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CAYLEY MAP AND HIGHER DIMENSIONAL REPRESENTATIONS OF ROTATIONS

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Abstract. The embeddings of the $\mathfrak{so}(3)$ Lie algebra and the Lie group $\mathrm{SO}(3)$ in higher dimensions is an important construction from both mathematical and physical viewpoint. Here we present results based on a program package for building the generating matrices of the irreducible embeddings of the $\mathfrak{so}(3)$ Lie algebra within $\mathfrak{so}(n)$ in arbitrary dimension $n \geq 3, n \neq 4k+2, k \in \mathbb{N}$ relying on the algorithm developed recently by Campoamor-Strursberg [3]. For the remaining cases $n = 4k+2$ embeddings of $\mathfrak{so}(3)$ into $\mathfrak{so}(n)$ are also constructed. Besides, we investigate the characteristic polynomials of these $\mathfrak{so}(n)$ elements. We show that the *Cayley* map applied to $\mathcal{C} \in \mathfrak{so}(n)$ is well defined and generates a subset of $\mathrm{SO}(n)$. Furthermore, we obtain explicit formulas for the images of the *Cayley* map. The so obtained $\mathrm{SO}(n)$ matrices are expressed as polynomials of $\mathcal{C}$ whose coefficients are rational functions of the norm of the vector-parameter $\mathbf{c}$. The composition laws are extracted for the cases $n = 4, 6$ and for the first case it is shown that via the *Cayley* map the isomorphism $\mathrm{SU}(2) \cong \operatorname{im}\mathrm{Cay}_{\operatorname{im} j_4} \cup \{-\mathcal{I}_4\}$ holds. Also, for $n = 4$ explicit formulas for the the angular velocity matrices are derived. Comparisons between the results obtained via the exponential map and the *Cayley* map are made as well. In contrast to the case of the *Cayley* map, the results for the exponential map include either irrational or transcendental functions of the module of the vector-parameter.

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Notation and Nomenclature

$\mathbb{N}$	the set of natural numbers
$\mathrm{H}\mathbb{N}$	the set of half-integers, i.e., $\mathrm{H}\mathbb{N} = \{\frac{2k-1}{2}\,;\ k \in \mathbb{N}\}$
j_n	real irreducible embedding $\mathfrak{so}(3) \hookrightarrow \mathfrak{so}(n)$
n	the dimension under consideration
J	spin number $n = 2J+1$, $J \in \mathbb{N} \cup \mathrm{H}\mathbb{N}$
$[x]$	the integer part of the number x
m, r	spin numbers in the different cases
i, i_t, j, s, k, l, m	natural numbers used as indices
δ_k^l	*Kronecker* symbol
$\epsilon_{i,j,k}$	permutation symbol
a_l^m	numerical coefficients
$\mathbf{J}_n = \{J_{n\|1}, J_{n\|2}, J_{n\|3}\}$	the generating set of $\mathfrak{so}(3) \simeq \operatorname{im} j_n \subset \mathfrak{so}(n)$
$\mathcal{C}_n = \mathcal{C} = \mathbf{c}.\mathbf{J}_n$	an arbitrary element of $\operatorname{im} j_n$
$\mathcal{A}$	an arbitrary element of $\operatorname{im} j_n$ or SU(2) element
$\mathcal{H}$	Lie algebra element of $\mathfrak{so}(6)$
$\mathcal{I}, \mathcal{I}_n$	the unit matrix of respective dimension
$\mathcal{O}, \mathcal{O}_n$	the zero matrix of respective dimension
$(\mathbf{x}, \mathbf{z}) = \mathbf{x}.\mathbf{z}, \quad \mathbf{u} \times \mathbf{v}$	scalar and vector products in $\mathbb{R}^3$
G	connected matrix *Lie* group
Π, π	representations of G and its *Lie* algebra $\mathfrak{g}$
$\mathfrak{sl}(n)$	the *Lie* algebra of $n \times n$traceless matrices
$\mathfrak{so}(n)$	the *Lie* algebra of $n \times n$anti-symmetric matrices
$\mathrm{SO}(n)$	special orthogonal group of order n
SU(2)	special unitary group in two dimantions
$\boldsymbol{a}, \mathbf{c}, \mathbf{c}_i, \widetilde{\mathbf{c}}$	SO(3) vector-parmeters of $\mathrm{SO}(3) \subset \mathrm{SO}(n)$ elements
a, c	norms of the vectors $\boldsymbol{a}$ and $\mathbf{c}$
$M_{i,j}$	(i,j)-th component of the matrix M
$p_n(\lambda), \mu_n(\lambda)$	the characteristic and minimal polynomials of $\mathbf{c}.\mathbf{J}_n$
$\mathcal{U}$	unitary matrix
F_n	anti-symmetric matrix
$\Lambda, \Lambda_i, \widetilde{\Lambda}$	complex diagonal matrices
$\boldsymbol{\omega}^{\times}, \boldsymbol{\omega}_n^{\times}, \boldsymbol{\Omega}^{\times}, \boldsymbol{\Omega}_n^{\times}$	angular velocity matrices
$\boldsymbol{\omega}, \boldsymbol{\omega}_n, \boldsymbol{\Omega}, \boldsymbol{\Omega}_n$	angular velocity vectors

CONTENTS

1. Introduction

The group $\mathrm{SO}(3)$ and its *Lie* algebra $\mathfrak{so}(3)$ are of great importance for modern physics [10]. Higher dimensional realizations of $\mathfrak{so}(3)$ and $\mathrm{SO}(3)$ play a huge role in many fundamental areas and applications. Rotations and their higher dimensional realizations are of interest also in crystallography [13], in the problems of image recognition [17] and many other areas of modern physics, especially mechanics [19]. In this paper we will use the *Cayley* map to produce three-dimensional submanifolds in $\mathrm{SO}(n)$. The so obtained $\mathrm{SO}(n)$ rotations are expressed as polynomials of the general *Lie* algebra element of the embeddings $\mathfrak{so}(3) \hookrightarrow \mathfrak{so}(n)$ with coefficients that are rational functions of the norm c of the vector-parameter $\mathbf{c}$.

2. The Irreducible Representations of $\mathfrak{so}(3)$

Recall the standard $\mathbb{R}$-basis $\mathbf{J}_3 = \{\mathrm{J}_{3|1}, \mathrm{J}_{3|2}, \mathrm{J}_{3|3}\}$ of $\mathfrak{so}(3)$

$$\mathrm{J}_{3|1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathrm{J}_{3|2} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad \mathrm{J}_{3|3} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The embedding $j : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ is called irreducible [7] if the lowest dimensional irreducible representation Γ of $\widetilde{\mathfrak{g}}$ remains irreducible when restricted to $\mathfrak{g}$. *Campoamor-Stursberg* [3] derived explicit formulas for real irreducible embeddings of the algebra $\mathfrak{so}(3)$ into $\mathfrak{so}(n)$ for $n \geq 3$ where $n \neq 4m+2, m \in \mathbb{N}$. To do this, he uses the explicit embedding $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{sl}(n,\mathbb{C})$, which maps the generators $\{h, e, f\}$ of $\mathfrak{sl}(2,\mathbb{C})$ to the generators $\{D_J(h), D_J(e), D_J(f)\}$ in dimension $n = 2J+1$ for $J = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ via the explicit formulas

$$(D_J(h))_{k,l} = \delta_k^l(2J+2-2k), \qquad (D_J(e))_{k,l} = \delta_{k+1}^l(2J+1-k)$$
$$(D_J(f))_{k,l} = \delta_k^{l+1}(k-1)$$

where $1 \leq k, l \leq n$ are indices, δ_k^l is the *Kronecker*'s symbol and $M_{i,j}$ denotes the (i,j)-th element of the matrix M. If the scalars are restricted to the real numbers, then $\{D_J(h), D_J(e), D_J(f)\}$ becomes $\mathbb{R}-$basis of $\mathfrak{sl}(2,\mathbb{R}) \hookrightarrow \mathfrak{sl}(n,\mathbb{R})$. From here complex representation of $\mathfrak{so}(3)$ can be built by the generators $\{D_J(X_1), D_J(X_2), D_J(X_3)\}$ where

$$D_J(X_1) = \frac{\mathrm{i}}{2}D_J(h), \qquad D_J(X_2) = \frac{1}{2}(D_J(e) - D_J(f))$$
$$D_J(X_3) = \frac{\mathrm{i}}{2}(D_J(e) + D_J(f)).$$

Conjugating the matrices $\{D_J(X_k)\}_{k=1,2,3}$ with a suitable unitary matrix $U = U_J$ the real embedding

$$j_n : \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(n) \tag{1}$$

is obtained. The explicit formulas of j_n depend on the parity of the number n. Let us denote the vector $\mathbf{J}_n = \{\mathrm{J}_{n|1}, \mathrm{J}_{n|2}, \mathrm{J}_{n|3}\}$ of obtained by the images of $\mathrm{J}_{3|i}, i = 1,2,3$ under j_n. Let us introduce

$$a_l^q = \sqrt{\frac{l(2q+1-l)}{4}}, \qquad 1 \leq l \leq q-1 \tag{2}$$

where l takes integer values and $q \in \mathbb{N} \cup \mathrm{H}\mathbb{N}$. Below we will present some refined formulas for computing $\mathrm{J}_{n|1}, \mathrm{J}_{n|2}, \mathrm{J}_{n|3}$ for any integer $n \geq 3$. All numbers $n \geq 2$

can be expressed in the form $n = 2J+1$ where $J = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$, i.e., $J \in \mathbb{N} \cup \mathrm{H}\mathbb{N}$. The series $J = 1, 2, 3, \ldots$ correspond to odd n. The series $J = \frac{3}{2}, \frac{7}{2}, \ldots$, i.e., of the type $J = \frac{4k-1}{2}$ for $k \in \mathbb{N}$ correspond to even $n = 4k, k \in \mathbb{N}$. The last series for J, i.e., of the type $J = \frac{4m+1}{2}, m \in \mathbb{N}$ correspond to the dimensions $n = 4m+2, m \in \mathbb{N}$ in which no real irreducible representation of $\mathfrak{so}(3)$ can be built. However, formulas for real reducible representations in these dimensions are also available. We consider two different cases for the parity of n

1. n is odd: in this case $n = 2m+1$ for integer m. We have

$$\begin{aligned}(\mathrm{J}_{n|1})_{k,l} = {} & \left(\frac{1+(-1)^k}{2}\right)(\delta^l_{k+1} a^m_{[\frac{k}{2}]} + \delta^{l+3}_k a^m_{[\frac{k-2}{2}]}) - \left(a^m_m + \sqrt{\frac{m^2+m}{2}}\right) \\ & \times (\delta^l_n \delta^{n-1}_k - \delta^l_{n-1}\delta^n_k) - \left(\frac{1+(-1)^{k-1}}{2}\right)(\delta^l_{k+3} a^m_{[\frac{k+1}{2}]} + \delta^{l+1}_k a^m_{[\frac{k-1}{2}]}) \\ (\mathrm{J}_{n|2})_{k,l} = {} & \left(a^m_m + \sqrt{\frac{m^2+m}{2}}\right)(\delta^l_n\delta^{n-2}_k - \delta^l_{n-2}\delta^n_k) - (\delta^l_{k+2} a^m_{[\frac{k+1}{2}]} + \delta^{l+2}_k a^m_{[\frac{k-1}{2}]}) \\ (\mathrm{J}_{n|3})_{k,l} = {} & \frac{(1+(-1)^k)\delta^{l+1}_k (n+1-k) - (1+(-1)^{k-1})\delta^{k+1}_l (n-k)}{4}\end{aligned} \tag{3}$$

where $1 \le k, l \le n$ and $[x]$ denotes the integer part of x.

2. n is even: in this case we make use of the parameter r: $n = 4r+2 = 2J+1$ where $r \in \mathbb{N} \cup \mathrm{H}\mathbb{N}$, $J \in \mathrm{H}\mathbb{N}$. Here $r = \dfrac{2s-1}{2} \in \mathrm{H}\mathbb{N}$ refers to the dimensions $n = 4s, n \in \mathbb{N}$ with irreducible real representations and $r = m \in \mathbb{N}$ corresponds to $n = 4m+2$

$$\begin{aligned}(\mathrm{J}_{n|1})_{k,l} &= \left(\frac{1+(-1)^k}{2}\right)(\delta^l_{k+1} a^r_{[\frac{k}{2}]} + \delta^{l+3}_k a^r_{[\frac{k-2}{2}]}) \\ &\quad - \left(\frac{1+(-1)^{k-1}}{2}\right)(\delta^l_{k+3} a^r_{[\frac{k+1}{2}]} + \delta^{l+1}_k a^r_{[\frac{k-1}{2}]}) \\ (\mathrm{J}_{n|2})_{k,l} &= \delta^{l+2}_k a^r_{[\frac{k-1}{2}]} - \delta^l_{k+2} a^r_{[\frac{k+1}{2}]} \\ (\mathrm{J}_{n|3})_{k,l} &= \frac{(1+(-1)^k)\delta^{l+1}_k (n+1-k) - (1+(-1)^{k-1})\delta^{k+1}_l (n-k)}{4}\end{aligned} \tag{4}$$

where $1 \le k, l \le n$ are matrix indices. For an arbitrary $n \in \mathbb{N}, n \ge 3$, $\mathbf{J}_n$ is the generating set of the Lie subalgebra $\mathfrak{so}(3) \simeq \operatorname{im} j_n$ in $\mathfrak{so}(n)$.

2.1. Examples

Let us consider a few examples in order to demonstrate the developed computer procedure for generating the matrices $\mathrm{J}_{n|1}, \mathrm{J}_{n|2}, \mathrm{J}_{n|3}$ in the cases

- $n = 4$ obtained for the spin number $r = \frac{1}{2}$ in formulas (4)
- $n = 5$ obtained for $m = 2$ via the formulas (3)
- $n = 6$ obtained for the integer number $r = 1$ in (4).

In all examples we will show the most arbitrary matrix $\mathbf{c}.\mathbf{J}_n$ and the matrices $\mathrm{J}_{n|1}, \mathrm{J}_{n|2}, \mathrm{J}_{n|3}$. For $n = 4$ we have

$$
\begin{aligned}
J_{4|1} &= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad
J_{4|2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
J_{4|3} &= \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad
\mathbf{c}.\mathbf{J}_4 = \frac{1}{2}\begin{pmatrix} 0 & -c_3 & -c_2 & -c_1 \\ c_3 & 0 & c_1 & -c_2 \\ c_2 & -c_1 & 0 & c_3 \\ c_1 & c_2 & -c_3 & 0 \end{pmatrix}.
\end{aligned}
\tag{5}
$$

Next, for $n = 5$ we get

$$
J_{5|1} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & -\sqrt{3} & 0 \end{pmatrix}, \qquad
J_{5|2} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & -\sqrt{3} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \end{pmatrix}
$$

$$
J_{5|3} = \begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad
\mathbf{c}.\mathbf{J}_5 = \begin{pmatrix} 0 & -2c_3 & c_2 & c_1 & 0 \\ 2c_3 & 0 & -c_1 & c_2 & 0 \\ -c_2 & c_1 & 0 & -c_3 & -\sqrt{3}c_2 \\ -c_1 & -c_2 & c_3 & 0 & \sqrt{3}c_1 \\ 0 & 0 & \sqrt{3}c_2 & -\sqrt{3}c_1 & 0 \end{pmatrix}
\tag{6}
$$

and for $n = 6$ the obtained matrix is

$$\mathbf{c}.\mathbf{J}_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\sqrt{2}c_3 & -c_2 & -c_1 & 0 & 0 \\ \sqrt{2}c_3 & 0 & c_1 & -c_2 & 0 & 0 \\ c_2 & -c_1 & 0 & 0 & -c_2 & -c_1 \\ c_1 & c_2 & 0 & 0 & c_1 & -c_2 \\ 0 & 0 & c_2 & -c_1 & 0 & \sqrt{2}c_3 \\ 0 & 0 & c_1 & c_2 & -\sqrt{2}c_3 & 0 \end{pmatrix}. \tag{7}$$

The matrices $J_{6|i}, i = 1, 2, 3$ can be obtained as $\frac{\partial(\mathbf{c}.\mathbf{J}_6)}{\partial c_i}\big|_{\mathbf{c}=\mathbf{0}}$.

2.2. Irreducible Representations of $\mathrm{SO}(3)$

Let us recall [14, Propositions 4.4 & 4.5] the following facts: If G is a connected matrix *Lie* group with *Lie* algebra $\mathfrak{g}$, Π is a representation of G and π is the associated representation of $\mathfrak{g}$ such that

$$\Pi(\exp X) = \exp(\pi(X)), \qquad \pi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\Pi(\exp(tX))\Big|_{t=0}, \qquad X \in \mathfrak{g}.$$

Then Π is irreducible if and only if π is irreducible.

Many years ago *Wageningen* [23] and *Fedorov* [9] have constructed irreducible representations of $\mathrm{SO}(3)$ relying on the exponential map. The first named author had obtained explicit formula for the n-dimensional representation of an $\mathrm{SO}(3)$ matrix as a polynomial of the *Lie* algebra element used in the exponentiation for $n \le 13$ whereas the second one did the same for $n \le 7$. The coefficients in *Fedorov*'s formulas are algebraic functions of the norm of the vector-parameter corresponding to the *Lie* algebra element whereas in the *Wageningen*'s formulas the coefficients involve the trigonometric functions sin and cos.

In contrast to *Wageningen* and *Fedorov*, we will apply the *Cayley* map to the elements $\mathcal{C} = \mathbf{c}.\mathbf{J}_n \in \operatorname{im} j_n$ and in this way we will obtain explicit formulas for the parametrized $\mathrm{SO}(n)$ matrices as a polynomials of $\mathcal{C}$ whose coefficients are rational functions of c ($c^2 = \mathbf{c}.\mathbf{c}$).

3. The Cayley Map for the Embedding $\mathfrak{so}(3) \hookrightarrow \mathfrak{so}(n)$

We will consider the *Cayley* map defined on im j_n, i.e.,

$$\mathrm{Cay}(\mathcal{C}) = (\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} \tag{8}$$

for an arbitrary $\mathcal{C} = \mathcal{C}_n = \mathbf{c}.\mathbf{J}_n = c_1.\mathrm{J}_{n|1} + c_2.\mathrm{J}_{n|2} + c_3.\mathrm{J}_{n|3} \in \mathrm{im}\, j_n$, where

$$\mathbf{c} = (c_1, c_2, c_3), \qquad c^2 = c_1^2 + c_2^2 + c_3^2 = \mathbf{c}.\mathbf{c} = |\mathbf{c}|^2 = c^2.$$

We will derive explicit formulas for (8) depending on the parity of $n \geq 3$.

Lemma 1. *If $\mathcal{C}$ is $n \times n$ skew-symmetric matrix then* $\mathrm{Cay}(\mathcal{C}) \in \mathrm{SO}(n)$. *Also, if* $\mathcal{R} = \mathrm{Cay}(\mathbf{c}.\mathbf{J}_n)$ *then* $\mathcal{R}^{-1} = \mathcal{R}^t = \mathrm{Cay}(-\mathbf{c}.\mathbf{J}_n)$.

Proof: Using the fact that $\mathcal{C}^t = -\mathcal{C}$ and the fact that the matrices $\mathcal{I} - \mathcal{C}$ and $\mathcal{I} + \mathcal{C}$ commute. We obtain

$$((\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1})^t(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = ((\mathcal{I}-\mathcal{C})^{-1})^t(\mathcal{I}+\mathcal{C})^t(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}$$
$$= (\mathcal{I}+\mathcal{C})^{-1}(\mathcal{I}-\mathcal{C})(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = (\mathcal{I}+\mathcal{C})^{-1}(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = \mathcal{I}.$$

Furthermore

$$\det\,(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1} = \frac{\det\,(\mathcal{I}+\mathcal{C})}{\det\,(\mathcal{I}-\mathcal{C})} = \frac{\det\,(\mathcal{I}+\mathcal{C})}{\det\,(\mathcal{I}+\mathcal{C})^t} = 1.$$

Thus, $\mathrm{Cay}(\mathcal{C}) \in \mathrm{SO}(n)$. The second statement follows immediatelly

$$\mathrm{Cay}(\mathcal{C})\mathrm{Cay}(-\mathcal{C}) = (\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}(\mathcal{I}-\mathcal{C})(\mathcal{I}+\mathcal{C})^{-1} = \mathcal{I}.$$

∎

Lemma 2. *If $\mathcal{C}$ is $n \times n$ skew-symmetric matrix and*

i) $(\mathrm{Cay}(\mathcal{C}))^2 = \mathcal{I}$ *if and only if* $\mathcal{C} = \mathcal{O}$

ii) $(\mathrm{Cay}(\mathcal{C}))^2 = -\mathcal{I}$ *if and only if* $\mathcal{C}^2 = -\mathcal{I}$.

Proof: i) Let $(\mathrm{Cay}(\mathcal{C}))^2 = \mathcal{I}$. Because $\mathcal{I} + \mathcal{C}$ and $\mathcal{I} - \mathcal{C}$ commute we have

$$(\mathcal{I}+\mathcal{C})^2((\mathcal{I}-\mathcal{C})^2)^{-1} = \mathcal{I} \Longleftrightarrow (\mathcal{I}+\mathcal{C})^2 = (\mathcal{I}-\mathcal{C})^2 \Longleftrightarrow \mathcal{C} = \mathcal{O}.$$

ii) We have

$$(\mathrm{Cay}(\mathcal{C}))^2 = -\mathcal{I} \Leftrightarrow (\mathcal{I}+\mathcal{C})^2 = -(\mathcal{I}-\mathcal{C})^2 \Leftrightarrow 2(\mathcal{C}^2 + \mathcal{I}^2) = \mathcal{O} \Leftrightarrow \mathcal{C}^2 = -\mathcal{I}.$$

∎

3.1. The Case of Odd Dimensions

Let $n = 2m+1, m \geq 1$. The characteristic (which is also a minimal) polynomial [9] of an arbitrary matrix $\mathcal{C} = \mathcal{C}_n = \mathbf{c}.\mathbf{J}_n$ is

$$\begin{aligned} p_{2m+1}(\lambda) &= -\lambda(\lambda^2 + 1^2c^2)\ldots(\lambda^2 + m^2c^2) = -\lambda\prod_{t=1}^{m}(\lambda^2 + t^2c^2) \\ &= -\lambda^{2m+1} - \alpha_{2m-1}c^2\lambda^{2m-1} - \ldots - \alpha_1 c^{2m}\lambda \\ &= -\lambda^{2m+1} - \sum_{t=1}^{m}\alpha_{2m+1-2t}c^{2t}\lambda^{2m+1-2t}. \end{aligned} \tag{9}$$

Because p_{2m+1} has only simple roots, it coincides with the minimal polynomial μ_{2m+1} of $\mathcal{C}$. Consider the polynomial

$$g_{2m+1}(\mu) = \mu^m + \alpha_{2m-1}\mu^{m-1} + \alpha_{2m-3}\mu^{m-2} + \ldots + \alpha_3\mu + \alpha_1$$

obtained by $\dfrac{-p_{2m+1}(\lambda)}{\lambda c^{2m}}$ after the substitution of $\dfrac{\lambda^2}{c^2}$ with μ. The so obtained polynomial $g_{2m+1}(\mu)$ is an unitary and of degree m with simple roots $-1^2, -2^2, \ldots -m^2$. Expressions for the coefficients $\alpha_{2m+1}, \alpha_{2m-1}, \ldots, \alpha_1$ can be obtained by *Vieta*'s formulas for g_{2m+1}

$$\alpha_{2m+1-2t} = \sum_{1\leq i_1<\ldots<i_t\leq m} i_1^2\ldots i_t^2, \qquad t = 1, 2, \ldots, m.$$

For example, the closed forms of α_{2m-1}, α_{2m-3} for $m \geq 2$ are

$$\alpha_{2m-1} = \frac{m(m+1)(2m+1)}{6}, \qquad \alpha_{2m-3} = \frac{m(m^2-1)(4m^2-1)(5m+6)}{180}$$

and $\alpha_1 = (m!)^2$. More explicit expressions and relations for the coefficients $\alpha_{2m+1-2t}, t = 1, \ldots, m$ can be given in terms of the *Bernouli*'s coefficients [2, 16] and the generalized harmonic coefficients $H_{m,2} = \sum\limits_{k=1}^{m}\dfrac{1}{k^2}\cdot$ For example, $\alpha_3 = (m!)^2 H_{m,2}$.

Theorem 3. *For an arbitrary* $n = 2m+1, m \geq 1$ *the Cayley map* (8) *is well-defined on* im j_n *and the following explicit formula holds true*

$$\mathrm{Cay}(\mathcal{C}) = \mathcal{I}_{2m+1} + 2\sum_{s=0}^{m-1}\frac{1 + \sum\limits_{k=1}^{m-s-1}\alpha_{2m+1-2k}c^{2k}}{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}}(\mathcal{C}^{2s+1} + \mathcal{C}^{2s+2}) \tag{10}$$

for all $\mathcal{C} \equiv \mathcal{C}_n = \mathbf{c}.\mathbf{J}_n \in$ im j_n. *Also, the map* Cay *takes values in* $\mathrm{SO}(2m+1)$.

Proof: We need to prove that $\mathcal{I}_{2m+1} - \mathcal{C}$ is invertable and to find an explicit formula for it and in this way for $\operatorname{Cay}(\mathcal{C}) = (\mathcal{I}_{2m+1}+\mathcal{C})(\mathcal{I}_{2m+1}-\mathcal{C})^{-1}$. From now on we will use $\mathcal{I}$ to denote the identity matrix of dimension coherent with the context. We will seek a formula for $(\mathcal{I} - \mathcal{C})^{-1}$ via the *ansatz*

$$(\mathcal{I} - \mathcal{C})^{-1} = \xi_0\mathcal{I} + \xi_1\mathcal{C} + \ldots \xi_{2m}\mathcal{C}^{2m}.$$

We want to find the coefficients $\xi_0, \ldots \xi_{2m}$ such that $\mathcal{I} = (\mathcal{I} - \mathcal{C})(\sum\limits_{k=0}^{2m} \xi_k\mathcal{C}^k)$. Taking into account the *Hamilton–Cayley* formula for $\mathcal{C}$ we calculate

$$\begin{aligned}
\mathcal{I} &= (\mathcal{I} - \mathcal{C})(\xi_0\mathcal{I} + \xi_1\mathcal{C} + \ldots + \xi_{2m}\mathcal{C}^{2m}) \\
&= \xi_0\mathcal{I} + (\xi_1 - \xi_0)\mathcal{C} + (\xi_2 - \xi_1)\mathcal{C}^2 + \ldots + (\xi_{2m} - \xi_{2m-1})\mathcal{C}^{2m} - \xi_{2m}\mathcal{C}^{2m+1} \\
&= \xi_0\mathcal{I} + (\xi_1 - \xi_0 + \xi_{2m}\alpha_1 c^{2m})\mathcal{C} + (\xi_2 - \xi_1)\mathcal{C}^2 + \ldots \\
&\quad + (\xi_{2m-1} - \xi_{2m-2} + \xi_{2m}\alpha_{2m-1}c^2)\mathcal{C}^{2m-1} + (\xi_{2m} - \xi_{2m-1})\mathcal{C}^{2m} \qquad (11)\\
&= \xi_0\mathcal{I} + \sum_{s=0}^{m-1}(\xi_{2s+1} - \xi_{2s} + \xi_{2m}\alpha_{2s+1}c^{2m-2s})\mathcal{C}^{2s+1} \\
&\quad + \sum_{s=0}^{m-1}(\xi_{2s+2} - \xi_{2s+1})\mathcal{C}^{2s+2}.
\end{aligned}$$

From (11) we obtain directly a linear system of equations for the unknown quantities $\xi_0, \ldots, \xi_{2m}$ consisting of $2m + 1$ equations which can be splited in the following two sets of equations

$$\begin{array}{rcl}
\xi_2 &=& \xi_1 \\
\xi_4 &=& \xi_3 \\
&\ldots& \\
\xi_{2m-2} &=& \xi_{2m-3} \\
\xi_{2m} &=& \xi_{2m-1}
\end{array}
\qquad
\begin{array}{rcl}
\xi_0 & = & 1 \\
\xi_1 - \xi_0 & = & -\xi_{2m}\alpha_1 c^{2m} \\
\xi_3 - \xi_2 & = & -\xi_{2m}\alpha_3 c^{2m-2} \\
\ldots & & \\
\xi_{2m-1} - \xi_{2m-2} & = & -\xi_{2m}\alpha_{2m-1}c^2.
\end{array} \qquad (12)$$

Resolving the system (12) step by step we obtain $\xi_0 = 1$

$$\begin{array}{rcl}
\xi_2 = \xi_1 &=& 1 - \xi_{2m}\alpha_1 c^{2m} \\
\xi_4 = \xi_3 &=& 1 - \xi_{2m}(\alpha_1 c^{2m} + \alpha_3 c^{2m-2}) \\
\ldots && \\
\xi_{2m} = \xi_{2m-1} &=& 1 - \xi_{2m}(\alpha_1 c^{2m} + \alpha_3 c^{2m-2} + \ldots + \alpha_{2m-1}c^2).
\end{array} \qquad (13)$$

Summing up all equations on the right hand side of (12), we obtain

$$\xi_{2m} = \xi_{2m-1} = 1 - \xi_{2m}(\alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}) = 1 + \xi_{2m}(p_{2m+1}(1) + 1)$$

and thus

$$\xi_{2m} = -\frac{1}{p_{2m+1}(1)} = \frac{1}{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}}.$$

Note also that

$$-p_{2m+1}(1) = p_{2m+1}(-1) = (1 + c^2)(1 + 4c^2)\ldots(1 + m^2c^2) > 0$$

for all $\mathbf{c} \in \mathbb{R}^3$. Substituting this result in (13) gives

$$\begin{aligned} \xi_2 = \xi_1 &= \frac{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_3 c^{2m-2}}{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}} \\ \xi_4 = \xi_3 &= \frac{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_5 c^{2m-4}}{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}} \\ &\ldots \\ \xi_{2m} = \xi_{2m-1} &= \frac{1}{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}}. \end{aligned} \tag{14}$$

In this way we have obtained that for all $\mathbf{c} \in \mathbb{R}^3$ the matrix $\mathcal{I} - \mathcal{C}$ is invertable and

$$(\mathcal{I} - \mathcal{C})^{-1} = \mathcal{I} + \sum_{s=0}^{m-1} \frac{1 + \sum\limits_{k=1}^{m-s-1} \alpha_{n-2k} c^{2k}}{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}} (\mathcal{C}^{2s+1} + \mathcal{C}^{2s+2}). \tag{15}$$

Now it is a straightforward, but tedious computation to express $(\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1}$ as a polynomial in $\mathcal{C}$ using the simple fact that

$$\mathrm{Cay}(\mathcal{C}) = (\mathcal{I} + \mathcal{C})(\mathcal{I} - \mathcal{C})^{-1} = \mathcal{I} + 2\mathcal{C}(\mathcal{I} - \mathcal{C})^{-1} \tag{16}$$

and this leads to the formula (10). It is curious that the above formulas (10) and (15) are so alike. Because of Lemma 1, $\mathrm{Cay}(\mathcal{C}) \in \mathrm{SO}(n)$. ■

3.1.1. The Case $n = 3$

In the case $n = 3$ the map j_3 coincides with the identity and the *Cayley* map reduces to the known *Gibbs* vector-parameter representation of SO(3). The characteristic polynomial of $\mathcal{C} \equiv \mathcal{C}_3 = \mathbf{c}.\mathbf{J}_3$ is $p_3(\lambda) = -\lambda^3 - c^2\lambda$ for all $\mathbf{c} \in \mathbb{R}^3$ and respectively the *Cayley* map is

$$\operatorname{Cay}(\mathcal{C}) = \mathcal{I}_3 + \frac{2}{1+c^2}\mathcal{C} + \frac{2}{1+c^2}\mathcal{C}^2 = \mathcal{I}_3 + \frac{2}{1+c^2}(\mathcal{C}+\mathcal{C}^2)$$
$$= \frac{2}{1+c^2}\begin{pmatrix} 1+c_1^2-c_2^2-c_3^2 & c_1c_2-c_3 & c_1c_3+c_2 \\ c_1c_2+c_3 & 1-c_1^2+c_2^2-c_3^2 & c_2c_3-c_1 \\ c_1c_3-c_2 & c_2c_3+c_1 & 1-c_1^2-c_2^2+c_3^2 \end{pmatrix}. \tag{17}$$

This leads to the composition law [9]

$$\mathcal{R}_3(\widetilde{\mathbf{c}}) = \mathcal{R}_{\boldsymbol{a}}(\boldsymbol{a})\mathcal{R}_{\mathbf{c}}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \langle \boldsymbol{a}, \mathbf{c}\rangle_{\mathrm{SO}(3,\mathbb{R})} = \frac{\boldsymbol{a}+\mathbf{c}+\boldsymbol{a}\times\mathbf{c}}{1-\boldsymbol{a}.\mathbf{c}} \tag{18}$$

where $\mathcal{R}(\boldsymbol{a}) = \operatorname{Cay}(\boldsymbol{a}.\mathbf{J}_3)$ and $\mathcal{R}(\mathbf{c}) = \operatorname{Cay}(\mathbf{c}.\mathbf{J}_3)$ are two proper rotations. If one wants to extend formula (18) in order to cover all possible scenarios, i.e., the cases where half-turns are involved in the composition or the result is a half-turn and to keep its intuitive nature and low computational complexity one could use the *Cayley* map for the covering group $\mathrm{SU}(2)$ and parametrize the half-turns as it was done in [6]. For the so extended composition law see Table 1 in Appendix A.

3.1.2. The Case $n = 5$

In the first non-trivial case $n = 5$ for the series of odd n we compute the characteristic polynomial of the matrix $\mathcal{C} \equiv \mathcal{C}_5 = \mathbf{c}.\mathbf{J}_5$

$$p_5(\lambda) = -\lambda^5 - 5c^2\lambda^3 - 4c^4\lambda.$$

The explicit formula for the *Cayley* map $\mathcal{R}_5(\mathbf{c}) = \operatorname{Cay}(\mathcal{C})$ reads as

$$\mathcal{R}_5(\mathbf{c}) = \mathcal{I} + 2\frac{5c^2+1}{4c^4+5c^2+1}(\mathcal{C}+\mathcal{C}^2) + \frac{2}{4c^4+5c^2+1}(\mathcal{C}^3+\mathcal{C}^4).$$

3.2. The Case of Even $n = 4s$ for Integer s

Let $r = \dfrac{2s-1}{2}$, $s \in \mathbb{N}$ be a positive half-integer. Then the number $n = 4\dfrac{2s-1}{2} + 2 = 4s$ is obviously divisible by 4. The characteristic polynomial of an arbitrary matrix $\mathcal{C}_n = \mathbf{c}.\mathbf{J}_n$ is

$$p_{4s}(\lambda) = \prod_{t=1}^{s}\Big(\lambda^2 + (\Big(\frac{2t-1}{2}\Big)^2 c^2)^2. \tag{19}$$

Lemma 4. *The minimal polynomial for the matrix* $\mathcal{C}_n \equiv \mathcal{C}_{4s} = \mathbf{c}.\mathbf{J}_{4s}$ *where* $s \in \mathbb{N}$, $\mathbf{c} \in \mathbb{R}^3$ *is*

$$\mu_{4s}(\lambda) = \prod_{t=1}^{s}(\lambda^2 + \Big(\frac{2t-1}{2}\Big)^2 c^2).$$

Proof: The polynomial μ_{4s} is the polynomial of least degree and can be seen as the minimal polynomial of $\mathcal{C}$ since it contains as simple roots all eigenvalues of $\mathcal{C}$ and is monic. The proof that $\mu_{4s}(\mathcal{C}_{4s}) = \mathcal{O}_{4m}$ is similar to the one done in Lemma 6 below and is omitted here. ■

Let us denote the coefficients of μ_{4s}

$$\mu_{4s}(\lambda) = \lambda^{2s} + \gamma_{2s-2}c^2\lambda^{2s-2} + \ldots + \gamma_2 c^{2s-2}\lambda^2 + \gamma_0 c^{2s}.$$

One can seek explicit and closed formulas for these coefficients via combinatorial means. For example

$$\gamma_{2s-2} = \sum_{t=1}^{s}\left(\frac{2t-1}{2}\right)^2 = \frac{s(2s-1)(2s+1)}{3}, \qquad \gamma_0 = \frac{(2s-1)!!}{2^s}.$$

By Lemma 4 we have that

$$\mathcal{C}_{4s}^{2s} = -\gamma_{2s-2}c^2\mathcal{C}_{4s}^{2s-2} - \ldots - \gamma_2 c^{2s-2}\mathcal{C}_{4s}^2 - \gamma_0 c^{2s}\mathcal{I}_{4s}. \tag{20}$$

Theorem 5. *For an arbitrary $n = 4s$ where $s \in \mathbb{N}$, the Cayley map* (8) *is well-defined on* im j_n *and the following explicit formula holds true*

$$\begin{aligned}\mathrm{Cay}(\mathcal{C}) &= \frac{1 + \gamma_{2s-2}c^2 + \ldots + \gamma_2 c^{2s-2} - \gamma_0 c^{2s}}{1 + \gamma_{2s-2}c^2 + \ldots + \gamma_2 c^{2s-2} + \gamma_0 c^{2s}}\mathcal{I}_{4s} + 2\frac{1 + \sum\limits_{k=1}^{s-1}\gamma_{2s-2k}c^{2k}}{\mu_{4s}(1)}\mathcal{C}\\ &\quad + 2\sum_{i=1}^{s-1}\frac{1 + \sum\limits_{k=1}^{s-i-1}\gamma_{2s-2k}c^{2k}}{\mu_{4s}(1)}(\mathcal{C}^{2i} + \mathcal{C}^{2i+1})\end{aligned} \tag{21}$$

for all $\mathcal{C} = \mathbf{c}.\mathbf{J}_n \in$ im j_n.

Proof: Similarly to the proof of Theorem 3 and taking into consideration (20) we use an ansatz for $(\mathcal{I} - \mathcal{C})^{-1}$ as a polynomial of degree $2s - 1$ in $\mathcal{C}$

$$(\mathcal{I} - \mathcal{C})^{-1} = \eta_0\mathcal{I}_n + \eta_1\mathcal{C} + \ldots + \eta_{2s-1}\mathcal{C}^{2s-1}.$$

This leads to the system of linear equations

$$\begin{array}{rclcrcrcl}\eta_1 &=& \eta_0 & \qquad & \eta_0 &-& 1 &=& -\eta_{2s-1}c^{2s}\gamma_0\\ \eta_3 &=& \eta_2 & & \eta_2 &-& \eta_1 &=& -\eta_{2s-1}c^{2s-2}\gamma_2\\ & \ldots & & & & \ldots & & & \\ \eta_{2s-1} &=& \eta_{2s-2} & & \eta_{2s-2} &-& \eta_{2s-3} &=& -\eta_{2s-1}c^2\gamma_{2s-2}\end{array}$$

which has an unique solution

$$\eta_1 = \eta_0 = \frac{1 + \gamma_{2s-2}c^2 + \ldots + \gamma_2 c^{2s-2}}{1 + \gamma_{2s-2}c^2 + \ldots + \gamma_0 c^{2s}}$$

$$\eta_3 = \eta_2 = \frac{1 + \gamma_{2s-2}c^2 + \ldots + \gamma_4 c^{2s-4}}{1 + \gamma_{2s-2}c^2 + \ldots + \gamma_0 c^{2s}}$$

$$\ldots$$

$$\eta_{2s-1} = \eta_{2s-2} = \frac{1}{1 + \gamma_{2s-2}c^2 + \ldots + \gamma_0 c^{2s}}$$

for all $\mathbf{c} \in \mathbb{R}^3$. We omit the details here. Finally for $(\mathcal{I} - \mathcal{C})^{-1}$ we obtain

$$(\mathcal{I} - \mathcal{C})^{-1} = \sum_{i=0}^{s-1} \frac{1 + \sum\limits_{k=1}^{s-i-1} \gamma_{2s-2k} c^{2k}}{\mu_{4s}(1)} (\mathcal{C}^{2i} + \mathcal{C}^{2i+1}).$$

Now it is a straightforward calculation of $(\mathcal{I}+\mathcal{C})(\mathcal{I}-\mathcal{C})^{-1}$ to establish the validity of formula (21). According to Lemma 1 the *Cayley* map takes values in $\mathrm{SO}(n)$. ■

The above realizations in the dimensions $n = 4s$, $s \in \mathbb{N}$ are quite interesting since the generating matrices $\mathcal{C} = \mathbf{c}.\mathbf{J}_{4s}$ are invertable for all $\mathbf{c} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$.

3.2.1. The Case $n = 4$

The *Hamilton–Cayley* theorem for $\mathcal{C}$ reads as

$$\mathcal{C}^4 + \frac{c^2}{2}\mathcal{C}^2 + \frac{c^4}{16}\mathcal{I} = \mathcal{O}_4 \Longrightarrow \mathcal{C}^4 = -\frac{c^2}{2}\mathcal{C}^2 - \frac{c^4}{16}\mathcal{I}$$

where $\mathcal{O}_k$ denotes the zero $k \times k$ matrix. Despite this fact, following Lemma 4 we take into account that $\mathcal{C}$ is a root of its minimal polynomial and so $\mathcal{C}^2 = -\frac{c^2}{4}\mathcal{I}$.

Recall [5] that if we consider an arbitrary $\mathfrak{su}(2)$ matrix

$$\mathcal{A} = \mathcal{A}(\mathbf{c}) = \frac{1}{2}\begin{pmatrix} -\mathrm{i}c_3 & -c_2 - \mathrm{i}c_1 \\ c_2 - \mathrm{i}c_1 & \mathrm{i}c_3 \end{pmatrix} = \mathbf{c}.\mathbf{s}$$

with coordinates $\mathbf{c} = (c_1, c_2, c_3)$ in the basis $\mathbf{s} = (s_1, s_2, s_3)$ chosen in [5] we have the same identity $\mathcal{A}^2 = -\frac{c^2}{4}\mathcal{I}_2$.

Direct algebraic simplification of (21) leads to the beautiful formula

$$\mathrm{Cay}(\mathcal{C}) = \mathcal{R}_4(\mathbf{c}) = \frac{4 - c^2}{4 + c^2}\mathcal{I} + \frac{8}{4 + c^2}\mathcal{C}. \tag{22}$$

This expression has the same form as that one for $\mathrm{Cay}_{\mathfrak{su}(2)}$ [5, Equation 2.29]. Notice also that

$$\mathcal{R}_4^t(\mathbf{c}) = \frac{4-c^2}{4+c^2}\mathcal{I} - \frac{8}{4+c^2}\mathcal{C}.$$

3.2.2. Extracting the Vector-Parameter From a Rotational Matrix For $n = 4$

Obviously, the *Cayley* map is defined for all $\mathbf{c} \in \mathbb{R}^3$. From $\mathcal{R}_4^t.\mathcal{R}_4 = \mathcal{I}_4$ it follows that $\operatorname{tr}\mathcal{R}_4(\mathbf{c}) < 4$. How do we extract the vector $\mathbf{c}$ from a given matrix $\mathcal{R}_4(\mathbf{c}) = \mathrm{Cay}(\mathcal{C})$? We have that

$$\operatorname{tr}\mathcal{R}_4(\mathbf{c}) = 4\frac{4-c^2}{4+c^2} \quad\Longrightarrow\quad \frac{1}{4+c^2} = \frac{4+\operatorname{tr}\mathcal{R}_4(\mathbf{c})}{32}$$

and thus if we consider $F_4 = \mathcal{R}_4(\mathbf{c}) - \mathcal{R}_4^t(\mathbf{c}) = \dfrac{16}{4+c^2}\mathcal{C}$ then we have

$$2\mathcal{C}(\mathbf{c}) = \frac{4}{4+\operatorname{tr}\mathcal{R}_4(\mathbf{c})}F_4$$

and $\mathbf{c} = -2(\mathcal{C}_{1,4}, \mathcal{C}_{1,3}, \mathcal{C}_{1,2})$, see equation (5).

3.2.3. The Angular Velocity Matrices in the Case $n = 4$

Let us calculate the angular velocity matrices

$$\boldsymbol{\omega}^\times \equiv \boldsymbol{\omega}_4^\times = \mathcal{R}_4^t(\mathbf{c})\dot{\mathcal{R}}_4(\mathbf{c}), \qquad \boldsymbol{\Omega}^\times \equiv \boldsymbol{\Omega}_4^\times = \dot{\mathcal{R}}_4(\mathbf{c})\mathcal{R}_4^t(\mathbf{c})$$

for a rigid body motion. The matrix $\boldsymbol{\omega}^\times$ corresponds to the angular velocity vector $\boldsymbol{\omega} \equiv \boldsymbol{\omega}_4$ in the body frame attached to the rigid body, whereas $\boldsymbol{\Omega}^\times$ corresponds to the angular velocity vector $\boldsymbol{\Omega} \equiv \boldsymbol{\Omega}_4$ in the inertial frame [21]. Recall [19] that in the three-dimensional case the formulas for $\boldsymbol{\omega}_3$ and $\boldsymbol{\Omega}_3$ are

$$\boldsymbol{\omega}_3 = \frac{2}{1+c^2}(\mathbf{c}\times\dot{\mathbf{c}} + \dot{\mathbf{c}}), \qquad \boldsymbol{\Omega}_3 = \frac{2}{1+c^2}(\dot{\mathbf{c}}\times\mathbf{c} + \dot{\mathbf{c}}).$$

We are going to compute $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ in the four dimensional case. Direct calculation gives

$$\frac{\partial}{\partial t}\left(\frac{4-c^2}{4+c^2}\right) = \frac{\partial}{\partial t}\left(-1+\frac{8}{4+c^2}\right) = -\frac{16\mathbf{c}.\dot{\mathbf{c}}}{(4+c^2)^2}$$

where $\dot{\mathbf{c}} = (\dot{c}_1, \dot{c}_2, \dot{c}_3)$. From (22) we have

$$\dot{\mathcal{R}}_4(\mathbf{c}) = -\frac{16\mathbf{c}.\dot{\mathbf{c}}}{(4+c^2)^2}\mathcal{I} - \frac{16\mathbf{c}.\dot{\mathbf{c}}}{(4+c^2)^2}\mathcal{C} + \frac{8}{4+c^2}\dot{C} = \frac{8}{4+c^2}\Big(-\frac{2\mathbf{c}.\dot{\mathbf{c}}}{4+c^2}(\mathcal{I}+\mathcal{C}) + \dot{\mathcal{C}}\Big).$$

Now taking into account $\mathcal{C}^2 = -\dfrac{c^2}{4}\mathcal{I}$ and (22) we obtain

$$\begin{aligned}
\frac{(4+c^2)^2}{8}\mathbf{\Omega}^\times &= \Big(-\frac{2\mathbf{c}.\dot{\mathbf{c}}}{4+c^2}(\mathcal{I}+\mathcal{C})+\dot{\mathcal{C}}\Big)\left((4-c^2)\mathcal{I}-8\mathcal{C})\right) \\
&= -\frac{2(4-c^2)\mathbf{c}.\dot{\mathbf{c}}}{4+c^2}(\mathcal{I}+\mathcal{C}) + (4-c^2)\dot{\mathcal{C}} + \frac{16\mathbf{c}.\dot{\mathbf{c}}}{4+c^2}\mathcal{C} - \frac{16\mathbf{c}.\dot{\mathbf{c}}}{4+c^2}\frac{c^2}{4}\mathcal{I} - 8\dot{\mathcal{C}}\mathcal{C} \\
&= -2\mathbf{c}.\dot{\mathbf{c}}\mathcal{I} + 2\mathbf{c}.\dot{\mathbf{c}}\mathcal{C} + (4-c^2)\dot{\mathcal{C}} - 8\dot{\mathcal{C}}\mathcal{C}.
\end{aligned}$$

The only difference in the result for $\boldsymbol{\omega}^\times$ is in the term $8\dot{\mathcal{C}}\mathcal{C}$ which appears as $8\mathcal{C}\dot{\mathcal{C}}$ in $\mathbf{\Omega}^\times$

$$\boldsymbol{\omega}^\times = \frac{8}{(4+c^2)}\left(-2\mathbf{c}.\dot{\mathbf{c}}\mathcal{I} + 2\mathbf{c}.\dot{\mathbf{c}}\mathcal{C} + (4-c^2)\dot{\mathcal{C}} - 8\mathcal{C}\dot{\mathcal{C}}\right) \tag{23}$$

$$\mathbf{\Omega}^\times = \frac{8}{(4+c^2)}\left(-2\mathbf{c}.\dot{\mathbf{c}}\mathcal{I} + 2\mathbf{c}.\dot{\mathbf{c}}\mathcal{C} + (4-c^2)\dot{\mathcal{C}} - 8\dot{\mathcal{C}}\mathcal{C}\right). \tag{24}$$

One can observe though, that the following identities hold true

$$8\mathcal{C}\dot{\mathcal{C}} = -2\mathbf{c}.\dot{\mathbf{c}}\mathcal{I} + 4(\mathbf{c}\times\dot{\mathbf{c}}).\mathbf{J}_4, \qquad 8\dot{\mathcal{C}}\mathcal{C} = -2\mathbf{c}.\dot{\mathbf{c}}\mathcal{I} - 4(\mathbf{c}\times\dot{\mathbf{c}}).\mathbf{J}_4$$

and from these and $\dot{\mathcal{C}} = \dot{\mathbf{c}}^\times$ we obtain that (23) and (24) simplify to

$$\boldsymbol{\omega}^\times = \boldsymbol{\omega}.\mathbf{J}_4 = \frac{8}{4+c^2}\left(2(\mathbf{c}.\dot{\mathbf{c}})\mathbf{c} + (4-c^2)\dot{\mathbf{c}} - 4\mathbf{c}\times\dot{\mathbf{c}}\right).\mathbf{J}_4 \tag{25}$$

$$\mathbf{\Omega}^\times = \mathbf{\Omega}.\mathbf{J}_4 = \frac{8}{4+c^2}\left(2(\mathbf{c}.\dot{\mathbf{c}})\mathbf{c} + (4-c^2)\dot{\mathbf{c}} + 4\mathbf{c}\times\dot{\mathbf{c}}\right).\mathbf{J}_4. \tag{26}$$

We can express formulas (25) and (26) in the following way

$$\begin{aligned}
\boldsymbol{\omega} &= -8\frac{\partial}{\partial t}\frac{\mathbf{c}}{4+c^2} + 32\frac{2\dot{\mathbf{c}} - 4\mathbf{c}\times\dot{\mathbf{c}}}{(4+c^2)^2} \\
\mathbf{\Omega} &= -8\frac{\partial}{\partial t}\frac{\mathbf{c}}{4+c^2} + 32\frac{2\dot{\mathbf{c}} + 4\mathbf{c}\times\dot{\mathbf{c}}}{(4+c^2)^2}.
\end{aligned}$$

Formulas (25) and (26) can be written as matrix equations in the form

$$\boldsymbol{\omega} = B_\omega\dot{\mathbf{c}}, \qquad \mathbf{\Omega} = B_\Omega\dot{\mathbf{c}}$$

where $B_\omega = B_\Omega^t$ and

$$B_\Omega = \frac{8}{(4+c^2)^2}\begin{pmatrix} 4+c_1^2-c_2^2-c_3^2 & 2c_1c_2-4c_3 & 2c_1c_3+4c_2 \\ 2c_1c_2+4c_3 & 4-c_1^2+c_2^2-c_3^2 & 2c_2c_3-4c_1 \\ 2c_1c_3-4c_2 & 2c_2c_3+4c_1 & 4-c_1^2-c_2^2+c_3^2 \end{pmatrix}.$$

The matrix B_Ω is invertible for all $\mathbf{c} \in \mathbb{R}^3$ and it is a straightforward to compute

$$B_\Omega^{-1} = \frac{(4+c^2)^2}{64} B_\Omega^t. \tag{27}$$

Respectively, the matrix

$$\mathcal{R}_\Omega := B_\Omega^{-1} B_\Omega^t = B_\Omega^t B_\Omega^{-1} \tag{28}$$

is the four dimensional analogue to the matrix defined in [21, Formula (32)] which has many applications in mechanics. From formula (27) follows also that

$$\mathcal{R}_\Omega = \frac{(4+c^2)^2}{64} (B_\Omega^t)^2 = \frac{(4+c^2)^2}{64} B_\omega^2.$$

The matrix form of $\dfrac{(4+c^2)^2}{8} \mathcal{R}_\Omega$ is

$$\begin{pmatrix} (4+c^2)^2 - 32(c_2^2+c_3^2) & 32c_1c_2 + 8c_3(4-c^2) & 32c_1c_3 - 8c_2(4-c^2) \\ 32c_1c_2 - 8c_3(4-c^2) & (4+c^2)^2 - 32(c_1^2+c_3^2) & 32c_2c_3 + 8c_1(4-c^2) \\ 32c_1c_3 + 8c_2(4-c^2) & 32c_2c_3 - 8c_1(4-c^2) & (4+c^2)^2 - 32(c_1^2+c_2^2) \end{pmatrix}.$$

It is worth noticing that the matrix $\mathcal{B}_\Omega = (4+c^2)B_\Omega$

$$\mathcal{B}_\Omega = \frac{1}{4+c^2} \begin{pmatrix} 4+c_1^2-c_2^2-c_3^2 & 2c_1c_2 - 4c_3 & 2c_1c_3 + 4c_2 \\ 2c_1c_2 + 4c_3 & 4-c_1^2+c_2^2-c_3^2 & 2c_2c_3 - 4c_1 \\ 2c_1c_3 - 4c_2 & 2c_2c_3 + 4c_1 & 4-c_1^2-c_2^2+c_3^2 \end{pmatrix}$$

is an $\mathrm{SO}(3)$ matrix which is surprisingly similar to the rotational matrix in equation (17), i.e., $\mathcal{B}_\Omega(\mathbf{c}) = \mathrm{Cay}(2\mathbf{c}.\mathbf{J}_3)$.

3.2.4. The Composition Law for $n = 4$

Let $\mathcal{A} = \boldsymbol{a}.\mathbf{J}_4$ and $\mathcal{C} = \mathbf{c}.\mathbf{J}_4$ be two arbitrary elements from im j_4. Let $\mathcal{R}(\boldsymbol{a})$ and $\mathcal{R}(\mathbf{c})$ be the images of these matrices under the *Cayley* map, i.e.,

$$\mathrm{Cay}(\mathcal{A}) = \mathcal{R}(\boldsymbol{a}), \qquad \mathrm{Cay}(\mathcal{C}) = \mathcal{R}(\mathbf{c}).$$

Let $\mathcal{R} = \mathcal{R}(\boldsymbol{a})\mathcal{R}(\mathbf{c})$ be their composition in $\mathrm{SO}(4)$. We want to find an element $\widetilde{\mathcal{C}} = \widetilde{\mathbf{c}}.\mathbf{J}_4$ such that $\mathrm{Cay}(\widetilde{\mathcal{C}}) = \mathcal{R} = \mathcal{R}(\widetilde{\mathbf{c}})$. Following the same arguments as in [5, Proposition 1] we obtain

$$\widetilde{\mathbf{c}} = \langle \boldsymbol{a}, \mathbf{c} \rangle_{\mathrm{SO}(4,\mathbb{R})} = \frac{\left(1 - \dfrac{\mathrm{c}^2}{4}\right)\boldsymbol{a} + \left(1 - \dfrac{a^2}{4}\right)\mathbf{c} + 4\dfrac{\boldsymbol{a}}{2} \times \dfrac{\mathbf{c}}{2}}{1 - 2\dfrac{\boldsymbol{a}}{2} \cdot \dfrac{\mathbf{c}}{2} + \dfrac{a^2}{4}\dfrac{\mathrm{c}^2}{4}} \tag{29}$$

provided that it is not true that either $\boldsymbol{a}.\mathbf{c} = 4$ and $\boldsymbol{a}||\mathbf{c}$ where $||$ denotes the collinearity relation. If the latter is fulfilled, it can be calculated directly that in this case $\mathcal{R}(\boldsymbol{a})\mathcal{R}(\mathbf{c}) = -\mathcal{I}_4$. Note that the matrix $-\mathcal{I}_4 \in \mathrm{SO}(4,\mathbb{R})$ is not in the image of the *Cayley* map applied to im j_4. Because the composition law is the same as for the group $\mathrm{SU}(2)$ we can conclude that as a *Lie* group

$$\mathrm{SU}(2) \cong \mathrm{im}\ \mathrm{Cay}_{\mathrm{im}\, j_4} \cup \{-\mathcal{I}_4\}.$$

The isomorphism is achieved via identification of the corresponding *Lie* algebra elements and identification of $-\mathcal{I}_2$ and $-\mathcal{I}_4$. The $\mathrm{SO}(3)$ half-turns $\mathcal{O}(\mathbf{n})$ about the arbitrary axes $\mathbf{n} \in \mathbb{R}^3, \mathbf{n}^2 = 1$ are associated with vector-parameters $\mathbf{c} = \pm 2\mathbf{n}$ of length 2 in $\mathrm{SU}(2)$ and represented as $\mathrm{SO}(4,\mathbb{R})$ elements by the matrices

$$\mathrm{Cay}(\pm 2\mathbf{n}) = \pm \begin{pmatrix} 0 & -n_3 & -n_2 & -n_1 \\ n_3 & 0 & n_1 & -n_2 \\ n_2 & -n_1 & 0 & n_3 \\ n_1 & n_2 & -n_3 & 0 \end{pmatrix}.$$

These are actually elements from both the Lie group $\mathrm{SO}(4,\mathbb{R})$ and its Lie algebra $\mathfrak{so}(4)$. Actually in this representation we have that the group element is a half-turn if and only it actually belongs to the *Lie* algebra, see Lemma 2, (ii). Let us note that the composition law $\langle \boldsymbol{a}, \mathbf{c} \rangle_{\mathrm{SO}(3)}$ works only "up to a sign" in *Fedorov* and *Wageningen*'s representations, i.e.,

$$\exp(\widetilde{\mathbf{c}}\,.\mathbf{J}_4) = \mathrm{sgn}(1 - \boldsymbol{a}.\mathbf{c}) \exp(\boldsymbol{a}.\mathbf{J}_4) \exp(\mathbf{c}.\mathbf{J}_4), \qquad \widetilde{\mathbf{c}} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a} \times \mathbf{c}}{1 - \boldsymbol{a}.\mathbf{c}}.$$

The reason for this is the following connection, see also [5, Proposition 2]

$$\exp(\mathbf{c}.\mathbf{J}_4) = \mathrm{Cay}(\boldsymbol{a}.\mathbf{J}_4), \qquad \boldsymbol{a} = \frac{2}{c^2}(\sqrt{1 + c^2} - 1)\mathbf{c}.$$

3.2.5. The Case $n = 8$

In the special case $n = 8$ the minimal polynomial of the matrix $\mathcal{C} \equiv \mathcal{C}_8 = \mathbf{c}.\mathbf{J}_8$ is

$$p_8(\lambda) = (\lambda^2 + \frac{1}{4}c^2)^2(\lambda^2 + \frac{9}{4}c^2)^2, \quad \mu_8(\lambda) = \sqrt{p_8(\lambda)} = \lambda^4 + \frac{5}{2}c^2\lambda^2 + \frac{9}{16}c^4$$

and the explicit formula for the *Cayley* map can be written as (see Theorem 5)

$$\mathrm{Cay}(\mathcal{C}) = \frac{16 + 40c^2 - 9c^4}{16 + 40c^2 + 9c^4}\mathcal{I} + \frac{16}{16 + 40c^2 + 9c^4}\left((2 + 5c^2)\mathcal{C} + 2\mathcal{C}^2 + 2\mathcal{C}^3\right).$$

3.3. The Case of Even $n = 4m + 2$ for Integer m

Let $n = 4m + 2, m \in \mathbb{N}$. The characteristic polynomial of an arbitrary matrix $\mathcal{C} \equiv \mathcal{C}_{4m+2} = \mathbf{c}.\mathbf{J}_{4m+2}$ is

$$\begin{aligned} p_{4m+2}(\lambda) &= \lambda^2(\lambda^2 + 1^2c^2)^2 \ldots (\lambda^2 + m^2c^2)^2 = \lambda^2 \prod_{t=1}^{m} (\lambda^2 + t^2c^2)^2 \\ &= \lambda^{4m+2} + \beta_{4r}c^2\lambda^{4r} + \ldots + \beta_2 c^{4m}\lambda^2 = \lambda^n + \sum_{t=1}^{2r} \beta_{n-2t}c^{2t}\lambda^{n-2t}. \end{aligned} \tag{30}$$

It should be noted that the polynomial in (30) does not coincide with the characteristic polynomials used in [9, 23]. The latter ones are complex.

One can derive formulas for the coefficients in (30) using *Vieta*'s formulas for the polynomial $h(\nu) = \nu^{2r} + \beta_{4r}\nu^{2\nu-1} + \ldots \beta_4\nu + \beta_2$ obtained from $\dfrac{p_{4r+2}(\lambda)}{\lambda^2}$ by substitution of $\dfrac{\lambda^2}{c^2}$ for ν. The distinct roots of h are $-1^2, -2^2, \ldots, -r^2$ and all of them are of double multiplicity. To obtain $\mathrm{Cay}(\mathcal{C}_{4m+2})$ in this case we can proceed as in Theorem 3. However, due to equations (9) and (30) and *Hamilton–Cayley*'s theorem it is clear that

$$p_{4m+2} = p_{2m+1}^2, \qquad p_{4m+2}(\mathcal{C}_n) = (p_{2m+1}(\mathcal{C}_n))^2 = M^2 = \mathcal{O}_{4m+2}$$

where $M = p_{2m+1}(\mathcal{C}_{4m+2})$ is $(4m + 2) \times (4m + 2)$ real matrix. Thus, either M is non-zero nilpotent matrix of order two or $M \equiv \mathcal{O}_n$. We will show that $M \equiv \mathcal{O}_{4m+2}$.

Lemma 6. *The minimal polynomial for the matrix* $\mathcal{C}_n = \mathcal{C}_{4m+2} = \mathbf{c}.\mathbf{J}_{4m+2}$ *where* $m \in \mathbb{N}$, $\mathbf{c} \in \mathbb{R}^3$ *is* p_{2m+1}, *i.e., the characteristic polynomial of* $\mathcal{C}_{2m+1} = \mathbf{c}.\mathbf{J}_{2m+1}$.

Proof: The polynomial $p_{2m+1}(\lambda) = \lambda \prod_{t=1}^{m} (\lambda^2 + t^2c^2)$ is the polynomial of least degree that is candidate to be the minimal polynomial of $\mathcal{C}_{4m+2}$ since it has all distinct eigenvalues of $\mathcal{C}_{4m+2}$ as simple roots and is monic. We will prove that $M = p_{2m+1}(\mathcal{C}_{4m+2}) = \mathcal{O}_{4m+2}$. Since $\mathcal{C}_{4m+2}$ is real skew-symmetric matrix it is similar to the diagonal matrix

$$\Lambda = \mathrm{diag}(0, 0, \mathrm{i}c, \mathrm{i}c, -\mathrm{i}c, -\mathrm{i}c, \ldots, , \mathrm{i}mc, \mathrm{i}mc, -\mathrm{i}mc, -\mathrm{i}mc)$$

i.e.,

$$\mathcal{C}_{4m+2} = \mathcal{U}^\dagger \Lambda \mathcal{U}, \qquad \mathcal{U}^\dagger \mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = \mathcal{I}_{4m+2} \tag{31}$$

where the † operator performs complex conjugation and transposition. Thus, for $\mathcal{C}^2$ we have $\mathcal{C}^2 = \mathcal{U}^\dagger \Lambda^2 \mathcal{U}$ and

$$\Lambda^2 = \mathrm{diag}(0, 0, -c^2, -c^2, -c^2, -c^2, \ldots, -m^2c^2, -m^2c^2, -m^2c^2, -m^2c^2).$$

Now for arbitrary $1 \leq k \leq m$ we have also

$$\mathcal{C}^2_{4m+2} + k^2c^2\mathcal{I} = \mathcal{U}^\dagger(\Lambda^2 + k^2c^2\mathcal{I})\mathcal{U} = \mathcal{U}^\dagger\Lambda_k\mathcal{U} \tag{32}$$

and that the elements at positions $4k-1, 4k, 4k+1, 4k+2$ on the main diagonal of Λ_k vanish. Now from (31) and (32) we obtain that

$$\begin{aligned} p_{2m+1}(\mathcal{C}_{4m+2}) &= \mathcal{C}_{4m+2}(\mathcal{C}^2_{4m+2} + c^2\mathcal{I})(\mathcal{C}^2_{4m+2} + 2^2c^2\mathcal{I})\ldots(\mathcal{C}^2_{4m+2} + m^2c^2\mathcal{I}) \\ &= \mathcal{U}^\dagger\Lambda\mathcal{U}\mathcal{U}^\dagger\Lambda_1\mathcal{U}\mathcal{U}^\dagger\ldots\mathcal{U}^\dagger\Lambda_m\mathcal{U} = \mathcal{U}^\dagger\Lambda\Lambda_1\ldots\Lambda_m\mathcal{U}. \end{aligned} \tag{33}$$

Consider the diagonal matrix $\widetilde{\Lambda} = \Lambda\Lambda_1\ldots\Lambda_m$. The first two elements on the main diagonal are zeros due to Λ. Also for all $1 \leq k \leq m$ the elements on the main diagonal with indexes $4k-1, 4k, 4k+1, 4k+2$ are also zero. Thus, $\widetilde{\Lambda} = \mathcal{O}_{4m+2}$ and $p_{2m+1}(\mathcal{C}_{4m+2}) = \mathcal{O}_{4m+2}$. ■

This lemma will allow us to express the $4m+2$ rotation matrix $\mathcal{R}(\mathbf{c}) = \mathrm{Cay}(\mathcal{C}_{4m+2})$ as a polynomial of degree $2m$ in the variable $\mathcal{C}_{4m+2}$ instead of $4m+1$.

Theorem 7. *For an arbitrary $n = 4m+2, m \in \mathbb{N}$ the Cayley map* (8) *is well-defined on* im j_{4m+2} *and the following explicit formula holds true*

$$\mathrm{Cay}(\mathcal{C}_{4m+2}) = \mathcal{I}_{4m+2} + 2\sum_{s=0}^{m-1} \frac{1 + \sum\limits_{k=1}^{m-s-1} \alpha_{2m+1-2k}c^{2k}}{1 + \alpha_{2m-1}c^2 + \ldots + \alpha_1 c^{2m}}(\mathcal{C}^{2s+1}_{4m+2} + \mathcal{C}^{2s+2}_{4m+2})$$

for all $\mathcal{C} \equiv \mathcal{C}_{4m+2} = \mathbf{c}.\mathbf{J}_{4m+2} \in$ im j_{4m+2} where the coefficients α_i are those of the characteristic polynomial of $\mathbf{c}.\mathbf{J}_{2m+1}$, *see* (9). *Also, the Cayley map takes values in* $\mathrm{SO}(4m+2)$.

Proof: From Lemma 6 we know that $p_{2m+1}(\mathcal{C}_{4m+2}) = 0$ where p_{2m+1} is the characteristic (and minimal) polynomial of $\mathcal{C}_{2m+1} = \mathbf{c}.\mathbf{J}_{2m+1}$. Thus the scheme and calculations in Theorem 3 are applicable here. The above formula and the fact that the *Cayley* map takes values in $\mathrm{SO}(4m+2)$ follow directly. ■

3.3.1. The Case $n = 6$

In this case the characteristic polynomial of the matrix $\mathcal{C} \equiv \mathcal{C}_6 = \mathbf{c}.\mathbf{J}_6$ is $p_6(\lambda) = \lambda^6 + 2c^2\lambda^4 + c^4\lambda^2 = (\lambda^3 + c^2\lambda)^2$. However, the minimal polynomial of $\mathcal{C}_6$ is p_3, the characteristic polynomial of $\mathcal{C}_3 = \mathbf{c}.\mathbf{J}_3$. Thus, by Theorem 7 we obtain

$$\operatorname{Cay}(\mathcal{C}) = \mathcal{I}_6 + \frac{2}{1+c^2}\mathcal{C} + \frac{2}{1+c^2}\mathcal{C}^2. \tag{34}$$

The matrix form of $(1+c^2)\mathcal{R}_6(\mathbf{c})$ is

$$\begin{pmatrix} 1-c_3^2 & -2c_3 & -\sqrt{2}(c_2+c_1c_3) & \sqrt{2}(-c_1+c_2c_3) & -c_1^2+c_2^2 & 2c_1c_2 \\ 2c_3 & 1-c_3^2 & \sqrt{2}(c_1-c_2c_3) & -\sqrt{2}(c_2+c_1c_3) & -2c_1c_2 & -c_1^2+c_2^2 \\ \sqrt{2}(c_2-c_1c_3) & -\sqrt{2}(c_1+c_2c_3) & 1-c_1^2-c_2^2+c_3^2 & 0 & \sqrt{2}(-c_2+c_1c_3) & -\sqrt{2}(c_1+c_2c_3) \\ \sqrt{2}(c_1+c_2c_3) & \sqrt{2}(c_2-c_1c_3) & 0 & 1-c_1^2-c_2^2+c_3^2 & \sqrt{2}(c_1+c_2c_3) & \sqrt{2}(-c_2+c_1c_3) \\ -c_1^2+c_2^2 & -2c_1c_2 & \sqrt{2}(c_2+c_1c_3) & \sqrt{2}(-c_1+c_2c_3) & 1-c_3^2 & 2c_3 \\ 2c_1c_2 & -c_1^2+c_2^2 & \sqrt{2}(c_1-c_2c_3) & \sqrt{2}(c_2+c_1c_3) & -2c_3 & 1-c_3^2 \end{pmatrix}. \tag{35}$$

If we know that a matrix $\mathcal{R} = \mathcal{R}(\mathbf{c}) \in \mathrm{SO}(6)$ belongs to the $\operatorname{im}\operatorname{Cay}[\operatorname{im} j_6]$ of the *Cayley* map we can obtain the vector-parameter $\mathbf{c}$ in the following way. Let us define $F \equiv F_6 = \mathcal{R}(\mathbf{c}) - \mathcal{R}^t(\mathbf{c}) = \frac{4}{1+c^2}\mathcal{C}_6$ and take into account that

$$\operatorname{tr} R(\mathbf{c}) = \frac{8}{1+c^2} - 2 > -2, \qquad \mathbf{c} \in \mathbb{R}^3. \tag{36}$$

One can retrieve $\mathbf{c}$ from $\mathcal{R}(\mathbf{c})$ via the formula

$$\mathbf{c} = \frac{3}{4+2\operatorname{tr}\mathcal{R}}(\sqrt{2}F(4,1), \sqrt{2}F(3,1), F(2,1)). \tag{37}$$

3.3.2. The Composition Law in the Case $n = 6$

Theorem 8. *Let $\mathcal{C}$ and $\mathcal{A}$ be two $\mathfrak{so}(6)$ matrices obtained via the j_6 map from the vector-parameters $\boldsymbol{a}, \mathbf{c} \in \mathbb{R}_3$. Let $\mathcal{R}(\boldsymbol{a})$ and $\mathcal{R}(\mathbf{c})$ are the $\mathrm{SO}(6,\mathbb{R})$ matrices obtained via the Cayley map from $\mathcal{A}$ and $\mathcal{C}$ and let $\mathcal{R}$ be their composition in $\mathrm{SO}(6,\mathbb{R})$. Then $\mathcal{R} = \mathcal{R}(\widetilde{\mathbf{c}})$ where*

$$\mathcal{R}(\widetilde{\mathbf{c}}) = \mathcal{R}(\boldsymbol{a})\mathcal{R}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \langle \boldsymbol{a}, \mathbf{c}\rangle_{\mathrm{SO}(6,\mathbb{R})} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a}\times\mathbf{c}}{1 - \boldsymbol{a}.\mathbf{c}} \tag{38}$$

provided that $\boldsymbol{a}.\mathbf{c} \neq 1$.

In other words, the composition law in this six dimensional realization of $\mathrm{SO}(3)$ is that one of the genuine $\mathrm{SO}(3)$ representation.

Proof: Let $\mathcal{R} = \mathcal{R}_{\boldsymbol{a}}.\mathcal{R}_{\mathbf{c}}$ and $F \equiv F_3 = \mathcal{R}_3 - \mathcal{R}_3^t$. Straightforward, but tedious calculation gives

$$\operatorname{tr}\mathcal{R}_3 \;=\; 8 + 8\frac{(1-\boldsymbol{a}.\mathbf{c})^2}{(1+c_2^2)(1+c_1^2)} \tag{39}$$

$$\begin{aligned}F(2,1) &= -4\frac{1-\boldsymbol{a}.\mathbf{c}}{(1+c_2^2)\big(1+c_1^2\big)}\big(c_3+a_3-c_1a_2+c_2a_1\big)\\ F(3,1) &= -2\sqrt{2}\frac{1-\boldsymbol{a}.\mathbf{c}}{(1+c_2^2)(1+c_1^2)}\big(c_2+a_2-c_3a_1+c_2a_3\big)\\ F(4,1) &= -2\sqrt{2}\frac{1-\boldsymbol{a}.\mathbf{c}}{(1+c_2^2)(1+c_1^2)}\big(c_1+a_1-c_2a_3+c_3a_2\big).\end{aligned} \tag{40}$$

If we suppose that there exists $\widetilde{\mathbf{c}} \in \mathbb{R}_3$ such that $\mathcal{R}_3 = \mathcal{R}_3(\widetilde{\mathbf{c}})$ and using equations (36) and (37) we obtain $\widetilde{\mathbf{c}} = \langle \boldsymbol{a}, \mathbf{c}\rangle_{\mathrm{SO}(3)}$. Now direct substitution proves (38). The details of these calculations are omitted here. ∎

What about half-turns in $\mathrm{SO}(6,\mathbb{R})$? We already understood that both the formula for the *Cayley* map and the composition law for $n = 6$ coincide with that one for $n = 3$. In SO(3) half-turns can be obtained by taking limits $\lim_{t\to\infty} t(n_1, n_2, n_3)$ where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector or by SU(2) vector-parameter $2\mathbf{n}$, see [6]). It is well known [14] that if G is a matrix Lie group and $\{g_l\}_{l=1}^{\infty}$ is a sequence in G that converges to g, then either $g \in G$ or g is not invertible.

Let $\mathbf{c}(t) = t\mathbf{n}$ where $\mathbf{n} = (n_1, n_2, n_3)$, $\mathbf{n}^2 = 1$. From the structure of the formula (34) it can immediately be concluded that

$$\lim_{t\to\infty} \mathcal{R}_6(t\mathbf{n}) = \mathcal{O}_6(\mathbf{n}) = \tag{41}$$

$$\begin{pmatrix} -n_3^2 & 0 & -\sqrt{2}n_1n_3 & \sqrt{2}n_2n_3 & -1+2n_2^2+n_3^2 & 2n_1n_2\\ 0 & -n_3^2 & -\sqrt{2}n_2n_3 & -\sqrt{2}n_1n_3 & -2n_1n_2 & -1+2n_2^2+n_3^2\\ -\sqrt{2}n_1n_3 & -\sqrt{2}n_2n_3 & -1+2n_3^2 & 0 & \sqrt{2}n_1n_3 & -\sqrt{2}n_2n_3\\ \sqrt{2}n_2n_3 & -\sqrt{2}n_1n_3 & 0 & -1+2n_3^2 & \sqrt{2}n_2n_3 & \sqrt{2}n_1n_3\\ -1+2n_2^2+n_3^2 & -2n_1n_2 & \sqrt{2}n_1n_3 & \sqrt{2}n_2n_3 & -n_3^2 & 0\\ 2n_1n_2 & -1+2n_2^2+n_3^2 & -\sqrt{2}n_2n_3 & \sqrt{2}n_1n_3 & 0 & -n_3^2 \end{pmatrix}$$

which is a true $\mathrm{SO}(6,\mathbb{R})$ matrix.

4. Concluding Remarks

Despite the nice formulas obtained for the $\mathrm{SO}(n)$ matrices in dimensions other than 3, 4 and 6 the subset $\operatorname{im} j_n$ is not a subgroup in general. We leave for a future research to find a way to fix this and to derive the correspondence between the *Cayley* and the $\exp$ realizations.

Appendix A. The Cayley and Exponential Parametrizations of SO(3)

The general element of the *Lie* algebra $\mathfrak{so}(3)$ is of the form

$$\mathcal{C} = \mathbf{c}.\mathbf{J} = \begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}.$$

The characteristic and the minimal polynomial of $\mathcal{C}$ coincide, i.e.,

$$p(\lambda) = \mu(\lambda) = -\lambda^3 - c^2\lambda = -\lambda(c^2 + \lambda^2).$$

The explicit form of the *Cayley* map is

$$\mathrm{Cay}(\mathbf{c}) = \mathcal{I} + \frac{2}{1+c^2}\mathcal{C} + \frac{2}{1+c^2}\mathcal{C}^2.$$

Composition law in the *Cayley* realization

$$\mathrm{Cay}(\widetilde{\mathbf{c}}) = \mathrm{Cay}(\boldsymbol{a})\mathrm{Cay}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \langle \boldsymbol{a}, \mathbf{c}\rangle_{\mathrm{SO}(3)} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a} \times \mathbf{c}}{1 - \boldsymbol{a}.\mathbf{c}}\cdot$$

The exponential map is

$$\exp(\mathcal{C}) = \mathcal{I} + \frac{\sin c}{c}\mathcal{C} + \frac{1-\cos c}{c^2}\mathcal{C}^2.$$

Composition law for the $\exp$ realization is given by the formulas [8]

$$BCH(\mathcal{A}, \mathcal{C}) = BCH(\boldsymbol{a}.\mathbf{J}, \mathbf{c}.\mathbf{J}) = \alpha\mathcal{A} + \beta\mathcal{C} + \gamma[\mathcal{A}, \mathcal{C}]$$

$$\alpha = \frac{\sin^{-1}(q)}{q}\frac{m}{c}, \qquad \beta = \frac{\sin^{-1}(q)}{q}\frac{n}{a}, \qquad \gamma = \frac{\sin^{-1}(q)}{q}\frac{p}{ac}$$

where $\phi = \angle(\boldsymbol{a}, \mathbf{c}) = \cos^{-1}\left(\frac{\boldsymbol{a}.\mathbf{c}}{ac}\right)$ and

$$\begin{aligned} m &= \sin(c)\cos^2(a/2) - \sin(a)\sin^2(c/2)\cos(\phi) \\ n &= \sin(a)\cos^2(c/2) - \sin(c)\sin^2(a/2)\cos(\phi) \\ p &= \frac{1}{2}\sin(a)\sin(c) - 2\sin^2(a/2)\sin^2(c/2)\cos(\phi) \\ q &= \sqrt{m^2 + n^2 + 2mn\cos(\phi) + p^2\sin^2(\phi)}. \end{aligned}$$

Connection between the $\exp$ and Cay realizations is

$$\exp\theta\mathbf{n}^\times = \mathbf{Cay}(\varphi\mathbf{n}^\times), \qquad \varphi = \tan\frac{\theta}{2}\cdot$$

The extended composition law in the *Cayley* realization is discussed at some length in [6] and summarized in Table 1.

Table 1. An extended vector-parameter composition law for the group SO(3) covering all possible scenarios for compositions. A proper rotation associated with the vector-parameter $\mathbf{c}$ is denoted by $\mathcal{R}(\mathbf{c})$. The improper rotation (half-turn) about an axis $\mathbf{n}$ is denoted by $\mathcal{O}(\mathbf{n})$ and is represented by the ray $[\mathbf{n}]$ of all non-zero vectors that are co-linear with $\mathbf{n}$.

Product of rotations	Result	Condition	Compound rotation
$\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_1)$	$\mathbf{c}_3 = \dfrac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1}{1 - \mathbf{c}_2.\mathbf{c}_1}$,	$\mathbf{c}_2.\mathbf{c}_1 \neq 1$	$\mathcal{R}(\mathbf{c}_3)$
	$[\mathbf{n}_3] = [\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \times \mathbf{c}_1]$,	$\mathbf{c}_2.\mathbf{c}_1 = 1$	$\mathcal{O}(\mathbf{n}_3)$
$\mathcal{R}(\mathbf{c}_2)\mathcal{O}(\mathbf{n}_1)$	$\mathbf{c}_3 = -\dfrac{\mathbf{n}_1 + \mathbf{c}_2 \times \mathbf{n}_1}{\mathbf{c}_2.\mathbf{n}_1}$,	$\mathbf{c}_2.\mathbf{n}_1 \neq 0$	$\mathcal{R}(\mathbf{c}_3)$
	$[\mathbf{n}_3] = [\mathbf{n}_1 + \mathbf{c}_2 \times \mathbf{n}_1]$,	$\mathbf{c}_2.\mathbf{n}_1 = 0$	$\mathcal{O}(\mathbf{n}_3)$
$\mathcal{O}(\mathbf{n}_2)\mathcal{R}(\mathbf{c}_1)$	$\mathbf{c}_3 = -\dfrac{\mathbf{n}_2 + \mathbf{n}_2 \times \mathbf{c}_1}{\mathbf{n}_2.\mathbf{c}_1}$,	$\mathbf{n}_2.\mathbf{c}_1 \neq 0$	$\mathcal{R}(\mathbf{c}_3)$
	$[\mathbf{n}_3] = [\mathbf{n}_2 + \mathbf{n}_2 \times \mathbf{c}_1]$,	$\mathbf{n}_2.\mathbf{c}_1 = 0$	$\mathcal{O}(\mathbf{n}_3)$
$\mathcal{O}(\mathbf{n}_2)\mathcal{O}(\mathbf{n}_1)$	$\mathbf{c}_3 = -\dfrac{\mathbf{n}_2 \times \mathbf{n}_1}{\mathbf{n}_2.\mathbf{n}_1}$,	$\mathbf{n}_2.\mathbf{n}_1 \neq 0$	$\mathcal{R}(\mathbf{c}_3)$
	$[\mathbf{n}_3] = [\mathbf{n}_2 \times \mathbf{n}_1]$,	$\mathbf{n}_2.\mathbf{n}_1 = 0$	$\mathcal{O}(\mathbf{n}_3)$

Appendix B. Realizations of $\mathfrak{so}(3)$ and $\mathrm{SO}(3)$ in Higher Dimensions

Here we present various explicit formulas derived via the *Cayley* map for realizations of SO(3) in SO(n) in this paper and that ones obtained using the exponential map in [9] and [23]. Let us remind that the matrices $J_{n|i}, i = 1, 2, 3$ can be obtained by the formulas

$$J_{n|i} = \frac{\partial(\mathbf{c}.\mathbf{J}_n)}{\partial c_i}\Big|_{\mathbf{c}=\mathbf{0}}, \qquad i = 1, 2, 3$$

and satisfy the commutation relations

$$[J_{n|i}, J_{n|j}] = \epsilon_{i,j,k} J_{n|k}$$

where $\epsilon_{i,j,k} \equiv 1$ for even permutation of $1, 2, 3$ and $\epsilon_{i,j,k} \equiv -1$ otherwise.

We will work with the following notation

$$\mathbf{c} \mapsto \mathcal{C} = c_1 J_{n|1} + c_2 J_{n|2} + c_3 J_{n|3} = \mathbf{c}.\mathbf{J}_n$$
$$\hat{\mathbf{c}} = \frac{\mathbf{c}}{c}, \qquad \hat{\mathcal{C}} = \hat{\mathbf{c}}.\mathbf{J}_n, \qquad \alpha = 2\arctan c$$

where

$$\mathbf{c} = (c_1, c_2, c_3), \qquad \mathbf{c}^2 := c_1^2 + c_2^2 + c_3^2 = \mathbf{c}.\mathbf{c} = |\mathbf{c}|^2 = c^2. \tag{42}$$

We will denote the *Fedorov* and *Wageningen*'s $\mathrm{SO}(n)$ elements corresponding to the vector-parameter $\mathbf{c}$ respectively by

$$\mathrm{Fed}(\mathbf{c}) \equiv \mathrm{Fed}(\mathcal{C}), \qquad \mathrm{Wag}(\mathbf{c}) \equiv \mathrm{Wag}(\alpha\hat{\mathcal{C}}) \equiv \exp(\alpha\hat{\mathcal{C}})$$

because $\mathrm{Fed}(\mathcal{C})$ are expressed in terms of the powers of $\mathcal{C}$ whereas $\mathrm{Wag}(\mathcal{C})$ are expressed in terms of the powers of $\hat{\mathcal{C}}$ and trigonometric functions of α.

B.1. $n = 4$

The general element of the embedded *Lie* algebra $\mathfrak{so}(3)$ is

$$\mathbf{c}.\mathbf{J}_4 = \frac{1}{2}\begin{pmatrix} 0 & -c_3 & -c_2 & -c_1 \\ c_3 & 0 & c_1 & -c_2 \\ c_2 & -c_1 & 0 & c_3 \\ c_1 & c_2 & -c_3 & 0 \end{pmatrix}.$$

The characteristic and the minimal polynomial of $\mathcal{C}$ are

$$p(\lambda) = (\lambda^2 + \frac{c^2}{4})^2 = \lambda^4 + \frac{c^2}{2}\lambda^2 + \frac{c^4}{16}, \qquad \mu(\lambda) = \lambda^2 + \frac{c^2}{4}.$$

The formula for the *Cayley* map is

$$\mathrm{Cay}(\mathcal{C}) = \mathcal{R}_4(\mathbf{c}) = \frac{4 - c^2}{4 + c^2}\mathcal{I} + \frac{8}{4 + c^2}\mathcal{C}.$$

The composition law in the *Cayley* realization

$$\begin{aligned}
\mathrm{Cay}(\widetilde{\mathbf{c}}) &= \mathrm{Cay}(\boldsymbol{a})\mathrm{Cay}(\mathbf{c})\\
\widetilde{\mathbf{c}} = \langle \boldsymbol{a}, \mathbf{c}\rangle_{\mathrm{SO}(4)} &= \frac{\left(1-\frac{\mathrm{c}^2}{4}\right)\boldsymbol{a} + \left(1-\frac{a^2}{4}\right)\mathbf{c} + 4\frac{\boldsymbol{a}}{2}\times\frac{\mathbf{c}}{2}}{1-2\frac{\boldsymbol{a}}{2}\cdot\frac{\mathbf{c}}{2}+\frac{a^2}{4}\frac{\mathrm{c}^2}{4}}.
\end{aligned}$$

Formulas for the exponential map

$$\begin{aligned}
\mathrm{Fed}(\mathcal{C}) &= \frac{1}{(1+c^2)^{3/2}}\left((1+\frac{3}{2}c^2)\mathcal{I} + (2+\frac{7}{3}c^2)\mathcal{C} + 2\mathcal{C}^2 + \frac{4}{3}\mathcal{C}^3\right)\\
\mathrm{Wag}(\alpha\hat{\mathcal{C}}) &= (\frac{9}{8}\cos\frac{1}{2}\alpha - \frac{1}{8}\cos\frac{3}{2}\alpha)\mathcal{I} + (\frac{9}{4}\sin\frac{1}{2}\alpha - \frac{1}{12}\sin\frac{3}{2}\alpha)\hat{\mathcal{C}}\\
&\quad + (\frac{1}{2}\cos\frac{1}{2}\alpha - \frac{1}{2}\cos\frac{3}{2}\alpha)\hat{\mathcal{C}}^2 + \big(\sin\frac{1}{2}\alpha - \frac{1}{3}\sin\frac{3}{2}\alpha\big)\hat{\mathcal{C}}^3\\
\mathrm{Fed}(\mathbf{c}) &= \mathrm{Wag}(\mathbf{c}).
\end{aligned}$$

Composition law in the $\exp$ realization

$$\begin{aligned}
\mathrm{Fed}(\widetilde{\mathbf{c}}) &= \mathrm{sgn}(1-\boldsymbol{a}.\mathbf{c})\mathrm{Fed}(\boldsymbol{a})\mathrm{Fed}(\mathbf{c}), &\qquad \widetilde{\mathbf{c}} &= \frac{\boldsymbol{a}+\mathbf{c}+\boldsymbol{a}\times\mathbf{c}}{1-\boldsymbol{a}.\mathbf{c}}\\
\mathrm{Wag}(\widetilde{\mathbf{c}}) &= \mathrm{sgn}(1-\boldsymbol{a}.\mathbf{c})\mathrm{Wag}(\boldsymbol{a})\mathrm{Wag}(\mathbf{c}), &\qquad \widetilde{\mathbf{c}} &= \frac{\boldsymbol{a}+\mathbf{c}+\boldsymbol{a}\times\mathbf{c}}{1-\boldsymbol{a}.\mathbf{c}}.
\end{aligned}$$

Connection between the $\exp$ and Cay realizations

$$\mathrm{Fed}(\mathbf{c}) = \mathrm{Wag}(\mathbf{c}) = \mathrm{Cay}(\mathbf{a}), \qquad \mathbf{a} = \frac{2}{c^2}(\sqrt{1+c^2}-1)\mathbf{c}.$$

The angular momentum matrices

$$\begin{aligned}
\boldsymbol{\omega}^{\times} = \boldsymbol{\omega}.\mathbf{J}_4 &= \frac{8}{4+c^2}\left(2(\mathbf{c}.\dot{\mathbf{c}})\mathbf{c} + (4-c^2)\dot{\mathbf{c}} - 4(\mathbf{c}\times\dot{\mathbf{c}})\right).\mathbf{J}_4\\
\boldsymbol{\Omega}^{\times} = \boldsymbol{\Omega}.\mathbf{J}_4 &= \frac{8}{4+c^2}\left(2(\mathbf{c}.\dot{\mathbf{c}})\mathbf{c} + (4-c^2)\dot{\mathbf{c}} + 4(\mathbf{c}\times\dot{\mathbf{c}})\right).\mathbf{J}_4.
\end{aligned}$$

The alternative formulas for $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ can be expressed readily as

$$\boldsymbol{\omega} = -8\frac{\partial}{\partial t}\frac{\mathbf{c}}{4+c^2} + 32\frac{2\dot{\mathbf{c}} - 4\mathbf{c}\times\dot{\mathbf{c}}}{(4+c^2)^2}, \qquad \boldsymbol{\Omega} = -8\frac{\partial}{\partial t}\frac{\mathbf{c}}{4+c^2} + 32\frac{2\dot{\mathbf{c}} + 4\mathbf{c}\times\dot{\mathbf{c}}}{(4+c^2)^2}.$$

B.2. $n = 5$

General element of the embedded *Lie* algebra $\mathfrak{so}(3)$

$$\mathcal{C} = \mathcal{C}_5 = \mathbf{c}.\mathbf{J}_5 = \begin{pmatrix} 0 & -2c_3 & c_2 & c_1 & 0 \\ 2c_3 & 0 & -c_1 & c_2 & 0 \\ -c_2 & c_1 & 0 & -c_3 & -\sqrt{3}c_2 \\ -c_1 & -c_2 & c_3 & 0 & \sqrt{3}c_1 \\ 0 & 0 & \sqrt{3}c_2 & -\sqrt{3}c_1 & 0 \end{pmatrix}.$$

The characteristic and the minimal polynomial of $\mathcal{C}$ coincide, i.e.,

$$p(\lambda) = \mu(\lambda) = -\lambda(c^2 + \lambda^2)(4c^2 + \lambda^2) = -\lambda^5 - 5c^2\lambda^3 - 4c^4\lambda.$$

Formula for the *Cayley* map

$$\mathrm{Cay}(\mathbf{c}) = \mathcal{I} + 2\frac{5c^2 + 1}{4c^4 + 5c^2 + 1}(\mathcal{C} + \mathcal{C}^2) + \frac{2}{4c^4 + 5c^2 + 1}(\mathcal{C}^3 + \mathcal{C}^4).$$

Formulas for the exponential map

$$\mathrm{Fed}(\mathcal{C}) = \mathcal{I} + \frac{2}{3(1 + c^2)^2}\big((3 + 5c^2)\mathcal{C} + (3 + 4c^2)\mathcal{C}^2 + 2\mathcal{C}^3 + \mathcal{C}^4\big) \tag{43}$$

$$\begin{aligned} \mathrm{Wag}(\alpha\hat{\mathcal{C}}) = \mathcal{I} &- \big(\frac{4}{3}\sin\alpha - \frac{1}{6}\sin 2\alpha\big)\hat{\mathcal{C}} - \big(\frac{4}{3}(\cos\alpha - 1) - \frac{1}{12}(\cos 2\alpha - 1)\big)\hat{\mathcal{C}}^2 \\ &- \big(\frac{1}{3}\sin\alpha - \frac{1}{6}\sin 2\alpha\big)\hat{\mathcal{C}}^3 + \big(\frac{1}{12}(\cos 2\alpha - 1) - \frac{1}{3}(\cos\alpha - 1)\big)\hat{\mathcal{C}}^4 \end{aligned}$$

$$\mathrm{Fed}(\mathbf{c}) = \mathrm{Wag}(\mathbf{c}).$$

Composition law in the $\exp$ realization

$$\mathrm{Fed}(\widetilde{\mathbf{c}}) = \mathrm{Fed}(\boldsymbol{a})\mathrm{Fed}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a} \times \mathbf{c}}{1 - \boldsymbol{a}.\mathbf{c}}$$

$$\mathrm{Wag}(\widetilde{\mathbf{c}}) = \mathrm{Wag}(\boldsymbol{a})\mathrm{Wag}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a} \times \mathbf{c}}{1 - \boldsymbol{a}.\mathbf{c}}.$$

B.3. $n = 6$

General element of the embedded *Lie* algebra $\mathfrak{so}(3)$

$$\mathbf{c}.\mathbf{J}_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\sqrt{2}c_3 & -c_2 & -c_1 & 0 & 0 \\ \sqrt{2}c_3 & 0 & c_1 & -c_2 & 0 & 0 \\ c_2 & -c_1 & 0 & 0 & -c_2 & -c_1 \\ c_1 & c_2 & 0 & 0 & c_1 & -c_2 \\ 0 & 0 & c_2 & -c_1 & 0 & \sqrt{2}c_3 \\ 0 & 0 & c_1 & c_2 & -\sqrt{2}c_3 & 0 \end{pmatrix}.$$

Characteristic and minimal polynomial of $\mathcal{C}$

$$p(\lambda) = \lambda^6 + 2c^2\lambda^4 + c^4\lambda^2 = (\lambda^3 + c^2\lambda)^2 = \lambda^2(\lambda^2 + c^2)^2$$
$$\mu(\lambda) = \lambda^3 + c^2\lambda = \lambda(\lambda^2 + c^2).$$

Formula for the *Cayley* map

$$\mathrm{Cay}(\mathcal{C}) = \mathcal{I}_6 + \frac{2}{1+c^2}\mathcal{C} + \frac{2}{1+c^2}\mathcal{C}^2.$$

Composition law in the *Cayley* realization coincides with that one for the standard representation

$$\mathrm{Cay}(\widetilde{\mathbf{c}}) = \mathrm{Cay}(\boldsymbol{a})\mathrm{Cay}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \langle \boldsymbol{a}, \mathbf{c} \rangle_{\mathrm{SO}(3,\mathbb{R})} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a} \times \mathbf{c}}{1 - \boldsymbol{a}.\mathbf{c}}.$$

Formulas for the exponential map are respectively

$$\begin{aligned} \mathrm{Fed}(\mathcal{C}) = \frac{1}{(1+c^2)^{5/2}} \Big(& (1 + \frac{5}{2}c^2 + \frac{15}{8}c^4)\mathcal{I} + (2 + \frac{13}{3}c^2 + \frac{149}{60}c^4)\mathcal{C} \\ & + (2 + \frac{11}{3}c^2)\mathcal{C}^2 + (\frac{4}{3} + 2c^2)\mathcal{C}^3 + \frac{2}{3}\mathcal{C}^4 + \frac{4}{15}\mathcal{C}^5 \Big) \end{aligned}$$

$$\begin{aligned} \mathrm{Wag}(\alpha\hat{\mathcal{C}}) = & \left(\frac{75}{64}\cos\frac{1}{2}\alpha - \frac{25}{128}\cos\frac{3}{2}\alpha + \frac{3}{128}\cos\frac{5}{2}\alpha \right)\mathcal{I} \\ & - \left(-\frac{75}{32}\sin\frac{1}{2}\alpha + \frac{25}{192}\sin\frac{3}{2}\alpha - \frac{3}{320}\sin\frac{5}{2}\alpha \right)\hat{\mathcal{C}} \\ & - \left(-\frac{17}{24}\cos\frac{1}{2}\alpha + \frac{13}{16}\cos\frac{3}{2}\alpha - \frac{5}{48}\cos\frac{5}{2}\alpha \right)\hat{\mathcal{C}}^2 \end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{17}{12}\sin\frac{1}{2}\alpha-\frac{13}{24}\sin\frac{3}{2}\alpha+\frac{1}{24}\sin\frac{5}{2}\alpha\right)\hat{\mathcal{C}}^3\\
&+\left(\frac{1}{12}\cos\frac{1}{2}\alpha-\frac{1}{8}\cos\frac{3}{2}\alpha+\frac{1}{24}\cos\frac{5}{2}\alpha\right)\hat{\mathcal{C}}^4\\
&-\left(-\frac{1}{6}\sin\frac{1}{2}\alpha+\frac{1}{12}\sin\frac{3}{2}\alpha-\frac{1}{60}\sin\frac{5}{2}\alpha\right)\hat{\mathcal{C}}^5.
\end{aligned}
$$

Remark 9. *We should mention that with the constructed inclusion* $\mathfrak{so}(3)\hookrightarrow\mathfrak{so}(6)$ (*i.e., using the Lie algebra elements from* (7)) *for which the formulas* $\mathrm{Fed}(\mathcal{C})$ *and* $\mathrm{Wag}(\mathcal{C})$ *coincide but do not generate* $\mathrm{SO}(6)$ *matrices. This is due to the different characteristic polynomials of the used Lie algebra elements* (*theirs are not real*).

Remark 10. *The Cowin-Mehrabadi theorem* [1, 4, 15] *concerning normals to the planes of symmetry of anisotropic material was generalized to the six dimensional case and leads to interesting realization of* $\mathrm{SO}(3)$ *in* $\mathrm{SO}(6)$. *The following* $\mathfrak{so}(6)$ *matrix has been derived in* [18]

$$
\mathcal{H}\equiv\mathcal{H}_6(\mathbf{c})=\frac{1}{\sqrt{2}}\begin{pmatrix}
0 & 0 & 0 & 0 & c_2 & -c_3\\
0 & 0 & 0 & -c_1 & 0 & c_3\\
0 & 0 & 0 & c_1 & -c_2 & 0\\
0 & c_1 & -c_1 & 0 & \sqrt{2}c_3 & -\sqrt{2}c_2\\
-c_2 & 0 & c_2 & -\sqrt{2}c_3 & 0 & \sqrt{2}c_1\\
c_3 & -c_3 & 0 & \sqrt{2}c_2 & -\sqrt{2}c_1 & 0
\end{pmatrix}.
$$

The characteristic and minimal polynomials of $\mathcal{H}$ are

$$
\begin{aligned}
p(\lambda)&=\lambda^2(c^2+\lambda^2)(4c^2+\lambda^2)=\lambda^6+5c^2\lambda^4+4c^4\lambda^2\\
\mu(\lambda)&=-\lambda(c^2+\lambda^2)(4c^2+\lambda^2)=-\lambda^5-5c^2\lambda^3-4c^4\lambda.
\end{aligned}
$$

The exponential map gives the $\mathrm{SO}(6)$ matrix

$$
\exp(\alpha\widehat{\mathcal{H}})=\mathcal{I}+\frac{2}{3(1+c^2)^2}\left((3+5c^2)\mathcal{H}+(3+4c^2)\mathcal{H}^2+2\mathcal{H}^3+\mathcal{H}^4\right).
$$

It is no surprise that the coefficients in the formulas $\exp(\alpha\widehat{\mathcal{H}})$ and $\mathrm{Fed}(\mathbf{c})\subset\mathrm{SO}(5)$ (see formula (43)) coincide. Besides the fact that they produce rotational matrices of different dimension, the minimal polynomials of *Lie* algebra elements $\mathcal{H}_6(\mathbf{c})$ and $\mathbf{c}.\mathbf{J}_5$ coincide. In this realization the composition law is the same as in (18).

B.4. $n = 7$

General element of the embedded *Lie* algebra $\mathfrak{so}(3)$

$$\mathbf{c}.\mathbf{J}_7 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -3\sqrt{2}c_3 & -\sqrt{3}c_2 & -\sqrt{3}c_1 & 0 & 0 & 0 \\ 3\sqrt{2}c_3 & 0 & \sqrt{3}c_1 & -\sqrt{3}c_2 & 0 & 0 & 0 \\ \sqrt{3}c_2 & -\sqrt{3}c_1 & 0 & -2\sqrt{2}c_3 & -\sqrt{5}c_2 & -\sqrt{5}c_1 & 0 \\ \sqrt{3}c_1 & \sqrt{3}c_2 & 2\sqrt{2}c_3 & 0 & \sqrt{5}c_1 & -\sqrt{5}c_2 & 0 \\ 0 & 0 & \sqrt{5}c_2 & -\sqrt{5}c_1 & 0 & -\sqrt{2}c_3 & 2\sqrt{3}c_2 \\ 0 & 0 & \sqrt{5}c_1 & \sqrt{5}c_2 & \sqrt{2}c_3 & 0 & -2\sqrt{3}c_1 \\ 0 & 0 & 0 & 0 & -2\sqrt{3}c_2 & 2\sqrt{3}c_1 & 0 \end{pmatrix}.$$

The characteristic and the minimal polynomial of $\mathcal{C}$ coincide

$$p(\lambda) = \mu(\lambda) = -\lambda(c^2+\lambda^2)(4c^2+\lambda^2)(9c^2+\lambda^2)$$
$$= -36c^6\lambda - 49c^4\lambda^3 - 14c^2\lambda^5 - \lambda^7.$$

Formula for the *Cayley* map

$$\mathrm{Cay}(\mathcal{C}) = \mathcal{I} + \frac{2}{(1+c^2)^3}\Big((1+\frac{8}{3}c^2+\frac{11}{5}c^4)\mathcal{C} + (1+\frac{7}{3}c^2+\frac{68}{45}c^4)\mathcal{C}^2 + \frac{2}{3}(1+2c^2)\mathcal{C}^3 + (\frac{1}{3}+\frac{5}{9}c^2)\mathcal{C}^4 + \frac{2}{15}\mathcal{C}^5 + \frac{2}{45}\mathcal{C}^6\Big).$$

Formulas for the exponential map are

$$\mathrm{Fed}(\mathcal{C}) = \mathcal{I} + \frac{2}{(1+c^2)^3}\Big((1+\frac{8}{3}c^2+\frac{11}{5}c^4)\mathcal{C} + (1+\frac{7}{3}c^2+\frac{68}{45}c^4)\mathcal{C}^2 + \frac{2}{3}(1+2c^2)\mathcal{C}^3 + (\frac{1}{3}+\frac{5}{9}c^2)\mathcal{C}^4 + \frac{2}{15}\mathcal{C}^5 + \frac{2}{45}\mathcal{C}^6\Big)$$

$$\begin{aligned}\mathrm{Wag}(\alpha\hat{\mathcal{C}}) \;&=\; \mathcal{I} - \left(-\frac{3}{2}\sin\alpha + \frac{3}{10}\sin 2\alpha - \frac{1}{30}\sin 3\alpha\right)\hat{\mathcal{C}} \\ &- \left(\frac{3}{2}(\cos\alpha-1) - \frac{3}{20}(\cos 2\alpha-1) + \frac{1}{90}(\cos 3\alpha-1)\right)\hat{\mathcal{C}}^2 \\ &+ \left(\frac{13}{24}\sin\alpha - \frac{1}{3}\sin 2\alpha + \frac{1}{24}\sin 3\alpha\right)\hat{\mathcal{C}}^3 \\ &+ \left(-\frac{13}{24}(\cos\alpha-1) + \frac{1}{6}(\cos 2\alpha-1) - \frac{1}{72}(\cos 3\alpha-1)\right)\hat{\mathcal{C}}^4\end{aligned}$$

$$
- \left(-\frac{1}{24} \sin \alpha + \frac{1}{30} \sin 2\alpha - \frac{1}{120} \sin 3\alpha \right) \hat{\mathcal{C}}^5
$$
$$
- \left(\frac{1}{24} (\cos \alpha - 1) - \frac{1}{60} (\cos 2\alpha - 1) + \frac{1}{360} (\cos 3\alpha - 1) \right) \hat{\mathcal{C}}^6.
$$

Besides

$$
\mathrm{Fed}(\mathbf{c}) = \mathrm{Wag}(\mathbf{c}).
$$

Composition law in the $\exp$ realization

$$
\mathrm{Fed}(\widetilde{\mathbf{c}}) = \mathrm{Fed}(\boldsymbol{a})\mathrm{Fed}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a} \times \mathbf{c}}{1 - \boldsymbol{a}\mathbf{c}}
$$
$$
\mathrm{Wag}(\widetilde{\mathbf{c}}) = \mathrm{Wag}(\boldsymbol{a})\mathrm{Wag}(\mathbf{c}), \qquad \widetilde{\mathbf{c}} = \frac{\boldsymbol{a} + \mathbf{c} + \boldsymbol{a} \times \mathbf{c}}{1 - \boldsymbol{a}.\mathbf{c}}.
$$

B.5. $n = 8$

The general element of the embedded *Lie* algebra $\mathfrak{so}(3)$ is

$$
\mathbf{c}.\mathbf{J}_8 = \frac{1}{2} \begin{pmatrix}
0 & -3c_3 & -\sqrt{3}c_2 & -\sqrt{3}c_1 & 0 & 0 & 0 & 0 \\
3c_3 & 0 & \sqrt{3}c_1 & -\sqrt{3}c_2 & 0 & 0 & 0 & 0 \\
\sqrt{3}c_2 & -\sqrt{3}c_1 & 0 & -c_3 & -2c_2 & -2c_1 & 0 & 0 \\
\sqrt{3}c_1 & \sqrt{3}c_2 & c_3 & 0 & 2c_1 & -2c_2 & 0 & 0 \\
0 & 0 & 2c_2 & -2c_1 & 0 & c_3 & -\sqrt{3}c_2 & -\sqrt{3}c_1 \\
0 & 0 & 2c_1 & 2c_2 & -c_3 & 0 & \sqrt{3}c_1 & -\sqrt{3}c_2 \\
0 & 0 & 0 & 0 & \sqrt{3}c_2 & -\sqrt{3}c_1 & 0 & 3c_3 \\
0 & 0 & 0 & 0 & \sqrt{3}c_1 & \sqrt{3}c_2 & -3c_3 & 0
\end{pmatrix}.
$$

The characteristic and the minimal polynomial of $\mathcal{C}$ are respectively

$$
p(\lambda) = (\lambda^2 + c^2)^2 (\lambda^2 + \frac{1}{4} c^2)^2
$$
$$
\mu(\lambda) = (\lambda^2 + c^2)(\lambda^2 + \frac{1}{4} c^2) = \lambda^4 + \frac{5}{2} c^2 \lambda^2 + \frac{9}{16} c^4.
$$

Finally the formula for the *Cayley* map is

$$
\mathrm{Cay}(\mathcal{C}) = \frac{16 + 40c^2 - 9c^4}{16 + 40c^2 + 9c^4} \mathcal{I} + \frac{16}{16 + 40c^2 + 9c^4} \left((2 + 5c^2)\mathcal{C} + 2\mathcal{C}^2 + 2\mathcal{C}^3 \right).
$$

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