Geometry,
Integrability
and
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STAR FUNCTIONS: EXAMPLES AND APPLICATIONS

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Abstract. We give a review on non formal star product and its star exponentials with concrete examples.

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1. Introduction

In this note, we discuss a non-formal deformation quantization or non-formal star product. Star products are already treated by Weyl [13], Wigner [14] and Moyal [10]. These can be regarded as a deformaion of functions of the usual multiplication of functions. For these, Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer proposed the concept of deformation quantization [1] in 1970's which introduces the new point of view for quantization.

Formal deformation quantization means that deformation is considered in the space of formal power series with deformation parameter, which is very successful. Any manifold is quantiziable in the sense of formal deformation quantization (Kontsevich [7], see also Sternheimer [12]).

Meanwhile non-formal deformation quatnization is to consider deformation with deformation parameter being a number. Then, a primitive question arises: Can we consider non formal deformation quantization on a manifold?

At present, we have no general theory for non-formal deformation quantization problem, and no idea at present either, but we have some examples.

In this note, we show some concrete examples on $\mathbb{R}^n$ and $\mathbb{C}^n$ which illustrate non-formal star product computation.

2. Moyal Product

The canonical commutation relation is a basic identity of quantum mechanics, which is given by a pair of operators such as

$$[\hat{p}, \hat{q}] = \hat{p}\,\hat{q} - \hat{q}\,\hat{p} = -\mathrm{i}\hbar$$

where $\hbar > 0$ is the Dirac constant.

The algebra generated by $\hat{p}$ and $\hat{q}$ is referred to as the *Weyl algebra* which plays a fundamental role in quantum mechanics.

We can realize the same algebra by using polynomials on $\mathbb{R}^2$ or $\mathbb{C}^2$ without using operators. We introduce an associative product $*_o$, called the *Moyal product*, into the space of polynomials. The product is different from the usual multiplication of functions, which provides a deformation of the usual multiplication in the following way.

The Moyal Product $*_o$ For any smooth functions f, g on $\mathbb{R}^2$, we have the canonical Poisson bracket

$$\{f, g\}(q, p) = \partial_q f \partial_p g - \partial_p f \partial_q g, \qquad (q, p) \in \mathbb{R}^2.$$

The Poisson bracket is written as a biderivation $\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}$ such that

$$\{f, g\} = f\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right) g = \partial_q f\,\partial_p g - \partial_p f\,\partial_q g\,.$$

For polynomials f and g, the Moyal product $f *_o g$ is given by a formal exponential of the biderivation $\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}$ such that

$$f *_o g = f\exp\left\{\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)\right\}g = f\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\mathrm{i}\hbar}{2}\right)^k\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)^k g$$

$$= fg + \frac{\mathrm{i}\hbar}{2} f\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right) g + \frac{1}{2!}\left(\frac{\mathrm{i}\hbar}{2}\right)^2 f\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)^2 g$$

$$+ \cdots + \frac{1}{k!}\left(\frac{\mathrm{i}\hbar}{2}\right)^k f\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)^k g + \cdots$$

where $\hbar$ is a positive number.

The product is well-defined on polynomials, then by a direct calculation we easily see that *the Moyal product is associative*. Besides, it is obvious that when $\hbar \to 0$, the product converges: $f *_o g = fg + \frac{\mathrm{i}\hbar}{2}\{f, g\} + \cdots \to fg$. Therefore the product $f *_o g$ is regarded as a deformation of the usual multiplication fg, which is an origin of the concept of deformation quantization.

Now we calculate the commutator of the functions p and q. We have

$$p *_o q = p \exp\left\{\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)\right\} q$$

$$= p \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\mathrm{i}\hbar}{2}\right)^k \left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)^k q$$

$$= pq + \frac{\mathrm{i}\hbar}{2} p \left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right) q = pq - \frac{\mathrm{i}\hbar}{2}.$$

Similarly we see that

$$q *_o p = pq + \frac{\mathrm{i}\hbar}{2}$$

and that the functions p and q satisfy the canonical commutation relation under the commutator of the product $*_o$

$$[p, q]_* = p *_o q - q *_o p = -\mathrm{i}\hbar.$$

Thus the associative product $*_o$ is satisfying the canonical commutation relations on polynomials, and then we obtain the Weyl algebra given by the ordinary polynomials with the product $*_o$, $(\mathbb{C}[q, p], *_o)$.

Using this Weyl algebra of polynomials $(\mathbb{C}[q, p], *_o)$, we can obtain some results of quantum mechanics, for example, the eigenvalues of harmonic oscillator which is explained in the next section.

3. Star Product Calculation of Eigenvalues

We can calculate the eigenvalues of the harmonic oscillator and also of the MIC-Kepler problem by means of the star product $*_o$.

3.1. Harmonic Oscillator

Eigenvalues. The Schrödinger operator of the harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2}\left(\frac{\partial}{\partial q}\right)^2 + \frac{1}{2}q^2.$$

The eigenvalues are

$$E_n = \hbar\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \cdots.$$

Parallel to the arguments in familiar text books, we can calculate the eigenvalues E_n by using the star product $*_o$ and its eigenfunctions of p and q in the following way.

The classical hamiltonian function is

$$H = \frac{1}{2}(p^2 + q^2).$$

We take functions such as

$$a = \frac{1}{\sqrt{2\,\hbar}}(q + \mathrm{i}p), \qquad a^\dagger = \frac{1}{\sqrt{2\,\hbar}}(q - \mathrm{i}p).$$

Then we see easily that

$$a^\dagger *_o a = \frac{1}{2\,\hbar}\left(p *_o p - \mathrm{i}[p, q]_* + q *_o q\right) = \frac{1}{2\,\hbar}\left\{p \cdot p - \mathrm{i} \cdot (-\mathrm{i}\hbar) + q \cdot q\right\}$$

and find also that $a^\dagger *_o a = \frac{1}{2\hbar}(p^2 + q^2) - \frac{1}{2}$. Then we have the identity of functions

$$H = \hbar(N + \frac{1}{2}), \qquad N := a^\dagger *_o a.$$

The commutator of the functions a and $a^\dagger$ with respect to the star product is easily seen to be

$$[a, a^\dagger]_* = a *_o a^\dagger - a^\dagger *_o a = \frac{\mathrm{i}}{2\,\hbar}2[p, q]_* = 1.$$

Let us note that $[a, a^\dagger]_* = 1$ is equivalent to $a *_o a^\dagger = a^\dagger *_o a + 1 = N + 1$, and then we have the following commutation relation

$$N *_o a^\dagger = (a^\dagger *_o a) *_o a^\dagger = a^\dagger *_o (a *_o a^\dagger) = a^\dagger *_o (N + 1).$$

Now we set a function called a *vacuum*

$$f_0 = \frac{1}{\pi\,\hbar} \exp\left(-2\,aa^\dagger\right) = \frac{1}{\pi\,\hbar} \exp\left\{-\frac{1}{\hbar}(p^2 + q^2)\right\}.$$

Since $\hbar$ is positive, the function f_0 is smooth at $\hbar = 0$ and all its Taylor coefficients are equal to 0. Remark here that the vacuum f_0 vanishes in the space of formal power series $C^\infty(\mathbb{R}^2)[[\hbar]]$.

We set also the function

$$f_n = \frac{1}{n!} \underbrace{a^\dagger *_o \cdots *_o a^\dagger}_{n} *_o f_0 *_o \underbrace{a *_o \cdots *_o a}_{n}.$$

By a direct calculation we see

$$a *_o f_0 = f_0 *_o a^\dagger = 0$$

and then we have $N *_o f_0 = a^\dagger *_o a *_o f = 0$.

Using the commutation relation $N *_o a^\dagger = a^\dagger *_o (N + 1)$, we have as well

$$N *_o f_1 = N *_o (a^\dagger *_o f_0 *_o a) = a^\dagger *_o (N + 1) *_o f_0 *_o a = f_1 .$$

In a similar manner we easily see that

$$N *_o f_n = f_n *_o N = n \, f_n \, .$$

Since $H = \hbar(N + \frac{1}{2})$ we have the solutions of the star eigenvalue problem

$$H *_o f_n = f_n *_o H = \hbar(n + \frac{1}{2}) f_n = E_n \, f_n, \qquad n = 0, 1, 2, \cdots.$$

Thus we obtain the eigenvalues of the harmonic oscillator.

3.2. MIC-Kepler Problem (cf [4])

We apply the same method to the MIC-Kepler problem, namely the Kepler-problem with Dirac's monopole field.

Background. McIntosh and Cisneros [7] studied the dynamical system describing the motion of a charged particle under the influence of Dirac's monopole field besides the Coulomb's potential. Iwai-Uwano [2] gives the Hamiltonian description for the MIC-Kepler problem. They have showed that the classical system of MIC-Kepler problem is obtained by the S^1-reduction or Marsden-Weinstein reduction on the phase space. Recently the MIC-Kepler problem is generalized by Meng [8].

A non-formal star product quantizes classical systems on the phase space only by means of deforming the algebra of functions on phase space with a deformed product. Then we expect that the non-formal star product is capable to quantize the MIC-Kepler problem by quantizing the Marsden-Weinstein reduction in a natural way. However, although formal star products quantize Marsden-Weinstein reduction (cf. Fedosov, etc), we have no theory for the non-formal star product.

MIC-Kepler Problem. Now we introduce the MIC-Kepler problem.
We consider a closed two form Ω on $\dot{\mathbb{R}}^3 = \mathbb{R}^3 - \{0\}$ such that

$$\Omega = (\, q_1 \, \mathrm{d}q_2 \wedge \mathrm{d}q_3 \,+\, q_2 \, \mathrm{d}q_3 \wedge \mathrm{d}q_1 \,+\, q_3 \, \mathrm{d}q_1 \wedge \mathrm{d}q_2)/r^3$$

where $q = (q_1, q_2, q_3) \in \mathbb{R}^3$ and $r = \sqrt{q_1^2 + q_2^2 + q_3^2}$.
We consider the cotangent bundle $T^*\dot{\mathbb{R}}^3$ and equiped with the symplectic form

$$\sigma_\mu = \mathrm{d}p_1 \wedge \mathrm{d}q_1 \,+\, \mathrm{d}p_2 \wedge \mathrm{d}q_2 \,+\, \mathrm{d}p_3 \wedge \mathrm{d}q_3 + \Omega_\mu$$

where $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, q_2, q_3, p_1, p_2, p_3) \in T^*\dot{\mathbb{R}}^3$ and the two-form

$$\Omega_\mu = -\mu \, \Omega$$

stands for Dirac's monopole field of strength $-\mu \in \mathbb{R}$. Then the MIC-Kepler problem is defined by the triple

$$(\, T^*\dot{\mathbb{R}}^3, \, \sigma_\mu \,, \, H_\mu \,)$$

where H_μ is the Hamiltonian function

$$H_\mu\left(\boldsymbol{q},\,\boldsymbol{p}\right) = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{\mu^2}{2r^2} - \frac{k}{r}$$

in which $k > 0$ describes the Coulomb's potential constant. When $\mu = 0$, σ_μ is the canonical symplectic form, and then the system is just the Kepler problem.

S^1-**Action.** The MIC-Kepler problem is induced from the harmonic oscillator on $T^*\dot{\mathbb{R}}^4$ by the Marsden-Weinstein reduction with S^1-action as follows (for details see Iwai-Uwano [2]).

We denote the points by $y \in \mathbb{R}^4$ and $(y, \eta) \in T^*\mathbb{R}^4$. We identify a point in $T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4)$ with a point of $\mathbb{C}^4$ by the rule

$$T^*\mathbb{R}^4 \ni (y_1, y_2, y_3, y_4, \eta_1, \eta_2, \eta_3, \eta_4) \mapsto (z_1, z_2, \zeta_1, \zeta_2) \in T^*\mathbb{C}^2 = \mathbb{C}^4$$

where

$$z_1 = y_1 + \mathrm{i}\,y_2, \qquad z_2 = y_3 + \mathrm{i}\,y_4, \qquad \zeta_1 = \eta_1 + \mathrm{i}\,\eta_2, \qquad \zeta_2 = \eta_3 + \mathrm{i}\,\eta_4.$$

Then the canonical one form θ on $T^*\mathbb{R}^4$ can be written as

$$\theta(z, \zeta) = \mathrm{Re}\,(\bar{\zeta} \cdot \mathrm{d}z).$$

The S^1 action on the cotangent bundle $T^*\dot{\mathbb{R}}^4 = T^*(\mathbb{R}^4 - \{0\})$ is given by

$$\varphi_t : (z, \zeta) \mapsto (\mathrm{e}^{\mathrm{i}t}z, \mathrm{e}^{\mathrm{i}t}\zeta), \qquad t \in \mathbb{R}$$

which obviously preserves the canonical one form θ, and then it is an exact symplectic action. The induced vector field $v(z, \zeta)$ on $T^*\dot{\mathbb{R}}^4$ of the action is

$$v(z, \zeta) = (\mathrm{i}z, \mathrm{i}\,\zeta).$$

Then a moment map ψ of the action is given by

$$\psi(z, \zeta) = \iota_v\theta(z, \zeta) = \mathrm{Im}\,\zeta \cdot \bar{z} = (\zeta \cdot \bar{z} - \bar{\zeta} \cdot z)/2\,\mathrm{i}.$$

S^1-**Reduction.** Following the Marsden-Weinstein reduction theory, we consider a level set of the moment map $\psi^{-1}(\mu)$ for $\mu \in \mathbb{R}$. Then S^1 acts on the level set $\psi^{-1}(\mu)$ and the induced S^1-bundle

$$\pi_\mu : \psi^{-1}(\mu) \to \psi^{-1}(\mu)/S^1$$

has the symplectic structure ω_μ such that

$$\iota_\mu^*\mathrm{d}\theta = \pi_\mu^*\omega_\mu$$

where $\iota_\mu : \psi^{-1}(\mu) \to T^*\dot{\mathbb{R}}^4$ is the inclusion map, and hence we have a reduced symplectic manifold

$$(\psi^{-1}(\mu)/S^1, \omega_\mu).$$

Hence we obtain

Proposition 1 (Iwai-Uwano [2]). *The reduced phase space is diffeomorphic to the symplectic manifold of the* MIC-Kepler *problem*

$$(\psi^{-1}(\mu)/S^1, \omega_\mu) \simeq (T^*\dot{\mathbb{R}}^3, \sigma_\mu)$$

Conformal Kepler Problem on $T^*\dot{\mathbb{R}}^4$**.** Now we consider a harmonic oscillator on $T^*\dot{\mathbb{R}}^4$

$$H(z, \zeta) = \frac{1}{2}|\zeta|^2 + \frac{1}{2}\omega^2|z|^2.$$

Iwai-Uwano [2] have introduced the conformal Kepler problem with the Hamiltonian

$$H_{CF}(z, \zeta) = \frac{1}{4|z|^2}(H(z, \zeta) - 4k) - \frac{1}{8}\omega^2 = \frac{1}{8|z|^2}|\zeta|^2 - \frac{k}{|z|^2}.$$

The MIC-Kepler problem is the reduced hamitonian system of the conformal Kepler problem, i.e.,

$$\pi_\mu^* H_\mu = \iota_\mu^* H_{CF}.$$

The conformal Kepler problem is related to the harmonic oscillator on $T^*\dot{\mathbb{R}}^4$ as

$$4|z|^2\left(H_{CF}(z, \zeta) + \frac{1}{8}\omega^2\right) = H(z, \zeta) - 4k$$

which induces a correspondence of energy surfaces such that

$$H_{CF} = -\frac{1}{8}\omega^2 \iff H = 4k.$$

Star Product Calculation of the Eigenvalues. On eight-dimensional phase space $T^*\dot{\mathbb{R}}^4$, we have the canonical Poisson bracket $\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}$. Then by the same way as the previous section, we have the following Moyal product $*_o$

$$f *_o g = f \exp\left\{\frac{i\hbar}{2}\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)\right\} g$$

$$= fg + \frac{i\hbar}{2}f\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)^2 g$$

$$+ \cdots + \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n f\left(\overleftarrow{\partial_q} \cdot \overrightarrow{\partial_p} - \overleftarrow{\partial_p} \cdot \overrightarrow{\partial_q}\right)^n g + \cdots$$

Furthermore we consider the following functions

$$b_1(z, \zeta) = \frac{1}{2} \left(\sqrt{\frac{\omega}{\hbar}}\, z_1 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\, \zeta_1 \right), \qquad b_1^\dagger(z, \zeta) = \overline{b_1(z, \zeta)}$$

$$b_2(z, \zeta) = \frac{1}{2} \left(\sqrt{\frac{\omega}{\hbar}}\, z_2 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\, \zeta_2 \right), \qquad b_2^\dagger(z, \zeta) = \overline{b_2(z, \zeta)}$$

$$b_3(z, \zeta) = \frac{1}{2} \left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_1 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\, \bar{\zeta}_1 \right), \qquad b_3^\dagger(z, \zeta) = \overline{b_3(z, \zeta)}$$

$$b_4(z, \zeta) = \frac{1}{2} \left(\sqrt{\frac{\omega}{\hbar}}\, \bar{z}_2 + \frac{\mathrm{i}}{\sqrt{\omega\hbar}}\, \bar{\zeta}_2 \right), \qquad b_4^\dagger(z, \zeta) = \overline{b_4(z, \zeta)}.$$

We see the commutators of these functions are

$$[b_j, b_l]_* = [b_j^\dagger, b_l^\dagger]_* = 0, \qquad [b_j, b_l^\dagger]_* = \delta_{jl} \qquad j, l = 1, 2, 3, 4.$$

Similarly as before we set

$$N = b_1^\dagger *_o b_1 + b_2^\dagger *_o b_2 + b_3^\dagger *_o b_3 + b_4^\dagger *_o b_4$$

and then we have

$$H = \hbar\omega\,(N + 2).$$

In terms of b_j and $b_j^\dagger$ the moment map $\psi(z, \zeta)$ is written as follows

$$\psi(z, \zeta) = \frac{\hbar}{2} \left(-b_1^\dagger *_o b_1 - b_2^\dagger *_o b_2 + b_3^\dagger *_o b_3 + b_4^\dagger *_o b_4 \right)$$

We put the following functions for all $j = 1, 2, 3, 4$ as before

$$f_{j,0}(z, \zeta) = \frac{1}{\pi\hbar}\, \mathrm{e}^{-2\,b_j^\dagger b_j}$$

$$f_{j,n_j}(z, \zeta) = \frac{1}{n_j!}\, (b_j^\dagger)_*^{n_j} *_o f_{j,0} *_o (b_j)_*^{n_j}, \qquad n_j = 0, 1, 2, \cdots$$

and introduce the following functions

$$f_n = f_{1,n_1} *_o f_{2,n_2} *_o f_{3,n_3} *_o f_{4,n_4}, \qquad n = n_1 + n_2 + n_3 + n_4.$$

Using these functions, we can calculate the eigenvalues of the MIC-Kepler problem as follows. Similarly as before we easily see

$$H *_o f_n = \hbar\omega(N + 2) *_o f_n = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2)\, f_n$$

and

$$\psi *_o f_n = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4)\, f_n.$$

Hence the energy level

$$H = 4k \quad \text{and} \quad \psi = \mu$$

is quantized respectively as

$$4k = \hbar\omega(n_1 + n_2 + n_3 + n_4 + 2) \tag{1}$$

and

$$\mu = \frac{\hbar}{2}(-n_1 - n_2 + n_3 + n_4). \tag{2}$$

Since the corresponding energy levels are corresponding as

$$H = 4k \Longleftrightarrow H_{CF} = -\frac{1}{8}\omega^2$$

then the relation (1) yields that the quantized energy level of H_{CF} is

$$-\frac{1}{8}\omega^2 = -\frac{2k^2}{\hbar^2(n_1 + n_2 + n_3 + n_4 + 2)^2}.$$

Moreover the strength of the magnetic monopole $-\mu$ is quantized by the condition (2).

Thus we have

Theorem 1. *The eigenvalues of the* MIC-Kepler *problem with the strength of magnetic monople* $-\hbar\frac{m}{2}$, $(m := -n_1 - n_2 + n_3 + n_4)$ *is*

$$E_n = -\frac{2k^2}{\hbar^2(n+2)^2}, \qquad n \geq |m| \qquad and \qquad n \pm m \equiv 0 \pmod 2.$$

The multiplicity of the eigenvalue E_n is

$$\frac{(n+m+2)(n-m+2)}{4}.$$

This is the same as the ones given in Iwai-Uwano [3] and Mladenov-Tsanov [9].

4. Star Products (cf [11])

4.1. Definiton of Star Product

Now by generalizing the Moyal product, we define a star product. Notice here that we consider on complex space $\mathbb{C}^n$.

Biderivation. Let Λ be an arbitrary $n \times n$ complex matrix. We consider a biderivation

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \cdots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \cdots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl} \overleftarrow{\partial_{w_k}} \overrightarrow{\partial_{w_l}}$$

where $(w_1, \cdots, w_n)$ is a generators of complex polynomials.

Now we define a star product similarly by the formula

Definition 1.
$$f *_\Lambda g = f \exp\left\{ \frac{i\hbar}{2} \left(\overleftarrow{\partial_w} \Lambda \overrightarrow{\partial_w} \right) \right\} g.$$

Then we see easily

Theorem 2. *For an arbitrary Λ, the star product $*_\Lambda$ is a well-defined associative product on complex polynomials.*

Remark 1 ($*_\Lambda$ covers various products)**.** i) *The star product $*_\Lambda$ is a generalization of the well-known star products. Actually*

- *if we put $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = J_0$, then we have the Moyal product*
- *if we choose $\Lambda = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ we have the normal product*
- *if we take $\Lambda = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$ we have the anti-normal product.*

ii) *If Λ is a symmetric matrix, the star product $*_\Lambda$ is commutative. Furthermore, if Λ is a zero matirx, then the star product is nothing but a usual multiplication product.*

Equivalence. For a complex matrix Λ we consider the decomposition
$$\Lambda = J + K$$
where J is the skew-symmetric part and K is the symmetric part of Λ repsectively. Let
$$\Lambda_1 = J + K_1, \qquad \Lambda_2 = J + K_2$$
be complex matrices with common skew-symmetric part. Then we have the following equivalence.

Proposition 2. *We have an Weyl algebra isomorphism $I_{K_1}^{K_2} : (\mathbb{C}[u,v], *_{\Lambda_1}) \to (\mathbb{C}[u,v], *_{\Lambda_2})$ given by the power series of the differential operator $\partial_w(K_2 - K_1)\partial_w$ such that*
$$I_{K_1}^{K_2}(f) = \exp\left\{ \frac{i\hbar}{4} \partial_w(K_2 - K_1)\partial_w \right\}(f)$$
where $\partial_w(K_2 - K_1)\partial_w = \sum_{k,l}(K_2 - K_1)_{kl}\,\partial_{w_k}\partial_{w_l}$.

By a direct calculation we have

Theorem 3. *Then isomorphisms satisfy the following chain rule*

1. $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = \mathrm{Id}$
2. $\left(I_{K_1}^{K_2} \right)^{-1} = I_{K_2}^{K_1}.$

4.2. Star Product Representation

Suppose $n = 2m$ and let J_0 be a skew-symmetric matrix such that

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us consider a complex matrix

$$\Lambda = J_0 + K$$

where K is an arbitrary $2m \times 2m$ complex symmetric matrix.

Since Λ is determined by the complex symmetric matrix K, we denote the star product by $*_K$ instead of $*_\Lambda$.

We denote the variables by $(w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$.

Then we obtain for an arbitrary K

Proposition 3.   i) *For a star product* $*_K$, *the generators* $(u_1, \ldots, u_m,$ $v_1, \ldots, v_m)$ *satisfy the canoical commutation relations*

$$[u_k, v_l]_* = -\mathrm{i}\hbar\delta_{kl}, \qquad [u_k, u_l]_* = [v_k, v_l]_* = 0, \qquad k, l = 1, 2, \ldots, m.$$

ii) *Then the algebra* $(\mathbb{C}[u, v], *_K)$ *is isomorphic to the Weyl algebra, and the algebra is regarded as a polynomial representation of the Weyl algebra.*

Remark 2. *By the previous proposition we see the algebras* $(\mathbb{C}[u, v], *_K)$ *are mutually isomorphic and isomorphic to the Weyl algebra. Then these are the same (i.e., isomorphic) at the algebra level. However when we consider exponetinal elements of this algebra, the difference of the expressions plays an important role.*

4.3. Star Exponentials

Idea of Definition. For a polynomial H_* in the star product algebra $(\mathbb{C}[w], *_\Lambda)$, we would like to define a star exponential $\mathrm{e}_*^{t\frac{H_*}{\mathrm{i}\hbar}}$. However, except special cases, the expansion

$$\sum_n \frac{t^n}{n!} \left(\frac{H_*}{\mathrm{i}\hbar}\right)^n = \sum_n \frac{t^n}{n!} \underbrace{\left(\frac{H_*}{\mathrm{i}\hbar}\right) *_\Lambda \cdots *_\Lambda \left(\frac{H_*}{\mathrm{i}\hbar}\right)}_{n}$$

is not convergent. Then we define a star exponential by means of the differential equation as follows.

Definition 2. *The star exponential* $\mathrm{e}_*^{t\frac{H_*}{\mathrm{i}\hbar}}$ *is given as a solution of the following differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t = H_* *_\Lambda F_t, \qquad F_0 = 1.$$

For the case where H_* is linear or quadratic polynomials, we have explicit solutions.

Example 1 (Linear case). We denote a linear polynomial by $l = \sum\limits_{j=1}^{2m} a_j w_j$. Then we have the following proposition.

Proposition 4. *For* $l = \sum\limits_j a_j w_j = \langle \mathbf{a}, \mathbf{w} \rangle$, *the star exponential with respect to the product* $*_\Lambda$ *is*

$$\mathrm{e}_*^{t\frac{l}{\mathrm{i}\hbar}} = \exp\left(t^2 \frac{\mathbf{a}K\mathbf{a}}{4\mathrm{i}\hbar}\right) \exp\left(t \frac{l}{\mathrm{i}\hbar}\right).$$

Example 2 (Quadratic case). We limit the case where $J = J_0$, namely the algebra $(\mathbb{C}[w], *_K)$ is the Weyl algebra.

Proposition 5. *For a quadratic polynomial* $Q_* = \langle \mathbf{w}A, \mathbf{w} \rangle_*$ *where* A *is a* $2m \times 2m$ *complex symmetric matrix*

$$\mathrm{e}_*^{t\frac{Q_*}{\mathrm{i}\hbar}} = \frac{2^m}{\sqrt{\det(I - \kappa + \mathrm{e}^{-2t\alpha}(I + \kappa))}} \, \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}\langle \mathbf{w} \frac{1}{I-\kappa+\mathrm{e}^{-2t\alpha}(I+\kappa)}(I - \mathrm{e}^{-2t\alpha})J_0, \mathbf{w}\rangle}$$

where $\kappa = KJ_0$ *and* $\alpha = AJ_0$.

4.4. Star Functions

Using star exponentials, we can consider several star functions following the standard method in text book for ordinary exponential functions.

In what follows, we consider the star product for the simplest case. We consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w)\, \mathrm{e}^{\frac{\tau}{2}\overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w).$$

4.4.1. Star Hermite Function

Recall the identity

$$\exp\left(\sqrt{2}\,tw - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w)\frac{t^n}{n!}$$

where $H_n(w)$ is an Hermite polynomial. By applying the explicit formula of linear case $\mathrm{e}_*^{t(l/\mathrm{i}\hbar)} = \mathrm{e}^{t^2 \mathbf{a}K\mathbf{a}/4\mathrm{i}\hbar}\, \mathrm{e}^{t(l/\mathrm{i}\hbar)}$ for $l = w$, we get

$$\exp_*(\sqrt{2}\,tw_*)_{\tau=-1} = \exp\left(\sqrt{2}\,tw - \frac{1}{2}t^2\right).$$

Since $\exp_*\left(\sqrt{2}\,tw_*\right) = \sum_{n=0}^{\infty}(\sqrt{2}\,w_*)^n\,\frac{t^n}{n!}$ we have

$$H_n(w) = (\sqrt{2}\,w_*)^n_{\tau=-1}\,.$$

We define $*$-Hermite function by

$$H_n(w,\tau) = (\sqrt{2}\,w_*)^n, \qquad n = 0,1,2,\cdots$$

with respect to $*_\tau$ product and then

$$\exp_*(\sqrt{2}\,tw_*) = \sum_{n=0}^{\infty} H_n(w,\tau)\,\frac{t^n}{n!}.$$

Identities. Trivial identity $\frac{d}{dt}\exp_*(\sqrt{2}\,tw_*) = \sqrt{2}\,w*\exp_*(\sqrt{2}\,tw_*)$ for the product $*_\tau$ yields the identity

$$\frac{\tau}{\sqrt{2}}H'_n(w,\tau) + \sqrt{2}\,wH_n(w,\tau) = H_{n+1}(w,\tau), \qquad n = 0,1,2,\cdots$$

for every $\tau \in \mathbb{C}$.

The exponential law

$$\exp_*\left(\sqrt{2}\,sw_*\right) * \exp_*\left(\sqrt{2}\,tw_*\right) = \exp_*\left(\sqrt{2}\,(s+t)w_*\right)$$

for the product $*_\tau$ yields the identity

$$\sum_{k+l=n} \frac{n!}{k!l!}\,H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau)$$

for every $\tau \in \mathbb{C}$.

4.4.2. Star Theta Function

We can express the Jacobi's theta functions by using star exponentials.

Using the formula for the linear case, a direct calculation gives

$$\exp_{*_\tau} itw = \exp\left(i\,tw - (\tau/4)t^2\right).$$

Hence for $\mathrm{Re}\,\tau > 0$, the star exponential $\exp_{*_\tau} ni\,w = \exp(ni\,w - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then the summation converges to give

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2ni\,w = \sum_{n=-\infty}^{\infty} \exp\left(2ni\,w - \tau\,n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni\,w}, \qquad q = e^{-\tau}$$

The exponential law of the star exponential yields trivial identities such that

$$\exp_{*_\tau} 2i\,w *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w), \qquad k = 2,3$$
$$\exp_{*_\tau} 2i\,w *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w), \qquad k = 1,4.$$

Then using $\exp_{*_\tau} 2\mathrm{i}\, w = \mathrm{e}^{-\tau}\mathrm{e}^{2\mathrm{i}\, w}$ and the product formula directly we deduce that the above identities are just

$$\mathrm{e}^{2\mathrm{i}\, w - \tau}\theta_{k*_\tau}(w + \mathrm{i}\,\tau) = \theta_{k*_\tau}(w), \qquad k = 2, 3$$

$$\mathrm{e}^{2\mathrm{i}\, w - \tau}\theta_{k*_\tau}(w + \mathrm{i}\,\tau) = -\theta_{k*_\tau}(w), \qquad k = 1, 4.$$

4.4.3. $*$-Delta Functions

Since the $*_\tau$-exponential $\exp_*(\mathrm{i}\, tw_*) = \exp(\mathrm{i}\, tw - \frac{\tau}{4}t^2)$ is raidly decreasing with respect to t when $\operatorname{Re} \tau > 0$. Then the integral of $*_\tau$-exponential

$$\int_{-\infty}^{\infty} \exp_*(\mathrm{i}t(w-a)_*)\mathrm{d}t = \int_{-\infty}^{\infty} \exp_*(\mathrm{i}t(w-a)_*)\mathrm{d}t = \int_{-\infty}^{\infty} \exp(\mathrm{i}t(w-a) - \frac{\tau}{4}t^2)\mathrm{d}t$$

converges for any $a \in \mathbb{C}$. We can introduce a star δ-function

$$\delta_*(w - a) = \int_{-\infty}^{\infty} \exp_*(\mathrm{i}\, t(w - a)_*)\, \mathrm{d}t$$

which has a meaning at τ with $\operatorname{Re} \tau > 0$. It is easy to see for any element $p_*(w) \in \mathcal{P}_*(\mathbb{C})$

$$p_*(w) * \delta_*(w - a) = p(a)\delta_*(w - a), \; w_* * \delta_*(w) = 0.$$

Using the Fourier transform we have

Proposition 6.

$$\theta_{1*}(w) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + \frac{\pi}{2} + n\pi)$$

$$\theta_{2*}(w) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \delta_*(w + n\pi)$$

$$\theta_{3*}(w) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta_*(w + n\pi)$$

$$\theta_{4*}(w) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta_*(w + \frac{\pi}{2} + n\pi).$$

Now, we consider the τ with the condition $\mathrm{Re}\,\tau > 0$. We calcultate the integral and obtain $\delta_*(w - a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w-a)^2\right)$. Then we have

$$\theta_3(w,\tau) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w + n\pi) = \sum_{n=-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}(w + n\pi)^2\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}\right)\sum_{n=-\infty}^{\infty}\exp\left(-2n\frac{1}{\tau}w - \frac{1}{\tau}n^2\tau^2\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{\tau}}\exp\left(-\frac{1}{\tau}\right)\theta_{3*}\left(\frac{2\pi w}{\mathrm{i}\tau}, \frac{\pi^2}{\tau}\right).$$

We also have similar identities for other $*$-theta functions by the similar way.

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