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BERTRAND SYSTEMS ON SPACES OF CONSTANT SECTIONAL CURVATURE. THE ACTION-ANGLE ANALYSIS. CLASSICAL, QUASI-CLASSICAL AND QUANTUM PROBLEMS

JAN JERZY SŁAWIANOWSKI and BARBARA GOŁUBOWSKA

Institute of Fundamental Technological Research, Polish Academy of Sciences
5^B *Pawińskiego Str., 02-106 Warsaw, Poland*

Abstract. Studied is the problem of degeneracy of mechanical systems the configuration space of which is the three-dimensional sphere, the elliptic space, i.e., the quotient of that sphere modulo the antipodal identification, and finally, the three-dimensional pseudo-sphere, namely, the Lobatchevski space. In other words, discussed are systems on groups $\mathrm{SU}(2)$, $\mathrm{SO}(3,\mathbb{R})$, and $\mathrm{SL}(2,\mathbb{R})$ or its quotient $\mathrm{SO}(1,2)$. The main subject are completely degenerate Bertrand-like systems. We present the action-angle classical description, the corresponding quasi-classical analysis and the rigorous quantum formulas. It is interesting that both the classical action-angle formulas and the rigorous quantum mechanical energy levels are superpositions of the flat-space expression, with those describing free geodetic motion on groups.

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1. Introduction

It belongs to the classics of analytical mechanics that in the flat Euclidean space there are two isotropic potential models with completely degenerate motion. By the "complete degeneracy" we mean that all bounded orbits are closed. The two completely degenerate potentials are the attractive Coulomb problem and isotropic harmonic oscillator. If the distance from the center of forces is denoted by r, then the corresponding potential energies are given by

$$V_{\mathrm{Coul}} = -\frac{\alpha}{r}, \qquad V_{\mathrm{osc}} = \frac{k}{2}r^2. \tag{1}$$

The center-attractive character of those potentials means that the constants α, k are positive. The motion in the central potential field is always flat and because of this it reduces to $\mathbb{R}^2$. This holds only on the classical level, but the quantum case is also canonical for those potentials, although there is no literal reduction to $\mathbb{R}^2$. Let us mention some peculiarity of the three-dimensional space $\mathbb{R}^3$. Namely, the attractive Coulomb potential turns out to be Green function for the Poisson equation

$$\Delta V = \delta^{ij}\partial_i\partial_j V = -f.$$

This is peculiarity of dimension n=3. Obviously, for the attractive Coulomb model the set of closed orbits coincides with those of the negative total energy, whereas for the attractive isotropic oscillator it is identical with the set of all orbits, i.e., with the general solution of equations of motion. Let us mention that in some sense the free motion in $\mathbb{R}^3$ is also Bertrand-like. Namely, if we project $\mathbb{R}^3$ to the projective space $\mathbb{RP}^3$, then every free motion, being described by straight line as a trajectory is obviously closed at infinity. In this sense all free motions are "periodic", however with the infinite period $T = \infty$. Nevertheless, this observation becomes literally true when we go to the motion in the spherical manifold $\mathrm{SU}(2)$, which may be naturally identified with the sphere $S^3(0,1) \subset \mathbb{R}^4$, or with the sphere of the general radius R, $S^3(0,R) \subset \mathbb{R}^4$. This brings about the question about the possible modification of the flat-space Bertrand potential, like (1) to some Bertrand-like potentials in other manifolds of constant sectional curvature, namely, to the sphere with the radius R or to the pseudosphere (Lobatchevski space) with the pseudoradius R. In principle the Bertrand-like potentials may be determined by following the method presented by Arnold [2] in the case of the usual Bertrand models in a flat Euclidean space. However, it is more convenient to find them by performing appropriate projective transformation, between $\mathbb{R}^3$ and the sphere and pseudosphere $S^3(0,R)$, $H^{3,2+}(0,R)$ (the symbols $2,+$ in last expression refer to the fact that the pseudosphere is a two-shell hyperboloid and we concentrate on the upper, positive leaf. One can easily show that the Bertrand potentials on those Riemannian manifolds are projective transformations of the usual Bertrand models in $\mathbb{R}^3$. Having in view quantum applications we concentrate on the dimension three, however , classically the same holds for $n = 2$ and for higher dimensions $n > 3$. As mentioned above, we discuss classical, quasi-classical and purely quantum aspects. One should mention that certain aspects of the problem were also investigated by other people, e.g., Schrödinger [21], Infeld [12], Kozlov [13], Shchepietilov [22] and certainly by some other persons. Let us also mention [26]. Here we discuss jointly the purely classical action-angle problems, quasiclassical Bohr-Sommerfeld aspect and rigorous quantum models. There is a very interesting result concerning energetic properties of our constant curvature Bertrand models. Namely, both the action-angle representation of Hamiltonians and quantum energy levels are superpositions of expressions characteristic for the flat-space formulas and for the geodetic, $R-$dependent expressions for the free motion on curved manifolds. Incidentltally, the second expressions differ in sign. It is perhaps a kind of insulent joke, but this may offer some possibility of the experimental checking what is the geometry and topology of the three-dimensional World: is it closed as Einstein seemed to suggest, or just the infinitely extended Lobatschevski space like Penrose wanted?

2. Constant-Curvature Hypersurfaces in $\mathbb{R}^4$ and Their Bertrand Potentials

Let us consider three hypersurfaces of $\mathbb{R}^4$ with the constant sectional curvature: the $S^3(0,R) \subset \mathbb{R}^4$, pseudosphere $H^{3,2,+}(0,R) \subset \mathbb{R}^4$ and the flat subspace $\mathbb{R}^3$. The sphere is given by the obvious equation

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 = R^2. \tag{2}$$

Analytically it is usually parametrized by the variables (r, ϑ, φ) in the following way

$$\begin{aligned} x^1 &= R \sin\left(\frac{r}{R}\right) \sin(\vartheta) \cos(\varphi) \\ x^2 &= R \sin\left(\frac{r}{R}\right) \sin(\vartheta) \sin(\varphi) \\ x^3 &= R \sin\left(\frac{r}{R}\right) \cos(\vartheta) \\ x^4 &= R \cos\left(\frac{r}{R}\right) \end{aligned} \tag{3}$$

where the range of coordinates is given by

$$r \in [0, \pi R], \qquad \vartheta \in [0, \pi], \qquad \varphi \in [0, 2\pi[.$$

The Euclidean metric in $\mathbb{R}^4$

$$\mathrm{d}S^2 = \left(\mathrm{d}x^1\right)^2 + \left(\mathrm{d}x^2\right)^2 + \left(\mathrm{d}x^3\right)^2 + \left(\mathrm{d}x^4\right)^2$$

after restriction to (2) becomes

$$\mathrm{d}s^2 = \mathrm{d}r^2 + R^2 \sin^2\left(\frac{r}{R}\right)\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right). \tag{4}$$

Very often in calculations one uses the two-dimensional subspace $\mathbb{R}^2,\ \vartheta = \frac{\pi}{2}$, with the induced metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + R^2 \sin^2\left(\frac{r}{R}\right)\mathrm{d}\varphi^2.$$

The hypersphere, i.e., the upper-shell component of the hyperbole (Lobatchevski space) is given by the following equation

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 - \left(x^4\right)^2 = -R^2, \qquad x^4 > 0 \tag{5}$$

is parametrically given by the radial hyperbolic functions

$$\begin{aligned} x^1 &= R\sinh\left(\frac{r}{R}\right)\sin(\vartheta)\cos(\varphi) \\ x^2 &= R\sinh\left(\frac{r}{R}\right)\sin(\vartheta)\sin(\varphi) \\ x^3 &= R\sinh\left(\frac{r}{R}\right)\cos(\vartheta) \\ x^4 &= R\cosh\left(\frac{r}{R}\right) \end{aligned} \tag{6}$$

with the corresponding range of variables

$$r \in [0, \infty], \qquad \vartheta \in [0, \pi], \qquad \varphi \in [0, 2\pi[.$$

Restricting the Euclidean metric of $\mathbb{R}^4$ to $H^{3,2,+}$we obtain

$$\mathrm{d}s^2 = \mathrm{d}r^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right). \tag{7}$$

Again the restriction to $H^{2,2,+}(0, R)$, i.e., to constraints given by $\vartheta = \frac{\pi}{2}$ gives us the metric tensor corresponding to the arc element

$$\mathrm{d}s^2 = \mathrm{d}r^2 + R^2 \sinh^2\left(\frac{r}{R}\right)\mathrm{d}\varphi^2. \tag{8}$$

Those are metrics corresponding to the constant positive and negative curvatures on the mentioned hypersurfaces in $\mathbb{R}^4$. The corresponding curvature scalars are given by

$$\mathcal{R} = \frac{2}{R^2}, \qquad \mathcal{R} = -\frac{2}{R^2}\cdot \tag{9}$$

Incidentally, let us mention that the a little embarrassing factor two follows from the definition of the curvature tensor we are using here. In another, perhaps more popular definition (difference by a constant factor) the curvature scalars would be simply given by

$$\mathcal{R} = \frac{1}{R^2}, \qquad \mathcal{R} = -\frac{1}{R^2}\cdot \tag{10}$$

Obviously, this is only the cosmetic difference and we are using rather the less popular expression (9).

Finally, let us stress the most important classical case of the vanishing curvature, $R = \infty$, $\mathcal{R} = 0$, i.e., the $\mathbb{R}^3-$space. Substituting this formally to (4), (8) we obtain the Euclidean metric of $\mathbb{R}^3 \subset \mathbb{R}^4$

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \left(\mathrm{d}\vartheta^2 + \sin^2(\vartheta)\,\mathrm{d}\varphi^2\right).$$

Similarly, in the limit $R \to \infty$ equations (2), (3) and (5), (6) reduce respectively to the following $R-$free from

$$x^1 = r\sin(\vartheta)\cos(\varphi), \qquad x^2 = r\sin(\vartheta)\sin(\varphi), \qquad x^3 = r\cos(\vartheta).$$

All this treatment was based on the restriction of the Euclidean $\mathbb{R}^4$-metric to the spherical or pseudo-spherical submanifolds. Let us observe, however that it is possible to use the description in terms of the $\mathbb{R}^3-$space with appropriately defined metric tensor. More precisely, one must say that the description is based on some appropriately defined, subsets of $\mathbb{R}^3$ or their quotients with respect to some identifications. Therefore, $S^3(0, R)$ is represented by the subset of $\mathbb{R}^3$ given by

$$\overline{r} = r(\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta))$$

where $r \in [0, \pi R]$, however with the proviso that all the points on the sphere $S^2(0, \pi R) \subset \mathbb{R}^3$ are identified. Then we can write the corresponding metric tensor Γ_{ij}

$$\mathrm{d}s^2 = \Gamma_{ij}\mathrm{d}r^i\mathrm{d}r^j$$

where

$$\Gamma_{ij} = \frac{R^2}{r^2}\sin^2\frac{r}{R}\delta_{ij} + \frac{1}{r^2}\left(1 - \frac{R^2}{r^2}\sin^2\frac{r}{R}\right) r_i r_j \tag{11}$$

where now $r^i = r_i$ are coordinates of the usual radius vector in $\mathbb{R}^3$. The indices are here moved in the sense of the usual metric tensor δ_{ij} in $\mathbb{R}^3$. Similarly, for the pseudosphere $H^{3,2,+}(0, R)$ we have the following parametrization in terms of $\mathbb{R}^3$ variables

$$\Gamma_{ij} = \frac{R^2}{r^2}\sinh^2\frac{r}{R}\delta_{ij} + \frac{1}{r^2}\left(1 - \frac{R^2}{r^2}\sinh^2\frac{r}{R}\right) r_i r_j \tag{12}$$

where now r^i become the components of the radius-vector in $\mathbb{R}^3$.

It is clear that in the limit transition $R \to \infty$ both become identical with the usual Euclidean form:

$$\Gamma_{ij} = \delta_{ij}. \tag{13}$$

This is identical with the usual radial expression of the metric in $\mathbb{R}^3$.

Obviously, the kinetic energy is given by the usual metric-based formula

$$T = \frac{m}{2}\Gamma_{ij}\frac{\mathrm{d}r^i}{\mathrm{d}t}\frac{\mathrm{d}r^j}{\mathrm{d}t}.$$

Let us also mention that the analytical formula is valid as well for the elliptic space, i.e., in the group language on the orthogonal group

$$\mathrm{SO}(3,\mathbb{R}) = \mathrm{SU}(2)/\mathbb{Z}_2, \qquad \mathbb{Z}_2 = \{I_2, -I_2\}.$$

The difference is that this group is doubly-connected. The parameter range is $r \in [0, \pi R/2]$ and the antipodal points on the sphere $S^2(0, \pi R/2)$ are identified. This leads to certain differences in phase portraits and quantum spectra, even in the case of Bertrand potentials.

For the usual "non-magnetic" Lagrangians of the form

$$L = T - V(\overline{r})$$

the corresponding Legendre transformation of TQ on the T^*Q is given by

$$p_i = \frac{\partial T}{\partial \dot{r}_i} = m\Gamma_{ij}\frac{\mathrm{d}r^j}{\mathrm{d}t}\cdot$$

In the case of the usual rigid body in $\mathbb{R}^3$, this formula becomes, after usual Kronecker-delta identification of the linear space $\mathbb{R}^3$ and its dual the following expression written down in the vector form

$$\overline{p} = m\frac{R^2}{r^2}\sin^2\frac{r}{R}\frac{\mathrm{d}\overline{r}}{\mathrm{d}t} + \frac{m}{r^2}\left(1 - \frac{R^2}{r^2}\sin^2\frac{r}{R}\right)\left(\overline{r}\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\right)\overline{r} \tag{14}$$

when $Q = S^3(0, R)$. In Lobatchevski space $H^{3,2+}(0, R)$ this becomes the same formula with trigonometric functions replaced by the corresponding hyperbolic functions

$$\overline{p} = m\frac{R^2}{r^2}\sinh^2\frac{r}{R}\frac{\mathrm{d}\overline{r}}{\mathrm{d}t} + \frac{m}{r^2}\left(1 - \frac{R^2}{r^2}\sinh^2\frac{r}{R}\right)\left(\overline{r}\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\right)\overline{r} \tag{15}$$

where for any $\mathbb{R}^3$ vectors $\overline{a}\cdot\overline{b}$ denotes the Cartesian scalar product

$$\overline{a}\cdot\overline{b} = \delta_{kl}a^k b^l.$$

It is clear that for $R \to \infty$ the formulas (14), (15) reduce to the usual Euclidean formula in $\mathbb{R}^3$

$$\overline{p} = m\frac{\mathrm{d}\overline{r}}{\mathrm{d}t}$$

because the both metrics reduce then to the Euclidean one just (13). The isometry groups of the metrics (11), (12), (13) are given by: $\mathrm{SO}(4,\mathbb{R})$, $\mathrm{SO}(1,3)$, $E(4,\mathbb{R}) = \mathrm{SO}(3,\mathbb{R}) \underset{\sim}{\times} \mathbb{R}^3$. The last, non-semisimple group is the semi-direct product of the rotation group and the translation group $\mathbb{R}^3$ acting in itself. In all cases the isotropy subgroup of the pole $r = 0$ is given by $\mathrm{SO}(3,\mathbb{R})$ as the group of rotations of the "rotation vector" $\overline{r}$. The corresponding generators of the phase space of variables $(\overline{r}, \overline{p})$ are given by the vector

$$\overline{L} = \overline{r} \times \overline{p}$$

where obviously the following Poisson brackets are satisfied

$$\{L_i, L_j\} = \varepsilon_{ikj} L_k$$

where obviously the moving of indices is meant in the sense of the Kronecker delta, just to be in accord with summation convention.

Expressing $\overline{L}$ in terms of velocities in the spherical and pseudosphical case we apparently loose the Euclidean structure

$$\overline{L} = m\overline{r} \times \overline{p}$$

because then we have respectively

$$\overline{L} = m\frac{R^2}{r^2}\sin^2\frac{r}{R}\overline{r} \times \frac{\mathrm{d}\overline{r}}{\mathrm{d}t}, \qquad \overline{L} = m\frac{R^2}{r^2}\sinh^2\frac{r}{R}\overline{r} \times \frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\cdot$$

Obviously, when $R \to \infty$, the both expressions asymptotically approach the usual Euclidean expression

$$m\overline{r} \times \frac{\mathrm{d}\overline{r}}{\mathrm{d}t}\cdot$$

From now on we concentrate on the spherically-invariant dynamical models, i.e., on the motion under isotropic, $\mathrm{SO}(3, \mathbb{R})$-invariant potentials. We mean of course the $\mathrm{SO}(3, \mathbb{R})$-action on the "radius vector" $\overline{r}$. Therefore, the Lagrangian will be given by

$$L = T - V(r)$$

where T is as in (7) and V depends only on the distance r of the center of forces. The generators $\overline{L}$ are again constants of motion and so is their direction. And although $\overline{r} \times \frac{\mathrm{d}\overline{r}}{\mathrm{d}t}$ is not any longer the constant of motion, its direction still is conserved, because the vector $\overline{L}$ is collinear with it. And this implies that the motion is planar in the space of $\overline{r}-s$, i.e., it is always placed within some plane depending on initial conditions. Therefore, for any central V the motion is at least once degenerate. Let us denote the canonical momenta to r, ϑ, φ, by $p_r, p_\vartheta, p_\varphi$ respectively. Performing the Legendre transformation and expressing all dynamical quantities by $(r, \vartheta, \varphi, p_r, p_\vartheta, p_\varphi)$ we obtain the following formulas for involutive system of constants of motion characteristic for the rotational invariance: In the spherical manifold $S^3(0,R)$ they are given by

$$\begin{aligned}
L_3 &= p_\varphi = mR^2 \sin^2 \frac{r}{R} \frac{\mathrm{d}\varphi}{\mathrm{d}t} \\
\overline{L}^2 &= \overline{L} \cdot \overline{L} = {p_\vartheta}^2 + \frac{{p_\varphi}^2}{\sin^2 \vartheta} = m^2 R^4 \sin^4 \frac{r}{R} \left(\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t} \right)^2 + \sin^2 \vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} \right)^2 \right) \\
H &= \frac{1}{2m} \left({p_r}^2 + \frac{L^2}{R^2 \sin^2 \frac{r}{R}} \right) + V(r) \\
&= \frac{m}{2} \left(\left(\frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 + \frac{L^2}{m^2 R^2 \sin^2 \frac{r}{R}} \right) + V(r) \, .
\end{aligned}$$

Similarly, in the pseudospherical case we replace the above trigonometric functions by the corresponding hyperbolic ones

$$\begin{aligned}
L_3 &= p_\varphi = mR^2 \sinh^2 \frac{r}{R} \frac{\mathrm{d}\varphi}{\mathrm{d}t} \\
\overline{L}^2 &= \overline{L} \cdot \overline{L} = {p_\vartheta}^2 + \frac{{p_\varphi}^2}{\sin^2 \vartheta} = m^2 R^4 \sinh^4 \frac{r}{R} \left(\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t} \right)^2 + \sin^2 \vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} \right)^2 \right) \\
H &= \frac{1}{2m} \left({p_r}^2 + \frac{L^2}{R^2 \sinh^2 \frac{r}{R}} \right) + V(r) \\
&= \frac{m}{2} \left(\left(\frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 + \frac{L^2}{m^2 R^2 \sinh^2 \frac{r}{R}} \right) + V(r) \, .
\end{aligned}$$

And finally, in the Euclidean case, when $R \to \infty$, those formulas asymptotically become

$$\begin{aligned}
L_3 &= p_\varphi = mr^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} \\
\overline{L}^2 &= \overline{L} \cdot \overline{L} = {p_\vartheta}^2 + \frac{{p_\varphi}^2}{\sin^2 \vartheta} = m^2 r^4 \left(\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}t} \right)^2 + \sin^2 \vartheta \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} \right)^2 \right) \\
H &= \frac{1}{2m} \left({p_r}^2 + \frac{L^2}{r^2} \right) + V(r) = \frac{m}{2} \left(\left(\frac{\mathrm{d}r}{\mathrm{d}t} \right)^2 + \frac{L^2}{m^2 r^2} \right) + V(r) \, .
\end{aligned}$$

Obviously, for any of those models we have the flowing integrability condition

$$\{L_3, L\} = 0, \qquad \{L_3, H\} = 0, \qquad \{L, H\} = 0.$$

Obviously, on the classical level any of the corresponding motions is flat in $\mathbb{R}^3$ and may be discussed, e.g., on the plane $\vartheta = \pi/2$, i.e., on the $x, y-$plane.

However, before going any further, let us stress the conformal flatness of both $\mathrm{SO}(3,R)$ and $H^{3,2+}(0,R)$. Namely, performing on them respectively the following transformations

$$\overline{\xi} = a\tan\left(\frac{r}{2R}\right)\frac{\overline{r}}{r}, \qquad \overline{\xi} = a\tanh\left(\frac{r}{2R}\right)\frac{\overline{r}}{r} \tag{16}$$

we obtain for the metric tensor the following formulas

$$\begin{aligned} S^3(0,R): \quad \mathrm{d}s^2 &= \frac{4R^2a^2}{(a^2+\xi^2)^2}\left(\mathrm{d}\xi^2+\xi^2\left(\mathrm{d}\vartheta^2+\sin^2(\vartheta)\,\mathrm{d}\varphi^2\right)\right) \\ H^{3,2,+}(0,R): \quad \mathrm{d}s^2 &= \frac{4R^2a^2}{(a^2-\xi^2)^2}\left(\mathrm{d}\xi^2+\xi^2\left(\mathrm{d}\vartheta^2+\sin^2(\vartheta)\,\mathrm{d}\varphi^2\right)\right). \end{aligned}$$

Substituting here the standard value $a = R$, we obtain respectively

$$\begin{aligned} S^3(0,R): \quad \mathrm{d}s^2 &= \frac{4}{\left(1+\frac{\xi}{R^2}^2\right)^2}\left(\mathrm{d}\xi^2+\xi^2\left(\mathrm{d}\vartheta^2+\sin^2(\vartheta)\,\mathrm{d}\varphi^2\right)\right) \\ H^{3,2,+}(0,R): \quad \mathrm{d}s^2 &= \frac{4}{\left(1-\frac{\xi}{R^2}^2\right)^2}\left(\mathrm{d}\xi^2+\xi^2\left(\mathrm{d}\vartheta^2+\sin^2(\vartheta)\,\mathrm{d}\varphi^2\right)\right). \end{aligned}$$

This makes obvious the mutual relationship between the three constant curvature spaces: $S^3(O,R)$, $H^{3,2,+}(O,R)$, and $\mathbb{R}^3$ in three dimensions. And the flatness of motion corresponds to the one-fold degeneracy of all systems with the radial potentials $V(r)$.

In the classical case, after the two-dimensional reduction to $\vartheta = \pi/2$ we obtain the following dynamical systems on the sphere, pseudosphere and the Euclidean plane

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{\frac{2}{m}\left(E-V(r)\right)-\frac{M^2}{m^2R^2}\sin^{-2}\left(\frac{r}{R}\right)}, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{M}{mR^2}\sin^{-2}\left(\frac{r}{R}\right) \tag{17}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{\frac{2}{m}\left(E-V(r)\right)-\frac{M^2}{m^2R^2}\sinh^{-2}\left(\frac{r}{R}\right)}, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{M}{mR^2}\sinh^{-2}\left(\frac{r}{R}\right) \tag{18}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(\frac{2}{m}\left(E-V\right)-\frac{M^2}{m^2r^2}\right)^{\frac{1}{2}}, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{M}{mr^2} \tag{19}$$

where, obviously, the parameters $E,\ M$ denote the fixed values of the constants of motion $H,\ L_3$.

Let us introduce the concept of the effective potential, i.e., superposition of the usual potential and centrifugal one

$$V_{\text{eff}} = V + \frac{M^2}{2mR^2} \sin^{-2}\left(\frac{r}{R}\right) \tag{20}$$

$$V_{\text{eff}} = V + \frac{M^2}{2mR^2} \sinh^{-2}\left(\frac{r}{R}\right) \tag{21}$$

$$V_{\text{eff}} = V + \frac{M^2}{2mr^2}. \tag{22}$$

Then just as in the usual central motion in $\mathbb{R}^3$ we obtain the following dynamical systems

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm \frac{1}{\sqrt{\frac{2}{m}\left(E - V_{\text{eff}}\right)}} \tag{23}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{M}{mR^2} \sin^{-2}\left(\frac{r}{R}\right), \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{M}{mR^2} \sinh^{-2}\left(\frac{r}{R}\right), \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{M}{mr^2} \tag{24}$$

respectively in the spherical cases (20)–(22).

Equation (23) may be in principle solved in terms of quadratures by inverting the obtained function and substituting it to (24) we can also in principle integrate it and find the time dependence of φ.

The information about the orbit itself, is contained in the product of equations (23), (24)

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \pm \frac{M}{mR^2} \sin^{-2}\left(\frac{r}{R}\right) \left(\frac{2}{m}\left(E - V_{\text{eff}}\right)\right)^{-\frac{1}{2}} \tag{25}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \pm \frac{M}{mR^2} \sinh^{-2}\left(\frac{r}{R}\right) \left(\frac{2}{m}\left(E - V_{\text{eff}}\right)\right)^{-\frac{1}{2}} \tag{26}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \pm \frac{M}{mr^2} \left(\frac{2}{m}\left(E - V_{\text{eff}}\right)\right)^{-\frac{1}{2}} \tag{27}$$

with appropriately substituted expressions (20)–(22). Let us stress that the $\pm$ signs in (17), (18), (19), (23), (25), (26), (27) depends on the phase motion. Let us observe that the main part of the dynamics is described by the radial equation

$$m\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{\mathrm{d}}{\mathrm{d}r} V_{\text{eff}}.$$

Therefore, for the exceptional, circular orbits we obtain the following relationship between their radius ρ and the angular momentum M

$$\frac{M^2 \cos\frac{\rho}{R}}{mR^3 \sin^3\frac{\rho}{R}} = V'(\rho), \qquad \frac{M^2 \cosh\frac{\rho}{R}}{mR^3 \sinh^3\frac{\rho}{R}} = V'(\rho), \qquad \frac{M^2}{m\rho^3} = V'(\rho)$$

respectively in the spherical, pseudospherical and planar case.

It is seen from the formula (23) that in the case of finite (bounded) motion, the radial period is given by

$$T_{\rm rad}(E, M) = 2\int_{r_{\min}}^{r_{\max}} \frac{\mathrm{d}r}{\sqrt{\frac{2}{m}\left(E - V_{\rm eff}(r, M)\right)}}$$

where $r_{\min},\ r_{\max}$ denote respectively the minimal and maximal value of r in a given motion. Similarly, the formulas (24) imply that the angular period equals respectively

$$\begin{aligned}
T_{\rm ang} &= \frac{mR^2}{M}\int_0^{2\pi} \sin^2\frac{r}{R}\mathrm{d}\varphi\\
T_{\rm ang} &= \frac{mR^2}{M}\int_0^{2\pi} \sinh^2\frac{r}{R}\mathrm{d}\varphi\\
T_{\rm ang} &= \frac{m}{M}\int_0^{2\pi} r^2\mathrm{d}\varphi
\end{aligned}$$

where the functions $r(\varphi)$ is obtained by integration of (25), (26), (27) and inversion of the resulting formula for $\varphi(r)$ to the function $r(\varphi)$. The motion, is periodic if $T_{\rm ang}$, $T_{\rm rad}$ are commensurable, i.e., if for some integrals $k, l \in \mathbb{Z}$ the following holds

$$kT_{\rm rad} + lT_{\rm ang} = 0.$$

This holds when $\Phi/2\pi \in \mathbb{Q}$, i.e., when it is a rational number.

It turns out that in spite of its obvious attractive features the conformal mapping (16) does not help us with finding the $S^3(0, R)$ and $H^{3,2,+}(0, R)$ counterparts of the Bertrand potentials (1) in $\mathbb{R}^3$. Nevertheless, there are another mappings which are helpful here. Namely, let us introduce the following new variables

$$\begin{aligned}
S^3(0, R): &\qquad y = \frac{1}{R}\cot\frac{r}{R}\\
H^{3,2,+}(0, R): &\qquad y = \frac{1}{R}\coth\frac{r}{R}.
\end{aligned}$$

In the first case, when r runs between $[0, \pi R]$, y runs over range $[+\infty, -\infty]$. In the second, hyperbolic case when $r \in [0, \infty]$, then y runs over the semi-infinite segment $[+\infty, \frac{1}{R}]$.

Then in spherical and pseudo-spherical case we obtain respectively

$$S^3(0,R): \quad \varphi(r) = \varphi[y] = \pm\frac{L}{m}\int\frac{\mathrm{d}y}{\sqrt{-\frac{M^2}{m^2}y^2+\frac{2}{m}\left(\mathcal{E}_s - V\right)}} \tag{28}$$

$$H^{3,2,+}(0,R): \quad \varphi(r) = \varphi[y] = \pm\frac{L}{m}\int\frac{\mathrm{d}y}{\sqrt{-\frac{M^2}{m^2}y^2+\frac{2}{m}\left(\mathcal{E}_h - V\right)}} \tag{29}$$

where the following abbreviation is used

$$\mathcal{E}_s = E - \frac{M^2}{2mR^2}, \qquad \mathcal{E}_h = E + \frac{M^2}{2mR^2}\cdot \tag{30}$$

Obviously, in the special case of the Euclidean space we have $\Gamma_{ij} = \delta_{ij}$, $R = \infty$, $y = 1/r$, and then

$$\varphi(r) = \varphi[y] = \pm\frac{L}{m}\int\frac{\mathrm{d}y}{\sqrt{-\frac{M^2}{m^2}y^2+\frac{2}{m}\left(E - V\right)}}\cdot \tag{31}$$

The three formulas (28), (29), (31) are almost identical apart that the energy variable E in the Euclidean case is replaced respectively by $\mathcal{E}_s,\ \mathcal{E}_h$.

Formally and locally the formulas (28), (29), (31) are identical, although of course there are serious differences from the global topological point of view. The energy variable E in (31) is to be replaced by the expression (30) depending on the curvature scalars and on the angular momentum variables. In the geodetic case, when $V = 0$ the formulas (28)–(31) establish the projective mappings acting between manifolds $\mathbb{R}^3,\ S^3(0,R),\ H^{3,2,+}(0,R)$. Let us stress that the mentioned projective mappings act locally, without preserving the affine parameter. They simply map the arcs of the geodetic curves onto the same arcs in other spaces, but, without preserving the affine parameters.

This means that we have the following Bertrand type potentials in $S^3(0,R)$ (more generally, in $S^n(0,R)$, and classically, as a matter of fact, on $S^2(0,R)$), and in $H^{3,2,+}(0,R)$ (more generally, in $H^{n,2,+}(0,R)$, and, as a matter of fact, in $H^{2,2,+}(0,R)$)

$$V_{\text{osc}} = \frac{kR^2}{2}\tan^2\frac{r}{R} = \frac{k}{2}\frac{1}{y^2}, \qquad V_{\text{Coul}} = -\frac{\alpha}{R}\cot\frac{r}{R} = -\alpha y$$

$$V_{\text{osc}} = \frac{kR^2}{2}\tanh^2\frac{r}{R} = \frac{k}{2}\frac{1}{y^2}, \qquad V_{\text{Coul}} = -\frac{\alpha}{R}\coth\frac{r}{R} = -\alpha y.$$

In the Euclidean space we have respectively

$$V_{\text{osc}} = \frac{kr^2}{2} = \frac{k}{2}\frac{1}{y^2}, \qquad V_{\text{Coul}} = -\frac{\alpha}{r} = -\alpha y.$$

3. Some General Features of Motion

In the spherical case $S^3(0,R)$ we can avoid using the statement "all bounded orbits". The point is that $S^3(0,R)$ is compact, and all orbits, moreover all curves are bounded. However, in non-compact Lobatchevski space $H^{3,2,+}(0,R)$ there exist upper bounds of the potential energy even in the isotropic degenerate oscillator case. In particular, there exists a kind of ionization threshold and continuum of non-bounded orbits above it. on the quantum level this leads to the existence of continuous spectrum above the discrete system of wave functions describing finite quantum motion. There are also some problems with the Coulomb-Kepler model in the elliptic space $\mathrm{SU}(2)/\mathbb{Z}_2$. Namely, it turns out that there are some doubtful points concerning sufficiently large orbits. In the case of Coulomb problem in $S^3(0,R)$ there is no need to assume that α is positive. The potential $V(r) = -\frac{\alpha}{R}\cot\frac{r}{R}$ is as Green function of the Laplace equation corresponding to the Laplace-Beltrami operator on $S^3(O,\mathbb{R}) \simeq \mathrm{SU}(2)$. When $\alpha > 0$, the northern pole $r = 0$ is an attractive singularity, whereas the southern pole $r = \pi R$ is the repulsive singularity. When $\alpha < 0$, their roles are reversed: $r = 0$ is repulsive, and $\alpha = \pi R$ is attractive. This has also some consequences in the existence of circular orbits. When α is positive, there exist circular orbits with $r < \frac{\pi R}{2}$, but not with $r > \frac{\pi R}{2}$. When α is negative, again we obtain opposite picture: there are closed circular orbits with $r > \frac{\pi R}{2}$, but there are no ones with $r < \frac{\pi R}{2}$. The opposite poles $r = 0$, $r = \pi R$, any other pair of opposite poles, form something like the electrostatic dipole. Let us remind the proof of Landau and Lifshitz that in a closed Universe the total electric charge must vanish.The dipole property of Green function is a good example.

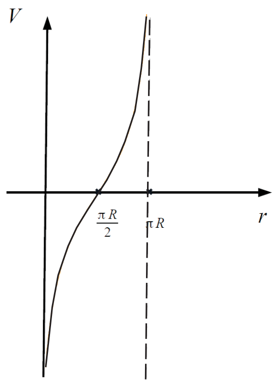

Figure 1. Coulomb potential.

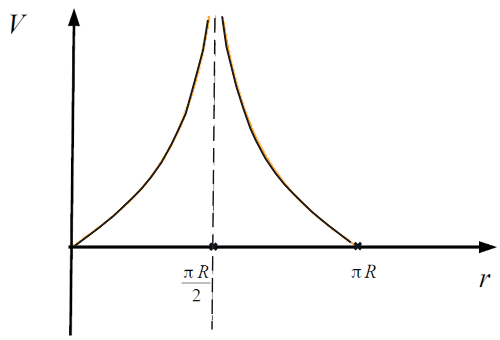

Figure 2. Isotropic oscillator potential.

It is clear from this picture (Fig. 1) that for $\alpha \neq 0$ the Columb potential is not smoothly projectable to the elliptic space $\mathrm{SU}(2)/\mathbb{Z}_2$.

The degenerate potential of the radial oscillator potential has an inpenetrable potential barrier at $r = \frac{\pi R}{2}\cdot$ It is invariant under $\mathbb{Z}_2$, i.e., under the antipodal identification. Therefore, the corresponding dynamical model projects smoothly to be Riemannian elliptic space $\mathrm{SU}(2)/\mathbb{Z}_2$. So, roughly speaking, for $k \neq 0$ it separates into two mutually isomorphic problems in $[0, \pi R/2[$ and $]\pi R/2, \pi R]$ (Fig. 2). The points $r = 0,\ r = \pi R$ are attractive centers if $k > 0$ and the sphere $k = \pi R/2$ in analogy to the harmonic oscillator in $\mathbb{R}^3$ or $\mathbb{R}^2$ corresponds to infinity.

The Coulomb potential on the pseudosphere (Fig. 3) has the diagram with negative vertical asymptote at $r = 0$ and the horizontal asymptote given by the value $-\alpha/R$.

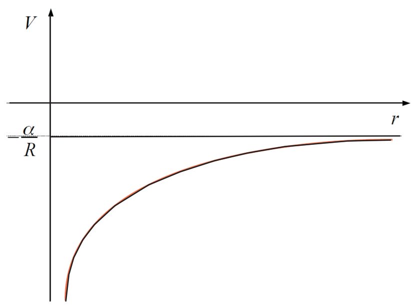

Figure 3. Coulomb potential on the pseudosphere.

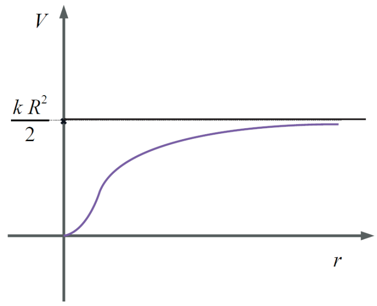

Figure 4. Degenerate oscillator on the pseudosphere.

The degenerate oscillator on the pseudosphere (Fig. 4)(Lobatchevski space) has the diagram parabolic at $r = 0$ and having the horizontal asymptote $kR^2/2$. We reject the apparently natural temptation of gauging the pseudospherical potentials by additive corrections which would make them functions vanishing in the infinite values of r, like

$$V(r) = -\frac{\alpha}{R}\coth\frac{r}{R} + \frac{\alpha}{R}, \qquad V(r) = \frac{kR^2}{2}\tanh^2\frac{r}{R} - \frac{kR^2}{2}\cdot$$

For certain reason it is better to preserve our earlier conventions.

One can show that for the Coulomb problem on $S^3(0, R)$ there is the following condition for E, L; let us notice that having in view the special case $n = 3$ we retain to the symbol L^2 instead of M^2

$$E \geq -\frac{m\alpha^2}{2L^2} + \frac{L^2}{2mR^2}\cdot \tag{32}$$

The special case of equality corresponds to circular orbits. The resulting function $L \to E(L)$ has in its diagram the negative vertical asymptote at $L = 0$, and becomes infinite when $L \to \infty$. The diagram of function intersects the L-axis at $L_0 = \sqrt{m\alpha R}$.

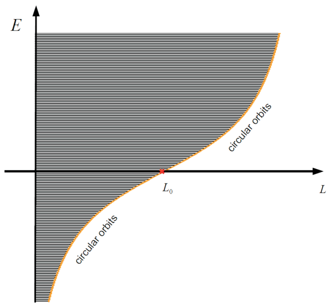

Figure 5. Spherical Coulomb problem.

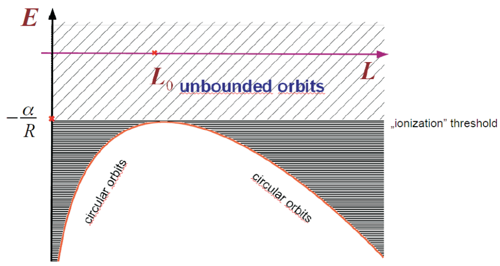

Figure 6. Pseudospherical Coulomb problem.

In the pseudospherical case instead of (32) we obtain

$$E \geq -\frac{m\alpha^2}{2L^2} - \frac{L^2}{2mR^2}.$$

Let us now describe the relationship for the degenerate isotropic oscillator. When the configuration space is spherical $S^3(0, R)$ then we obtain

$$E \geq L\omega_0 + \frac{L^2}{2mR^2}, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

The limiting case of equality we obtain here as the condition for circular orbits

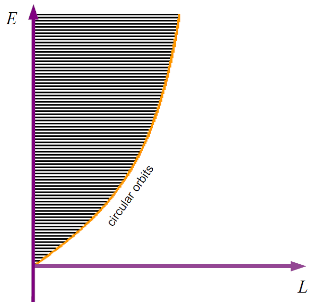

Figure 7. Spherical oscillator.

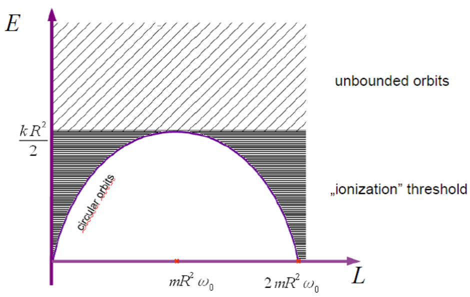

Figure 8. Pseudospherical oscillator.

In the pseudospherical configuration space $H^{3,2,+}(0, R)$ one obtains a more complex picture. Namely, there is a phenomenon of "saturation". Analytically, we obtain then inequality

$$E \geq L\omega_0 - \frac{L^2}{2mR^2}, \qquad \omega_0 = \sqrt{\frac{k}{m}}\cdot$$

Therefore, it turns out that there appears a continuum of unbounded classical orbits above the threshold (Fig. 8)

$$E = \frac{kR^2}{2}\cdot$$

One the quantum level of description, surprisingly enough, there appears a continuous spectrum above the "usual" discrete oscillator spectrum.

It is interesting to notice that the oscillatory and Coulomb potentials V_{Coul}, V_{osc} are related to each other just as their Euclidean counterparts, and namely

$$V_{\mathrm{osc}}\, {V_{\mathrm{Coul}}}^2 = \frac{\varkappa\alpha^2}{2}\cdot$$

Nevertheless, it must be stressed that the corresponding projective mapping fails to be an isomorphism of the corresponding dynamical systems, because the Killing metrics on non-semi-simple groups have a non-vanishing curvature.

4. Hamilton-Jacobi Equations, Action-Angle Variables and the Bohr-Sommerfeld Quantization Rule

It is clear that the Hamilton-Jacobi equations for the isotropic systems in spherical, pseudospherical and flat three-dimensional universes have respectively the following forms

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{R^2}\sin^{-2}\frac{r}{R}\left(\frac{\partial S}{\partial \vartheta}\right)^2 + \frac{1}{R^2}\sin^{-2}\frac{r}{R}\sin^{-2}\vartheta\left(\frac{\partial S}{\partial \varphi}\right)^2 = 2m\,(E - V) \tag{33}$$

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{R^2}\sinh^{-2}\frac{r}{R}\left(\frac{\partial S}{\partial \vartheta}\right)^2 + \frac{1}{R^2}\sinh^{-2}\frac{r}{R}\sin^{-2}\vartheta\left(\frac{\partial S}{\partial \varphi}\right)^2 = 2m\,(E - V) \tag{34}$$

$$\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \vartheta}\right)^2 + \frac{1}{r^2}\sin^{-2}\vartheta\left(\frac{\partial S}{\partial \varphi}\right)^2 = 2m\,(E - V)\,. \tag{35}$$

Using the Stäckel theorem one can postulate a general class of potentials for which those equations may be solved by the separation of variables procedure. However, in this paper we are interested only in the spherical potential, when V depends on the vector $\overline{r}$ only through its magnitude, $V(r)$. And now we are looking for the solutions given by

$$S(r,\vartheta,\varphi) = S_r(r) + S_\vartheta(\vartheta) + S_\varphi(\varphi).$$

The separated equations have the form

$$\left(\frac{\mathrm{d}S_\vartheta}{\mathrm{d}\vartheta}\right)^2 + \sin^{-2}(\vartheta)\left(\frac{\partial S_\varphi}{\partial\varphi}\right)^2 = {\alpha_\vartheta}^2 = L^2, \qquad \frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi} = \alpha_\varphi = M$$

and the following radial equations for three cases mentioned above, i.e., (33), (34), (35)

$$\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 - 2m\,(E - V) = -\frac{\alpha_\vartheta^2}{R^2\sin^2\left(\frac{r}{R}\right)} \tag{36}$$

$$\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 - 2m\,(E - V) = -\frac{\alpha_\vartheta^2}{R^2\sinh^2\left(\frac{r}{R}\right)} \tag{37}$$

$$\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 - 2m\,(E - V) = -\frac{\alpha_\vartheta^2}{r^2}. \tag{38}$$

Obviously, those are ordinary equations written in the quadrature forms. The symbols $M,\ L$ are just the previously introduced length of the angular momentum and the projection of angular momentum on the z-axis. They are related to the integration constants $\alpha_\varphi,\ \alpha_\vartheta$ and to the action variables $J_\varphi,\ J_\vartheta$ as follows

$$J_\varphi = \oint p_\varphi \mathrm{d}\varphi = \oint M\mathrm{d}\varphi = 2\pi M = 2\pi\alpha_\varphi$$

$$J_\vartheta = \oint p_\vartheta \mathrm{d}\vartheta = \oint \pm\sqrt{\alpha_\vartheta^2 - \frac{{\alpha_\varphi}^2}{\sin^2(\vartheta)}}\mathrm{d}\vartheta = 2\pi\left(\alpha_\vartheta - \alpha_\varphi\right) = 2\pi\left(L - M\right).$$

Therefore

$$J_\vartheta + J_\varphi = 2\pi L = 2\pi\alpha_\vartheta.$$

Substituting this to (36), (37), (38) we obtain respectively

$$\begin{aligned} J_r &= \oint p_r \mathrm{d}r = \oint \pm\sqrt{2m\left(E - V(r)\right) - \frac{\left(J_\vartheta + J_\varphi\right)^2}{4\pi^2R^2\sin^2\left(\frac{r}{R}\right)}}\mathrm{d}r \\ J_r &= \oint p_r \mathrm{d}r = \oint \pm\sqrt{2m\left(E - V(r)\right) - \frac{\left(J_\vartheta + J_\varphi\right)^2}{4\pi^2R^2\sinh^2\left(\frac{r}{R}\right)}}\mathrm{d}r \\ J_r &= \oint p_r \mathrm{d}r = \oint \pm\sqrt{2m\left(E - V(r)\right) - \frac{\left(J_\vartheta + J_\varphi\right)^2}{4\pi^2 r^2}}\mathrm{d}r. \end{aligned} \tag{39}$$

As usual in the radial action-angle variables, the use of the $\pm$ signs depends on the segment of integration line. Let us observe that the action variables J_φ, J_ϑ

enter here only trough their sum $J_\varphi + J_\vartheta = 2\pi L$. This corresponds exactly to the one-fold degeneracy of all spherically-invariant ($\mathrm{SO}(3,\mathbb{R})$-invariant) dynamical models. In any case we can write for all such systems the following formulas

$$J_r = \oint p_r \mathrm{d}r = \oint \pm\sqrt{2m\,(E - V(r)) - \frac{L^2}{R^2\sin^2\left(\frac{r}{R}\right)}}\mathrm{d}r$$

$$J_r = \oint p_r \mathrm{d}r = \oint \pm\sqrt{2m\,(E - V(r)) - \frac{L^2}{R^2\sinh^2\left(\frac{r}{R}\right)}}\mathrm{d}r$$

$$J_r = \oint p_r \mathrm{d}r = \oint \pm\sqrt{2m\,(E - V(r)) - \frac{L^2}{r^2}}\mathrm{d}r.$$

This one-fold degeneracy corresponds exactly to the flatness of motion resulting from the conservation of angular momentum. Let us now discuss the doubled degeneracy of our Bertrand potentials. We begin from the classical Bertrand models in the flat space. Then, as known for ages, substituting in (39) the well-known Bertrand potentials in $\mathbb{R}^3$, we obtain respectively after calculating the J_r-integral and solving the result with respect to E the following results

$$E_{\text{Coul}} = -\frac{2m\pi^2\alpha^2}{(J_r + J_\vartheta + J_\varphi)^2} \tag{40}$$

$$E_{\text{osc}} = \nu\,(2J_r + J_\vartheta + J_\varphi)^2\,, \qquad \nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\cdot \tag{41}$$

Obviously, the second formula is always valid, whereas the first one only below the dissociation threshold, when r undergoes a finite motion. It is seen that as usual in the case of total degeneracy, there is only one essential action variable, proportional to the combination of primary action variables with integer coefficients. But again there is an essential differences between two cases. Indeed, in the Coulomb and isotropic oscillator problems two different combination occur, namely

$$J = J_r + J_\vartheta + J_\varphi, \qquad J = 2J_r + J_\vartheta + J_\varphi.$$

It is not accidental, namely it follows from some deeply geometric differences. In the Coulomb case one radial period corresponds to one angular period. The reason is that the non-moving centers of orbits are placed at their foci. In oscillatory case, orbits have the non-moving centers just in their geometric centers of symmetry. In spite that the orbits in both models are ellipses (only in special cases they are circles), this is an evident distinction.

Let us now present the results of the similar procedure in the spherical Coulomb problem and spherical oscillator; obviously, by “spherical” we mean the model on $S^3(0,R)$. Performing a similar, although much more complicated, calculation as the above - mentioned for $\mathbb{R}^3$, we obtain the following statement

$$E = -\frac{2\pi^2 m\alpha^2}{\left(J_r + J_\vartheta + J_\varphi\right)^2} + \frac{\left(J_r + J_\vartheta + J_\varphi\right)^2}{8\pi^2 mR^2}\cdot \tag{42}$$

Let us remind, the Kepler-Coulomb problem is based on the potential

$$V = -\frac{\alpha}{R}\cot\left(\frac{r}{R}\right)\cdot$$

Obviously, for the geodetic problem on $S^3(0, R)$, when $\alpha = 0$ we obtain

$$E = \frac{\left(J_r + J_\vartheta + J_\varphi\right)^2}{8\pi^2 mR^2}\cdot \tag{43}$$

Just as in the flat-space Coulomb problem, there is exactly one radial turning point for one angular $\varphi-$period. Because of this one can introduce the following effective action variable

$$J = J_r + J_\vartheta + J_\varphi$$

and

$$E = -\frac{2\pi^2 m\alpha^2}{J^2} + \frac{J^2}{8\pi^2 mR^2}\cdot$$

For the isotropic degenerate oscillator

$$V = \frac{kR^2}{2}\tan^2\left(\frac{r}{R}\right)$$

we obtain

$$E = \frac{1}{2\pi}\omega_0\left(2J_r + J_\vartheta + J_\varphi\right) + \frac{\left(2J_r + J_\vartheta + J_\varphi\right)^2}{8\pi^2 mR^2} \tag{44}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} = 2\pi\nu_0.$$

Here, just as in the Euclidean case, there is one turning point of each kind per one angular circulation. This is the reason that the coefficient at J_r equals the doubled coefficients at $J_\vartheta,\ J_\varphi$. According to the standard theory of multiply periodic systems, one can introduce new globally valid action-angle variables such that E depends only on the one action; just the typical property of the completely degenerate systems. The essential action variable is given by

$$E = \nu_0 J + \frac{J^2}{8\pi^2 mR^2}, \qquad J_1 = J = 2J_r + J_\vartheta + J_\varphi \tag{45}$$

This oscillator is roughly speaking , as harmonic as possible, nevertheless, the very topology of degrees of freedom forces it to be anharmonic. There is no isochronic property

$$\nu = \frac{dE}{dJ} = \nu_0 + \frac{J}{4\pi^2 mR^2}$$

i.e., expressing this through E

$$\nu(E) = \sqrt{\nu_0^2 + \frac{E}{2\pi^2 m R^2}}, \qquad \omega(E) = \sqrt{\omega_0^2 + \frac{2E}{mR^2}}.$$

It is seen that anharmonicity becomes relevant for the large values of E, i.e., as expected, for highly excited vibrations.

Let us now discuss the pseudospherical Bertrand models. We begin with the Coulomb problem, when

$$V = -\frac{\alpha}{R}\coth\left(\frac{r}{R}\right).$$

After relatively complicated calculations and after the substitution

$$J = J_r + J_\vartheta + J_\varphi$$

just like in Coulomb problems, we obtain the following expression of energy through the action variables

$$E = -\frac{2\pi^2 m\alpha^2}{J^2} - \frac{J^2}{8\pi^2 mR^2}. \tag{46}$$

Obviously, this is valid below the dissociation threshold

$$\sup V = -\frac{\alpha}{R}.$$

Let us now see what result when we consider the isotropic degenerate oscillator problem,

$$V = \frac{kR^2}{2}\tanh^2\left(\frac{r}{R}\right).$$

It is relatively surprising that this potentials is bounded, namely

$$\sup V = \frac{kR^2}{2} \tag{47}$$

and because this there exists dissociation threshold.

One can show that

$$E = \nu_0 J - \frac{J^2}{8\pi^2 mR^2}, \qquad J = 2J_r + J_\vartheta + J_\varphi \tag{48}$$

where

$$\nu_0 = \frac{1}{2\pi}\omega_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

Just as in $\mathbb{R}^3$and $S^3(0,R)$ the mutual weights of $J_r,\ J_\vartheta,\ J_\varphi$ in J are $2:1:1$. There is no isochronism property and one can show that the frequency ν depends on the energy E as follows

$$\nu(E) = \sqrt{\nu_0^2 - \frac{E}{2\pi^2 m R^2}}, \qquad \omega(E) = \sqrt{\omega_0^2 - \frac{2E}{mR^2}}.$$

Obviously, all those statements are valid only for energy values below the dissociation $\sup V = kR^2/2$. Let us now compare qualitatively the formulas (40), (41) with (42), (43), (44) and with (43), (46), (48).

Something really strange, mysterious and surprising appears. This may be perhaps a special case of some more general regularity. Those are some details of the dependence of energy on the action variables. Namely, it turns that for the constant curvature spaces $S^3(0,R)$, $H^{3,2+}(0,R)$ the functions $E(J)$ are just the sums of the corresponding purely geodetic (potential-free) terms in manifolds, linear in their curvature scalars (10). Indeed, (42), (44), (46), (48) may be written down respectively as follows

$$\begin{aligned} E[S^3(0,R),\ \mathrm{Coul}] &= E[\mathbb{R}^3,\ \mathrm{Coul}] + E[S^3(0,R),\ \mathrm{Coul, geod}] \\ E[S^3(0,R),\ \mathrm{osc}] &= E[\mathbb{R}^3,\ \mathrm{osc}] + E[S^3(0,R),\ \mathrm{osc, geod}] \\ E[H^{3,2,+}(0,R),\ \mathrm{Coul}] &= E[\mathbb{R}^3,\ \mathrm{Coul}] + E[H^{3,2,+},\ \mathrm{Coul, geod}] \\ E[H^{3,2,+},\ \mathrm{osc}] &= E[\mathbb{R}^3,\ \mathrm{osc}] + E[H^{3,2,+},\ \mathrm{osc, geod}]. \end{aligned}$$

Obviously, all the E-symbols are analytically functions on $\mathbb{R}^3$, at least locally, are roughly speaking, functions of $(L_r,\ L_\vartheta,\ L_\varphi)$

$$\begin{aligned} E[\mathbb{R}^3,\ \mathrm{Coul}](J_r, J_\vartheta, J_\varphi) &= -\frac{2\pi^2 m\alpha^2}{(J_r + J_\vartheta + J_\varphi)^2} \\ E[\mathbb{R}^3,\ \mathrm{osc}](J_r, J_\vartheta, J_\varphi) &= \frac{1}{2\pi}\sqrt{\frac{k}{m}}\,(2J_r + J_\vartheta + J_\varphi) \\ E[S^3(0,R),\ \mathrm{Coul},\ \mathrm{geod}] &= \frac{(J_r + J_\vartheta + J_\varphi)^2}{8\pi^2 mR^2} \\ E[S^3(0,R),\ \mathrm{osc},\ \mathrm{geod}] &= \frac{(2J_r + J_\vartheta + J_\varphi)^2}{8\pi^2 mR^2} && (49) \\ E[H^{3,2,+}(0,R),\ \mathrm{Coul},\ \mathrm{geod}] &= -\frac{(J_r + J_\vartheta + J_\varphi)^2}{8\pi^2 mR^2} && (50) \\ E[H^{3,2,+}(0,R),\ \mathrm{osc},\ \mathrm{geod}] &= -\frac{(2J_r + J_\vartheta + J_\varphi)^2}{8\pi^2 mR^2}\cdot && (51) \end{aligned}$$

It may be easily see that the difference in sign between (47) and (50), and similarly between (49) and (51), apparently vanishes when we introduce the curvature scalars. Namely, in the convention (9) those formulas become respectively

$$\begin{aligned} &\frac{1}{16\pi^2 m}\ \mathcal{R}\ [S^3(0,R)\,(J_r + J_\vartheta + J_\varphi)^2 \\ &\frac{1}{16\pi^2 m}\ \mathcal{R}\ [H^{3,2,+}(0,R)\,(2J_r + J_\vartheta + J_\varphi)^2\,. \end{aligned}$$

It is interesting that there is no continuous transition between the $S^3(0,R)$-harmonic oscillator and its projection to the quotient space $S^3(0,R)/\mathbb{Z}_2$. The geodetic problem is then smoothly projected, but the range of angular variables is reduced from $[0,\ 2\pi]$ to $[0,\ \pi]$. Because of this the action variables are doubled and we obtain finally

$$E = \frac{(J_r + J_\vartheta + J_\varphi)^2}{2\pi^2 m R^2} = \frac{J^2}{2\pi^2 m R^2}.$$

Before going any further with the quantum formulas, let us first discuss what the Bohr-Sommerfeld quantum conditions

$$J = nh = 2\pi n\hbar$$

seem to tell to us. Let us stress that there is only one action variable J to be quantized in the case of Bertrand systems, i.e., completely degenerate ones. When in the system with n degrees of freedom an $(n-k)$-fold degeneracy occurs, i.e., when it is k-periodic, then one can introduce new action-angle variables $\theta^1, \ldots, \theta^n$ and $J_1, \ldots, J_n$ such that the Hamiltonian depends exactly on k of them

$$H = E(J_1, \ldots, J_k)$$

and it is impossible to find any new system of action-angle variables for which E would be dependent on a smaller number of J-s. The basic frequencies

$$\nu^a = \frac{\partial E}{\partial J_a}$$

do not satisfy any system of identities

$$\sum_{a=1}^{k} m_a \nu^a = 0$$

with integer coefficients $m_a \in \mathbb{Z}$.

The action variables $J_1, \ldots, J_k$ are adiabatic invariants, i.e., their increments are small of higher-order with comparison to the time rates of structural parameters defining the system. Unlike this, the remaining action variables J_r, $r = k+1, \ldots, n$ do not have this property. The Bohr-Sommerfeld quantization rules are imposed on J_k and not on J_r. The energy levels are then given by

$$E_{n_1,\ldots n_k} = E(n_1 h, \ldots n_k h)$$

where $n_1, \ldots n_k$ are integers.

This was a small digression, but here we concentrate only on periodic, i.e., twice degenerate Hamiltonian systems.

If we use the former definition of J-variables in our problem, then we have the following result. On $S^3(0, R)$ the Bohr-Sommerfeld spectrum for the isotropic oscillator is given by

$$E_n = n\nu_0 h + \frac{n^2 h^2}{8\pi^2 m R^2} = n\omega_0 \hbar + \frac{n^2 \hbar^2}{2mR^2}, \qquad n \in \{0\} \cup \mathbb{N}. \tag{52}$$

The Coulomb problem on $S^3(0,R)$ has the following Bohr-Sommerfeld spectrum

$$E_n = -\frac{2\pi^2 m\alpha^2}{n^2h^2} + \frac{n^2h^2}{8\pi^2 mR^2} = -\frac{m\alpha^2}{2n^2\hbar^2} + \frac{n^2\hbar^2}{2mR^2}, \qquad n \in \mathbb{N}.$$

The free geodetic motion has obviously the following Bohr-Sommerfeld energy level:

$$E_n = \frac{n^2h^2}{8\pi^2 mR^2} = \frac{n^2\hbar^2}{2mR^2}, \qquad n \in \{0\} \cup \mathbb{N}.$$

It is instructive to write down this formula with the use of half-integer quantum numbers

$$E_n = \frac{2n^2\hbar^2}{mR^2}, \qquad n \in \{0\} \cup \mathbb{N}/2, \qquad \text{i.e.,} \qquad n = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots. \tag{53}$$

It is interesting to note that the free, i.e., geodetic, spectrum is not a limiting case of Coulombian model with $\alpha \to 0$. Indeed, the ground state of the geodetic case has the vanishing energy value, white for the Coulombian ground state one obtains then for finite values of α the quantity

$$\frac{\hbar^2}{2mR^2} - \frac{m\alpha^2}{2\hbar^2}$$

which for finite values of $\hbar$, R does approach zero when $\alpha \to 0$. This fact is astonishing enough when comparing it with the corresponding classical situation, because for the Bohr-Sommerfeld oscillatory case we have good limiting property. So, the classically-mechanical phase portraits seem to suggested something incompatible even with the quasi-classical Bohr-Sommerfeld case.

Let as also mentioned, it is interesting to note that for the geodetic spectrum on the quotient manifold $S^3(0,R)/\mathbb{Z}_2$ the Bohr-Sommerfeld formula (53) seems to work, however with the proviso that only the integer values of n are admitted as quantum numbers.

Let us now discuss briefly the Bertrand models on the Lobatchevski space, i.e., $H^{3,2,+}(0,R)$. In this pseudosphere we have the natural counterparts of the spherical degenerate oscillator and Coulomb problems

$$E_n = n\nu_0 h - \frac{n^2h^2}{8\pi^2 mR^2} = n\omega_0\hbar - \frac{n^2\hbar^2}{2mR^2}, \qquad n \in \{0\} \cup \mathbb{N}. \tag{54}$$

The attractive Coulomb problem has the following Bohr-Sommerfeld spectrum

$$E_n = -\frac{2\pi^2 m\alpha^2}{n^2h^2} - \frac{n^2h^2}{8\pi^2 mR^2} = -\frac{m\alpha^2}{2n^2\hbar^2} - \frac{n^2\hbar^2}{2mR^2}, \qquad n \in \mathbb{N}.$$

Let us observe that those levels automatically are compatible with condition

$$E_n < -\frac{\alpha}{R} = \sup V.$$

Unlike this, in the oscillatory case we must in addition to (54) assume also that n is restricted by

$$n \leq 2\frac{R^2}{\hbar}\sqrt{km}.$$

This is to be compatible with the positive sign of energy. It is interesting that the number of admitted discrete energy levels is finite.

As it was mentioned above, the very fact that the formulas for the $S^3(0,R)$- and $H^{3,2,+}(0,R)$-Bertrand potentials and for the corresponding Bohr-Sommerfeld spectra are so deeply interrelated with the well-known for ages Euclidean expressions and with the geodetic (potential-free) formulas for $S^3(0,R)$, $H^{3,2,+}(0,R)$ as Riemannian spaces, is surprising and marvelous in itself. In particular, it is so due to the additive combination of the Euclidean and geodetic constant-curvature terms. But there is also an additional, philosophical reason for which this is interesting. Namely, the mentioned geodetic terms differ in sign, because they are proportional to the sectional curvatures, $\mathcal{R} = \pm 2/R^2$. Therefore, in principle the hydrogen energy levels, when ideally measured, may contain some information about the global structure of Universe: is it spherical or pseudospherical? (the second possibility would be pleasing for R. Penrose). Of course, this statement, when literally understood may look a bit comic. The value of R, even if finite, is very large, and the resulting effect very small in comparison with any other physical perturbation. Nevertheless, even the very "ideological" link between local and global problems is very exciting.

5. Rigorous Quantization in the Schrödinger Language

First of all, let us remind that in any Riemann manifold (M,g) we are given the canonical measure μ which in coordinates is given by the following local expression:

$$\mathrm{d}\mu(q) = \sqrt{|\det[g_{ij}]|}\mathrm{d}q^1 \ldots \mathrm{d}q^n.$$

Therefore, on the sphere $S^3(0,R)$ and pseudosphere $H^{3,2,+}(0,R)$ the volume elements are given by

$$\begin{aligned}
\mathrm{d}\mu(r,\vartheta,\varphi) &= R^2 \sin^2\frac{r}{R}\sin\vartheta \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi \\
\mathrm{d}\mu(r,\vartheta,\varphi) &= R^2 \sinh^2\frac{r}{R}\sin\vartheta \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\varphi.
\end{aligned}$$

When the pure states are describing by wave functions on the configuration spaces, then the scalar products are given by

$$\langle\Psi|\varphi\rangle = \int \overline{\Psi}\varphi \mathrm{d}\mu.$$

The Hamilton operator is given by

$$\widehat{H} = -\frac{\hbar^2}{2m}\Delta + V(r)$$

where, as usual, Δ denotes the Laplace-Beltrami operator corresponding to the metric tensor g. Denoting by ∇_i the Levi-Civita covariant differentiation induced by g_{ij}, we have obviously

$$\Delta = g^{ij}\nabla_i\nabla_j.$$

As usual in the spherically invariant systems in $\mathbb{R}^3$, we have the following system of commuting observables: $\widehat{H},\ \widehat{\overline{L}}^2,\ \widehat{L}_3$ (by convention, any other component $\widehat{L}_1,\ \widehat{L}_2$, might be chosen). Here $\widehat{L}_a$ are generators of rotations

$$\widehat{L}_a = \frac{\hbar}{\mathrm{i}}\varepsilon_{abc}r^b\frac{\partial}{\partial r^c}$$

$\widehat{\overline{L}}^2$, the square of magnitude, i.e., the square of "length" of the vector $\widehat{\overline{L}}$ is given by

$$\widehat{\overline{L}}^2 = \sum_{a=1}^{3}\left(\widehat{L}_a\right)^2.$$

Obviously, the components $\widehat{L}_a$ satisfy the obvious commutation rules

$$\frac{1}{\hbar\mathrm{i}}\left[\widehat{L}_a, \widehat{L}_b\right] = \varepsilon_{ab}{}^c\widehat{L}_c$$

and therefore

$$\left[\widehat{\overline{L}}^2, \widehat{L}_a\right] = 0$$

i.e., $\widehat{\overline{L}}^2$ is their Casimir operator. Therefore, their spectra are obvious, i.e., $\hbar^2 l(l+1)$ for $\widehat{\overline{L}}^2$, $a = 0, 1, 2, \ldots$, i.e., $a \in \{0\} \cup \mathbb{N}$ and $m = -l, -l+1, \ldots, 0, \ldots l-1, l$. And we can use the usual separation of variables for radial systems

$$\Psi_{nlm}(r, \vartheta, \varphi) = f_{nl}(r)Y_{lm}(\vartheta, \varphi).$$

Therefore, substituting this to the Schrödinger equations and making use of the standard properties of spherical functions we obtain the following ordinary radial equations

$$\frac{\mathrm{d}^2 f_{nl}}{\mathrm{d}r^2} + \frac{2}{R}\cot\frac{r}{R}\frac{\mathrm{d}f_{nl}}{\mathrm{d}r} - \frac{l(l+1)}{R^2\sin^2\frac{r}{R}}f_{nl} - \frac{2mV}{\hbar^2}f_{nl} + \frac{2mE}{\hbar^2}f_{nl} = 0 \quad (55)$$

$$\frac{\mathrm{d}^2 f_{nl}}{\mathrm{d}r^2} + \frac{2}{R}\coth\frac{r}{R}\frac{\mathrm{d}f_{nl}}{\mathrm{d}r} - \frac{l(l+1)}{R^2\sinh^2\frac{r}{R}}f_{nl} - \frac{2mV}{\hbar^2}f_{nl} + \frac{2mE}{\hbar^2}f_{nl} = 0. \quad (56)$$

Obviously, (55) is an equation for the sphere and (56) for the pseudosphere. It is clear that for any radial system there is a degeneracy with respect to m and for the Bertrand models an additional degeneracy with respect to l. This additional, not a priori obvious degeneracy, corresponds exactly to the total, i.e., two-fold degeneracy on the classical level. Such a correspondence is typical for systems with high symmetry group.

Without reporting here the details of derivation, based on the Sommerfeld polynomial method, let us restrict ourselves to presenting the final formulas for the quantum-mechanical energy spectrum.

For the Coulomb problem on $S^3(0,R)$ the energy levels are given by

$$E_n = -\frac{m\alpha^2}{2n^2\hbar^2} + \frac{(n-1)\,(n+1)\,\hbar^2}{2mR^2} \tag{57}$$

where $n \in \mathbb{N}$, i.e., $n = 1,2,3,\ldots$. For any fixed value of n, l takes on the values $l = 0,1,2,\ldots,n-1$.

Similarly, for the degenerate oscillator in $S^3(0,R)$ we obtain

$$E_n = \left(n + \frac{3}{2}\right)\hbar\Omega + \frac{(n+1)\,(n+3)\,\hbar^2}{2mR^2}$$

where the modified frequency is given by

$$\Omega = \frac{\hbar}{2mR^2}\left(\sqrt{1 + 4m^2R^4\omega_0^2\hbar^{-2}} - 1\right) \tag{58}$$

and $n = 0,1,2,\ldots$ $(n \in \{0\} \cup \mathbb{N})$.

For the free geodetic motion we obtain

$$E_n = \frac{n\,(n+2)\,\hbar^2}{2mR^2}, \qquad n = 0,1,2,\ldots, \quad \text{i.e.,} \quad n \in \{0\} \cup \mathbb{N}. \tag{59}$$

It is however, more convenient to re-gauge the quantum number n by -1 so that (59) becomes

$$E_n = \frac{(n-1)\,(n+1)\,\hbar^2}{2mR^2}, \qquad n \in \mathbb{N}, \quad \text{i.e.,} \quad n = 1,2,3,\ldots.$$

This expression is more suitable from the point of view of the comparison with the Coulomb problem, or rather with its second term in (57).

It is clear that quite as expected, those formulas are not identical with the classical Bohr-Sommerfeld and action-angle expressions. Nevertheless, they have in principle the same general structure of superposing in an additive way the free-motion spectrum with the Coulomb and oscillator spectra in the $\mathbb{R}^3$-models.

The only, relatively astonishing property is one concerning the oscillator spectra. Namely, in the Euclidean sector we would rather expect instead of Ω the usual quantity ω_0. It turns out that, incidentally, it is the case. Namely, for the large values of the curvature radius R, it turns out that $\Omega \to \omega_0$. And therefore, the formula (58) becomes compatible with (52).

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