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WEYL MANIFOLD: A QUANTIZED SYMPLECTIC MANIFOLD

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Abstract. We give a brief review on Weyl manifold as a quantization of symplectic manifold, equipped with a system of quantized canonical charts and quantized canonical transformations among them called Weyl diffeomorphism. Weyl manifold is deeply related to deformation quantization on symplectic manifolds. We explain a relation between Weyl manifolds and deformation quantization.

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1. Introduction

In this note, we discuss Weyl manifold defined by Omori-Maeda-Yoshioka [5] (see also Yoshioka [7]). Weyl manifold is regarded as a quantized symplectic manifold and has a structure of fiber bundle over a symplectic manifold with fiber consisting of Weyl algebra, which is deeply related to deformation quantization on a symplectic manifold. The concept of deformation quantization was given by Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer [1], and the existence on symplectic manifold is established independently with different methods, first by Dewilde-Lecomte [2], then [5] and Fedosov [3]. The existence of deformation quantization on general Poisson manifolds are finally proved by Kontsevich [4].

A Weyl manifold has quantized canonical charts or quantized Darboux charts, glued by quantized canonical transformations, called Weyl diffeomorphisms. From a Weyl manifold over a symplectic manifold M we can construct a deformation quantization on M and also from a deformation quantization on M we obtain a

Weyl manifold over M. Fedosov [3] and Omori-Maeda-Yoshioka [5] took a similar way, namely, they both consider the Weyl algebraic bundle over a symplectic manifold, and using an isomorphism between formal power series of functions on the symplectic manifold and the space of sections of the bundle, then they constructed a formal deformation quantization on the symplectic manifold by means of the algebra structure of sections.
We emphasize here that despite of the similarity, the basic idea is slightly different. Fedosov used his connection on the Weyl algebraic bundle, and Omori-Maeda-Yoshioka consider to quantize Darboux coordinates by means of a quantization of canonical transformation. The latter used the quantized contact structure to manipulate the Weyl algebraic bundle (cf. [7]) while Fedosov used the connection, thus the idea of Omori-Maeda-Yoshioka is more closely related to quantization of Hamiltonian systems, or quantization of symplectic manifold.

In this note we give a brief review on Weyl manifolds and deformation quantization based on the idea of [5].

2. Star Product, Deformation Quantization

2.1. Example: Moyal Product

We start by the well-known example of star product. Let M be a $2n$ dimensional euclidean space $\mathbb{R}^{2n}$. We write the coordinate as $(x_1, \ldots, x_n, y_1, \ldots, y_n) \in \mathbb{R}^{2n}$, and the canonical symplectic structure as $\omega = \sum_{k=1}^{n} \mathrm{d}y_k \wedge \mathrm{d}x_n$. Its Poisson bracket is given as the following biderivation.

$$\begin{aligned}\{f, g\} &= \sum_{k=1}^{n} (\partial_{x_k} f\, \partial_{y_k} g - \partial_{y_k} f\, \partial_{x_k} g) = \sum_{k=1}^{n} (f \overleftarrow{\partial}_{x_k} \overrightarrow{\partial}_{y_k} g - f \overleftarrow{\partial}_{y_k} \overrightarrow{\partial}_{x_k} g) \\ &= f \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y g - f \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x g = f(\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x) g = f \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y g.\end{aligned}$$

The l^{th} power of the biderivation is calculated by means of the binomial theorem such as

$$\left(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right)^l = \sum_{k=0}^{l} \binom{l}{k} (-1)^k (\overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y)^{l-k} (\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x)^k$$

which defines a bidifferential operator.
The Moyal product $*_0$ is given by a formal power series of the biderivation of exponential type such that

$$\begin{aligned}f *_0 g &= fg + (\tfrac{\nu}{2}) f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y) g + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l g + \cdots \\ &= f \exp\left(\tfrac{\nu}{2} \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g\end{aligned}$$

for any $f, g \in C^\infty(M)$, $M = \mathbb{R}^{2n}$, where ν is a formal parameter. The Moyal product is extended naturally to the formal power series such as $f, g \in C^\infty(M)[[\nu]]$. Then it is easy to see that the Moyal product is an associative product on the space of formal power series $C^\infty(M)[[\nu]]$.

We sometimes write the Moyal product in general form such as

$$f *_0 g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots$$

where $C_l(f,g) = \frac{1}{l!}(\frac{1}{2})^l(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l$, $l = 1, 2, \ldots$.

2.2. Star Product

The definition of star product is direct from the Moyal product.

For a manifold M, we consider a binary product on the space of formal power series $C^\infty(M)[[\nu]]$ such that

$$f * g = fg + \nu C_1(f,g) + \nu^2 C_2(f,g) + \cdots + \nu^l C_l(f,g) + \cdots$$

where $C_l(\cdot,\cdot)$ is a bilinear map from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$.

Definition 1. *A product $f * g$ is called a star product when it is associative.*

Then for a star product $*$ on M, we have an associative algebra $(\mathcal{A}_\nu(M), *)$, called a star product algebra.

Poisson Structure. We see that the star product naturally induces a Poisson structure on the manifold M.

Consider skewsymmetric part C_1^- of C_1, namely, $C_1^-(f,g) = \frac{1}{2}(C_1(f,g) - C_1(g,f))$, for all $f, g \in C^\infty(M)$. Then, the associative product satisfies

- $[f, g * h]_* = [f,g]_* * h + g * [f,h]_*$
- $[f, [g,h]_*]_* + (\text{cyclic}) = 0$

where $[f,g]_* = f * g - g * f$, and the expansion of the above gives

Proposition 1. *The skew symmetric part of C_1 is a Poisson bracket on M.*

Suppose we have a Poisson structure $\{\cdot,\cdot\}$ on M.

Definition 2. *A star product $*$ on M is called a deformation quantization of the Poisson manifold $(M, \{\cdot,\cdot\})$ when the skew symmetric part of C_1 is equal to $\{\cdot,\cdot\}$.*

Equivalence. Suppose we have star products $*$, $*'$ on a manifold M. Then we have star product algebras $(\mathcal{A}_\nu(M), *)$, $(\mathcal{A}_\nu(M), *')$.

Definition 3. *The star products $*$, $*'$ are equivalent if there exists an algebra isomorphism $T : (\mathcal{A}_\nu(M), *) \to (\mathcal{A}_\nu(M), *')$ of the form*

$$T(f) = f + \nu T_1(f) + \nu^2 T_2(f) + \cdots + \nu^l T_l(f) + \cdots$$

where T_l is a linear map of $C^\infty(M)$, $l = 1, 2, \ldots$.

We have the following proposition [5].

Proposition 2. *For every equivalence class of star product on M, there is a representative $f * g = fg + \nu C_1(f, g) + \cdots + \nu^l C_l(f, g) + \cdots$, $\forall f, g \in \mathcal{A}_\nu(M)$ such that C_1 is a Poisson bracket, namely, its symmetric part is zero. Moreover, we can take every C_l $(l = 1, 2, \ldots)$ is local, namely, a differential operator on M.*

Back Ground. Star products are already treated by Weyl, Wigner, Moyal. These can be regarded as a deformation of the usual multiplication of functions. For these, Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer proposed a concept of deformation quantization on a manifold.

By many people's efforts, the existence and classification problem became clear and was established. Kontsevich proved that there is a deformation quantiaztion on every Poisson manifold.

Symplectic Manifold. When M is symplectic, star products have a geometric picture which we call a Weyl manifold. From a Weyl manifold W_M over M, we can obtain a deformation quantization of the symplectic manifold M.

3. Weyl Manifold

In what follows, we will explain the construction of Weyl manifold on arbitrary symplectic manifold (M, ω) and also explain that from Weyl manifold we can obtain a deformation quantization of (M, ω). This section is based on [5].

Let (M, ω) be a $2n$ dimensional symplectic manifold.

Weyl manifold W_M is a Weyl algebra bundle over (M, ω) with certain properties. Weyl manifold has a deep relationship with deformation quantization of symplectic manifold.

3.1. Idea

We can show the existence of deformation quantization on a symplectic manifold by showing that every symplectic manifold has Weyl manifold over it from which one can obtain a deformation quantization.

The basic idea of the construction of Weyl manifold is to embed local functions on a Darboux chart into Weyl algebra, whose embedded image is called Weyl functions.

Quantized Darboux Chart. For any point $p \in M$, there exists a coodinate neighborhood $(U, (x_1, \ldots, x_n, y_1, \ldots, y_n))$ such that $\omega = \sum_{j=1}^n \mathrm{d}y_j \wedge \mathrm{d}x_j$ by Darboux theorem, which is called a canonical coordinate or a Darboux chart. Here we sometimes write as $z = (x, y)$ for simplicity. With respect to this coordinate system the Poisson bracket of (M, ω) is written as $\{\cdot,\cdot\} = \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y$, then we have the Moyal product on U

$$\begin{aligned} f *_0 g &= fg + (\tfrac{\nu}{2}) f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y) g + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} f(\overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y)^l g + \cdots \\ &= f \exp\left(\tfrac{\nu}{2} \overleftarrow{\partial}_x \wedge \overrightarrow{\partial}_y\right) g, \quad f, g \in \mathcal{A}_\nu(U) = C^\infty(U)[[\nu]]. \end{aligned}$$

Hence, the triplet $(U, (x, y), *_0)$ is regarded as a quantized Darboux chart.

Quantized Darboux Theorem. Suppose we have a deformation quantization $*$ of the symplectic manifold (M, ω)

$$f * g = fg + \nu C_1(f, g) + \nu^2 C_2(f, g) + \cdots + \nu^l C_l(f, g) + \cdots .$$

By the Proposition 2, we can assume $C_1 = \frac{1}{2}\{\cdot,\cdot\}$ and C_l $(l = 1, 2, \ldots)$ are bidifferential operators. Since the star product $*$ is local we can restrict to every $(U, (x, y), *)$. Then we have a "Quantized Darboux theorem" as follows.

Proposition 3. *On every U, the star product $*$ is equivalent to the Moyal product $*_0$. Hence, the star product $*$ has a local coordinate expression of quantized Darboux chart $(U, (x, y), *_0)$ for every point.*

Quantized Symplectic Atlas. Suppose we have a symplectic atlas $\{(U_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$, then quantized Darboux theorem shows that the star product has a quantized symplectic atlas $\{(U_\alpha, z_\alpha, *_0)\}_{\alpha\in\Lambda}$. And local star product algebras $\{(\mathcal{A}_\nu(U_\alpha), *_0)\}_{\alpha\in\Lambda}$ are glued together to be an original star product by algebra isomorphisms

$$T_{\beta\alpha} : (\mathcal{A}_\nu(U_\alpha), *_0)|_{U_\alpha\cap U_\beta} \to (\mathcal{A}_\nu(U_\beta), *_0)|_{U_\beta\cap U_\alpha}.$$

These isomorphisms obviously satisfy the following lemma.

Lemma 1.

i) $T_{\alpha\gamma}T_{\gamma\beta}T_{\beta\alpha} = 1$ *for* $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$.

ii) $T^{-1}_{\beta\alpha} = T_{\alpha\beta}$ *for* $U_\alpha \cap U_\beta \neq \emptyset$.

Weyl Manifold and Deformation Quantization. For any symplectic manifod (M, ω), we construct a deformation quantization of (M, ω) by gluing local Moyal algebras, or Quantized Darboux charts by algebra isomorphisms. For this purpose, we first construct a Weyl algebra bundle over (M, ω) from which we can obtain a deformation quantization.

3.2. Weyl Manifold

In a word, Weyl manifold is a locally trivial Weyl algebra bundle over a symplectic manifold whose gluing maps are Weyl diffeomorphisms of local trivial bundles. Here Weyl diffeomorphism is an isomorphism of Weyl algebra bundles which preserves Weyl functions. The Weyl functions are the key concept, or the geometric structure, of quantized symplectic manifold or Weyl manifold.

Formal Weyl Algebra. A formal Weyl algebra W is an associative algebra, with the multiplication denoted by $\hat{*}$, formally generated over $\mathbb{R}$ or $\mathbb{C}$ by elements $\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n$ where ν commutes with any elements and satisfy the canonical commutation relation

$$[X_j, Y_k]_* = \nu\delta_{jk}, \qquad [X_j, X_k]_* = [Y_j, Y_k]_* = 0, \qquad j, k = 1, 2, \ldots, n.$$

Here the bracket $[\cdot,\cdot]_*$ is the commutator of W; $[F, G]_* = F\hat{*}G - G\hat{*}F$, $F, G \in W$. For simplicity, instead of $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ we sometimes use a notation $(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = (Z_1, \ldots, Z_{2n})$.

Weyl Ordered Expression and Moyal Product Formula. Using the basis of monomials by completely symmetric polynomials such as $X_1\hat{*}X_2 + X_2\hat{*}X_1/2$, etc.

Using this basis, the formal Weyl algebra W is expressed as the formal power series of the generators with the Moyal product formula. Namely, we have a linear isomorphism

$$\sigma : W \to \mathbb{C}[[\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n]].$$

With this linear isomorphism we identify W with $\mathbb{C}[[\nu, X_1, \ldots, X_n, Y_1, \ldots, Y_n]]$ and the multiplication $\hat{*}$ is given as the Moyal product, namely any elements are expressed as a formal power series, $F = \sum_{l\alpha} a_{l\alpha}\nu^l Z^\alpha$, $G = \sum_{m\beta} b_{m\beta}\nu^m Z^\beta$, and we have

Lemma 2.

$$\begin{aligned} F\hat{*}G &= F \exp\left(\tfrac{\nu}{2}\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y\right) G \\ &= FG + (\tfrac{\nu}{2})F(\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y)G + \cdots + (\tfrac{\nu}{2})^l \tfrac{1}{l!} F(\overleftarrow{\partial}_X \wedge \overrightarrow{\partial}_Y)^l G + \cdots . \end{aligned}$$

Weyl Function. Let U be an open subset of $\mathbb{R}^{2n}$. We consider to embed a function f on U into a formal Weyl algebra W. The embedding is called a Weyl continuation of function denoted by $f^\#$ such that

$$f^\#(z) = \sum_\alpha \tfrac{1}{\alpha} f^{(\alpha)}(z) Z^\alpha, \qquad z \in U$$

where $f^{(\alpha)}(z) = \partial_z^\alpha f(z)$.

The Weyl continuation is obviously extended to the formal power series $\mathcal{A}_\nu(U) = C^\infty(U)[[\nu]]$, and gives a section of the trivial Weyl algebra bundle $U \times W = W_U$, namely, $f^\# \in \Gamma(W_U)$. We denote the image of $\#$ by $\mathcal{F}(W_U) = \mathcal{A}_\nu(U)^\# \subset \Gamma(W_U)$.

It is direct to see that the products $*_0$ and $\hat{*}$ commute with the Weyl continuation $\#$, namely we have

Proposition 4.

$$(f *_0 g)^\# = f^\# \hat{*} g^\#, \qquad f, g \in \mathcal{A}_\nu(U).$$

Then we have

Corollary 1.

i) *The space of the Weyl functions is an associative algebra under the multiplication $\hat{*}$, namely $(\mathcal{F}(W_U), \hat{*})$ is an associative algebra.*

ii) *The Weyl continuation is an algebra isomorphism*

$$\# : (\mathcal{A}_\nu(U), *_0) \to (\mathcal{F}(W_U), \hat{*}).$$

3.3. Weyl Diffeomorphism

Instead of gluing local Moyal algebras $(\mathcal{A}_\nu(U), *_0)$, we glue the isomorphic Weyl function algebras $(\mathcal{F}(W_U), \hat{*})$. Since $\mathcal{F}(W_U)$ is a certain class of sections of the trivial bundle W_U, then we gain a bundle picture for star products.

Consider trivial bundles $W_U = U \times W$, and $W_{U'}$ for open subsets $U, U' \subset \mathbb{R}^{2n}$.

Definition 4. *A bundle isomorphism $\Phi : W_U \to W_{U'}$ is called a Weyl diffeomorphism when*

i) $\Phi(\nu) = \nu$.

ii) $\Phi^*(\mathcal{F}(W_{U'}) = \mathcal{F}(W_U)$.

iii) $\Phi^* f^\# = (\phi^* f)^\# + O(\nu^2), \ \ f \in \mathcal{A}_\nu(U')$.

iv) $\overline{\Phi(F)} = \Phi(\overline{F})$, $F \in W_U$, *where $\overline{F}$ is a conjugation in W.*

Remark 1.

- *The condition* i) *is natural which means that Φ is $\mathbb{C}[[\nu]]$-linear.*
- *The condition* ii) *is essential to our theory. We regard the Weyl functions $\mathcal{F}(W_U)$ as the geometric structure of Weyl manifold W_M, or quantized symplectic manifold.*
- *The conditions* iii) *and* iv) *are optional. The condition* iii) *corresponds that the symmetric part C_1^+ vanishes, and the condition* iv) *means that the obtained star product has the conjugation operation, or so called a parity.*

A bundle map naturally induces a map between the base space. As to Weyl diffeomorpism we have the following lemma.

Lemma 3. *The induced map $\phi : U \to U'$ of a Weyl differomorphism $\Phi : W_U \to W_{U'}$ is a symplectic diffeomorphism.*

On the other hand, we have

Theorem 1. *For a symplectic diffeomorphism $\phi : U \to U'$, there exists a Weyl diffeomorphism $\Phi : W_U \to W_{U'}$ whose induced map is ϕ.*

Gluing and Contact Algebra. Let $\{U_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ be a symplectic atlas of (M, ω). Then (M, ω) is given by patching together $\{(U_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ by canonical transformations $\phi_{\alpha\beta}$ between U_α and U_β. Then we can take Weyl diffeomorphisms $\Phi_{\alpha\beta}$ between trivial bundles W_{U_α} and W_{U_β} by quantizing the canoical transformations $\phi_{\alpha\beta}$. We consider to glue local Weyl functions $\{\mathcal{F}(W_{U_\alpha})\}_{\alpha\in\Lambda}$ by Weyl diffeomorphisms.

We remark here the structure of a Weyl diffeomorphism $\Phi : W_U \to W_{U'}$ is roughly $\Phi = \mathrm{d}\phi \circ \exp\left(\frac{1}{\nu}\mathrm{ad}(f^{\#})\right)$, where $\mathrm{d}\phi$ is the tangent map of the induced symplectic diffeomorphism and $f^{\#}$ is a certain Weyl function on U. Hence, for a given symplectic diffeomorphism ϕ, a Weyl diffeomorphism with induced map ϕ, a quantization of ϕ, is not unique. For each canonical transformation $\phi_{\alpha\beta} : U_\alpha \to U_\beta$, we take a quantized canonical transformation, a Weyl diffeomorphism $\Phi_{\alpha\beta} : W_{U_\alpha} \to W_{U_\beta}$ with induced map $\phi_{\alpha\beta}$. For $U_{\alpha\beta} \cap U_{\beta\gamma} \cap U_{\gamma\alpha} \neq \emptyset$, the composition $\Phi_{\gamma\alpha} \circ \Phi_{\beta\gamma} \circ \Phi_{\alpha\beta}$ is not equal to the identity in general. So in order to adjust Weyl diffeomorphisms to satisfy transition function rule of the bundles, we have an idea to control the center. The idea is what we call a contact Lie algebra, which can be regarded as a quantized contact structure in some sense.

We introduce a degree d of the elements of the Weyl algebra W by setting

$$d(\nu) = 2, \qquad d(X_j) = d(Y_j) = 1, \qquad j = 1, 2, \dots, n.$$

Then the degree is well-defined for the Weyl algebra since it is of no contradiction with the relation $[X_j, Y_k]_* = \nu\delta_{jk}$. For exmaple we see $d(\nu X_1 \hat{*} Y_2) = 4$, etc.

Using the degree we can introduce a derivation of the Weyl algebra $D : W \to W$ such that

$$D(\nu) = 2\nu, \qquad D(X_j) = \nu X_j, \qquad D(Y_j) = \nu Y_j, \qquad j = 1, 2, \dots, n.$$

Notice that the center of W is equal to $\mathbb{C}[[\nu]]$ and D does not vanish on the center.

We introduce an element τ such that

$$[\tau, F] = -[F, \tau] = D(F), \qquad F \in W.$$

We consider a direct sum

$$\mathfrak{g} = \mathbb{C}\tau \oplus W$$

and then we can define a Lie algebra, called a contact Lie algebra, by putting

$$[\lambda\tau + a, \mu\tau + b] = \lambda[\tau, a] + \mu[a, \tau] + [a, b]_*, \qquad \lambda, \mu \in \mathbb{C}, \qquad a, b \in W.$$

Now we extend the Weyl diffeomorphism to a contact Weyl diffeomorphism. We consider a locally trivial Lie algebra bundle $\mathfrak{g}_U = U \times U$. We define

Definition 5. *A contact Lie algebra bundle isomorphism $\Psi : \mathfrak{g}_U \to \mathfrak{g}_{U'}$ is called a contact Weyl diffeomorphism when it satisfies*

i) $\Psi^*\tau = \tau + f^\#, \quad f \in C^\infty(U)[[\nu]]$.
ii) *The restriction to the Weyl algebra bundle $\Psi|W_U$ induces a Weyl diffeomorphism $\Psi|W_U : W_U \to W_{U'}$.*

We have

Proposition 5.

i) *For a Weyl diffeomorphism $\Phi : W_U \to W_{U'}$, there exists a contact Weyl diffeomorphism $\Psi : \mathfrak{g}_U \to \mathfrak{g}_{U'}$ such that the restriction $\Psi|W_U$ is equal to Φ.*
ii) *For contact Weyl diffeomorphisms $\Psi, \Psi' : \mathfrak{g}_U \to \mathfrak{g}_{U'}$ having the same restriction $\Psi|W_U = \Psi'|W_U$ there exists uniquely a central element $c = c_0 + c_1\nu^2 + \cdots + c_k\nu^{2k} + \cdots$ such that $\Psi' = \Psi \exp\big(\mathrm{ad}(\frac{1}{\nu}c)\big)$.*
Especially, a contact Weyl diffeomorphism which induces an identity Weyl diffeomorphism is uniquely written as $\Psi = \exp\big(\mathrm{ad}(\frac{1}{\nu}c)\big)$, $c \in \mathbb{C}[[\nu^2]]$.

Using contact Weyl diffeomorphisms to control central elements, and we can glue a system of trivial contact Lie algebra bundles $\{\mathfrak{g}_{U_\alpha}\}_{\alpha\in\Lambda}$ by contact Weyl diffeomorphisms and we obtain (see [5], [7])

Theorem 2. *For any symplectic manifold, there exists a contact Lie algebra bundle $\mathfrak{g}_M$. Restricting the fiber $\mathfrak{g}$ to the Weyl algebra W, we obtain a Weyl manifold W_M.*

4. Deformation Quantization

Using a Weyl diffeomorphism we can obtain a deformation quantization of the symplectic manifold in the following way.

By a transition functions, that is gluing Weyl diffeomorphisms, the local Weyl functions are also glued together to give a global Weyl functions. We denote this algebra by $(\mathcal{F}(W_M), \hat{*})$ called a space of Weyl functions on M.

Theorem 3. *We have a $\mathbb{C}[[\nu]]$-linear map $\sigma : C^\infty(M)[[\nu]] \to \mathcal{F}(W_M)$.*

By means of this linear isomorphism we can define an associative product on $C^\infty(M)[[\nu]]$ by

$$f * g = \sigma^{-1}(\sigma(f)\hat{*}\sigma(g)).$$

By expanding this product in the power of ν we see that the product $*$ is a deformation quantization of (M, ω).

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