Now notice that, by an appropriate arrangement of $b_2^+(B)$ and the Chern class k , it is entirely possible for the dimension (4.9) of the moduli space to come out zero. In this case, $\mathcal{M}(P_k, \mathfrak{g})$ is a 0-dimensional, oriented manifold (given an orientation of $H_+^2(B; \mathbb{R})$), i. e., it is a set of isolated points $[\omega]$, each of which is equipped with a sign $\mathcal{E}([\omega]) = \pm 1$. As it happens, the moduli space is necessarily compact in this case so we can add these signs and get an integer

$$q(B) = \sum_{[\omega] \in \mathcal{M}(P_k, \mathfrak{g})} \mathcal{E}([\omega]) . \quad (4.10)$$

One can show that if $b_2^+(B) > 1$, then this integer does not depend on the choice of the (generic) metric \mathfrak{g} and is, in fact, an orientation preserving diffeomorphism invariant of B (assuming the orientation of $H_+^2(B; \mathbb{R})$ is fixed). The integer $q(B)$ is called the 0-dimensional Donaldson invariant of B .

Our concern here is not with computing the Donaldson invariant, nor with using it to obtain results on the topology of 4-manifolds. What we would like to do is show that, by adopting a somewhat different perspective, $q(B)$ can be viewed as an “intersection number” for a section of some (infinite-dimensional) vector bundle with the 0-section of that bundle (analogous to the Poincaré–Hopf version of the Euler number). This then suggests the possibility of some integral representation of the invariant. To see that this is, indeed, possible (at least formally since the “integrations” must take place over infinite-dimensional spaces) we will, in the remaining sections, follow Atiyah and

Jeffrey [1] and show that the Mathai–Quillen expression (3.24) for the Euler form can be adapted to the infinite-dimensional situation to provide a formal integral representation for $q(B)$ which, remarkably, coincides with the partition function for Witten’s topological quantum field theory (defined by the action (1.1)).

The “different perspective” on $q(B)$ to which we alluded above arises in the following way. The group \mathcal{G} of gauge transformations does not act freely on the space $\hat{\mathcal{A}}$ of irreducible connections since even irreducible connections have a \mathbb{Z}_2 stabilizer. However, $\hat{\mathcal{G}} = \mathcal{G}/\mathbb{Z}_2$ does act freely on $\hat{\mathcal{A}}$ so we have an infinite-dimensional principal bundle

$$\hat{\mathcal{G}} \hookrightarrow \hat{\mathcal{A}} \rightarrow \hat{\mathcal{B}} \quad (4.11)$$

over the Banach manifold $\hat{\mathcal{B}}$. We build a vector bundle associated to this principal bundle as follows. Consider the (infinite-dimensional) vector space $\Omega_+^2(B, \text{ad } P)$ of 1-forms on B with values in the adjoint bundle. We claim that there is a smooth left action of $\hat{\mathcal{G}}$ on $\Omega_+^2(B, \text{ad } P)$. To see this we think of \mathcal{G} as the group of sections of the nonlinear adjoint bundle $P \times_{\text{Ad}} \text{SU}(2)$ under pointwise multiplication. Since the elements of $\Omega_+^2(B, \text{ad } P)$ take values in $\mathfrak{su}(2)$, \mathcal{G} acts on these values by conjugation. Moreover, conjugation takes the same value at $\pm f \in \mathcal{G}$ so this \mathcal{G} -action on $\Omega_+^2(B, \text{ad } P)$ descends to a $\mathcal{G}/\mathbb{Z}_2 = \hat{\mathcal{G}}$ -action. Thus, we have an associated vector bundle

$$\hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega_+^2(B, \text{ad } P) \quad (4.12)$$

the elements of which are equivalence classes $[\omega, \gamma] = [\omega \circ f, f^{-1} \circ \gamma]$ with $\omega \in \hat{\mathcal{A}}$, $\gamma \in \Omega_+^2(B, \text{ad } P)$, and $f \in \hat{\mathcal{G}}$.

Now, recall that sections of associated vector bundles can be identified with equivariant maps from the principal bundle space into the typical fiber. In our case we have an obvious map of $\hat{\mathcal{A}}$ into $\Omega_+^2(B, \text{ad } P)$:

$$\begin{aligned} F^+ : \hat{\mathcal{A}} &\rightarrow \Omega_+^2(B, \text{ad } P) \\ F^+(\omega) &= F_\omega^+ = \frac{1}{2} (F_\omega + *F_\omega) \end{aligned}$$

(the self-dual curvature map). Since the action of $\hat{\mathcal{G}}$ on $\hat{\mathcal{A}}$ is by conjugation and curvatures transform by conjugation under a gauge transformation, F^+ is equivariant:

$$F^+(\omega \circ f) = F_{\omega \circ f}^+ = f^{-1} F_\omega^+ f = f^{-1} \circ F_\omega^+ = f^{-1} \circ F^+(\omega)$$

F^+ can therefore be identified with a section

$$s_+ : \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \times_{\hat{\mathcal{G}}} \Omega_+^2(B, \text{ad } P) \quad (4.13)$$

of our vector bundle, given explicitly by

$$s_+([\omega]) = [\omega, F_\omega^+] \quad (4.14)$$

for every $[\omega] \in \hat{\mathcal{B}}$. Notice now that the moduli space $\hat{\mathcal{M}}$ of anti-self-dual connections ($F_\omega^+ = 0$) is precisely the *zero set* of the section s_+ .

Now, the image of any section of a vector bundle is a diffeomorphic copy of the base manifold so we may identify $\hat{\mathcal{B}}$ with $s_+(\hat{\mathcal{B}})$. The moduli space $\hat{\mathcal{M}}$ is then identified with

$$\hat{\mathcal{M}} = s_0(\hat{\mathcal{B}}) \cap s_+(\hat{\mathcal{B}}),$$

where s_0 is the 0-section. In the case in which $\hat{\mathcal{M}}$ is 0-dimensional and compact we may therefore view the Donaldson invariant $q(B)$ as an infinite-dimensional analogue of the Poincaré–Hopf version of the Euler number.

This interpretation of $q(B)$ suggest the possibility of a Gauss–Bonnet–Chern type integral representation of $q(B)$. Notice, however, that, in this case, the “integral” would necessarily be over the infinite-dimensional moduli space $\hat{\mathcal{B}}$ and such integrals are mathematically very difficult to define rigorously. Physicists, however, are not in the least deterred by such minor technical difficulties and compute with such “Feynman”, or “path” integrals very effectively. Indeed, it was Edward Witten who first found an integral representation for $q(B)$ of the desired type. He constructed this representation, however, not directly, but rather as what is called the “partition function” for a certain quantum field theory (the one whose classical action is (1.1)). The integrals are purely formal in the mathematical sense and the arguments leading to the choice of (1.1), the calculation of the partition function, and its identification with $q(B)$ are physical. Our next objective is to follow Atiyah and Jeffrey [1] to see that the same integral representation, and, in particular, Witten’s action (1.1), can be arrived at (formally) by taking the analogy between $q(B)$ and the Euler number seriously and writing out what the Mathai–Quillen Euler form “should” be for the infinite-dimensional vector bundle (4.12) and the section(4.13). In order to do this, however, we will have to re-write the Mathai–Quillen form.

5. The Mathai–Quillen Form Revisited

We begin with the Gaussian representative of the Thom class of $P \times_\rho V$ given in (3.22), which we repeat here for convenience.

$$U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^\top du + \frac{1}{2}\psi^\top(\rho_*(\Omega))\psi\right) \mathcal{D}\psi \quad (5.1)$$

(evaluated on horizontal parts)

Recall that in this formula one may use the curvature Ω of any connection on the initial principal bundle $G \hookrightarrow P \xrightarrow{\pi_P} X$, but now we make a specific choice. Because G is assumed compact we may select a Riemannian metric on P relative to which G acts on P by isometries (just average any Riemannian metric on P with respect to the Haar measure on G). Now, at each point $p \in P$ this Riemannian metric defines an orthogonal complement to the vertical subspace of $T_p(P)$ (tangent space to the G -orbit at p) and, since G acts by isometries, these orthogonals are invariant under the action of G and so they determine a connection ω on P . Henceforth, we will use this connection on P exclusively. We proceed with the cosmetic surgery on (5.1). First note that, by the Cartan Formula, $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ and the second term vanishes on horizontal vectors. Since (5.1) is to be evaluated on horizontal parts the result will be the same whether or not $\frac{1}{2}[\omega, \omega]$ is present. Thus, we may write

$$U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^\top du + \frac{1}{2}\psi^\top(\rho_*(d\omega))\psi\right) \mathcal{D}\psi \quad (5.2)$$

(evaluated on horizontal parts)

The next manipulation requires a few preliminaries. Begin by defining, at each $p \in P$, a linear map

$$C_p: \mathcal{G} \rightarrow \text{Vert}_p(P) \subseteq T_p(P)$$

from the Lie algebra into the space of vertical vectors at p (tangents to the fiber of π_P through p) by

$$C_p(\xi) = \xi^\#(p) = \left. \frac{d}{dt}(p \cdot \exp(t\xi)) \right|_{t=0}.$$

C_p is actually an isomorphism of \mathcal{G} onto $\text{Vert}_p(P)$, but we wish to regard it as a map into $T_p(P)$. Now, choose some ad G -invariant inner product (\cdot, \cdot) on \mathcal{G} . $T_p(P)$ has an inner product $\langle \cdot, \cdot \rangle$ arising from the Riemannian metric on P . Thus, C_p has an adjoint

$$C_p^*: T_p(P) \rightarrow \mathcal{G}$$

defined by

$$\langle w, C_p(\eta) \rangle = (C_p^*(w), \eta)$$

for all $w \in T_p(P)$ and $\eta \in \mathcal{G}$. In particular,

$$\langle C_p(\xi), C_p(\eta) \rangle = (C_p^*(C_p(\xi)), \eta) = (R_p(\xi), \eta)$$

where

$$R_p = C_p^* \circ C_p: \mathcal{G} \rightarrow \mathcal{G}.$$

It is easy to see that R_p is self-adjoint and has trivial kernel so we have an inverse

$$R_p^{-1}: \mathcal{G} \rightarrow \mathcal{G}.$$

Now, since C_p carries \mathcal{G} isomorphically onto $\text{Vert}_p(P)$, there is an inverse

$$C_p^{-1}: \text{Vert}_p(P) \rightarrow \mathcal{G}$$

and we claim that this agrees with ω_p on $\text{Vert}_p(P)$

$$C_p^{-1}(w) = \omega_p(w), \quad w \in \text{Vert}_p(P). \quad (5.3)$$

Indeed, if $w \in \text{Vert}_p(P)$, then $w = C_p(\eta) = \eta^\#(p)$ for a unique $\eta \in \mathcal{G}$. But ω is a connection form and one of its defining properties is that $\omega_p(\eta^\#(p)) = \eta$ for every $\eta \in \mathcal{G}$. Thus, $\omega_p(w) = C_p^{-1}(w)$ for $w \in \text{Vert}_p(P)$.

Next we define a 1-form $\theta \in \Omega^1(P, \mathcal{G}^*)$ on P with values in the dual \mathcal{G}^* of \mathcal{G} that is canonically associated with the action of G on the Riemannian manifold P as follows. For $w \in T_p(P)$, $\theta_p(w) \in \mathcal{G}^*$ is the map

$$\theta_p(w): \mathcal{G} \rightarrow \mathbb{R}$$

given by

$$\theta_p(w)(\xi) = \langle C_p(\xi), w \rangle.$$

We note that θ_p vanishes on horizontal vectors because ω -horizontal means $\langle \cdot, \cdot \rangle$ -orthogonal to the G -orbits, i. e. to $\text{Vert}_p(P)$, and $C_p(\xi) \in \text{Vert}_p(P)$ for each $\xi \in \mathcal{G}$. Now use the inner product (\cdot, \cdot) to identify \mathcal{G}^* with \mathcal{G} and so regard θ as a \mathcal{G} -valued 1-form on P . Specifically, we identify $\theta_p(w)$ with “ (\cdot, \cdot) -dotting with” some element of \mathcal{G} . But

$$\theta_p(w)(\xi) = \langle C_p(\xi), w \rangle = (\xi, C_p^*(w))$$

so

$$\theta_p(w) = (\cdot, C_p^*(w)).$$

Thus, the \mathcal{G} -valued 1-form on P corresponding to $\theta \in \Omega^1(P, \mathcal{G}^*)$ is just $C^* \in \Omega^1(P, \mathcal{G}^*)$. In particular, C^* vanishes on horizontal vector at each point of P . We claim that

$$C^* = R \circ \omega, \quad (5.4)$$

i. e., that $C_p^*(w) = R_p(\omega_p(w))$ for every $p \in P$ and $w \in T_p(P)$. Since both vanish on horizontal vectors we need only verify this when $w \in \text{Vert}_p(P)$. But then (5.3) gives

$$R_p(\omega_p(w)) = R_p(C_p^{-1}(w)) = (C_p^* \circ C_p)(C_p^{-1}(w)) = C_p^*(w).$$

Fixing a basis for \mathcal{G} we can identify each R_p with an invertible matrix and ω with a matrix of real-valued 2-forms on P so, (5.4) is a matrix equation

$$C^* = R\omega$$

which we write

$$\omega = R^{-1}C^*$$

and from which we compute

$$d\omega = R^{-1}dC^* + dR^{-1} \wedge C^*.$$

The second term vanishes on horizontal vectors so, in our expression (5.2) for U , we may replace $d\omega$ by $R^{-1}dC^*$.

$$U = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^\top du + \frac{1}{2}\psi^\top (\rho_*(R^{-1}dC^*))\psi\right) \mathcal{D}\psi \quad (5.5)$$

(evaluated on horizontal parts)

The next objective is to remove the explicit appearance of the inverse in (5.5) by using the Fourier inversion formula. We begin with a brief review of the Fourier transform. Let W be an oriented real vector space of dimension n with volume element $dw \in \wedge^n W^*$ and let w_1, \dots, w_n be coordinates on W with $dw = dw_1 \cdots dw_n$. Let y_1, \dots, y_n be coordinates on W^* dual to w_1, \dots, w_n and $dy = dy_1 \cdots dy_n \in \wedge^n W$ the volume element for W^* . Let $\mathcal{S}(W)$ and $\mathcal{S}(W^*)$ be the Schwartz spaces of rapidly decreasing functions in w_1, \dots, w_n and y_1, \dots, y_n , respectively. Finally, let $\langle \cdot, \cdot \rangle$ denote the natural pairing between W and W^* . The Fourier transform of $f \in \mathcal{S}(W)$ is $\hat{f} \in \mathcal{S}(W^*)$ defined by

$$\hat{f}(y) = (2\pi)^{-n/2} \int_W e^{-i\langle w, y \rangle} f(w) dw.$$

The Fourier inversion formula then asserts that

$$f(w) = (2\pi)^{-n/2} \int_{W^*} e^{i\langle w, y \rangle} \hat{f}(y) dy.$$

Combining these two formulas gives

$$f(w) = (2\pi)^{-n} \int_{W^*} \int_W e^{i\langle w, y \rangle} e^{-i\langle z, y \rangle} f(z) dz dy.$$

Assuming now that W and W^* are identified via some inner product we will write this simply as

$$f(w) = (2\pi)^{-n} \iint e^{i\langle w, y \rangle} e^{-i\langle z, y \rangle} f(z) dz dy. \quad (5.6)$$

with the understanding that both integrations are over W and the exponents are inner products.

The situation to which we would like to apply (5.6) is as follows. If R is a self-adjoint matrix with positive determinant, then one can use the formula to compute $f(R^{-1}w)$. To get an integral that does not explicitly involve the inverse, however, we also make the change of variable $y \rightarrow Ry$. Then $\langle R^{-1}w, Ry \rangle = \langle w, y \rangle$ and $d(Ry) = \det R dy$ so

$$f(R^{-1}w) = (2\pi)^{-n} \iint e^{i\langle w, y \rangle} e^{-i\langle z, Ry \rangle} f(z) \det R dz dy. \quad (5.7)$$

Now we return to our last expression (5.5) for U . Letting $\phi = (\phi_1, \dots, \phi_n)$ denote a Lie algebra variable in \mathcal{G} and consider the function

$$f(\phi) = (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \int \exp\left(i\psi^\top du + \frac{1}{2}\psi^\top(\rho_*(\phi))\psi\right) \mathcal{D}\psi.$$

Each value of f is an element of $\Omega^*(V)$. The components (relative to du_1, \dots, du_r) are real-valued functions of ϕ . Letting $\lambda = (\lambda_1, \dots, \lambda_n)$ be another Lie algebra variable in \mathcal{G} we apply (5.7) with $w = dC^*$ to get

$$\begin{aligned} U &= f(R^{-1}dC^*) = (2\pi)^{-n} \iint e^{i\langle dC^*, \lambda \rangle} e^{-i\langle \phi, R\lambda \rangle} f(\phi) \det R d\lambda d\phi \\ &= (2\pi)^{-n} (2\pi)^{-k} \iint e^{i\langle dC^*, \lambda \rangle} e^{-i\langle \phi, R\lambda \rangle} e^{-\frac{1}{2}\|u\|^2} \\ &\quad \times \int \exp\left(i\psi^\top du + \frac{1}{2}\psi^\top(\rho_*(\phi))\psi\right) \mathcal{D}\psi \det R d\lambda d\phi \\ U &= (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2}\|u\|^2} \iiint \exp\left(\frac{1}{2}\psi^\top(\rho_*(\phi))\psi\right. \\ &\quad \left.+ i\psi^\top du + i\langle dC^*, \lambda \rangle - i\langle \phi, R\lambda \rangle\right) \det R \mathcal{D}\psi d\lambda d\phi \end{aligned} \quad (5.8)$$

(evaluated on horizontal parts)

Remark. Notice that this expression contains one fermionic and two ordinary “bosonic” integrals. Note also that we are being rather sloppy about our use of the Fourier transform since the components of $f(\phi)$ are polynomials in ϕ and therefore not in the Schwartz space. This can be made more precise by inserting a rapidly decaying test function $e^{-\epsilon\langle \phi, \phi \rangle}$ and taking the limit as $\epsilon \rightarrow 0$. Since our objective is a formula to be applied formally to an infinite-dimensional situation in which complete rigor is (for the time being) out of the question anyway, we will not be scrupulous about such details.

Remark. Next we would like to include the parenthetical remark “evaluated on horizontal parts” directly into the integral expression for U . For this we require the notion of a “normalized vertical volume form” on a principal bundle $G \hookrightarrow P \xrightarrow{\pi_P} X$, which is essentially an analogue of a Thom form for a vector bundle. More precisely, if the dimension of G is n , then a normalized **vertical volume form** for $G \hookrightarrow P \xrightarrow{\pi_P} X$ is an n -form W on P such that, if $\iota_X: \pi_P^{-1}(x) \hookrightarrow P$ is the inclusion of a fiber, then

$$\int_{\pi_P^{-1}(x)} \iota_X^* W = 1. \quad (5.9)$$

It is not difficult to show that one can construct such a form W as follows. Choose a positive definite ad G -invariant inner product (\cdot, \cdot) on \mathcal{G} , normalized so that the volume of G (arising from the corresponding bi-invariant Riemannian metric on G) is 1. Let $\{\xi_1, \dots, \xi_n\}$ be an orthonormal basis for \mathcal{G} relative to (\cdot, \cdot) and consistent with the orientation G inherits as a fiber in P . Choose a connection ω on P and write $\omega = \omega^a \xi_a$, where $\omega^a \in \Omega^1(P)$, $a = 1, \dots, n$. Then

$$W = \omega^1 \wedge \dots \wedge \omega^n \quad (5.10)$$

is a normalized vertical volume form on P . Our interest in such a form stems from the fact that, for any integrable top rank form β on X ,

$$\int_X \beta = \int_P \pi_P^* \beta \wedge W \quad (5.11)$$

(essentially Fubini’s Theorem together with (5.9)). Now, the point to evaluating an element of $\Omega^*(P)$ on horizontal parts is to kill any terms with a vertical part (fiber coordinates in a local coordinate representation), thus yielding a horizontal form. Since W is, by definition, vertical and of rank $n = \dim G$, this same purpose can be accomplished as follows. Wedging an element of $\Omega^*(P)$ with W will kill any terms with a vertical part because W already has a full contingent of n vertical coordinates. Those terms which did not have a vertical part are now the same except that they all have a factor of W . If one now defines a new form by integrating out the vertical part of these remaining terms the result is, by (5.9), just the horizontal part of the original element of $\Omega^*(P)$. Thus, evaluating on horizontal parts can be accomplished by wedging with W and then integrating over the fibers. We now show that W can be written as a fermionic integral so that “wedging with W ” can be achieved by including another integration in (5.8).

Let $\{\eta_1, \dots, \eta_n\}$ be an orthonormal basis for \mathcal{G} relative to the normalized G -invariant inner product $(,)$ on \mathcal{G} (we will be more specific about the choice of basis shortly). Regard these as odd generators for $\wedge\mathcal{G}$ and consider the following element of $\Omega^*(P) \otimes \wedge\mathcal{G}$.

$$e^{\omega \otimes \eta} = e^{\sum_i \omega_i \eta_i} = e^{\omega_1 \eta_1} \dots e^{\omega_n \eta_n} = (1 + \omega_1 \eta_1) \dots (1 + \omega_n \eta_n) \quad (5.12)$$

Performing a fermionic integration with respect to η gives

$$\begin{aligned} \int e^{\omega \otimes \eta} \mathcal{D}\eta &= \int (1 + \omega_1 \eta_1) \dots (1 + \omega_n \eta_n) \mathcal{D}\eta = \int \omega_1 \eta_1 \dots \omega_n \eta_n \mathcal{D}\eta \\ &= (-1)^{n(n-1)/2} \int \omega_1 \dots \omega_n \eta_1 \dots \eta_n \mathcal{D}\eta. \end{aligned}$$

Now, if we choose $\{\eta_1, \dots, \eta_n\}$ to be the same as $\{\xi_1, \dots, \xi_n\}$ when $n(n-1)/2$ is even and an odd permutation of $\{\xi_1, \dots, \xi_n\}$ when $n(n-1)/2$ is odd, this gives

$$\int e^{\omega \otimes \eta} \mathcal{D}\eta = \omega_1 \wedge \dots \wedge \omega_n = W \quad (5.13)$$

for the normalized vertical volume form of $G \hookrightarrow P \rightarrow X$.

We prefer to include the vertical volume form W in our integral (5.8) in a form which does not specifically refer to the connection ω so our next objective is to explain and justify the equality

$$W = \int e^{\omega \otimes \eta} \mathcal{D}\eta = (\det R)^{-1} \int e^{(C^*, \eta)} \mathcal{D}\eta. \quad (5.14)$$

Recall that, for each $\eta \in \mathcal{G}$, $\langle \cdot, C\eta \rangle$ is a 1-form on P whose value at the vector field T on P is

$$\langle T, C\eta \rangle = (C^*T, \eta).$$

Thus,

$$\langle \cdot, C\eta \rangle = (C^*, \eta)$$

is a linear function of $\eta \in \mathcal{G}$ whose value is a 1-form on P , i.e. it is an element of $\mathbb{C}[\mathcal{G}] \otimes \Omega^*(P)$. We describe this element more explicitly as follows. Let $\{A_1, \dots, A_n, A_{n+1}, \dots, A_{n+m}\}$ be local coordinates on P with A_1, \dots, A_n vertical and A_{n+1}, \dots, A_{n+m} horizontal. Then, since $C\eta$ is vertical for every $\eta \in \mathcal{G}$,

$$\begin{aligned} (C^*, \eta) &= \langle \cdot, C\eta \rangle = \sum_{i=1}^{n+m} \left\langle \frac{\partial}{\partial A_i}, C\eta \right\rangle dA_i \\ &= \sum_{i=1}^n \left\langle \frac{\partial}{\partial A_i}, C\eta \right\rangle dA_i = \sum_{i=1}^n \left(C^* \left(\frac{\partial}{\partial A_i} \right), \eta \right) dA_i. \end{aligned} \quad (5.15)$$

Write $C^*\left(\frac{\partial}{\partial A_i}\right) = \sum_{j=1}^n a_i^j \eta_j$ so that

$$\begin{aligned} (C^*, \eta) &= \sum_{i=1}^n \sum_{j=1}^n a_i^j(\eta_j, \eta) dA_i = \sum_{j=1}^n \left(\sum_{i=1}^n a_i^j dA_i \right) (\eta_j, \eta) \\ &= \sum_{j=1}^n \beta_j \eta^j(\eta) \end{aligned} \quad (5.16)$$

where

$$\beta_j = \sum_{i=1}^n a_i^j dA_i \in \Omega^*(P)$$

and

$$\{\eta^1, \dots, \eta^n\} = \{(\eta_1, \cdot), \dots, (\eta_n, \cdot)\}$$

is the basis for \mathcal{G}^* dual to $\{\eta_1, \dots, \eta_n\}$. Identifying \mathcal{G}^* and \mathcal{G} via (\cdot) we consider the element $\sum_{j=1}^n \beta_j \eta_j$ of $\Omega^*(P) \otimes \wedge \mathcal{G}$ and perform the fermionic integration

$$\begin{aligned} \int e^{\sum_{j=1}^n \beta_j \eta_j} \mathcal{D}\eta &= (-1)^{n(n-1)/2} \beta_1 \wedge \dots \wedge \beta_n \\ &= (-1)^{n(n-1)/2} (a_1^1 dA_1 + \dots + a_n^1 dA_n) \wedge \dots \\ &\quad \dots \wedge (a_1^n dA_1 + \dots + a_n^n dA_n) \\ &= (-1)^{n(n-1)/2} \det(a_i^j) dA_1 \wedge \dots \wedge dA_n. \end{aligned} \quad (5.17)$$

Now we compute $\det(a_i^j)$ as follows. $C^*\left(\frac{\partial}{\partial A_i}\right) = \sum_{j=1}^n a_i^j \eta_j$ implies

$$\begin{aligned} a_i^j &= \left(C^* \left(\frac{\partial}{\partial A_i} \right), \eta_j \right) = \left\langle \frac{\partial}{\partial A_i}, C\eta_j \right\rangle \\ &= \left\langle C \left(\omega \left(\frac{\partial}{\partial A_i} \right) \right), C\eta_j \right\rangle \quad (\text{by (5.3)}) \\ &= \left(C^* \circ C \left(\omega \left(\frac{\partial}{\partial A_i} \right) \right), \eta_j \right) \\ &= \left(R \left(\omega \left(\frac{\partial}{\partial A_i} \right) \right), \eta_j \right). \end{aligned} \quad (5.18)$$

Now let us write

$$\omega = \underline{\omega}^1 \eta_1 + \dots + \underline{\omega}^n \eta_n.$$

Recall that, by the way we have chosen $\{\eta_1, \dots, \eta_n\}$, $\{\underline{\omega}^1, \dots, \underline{\omega}^n\}$ is at worst a permutation of $\{\omega^1, \dots, \omega^1\}$ and, in any case,

$$\underline{\omega}^1 \wedge \dots \wedge \underline{\omega}^n = (-1)^{n(n-1)/2} \omega^1 \wedge \dots \wedge \omega^n = (-1)^{n(n-1)/2} W. \quad (5.19)$$

Thus,

$$\omega \left(\frac{\partial}{\partial A_i} \right) = \underline{\omega}^1 \left(\frac{\partial}{\partial A_i} \right) \eta_1 + \cdots + \underline{\omega}^n \left(\frac{\partial}{\partial A_i} \right) \eta_n$$

and

$$R \left(\omega \left(\frac{\partial}{\partial A_i} \right) \right) = \underline{\omega}^1 \left(\frac{\partial}{\partial A_i} \right) R(\eta_1) + \cdots + \underline{\omega}^n \left(\frac{\partial}{\partial A_i} \right) R(\eta_n)$$

so that

$$a_i^j = \underline{\omega}^1 \left(\frac{\partial}{\partial A_i} \right) (R(\eta_1), \eta_j) + \cdots + \underline{\omega}^n \left(\frac{\partial}{\partial A_i} \right) (R(\eta_n), \eta_j) .$$

Consequently,

$$(a_i^j) = \begin{pmatrix} (R(\eta_1), \eta_1) & \cdots & (R(\eta_n), \eta_1) \\ \vdots & & \vdots \\ (R(\eta_1), \eta_n) & \cdots & (R(\eta_n), \eta_n) \end{pmatrix} \begin{pmatrix} \underline{\omega}^1 \left(\frac{\partial}{\partial A_1} \right) & \cdots & \underline{\omega}^1 \left(\frac{\partial}{\partial A_n} \right) \\ \vdots & & \vdots \\ \underline{\omega}^n \left(\frac{\partial}{\partial A_1} \right) & \cdots & \underline{\omega}^n \left(\frac{\partial}{\partial A_n} \right) \end{pmatrix}$$

and so

$$\det (a_i^j) = \det R \det (\underline{\omega}_i^j) \quad (5.20)$$

where

$$\underline{\omega}^j = \sum_{i=1}^n \underline{\omega}_i^j dA_i = \sum_{i=1}^n \underline{\omega}^j \left(\frac{\partial}{\partial A_i} \right) dA_i .$$

Substituting (5.20) into (5.17) and using (5.19) gives

$$\begin{aligned} \int e^{\sum_j \beta_j \eta_j} \mathcal{D}\eta &= (-1)^{n(n-1)/2} \det R \det (\underline{\omega}_i^j) dA_1 \wedge \cdots \wedge dA_n \\ &= (-1)^{n(n-1)/2} \det R (\underline{\omega}_1^1 dA_1 + \cdots + \underline{\omega}_n^1 dA_n) \wedge \cdots \\ &\quad \cdots \wedge (\underline{\omega}_1^n dA_1 + \cdots + \underline{\omega}_n^n dA_n) \\ &= (-1)^{n(n-1)/2} \det R \underline{\omega}^1 \wedge \cdots \wedge \underline{\omega}^n \\ &= \det R \omega^1 \wedge \cdots \wedge \omega^n = \det RW . \end{aligned}$$

From (5.16) and our identification of \mathcal{G}^* with \mathcal{G} via $(,)$ we also identify $(C^*, \cdot) = \sum_j \beta_j \eta_j$. As a matter of notation, however, it is customary to retain the reference to the fermionic Lie algebra variable η when writing the integral representation of the vertical volume form so we too will write

$$W = (\det R)^{-1} \int e^{\sum_j \beta_j \eta_j} \mathcal{D}\eta = (\det R)^{-1} \int e^{(C^*, \eta)} \mathcal{D}\eta$$

and this gives (5.14).

Now we return to our last expression (5.8) for U and enforce the horizontal projection by multiplying through by W , written as in the second equality in (5.14). Absorbing this into the integral adds the extra term (C^*, η) to the exponent and cancels the $\det R$, thus yielding

$$U = (2\pi)^{-n} (2\pi)^{-k} e^{-\frac{1}{2} \|u\|^2} \iiint \exp \left(\frac{1}{2} \psi^\top (\rho_*(\phi)) \psi + i\psi^\top du + i(dC^*, \lambda) - i(\phi, R\lambda) + (C^*, \eta) \right) \mathcal{D}\eta \mathcal{D}\psi d\lambda d\phi. \tag{5.21}$$

Remark. *One should keep in mind that the horizontal projection here refers to the principal bundle $G \hookrightarrow P \times V \rightarrow P \times_\rho V$ and that, in (5.21), there is still an implicit integration over the fibers to remove the vertical volume form whose purpose was to “kill vertical parts”.*

The form U in (5.21), after integrating out the vertical parts, is a basic form on $P \times V$ which, regarded as a form on $P \times_\rho V$, represents the Thom class. Pulling U back by a section of $P \times_\rho V$ then gives a representative of the Euler class which, when integrated over X , gives the Euler number. Now, any section of the associated bundle $P \times_\rho V$ can be written in the form $x \rightarrow [s(x), S(s(x))]$, where s is a section of P and $S: P \rightarrow V$ is an equivariant map ($S(p \cdot g) = \rho(g^{-1})(S(p))$) and pulling back by such a section amounts to pulling back the V -parts of U (prior to the fiber integration) by S and then pulling back the resultant form on P by s . Thus, our Euler form is the pullback by s of the form

$$(2\pi)^{-n} (2\pi)^{-k} \iiint \exp \left\{ -\frac{1}{2} \|S\|^2 + \frac{1}{2} \psi^\top (\rho_*(\phi)) \psi + i\psi^\top dS + i(dC^*, \lambda) - i(\phi, R\lambda) + (C^*, \eta) \right\} \mathcal{D}\eta \mathcal{D}\psi d\lambda d\phi. \tag{5.22}$$

If we refrain from pulling back by s we have a form on P whose integral over P is also the Euler number (because of (5.11)).

We have one last bit of cosemantic surgery to perform on U . There is a common notational device in (supersymmetric) physics whereby the integral of a top rank form on a manifold is written as two successive integrations, one fermionic and one bosonic. Recall that the integral of a (properly decaying) *function* φ on an oriented, Riemannian manifold P is defined by multiplying the volume form $d\omega$ of P by φ and integrating this over P .

$$\int_P \varphi d\omega.$$

Now, if α is any (properly decaying) form on P written in terms of local coordinates x^i on P ($\alpha = \alpha(x^i, dx^i)$) and if one introduces odd variables χ^i

(generators for some exterior algebra), then one can define an element $\alpha(x^i, \chi^i)$ of this exterior algebra by formally making the substitutions $dx^i \rightarrow \chi^i$. Then the fermionic integral

$$\int \alpha(x^i, \chi^i) \mathcal{D}\chi$$

is precisely the function one integrates (next to $d\omega$ as above) to get the integral of α over P :

$$\int_P \alpha = \int_P \int \alpha(x^i, \chi^i) \mathcal{D}\chi d\omega.$$

Applying this convention to the expression for the Euler number obtained by integrating (5.22) over P gives the final formula toward which all of this has been leading us. For this we will explicitly indicate all of the dependences on the three fermionic (χ, η, ψ) and three bosonic (λ, ϕ, ω) variables (in particular, all of the terms giving rise to forms on P ($i\psi^\top dS$, $i(dC^*, \lambda)$, and (C^*, η)) are regarded as functions of the new fermionic variable χ). We will also (finally!) suppress all but one of the integral signs.

$$\begin{aligned} (2\pi)^{-n} (2\pi)^{-k} \int \exp \left\{ -\frac{1}{2} \|S(\omega)\|^2 + \frac{1}{2} \psi^\top (\rho_*(\phi)) \psi \right. \\ \left. + i\psi^\top dS_\omega(\chi) + i(dC_\omega^*(\chi, \chi), \lambda) \right. \\ \left. - i(\phi, R_\omega \lambda) + (C_\omega^* \chi, \eta) \right\} \mathcal{D}\chi \mathcal{D}\eta \mathcal{D}\psi d\lambda d\phi d\omega. \end{aligned} \quad (5.23)$$

This is the **Atiyah–Jeffrey formula** for the Euler number of $P \times_\rho V$ and in the next section we will formally apply it to the infinite-dimensional vector bundle $\hat{A} \times_{\hat{g}} \Omega_+^2(B, \text{ad } P)$ of Section 4 with $S = F^+$ as the equivariant map. The result will be, formally at least, an expression for an “Euler number” for this bundle and also, as it happens, the partition function for Witten’s topological quantum field theory (i. e., the 0-dimensional Donaldson invariant).

6. The Donaldson Invariant as an Euler Number: Witten’s Partition Function

We must emphasize at the outset that what we intend to do in this section is not mathematics (and certainly not physics). Our objective is to find, within the context of the infinite-dimensional vector bundle (4.12) associated with the Donaldson invariant, formal field-theoretic analogues of the various bosonic and fermionic variables appearing in (5.23) and natural identifications of the terms in the exponent of (5.23) with functions of these variables. In the process the (perfectly well-defined) bosonic and fermionic integrals in (5.23) will metamorphose into Feynman integrals over spaces of fields with all of their

attendant mathematical difficulties. The purist will argue that this is meaningless manipulation of symbols and we can offer no credible defense against the charge. The only mitigating circumstance is that such formal manipulations have proved extraordinarily productive for both physics and mathematics and promise to be even more so in the future as the two subjects continue to re-establish lines of communication.

We begin by recalling that our derivation of the Atiyah–Jeffrey formula (5.23) assumes the existence of a Riemannian metric on the principal bundle space (in our case, $\hat{\mathcal{A}}$) for which the group $(\hat{\mathcal{G}})$ acts by isometries. Such a metric is easy to produce. Since $\hat{\mathcal{A}}$ is open in \mathcal{A} ,

$$T_\omega(\hat{\mathcal{A}}) = \Omega^1(B, \text{ad } P)$$

for each $\omega \in \hat{\mathcal{A}}$. Now, all of the vector spaces $\Omega^k(B, \text{ad } P)$ have natural L^2 -inner products arising from the (generic) metric g on B (and the corresponding Hodge star $*$) and an invariant inner product on the Lie algebra. Taking the inner product on $\mathfrak{su}(2)$ to be $(A, B) = -\text{Tr}(AB)$ this is given by

$$\langle \alpha, \beta \rangle_k = - \int_B \text{Tr}(\alpha \wedge * \beta).$$

In particular, this is true for $T_\omega(\hat{\mathcal{A}})$. Since \mathcal{A} is an affine space and $\hat{\mathcal{A}}$ is open in \mathcal{A} this defines a metric on $\hat{\mathcal{A}}$ and, since the inner product is invariant under the action of the gauge group $\hat{\mathcal{G}}$ (pointwise conjugation by a section of $P \times_{Ad} \mathbb{S}\mathbb{U}(2)$), $\hat{\mathcal{G}}$ acts by isometries on $\hat{\mathcal{A}}$.

This metric defines a connection on $\hat{\mathcal{A}}$ whose horizontal spaces are the orthogonal complements to the gauge orbits. The decomposition (4.4) and the fact that the tangent space to the orbit at ω is $\text{im}(d^\omega: \Omega^0(B, \text{ad } P) \rightarrow \Omega^1(B, \text{ad } P))$ implies that the horizontal space at ω is just $\ker(\delta^\omega: \Omega^1(B, \text{ad } P) \rightarrow \Omega^0(B, \text{ad } P))$.

Applying (5.23) also requires the choice of a section, i. e., an equivariant map $S: \hat{\mathcal{A}} \rightarrow \Omega^1(B, \text{ad } P)$, and for us there is only one reasonable choice, namely, the section whose intersection number with the 0-section is the Donaldson invariant. Thus, we take

$$S = F^+ : \hat{\mathcal{A}} \rightarrow \Omega_+^2(B, \text{ad } P)$$

$$S(\omega) = F_\omega^+.$$

The first term in the exponent of (5.23) is $-1/2\|S(\omega)\|^2$ which we now interpret as the squared norm of F_ω^+ arising from the inner product $\langle \cdot, \cdot \rangle_2$ on $\Omega^2(B, \text{ad } P)$.

Thus,

$$\|S(\omega)\|^2 = \|F_\omega^+\|^2 = \int_B |F_\omega^+|^2 \text{vol}_g = - \int_B \text{Tr}(F_\omega^+ \wedge *F_\omega^+).$$

Noting that

$$\|F_\omega\|^2 = \int_B |F_\omega^+|^2 + |F_\omega^-|^2 \text{vol}_g = - \int_B \text{Tr}(F_\omega \wedge *F_\omega)$$

and

$$\int_B |F_\omega^+|^2 - |F_\omega^-|^2 \text{vol}_g = - \int_B \text{Tr}(F_\omega \wedge F_\omega)$$

(the latter being a multiple of the Chern class), we find that

$$-\frac{1}{2} \|S(\omega)\|^2 = \frac{1}{4} \int_B \text{Tr}(F_\omega \wedge *F_\omega) + \frac{1}{4} \int_B \text{Tr}(F_\omega \wedge F_\omega). \quad (6.1)$$

Note that, except for the sign, this reproduces the first two terms of Witten's Lagrangian (1.1). The first term is of the typical **Yang-Mills variety** for a classical gauge theory, whereas the second Witten calls a **topological term** (because it is basically the Chern class of the underlying principal $\text{SU}(2)$ -bundle).

To proceed we must sort out the appropriate analogues, in the Donaldson theory context, of the maps C , C^* , and R of Section 5. At each point ω in the principal bundle space $\hat{\mathcal{A}}$, C_ω is the map from the Lie algebra of $\hat{\mathcal{G}}$ to the tangent space $T_\omega(\hat{\mathcal{A}}) = \Omega^1(B, \text{ad } P)$. The Lie algebra of $\hat{\mathcal{G}}$ is $\Omega^0(B, \text{ad } P)$ (these can be exponentiated pointwise to produce sections of $P \times_{\text{Ad}} \text{SU}(2)$) so

$$C_\omega: \Omega^0(B, \text{ad } P) \rightarrow \Omega^1(B, \text{ad } P).$$

For each $\xi \in \Omega^0(B, \text{ad } P)$, $C_\omega(\xi)$ is defined by

$$C_\omega(\xi) = \left. \frac{d}{dt} (\omega \cdot \exp(t\xi)) \right|_{t=0}.$$

Computing this derivative locally shows that

$$C_\omega(\xi) = d^\omega \xi. \quad (6.2)$$

Consequently, C_ω^* is the formal adjoint

$$C_\omega^* = \delta^\omega: \Omega^1(B, \text{ad } P) \rightarrow \Omega^0(B, \text{ad } P) \quad (6.3)$$

of d^ω relative to the natural inner products on the spaces of forms and

$$R_\omega = C_\omega^* \circ C_\omega = \delta^\omega \circ d^\omega = \Delta_0^\omega: \Omega^0(B, \text{ad } P) \rightarrow \Omega^0(B, \text{ad } P) \quad (6.4)$$

is the scalar Laplacian corresponding to ω .

With this information in hand we consider the term $-i(\phi, R_\omega \lambda)$ in (5.23). Both ϕ and λ are in the Lie algebra so we introduce two bosonic fields

$$\phi, \lambda \in \Omega^0(B, \text{ad } P)$$

and interpret $(,)$ as the natural inner product \langle , \rangle_0 on $\Omega^0(B, \text{ad } P)$.

Remark. We apply the adjectives “bosonic” and “fermionic” to the fields we introduce only because of the type of integral these variables correspond to in (5.23). We do not claim to have justified any corresponding physical implications these terms may connote.

Thus, the term $-i(\phi, R_\omega \lambda)$ is to be interpreted as

$$\begin{aligned} -i(\phi, R_\omega \lambda) &= -\langle \phi, \Delta_0^\omega \lambda \rangle_0 = i \int_B \text{Tr} (\phi \wedge * (\Delta_0^\omega \lambda)) \\ &= i \int_B \text{Tr} (* (\phi \Delta_0^\omega \lambda)) . \end{aligned} \tag{6.5}$$

Next we consider the term $(C_\omega^* \chi, \eta)$ in (5.23). Since C_ω^* maps from $\Omega^1(B, \text{ad } P)$ to $\Omega^0(B, \text{ad } P)$ we will need two fermionic fields

$$\eta \in \Omega^0(B, \text{ad } P)$$

and

$$\chi \in \Omega^1(B, \text{ad } P)$$

and must interpret $(,)$ as the natural inner product \langle , \rangle_0 on $\Omega^0(B, \text{ad } P)$. Thus, we find that

$$\begin{aligned} (C_\omega^* \chi, \eta) &= \langle \delta^\omega \chi, \eta \rangle_0 = \langle \chi, d^\omega \eta \rangle_1 \\ &= - \int_B \text{Tr} (\chi \wedge * d^\omega \eta) . \end{aligned} \tag{6.6}$$

The term $i(dC_\omega^*(\chi, \chi), \lambda)$ in (5.23) requires a bit more work, even though we have already identified all of the relevant fields. C^* is a 1-form on $\hat{\mathcal{A}}$ with values in the Lie algebra $\Omega^0(B, \text{ad } P)$ of $\hat{\mathcal{G}}$ so dC^* is a 2-form on $\hat{\mathcal{A}}$ with values in $\Omega^0(B, \text{ad } P)$. We compute dC^* at $\omega \in \hat{\mathcal{A}}$ as follows. Fix $\chi_1, \chi_2 \in T_\omega(\hat{\mathcal{A}})$. Since \mathcal{A} is an affine space and $\hat{\mathcal{A}}$ is open in \mathcal{A} we may regard χ_1 and χ_2 as constant vector fields on $\hat{\mathcal{A}}$. Thus,

$$\begin{aligned} dC_\omega^*(\chi_1, \chi_2) &= \chi_1 (C^* \chi_2) - \chi_2 (C^* \chi_1) - C^*([\chi_1, \chi_2]) \\ &= \chi_1 (C^* \chi_2) - \chi_2 (C^* \chi_1) , \end{aligned}$$

where $C^*\chi_1$ and $C^*\chi_2$ stand for the functions on \hat{A} defined by $\nu \rightarrow C^*_\nu\chi_i = \delta^\nu\chi_i$, $i = 1, 2$. Now,

$$\begin{aligned} (\chi_1 (C^*\chi_2)) (\omega) &= \chi_1(\omega) [C^*\chi_2] = \chi_1 [C^*\chi_2] \\ &= \frac{d}{dt} ((C^*\chi_2) (\omega + t\chi_1)) \Big|_{t=0} \\ &= \frac{d}{dt} C^*_{\omega+t\chi_1}(\chi_2) \Big|_{t=0} \\ (\chi_1 (C^*\chi_2)) (\omega) &= \frac{d}{dt} \delta^{\omega+t\chi_1}(\chi_2) \Big|_{t=0}. \end{aligned} \quad (6.7)$$

We compute $\delta^{\omega+t\chi_1}(\chi_2)$ as follows. For any $\lambda \in \Omega^0(B, \text{ad } P)$,

$$d^{\omega+t\chi_1}(\lambda) = d^\omega\lambda + t[\chi_1, \lambda] = d^\omega\lambda + tB_{\chi_1}(\lambda),$$

where $B_{\chi_1} : \Omega^0(B, \text{ad } P) \rightarrow \Omega^1(B, \text{ad } P)$ is defined by $B_{\chi_1}(\lambda) = [\chi_1, \lambda]$. Thus,

$$\delta^{\omega+t\chi_1}(\chi_2) = \delta^\omega\chi_2 + B_{\chi_1}^*(\chi_2),$$

where $B_{\chi_1}^* : \Omega^1(B, \text{ad } P) \rightarrow \Omega^0(B, \text{ad } P)$ is the adjoint of B_{χ_1} . We claim that,

$$B_{\chi_1}^*(\chi_2) = - * [\chi_1, *\chi_2]. \quad (6.8)$$

Indeed for any $\lambda \in \Omega^0(B, \text{ad } P)$,

$$\begin{aligned} \langle B_{\chi_1}(\lambda), \chi_2 \rangle_1 &= \langle [\chi_1, \lambda], \chi_2 \rangle_1 = - \int_B \text{Tr} ([\chi_1, \lambda] \wedge *\chi_2) \\ &= \int_B \text{Tr} ([\lambda, \chi_1] \wedge *\chi_2) = \int_B \text{Tr} (\lambda \wedge [\chi_1, *\chi_2]) \\ &\quad \text{any invariant inner product } \langle \cdot, \cdot \rangle \\ &\quad \text{on a Lie algebra satisfies } \langle x, [y, z] \rangle = \langle [x, y], z \rangle. \\ &= \int_B \text{Tr} (\lambda \wedge ** [\chi_1, *\chi_2]) \\ &= - \langle \lambda, * [\chi_1, *\chi_2] \rangle_0 = \langle \lambda, - * [\chi_1, *\chi_2] \rangle_0 \end{aligned}$$

which establishes (6.8). Thus,

$$\delta^{\omega+t\chi_1}(\chi_2) = \delta^\omega\chi_2 - t * [\chi_1, *\chi_2]$$

and computing the derivative at $t = 0$ gives

$$(\chi_1 (C^*\chi_2)) (\omega) = - * [\chi_1, *\chi_2] = * [\chi_2, *\chi_1]$$

(from (6.7)). Since the result is independent of ω ,

$$\chi_1 (C^* \chi_2) = * [\chi_2, * \chi_1] .$$

Similarly,

$$\chi_2 (C^* \chi_1) = * [\chi_2, * \chi_1] .$$

and so

$$dC_\omega^*(\chi_1, \chi_2) = 2 * [\chi_2, * \chi_1] . \quad (6.9)$$

Thus,

$$\begin{aligned} i \langle dC_\omega^*(\chi_1, \chi_2), \lambda \rangle_0 &= i \langle \lambda, dC_\omega^*(\chi_1, \chi_2) \rangle_0 \\ &= -i \int_B \text{Tr} (\lambda \wedge * (2 * [\chi_2, * \chi_1])) \\ &= -2i \int_B \text{Tr} ([\chi_2, * \chi_1] \lambda) \end{aligned}$$

and so we interpret the term $i(dC_\omega^*(\chi, \chi), \lambda)$ as

$$i (dC_\omega^*(\chi, \chi), \lambda) = -2i \int_B \text{Tr}([\chi, * \chi] \lambda) . \quad (6.10)$$

The two remaining terms ($i\psi^\top dS_\omega(\chi)$ and $\frac{1}{2}\psi^\top(\rho_*(\phi))\psi$) both involve the variable ψ which arises in the Mathai–Quillen formalism from the odd generators of the exterior algebra of the fiber vector space V ($\Omega_+^2(B, \text{ad } P)$ in our case). Thus, we introduce a fermionic field

$$\psi \in \Omega_+^2(B, \text{ad } P) .$$

Now we consider the term $i\psi^\top dS_\omega(\chi)$. S is the self-dual curvature map $S = F^+ : \hat{A} \rightarrow \Omega_+^2(B, \text{ad } P)$ and we have already noted that the derivative of this map at $\omega \in \hat{A}$ is $d_+^\omega : \Omega^1(B, \text{ad } P) \rightarrow \Omega^2(B, \text{ad } P)$. Thus,

$$dS_\omega(\chi) = d_+^\omega \chi \in \Omega_+^2(B, \text{ad } P) .$$

We will interpret finite-dimensional expressions such as

$$A^\top B = (A^1 \cdots A^r) \begin{pmatrix} B^1 \\ \vdots \\ B^r \end{pmatrix} = A^1 B^1 + \cdots + A^r B^r$$

in terms of the appropriate field-theoretic inner product so that

$$\begin{aligned}
i\psi^\top dS_\omega(\chi) &= i\langle \psi, d_+^\omega \chi \rangle_2 \\
&= i\langle \psi, d^\omega \chi \rangle_2 \quad (\psi \text{ is self-dual}) \\
&= i\langle d^\omega \chi, \psi \rangle_2 \\
&= -i \int_B \text{Tr} (d^\omega \chi \wedge * \psi) \\
i\psi^\top dS_\omega(\chi) &= -i \int_B \text{Tr} (d^\omega \chi \wedge \psi) . \tag{6.11}
\end{aligned}$$

Finally, we consider $\frac{1}{2}\psi^\top(\rho_*(\phi))\psi$. In the Mathai–Quillen form, the representation ρ corresponds to the action of G on V that gives rise to the vector bundle. In our case, \hat{G} (regarded as sections of the nonlinear adjoint bundle) acts on $\Omega_+^2(B, \text{ad } P)$ pointwise by conjugation. At each point this is just the ordinary adjoint action of G on its Lie algebra so the corresponding infinitesimal action is bracket. Thus, for each $\phi \in \Omega^0(B, \text{ad } P)$, $\rho_*(\phi)$ acts on $\psi \in \Omega_+^2(B, \text{ad } P)$ by

$$(\rho_*\phi)\psi = [\phi, \psi].$$

Thus, $\psi^\top(\rho_*(\phi))\psi$ is interpreted as

$$\begin{aligned}
\psi^\top(\rho_*(\phi))\psi &= \langle \psi, [\phi, \psi] \rangle_2 = \langle [\phi, \psi], \psi \rangle_2 \\
&= - \int_B \text{Tr} ([\phi, \psi] \wedge * \psi) = - \int_B \text{Tr} ([\phi, \psi] \wedge \psi) \\
&= \int_B \text{Tr} ([\phi, \psi] \wedge \psi) = \int_B \text{Tr} (\psi \wedge [\phi, \psi])
\end{aligned}$$

so

$$\frac{1}{2}\psi^\top(\rho_*(\phi))\psi = \frac{1}{2} \int_B \text{Tr} (\psi \wedge [\phi, \psi]) . \tag{6.12}$$

Putting together all of the terms (6.1), (6.5), (6.6), (6.10), (6.11), and (6.12) in the exponent of (5.23) we obtain

$$\begin{aligned}
\int_B \text{Tr} \left\{ \frac{1}{4} F_\omega \wedge * F_\omega + \frac{1}{4} F_\omega \wedge F_\omega + \frac{1}{2} \psi \wedge [\phi, \psi] - i d^\omega \chi \wedge \psi \right. \\
\left. - 2i [\chi, * \chi] \lambda + i * (\phi \Delta_0^\omega \lambda) - \chi \wedge * d^\omega \eta \right\} . \tag{6.13}
\end{aligned}$$

The Donaldson–Witten action S_{DW} is therefore (proportional to) minus this exponent. We introduce the **Donaldson–Witten Lagrangian**

$$\begin{aligned} \mathcal{L}_{\text{DW}} = & -\frac{1}{4}F_\omega \wedge *F_\omega - \frac{1}{4}F_\omega \wedge F_\omega - \frac{1}{2}\psi \wedge [\phi, \psi] + i d^\omega \chi \wedge \psi \\ & + 2i[\chi, *\chi] \lambda - i * (\phi \Delta_0^\omega \lambda) + \chi \wedge *d^\omega \eta. \end{aligned}$$

Then the exponent (6.13) becomes $-\int_B \text{Tr } \mathcal{L}_{\text{DW}}$ and the Atiyah–Jeffrey formula (5.23) for the “Euler number” of our vector bundle $\hat{A} \times_{\hat{g}} \Omega_+^2(B, \text{ad } P)$ is a constant times

$$\int e^{-\int_B \text{Tr } \mathcal{L}_{\text{DW}}} \mathcal{D}\chi \mathcal{D}\eta \mathcal{D}\psi \, d\lambda \, d\phi \, d\omega. \quad (6.14)$$

Notice that we have omitted the $(2\pi)^{-n}(2\pi)^{-k}$ in (5.23) since, in our present circumstances, both n and k would have to be infinite. The appropriate constant in (6.14) (and in (1.1)) is an issue for physics to decide since (6.14) is, in fact, the *partition function* for the field theory whose action is S_{DW} . We have also been cavalier about another constant (the so-called **coupling constant** of the theory) which the physicists must take very seriously (and could have been included from the outset in our definition of the universal Thom form (3.16)). For these issues we suggest Witten’s paper [18].

Let us recapitulate. The 0-dimensional Donaldson invariant of a smooth 4-manifold can be viewed as an intersection number for a section of an infinite-dimensional vector bundle and is therefore analogous to the Poincaré–Hopf version of an Euler number. Taking the analogy one step further one might hope for a Gauss–Bonnet–Chern type integral representation of the invariant. Atiyah and Jeffrey have shown that a formal application of the Mathai–Quillen formula for the Euler number to this infinite-dimensional vector bundle gives an integral which “should” represent this Euler number (if only the integrals, which are over infinite-dimensional spaces of fields, made sense). Whether it makes rigorous mathematical sense or not this integral representation coincides with the partition function of a topological quantum field theory devised by Witten with the sole purpose of capturing the Donaldson invariants as physical expectation values. The fact that two such radically different approaches to the problem of finding an integral representation of the Donaldson invariant should lead to the same result is heartening and highly suggestive of a much deeper connection between what topologists and theoretical physicists do.

But still, the result we have outlined here could be viewed as merely a curiosity if there were not a great deal more to this story. There are many Donaldson invariants besides the one we have described here and they all fit within this framework, as does the Casson invariant of homology 3-spheres, as well as the now famous Seiberg–Witten invariants. The reader inclined to do so can follow the story further in [1], [3], [5], [6], [11], [12], [16] and [18].

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