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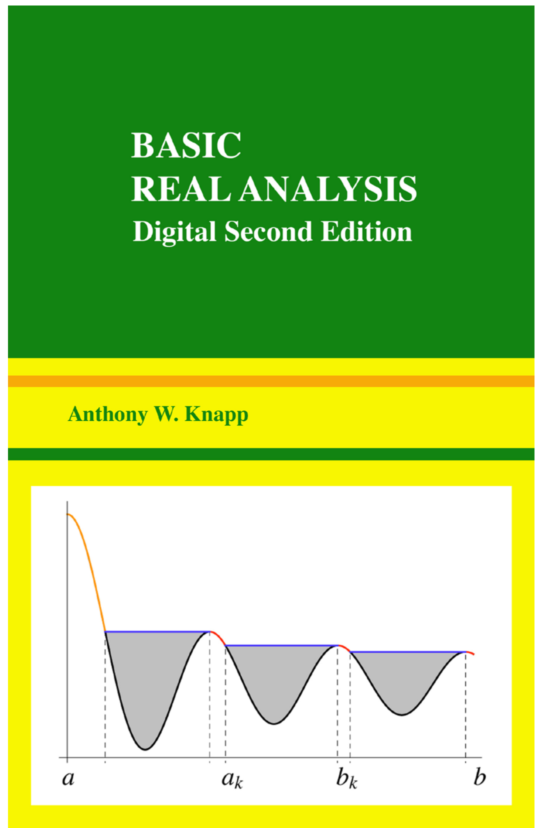

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APPENDIX B

Elementary Complex Analysis

Abstract. This appendix treats some aspects of elementary complex analysis that are useful as tools in real analysis. It assumes knowledge of Appendix A and much of Chapters I to III.

Section B1 Introduces the complex derivative of a complex-valued function defined on an open subset of $\mathbb{C}$, and it relates the notion to differentiability in the sense of Chapter III. The Cauchy–Riemann equations are part of this relationship. An analytic function on a region in $\mathbb{C}$ is a function with a complex derivative at each point.

Section B2 introduces complex line integrals and relates them to the traditional line integrals in the last three sections of Chapter III. An important result is that a continuous complex-valued function on a region in $\mathbb{C}$ is the complex derivative of an analytic function if and only if its complex line integral over every piecewise C^1 closed curve in the region is zero.

Section B3 proves Goursat's Lemma and a local form of the Cauchy Integral Theorem. Goursat's Lemma says that the complex line integral of a function over a rectangle is 0 if the function is analytic on a region containing the rectangle and its inside. The local form of the Cauchy Integral Theorem that follows says that for an analytic function in an open disk, the complex line integral is zero over every piecewise C^1 closed curve.

Section B4 obtains a simple form of the Cauchy Integral Formula for a disk and derives from it the corresponding formula for complex derivatives, Morera's Theorem, Cauchy's estimate, Liouville's Theorem, and the Fundamental Theorem of Algebra.

Section B5 establishes two versions of the complex-variable form of Taylor's Theorem. The first form includes a remainder term, and the second form asserts a convergent power series expansion.

Section B6 treats various local properties of analytic functions in regions. If the complex derivatives of all orders of such a function are zero, then the function is 0. Consequently if the function is not identically 0, then any zero has a nonnegative integer order, and the zeros of the function are isolated. Other consequences are the Maximum Modulus Theorem, a description of the behavior at poles, Weierstrass's result on essential singularities, and the Inverse Function Theorem.

Section B7 examines the exponential function and its local invertibility. This examination leads to the definition of winding number for a closed curve about a point, and the general form of the Cauchy Integral Formula for a disk follows.

Section B8 discusses operations on Taylor series and methods for computing such series.

Section B9 gives a first form of the Argument Principle relating the integral of $f'(z)/f(z)$ to the zeros and poles of $f(z)$.

Section B10 states and proves a first form of the Residue Theorem for evaluating the complex line integral of a function analytic except for poles.

Section B11 uses the the first form of the Residue Theorem to evaluate a number of examples of real definite integrals.

Section B12 extends the Cauchy Integral Theorem from closed curves in disks to cycles in simply connected regions, and it derives a corresponding version of the Residue Theorem.

Section B13 examines the extent to which the results of Section B12 extend to general regions when the cycle is assumed to be a boundary cycle.

Section B14 develops the Laurent series expansion of a function analytic in an annulus (washer). As a consequence the nature of essential singularities becomes a little clearer.

Section B15 introduces holomorphic functions of several variables, showing the equivalence of various definitions of such functions. This material is not used until *Advanced Real Analysis*.

B1. Complex Derivative and Analytic Functions

A broad treatment of complex analysis would view complex variable theory as a subject in its own right, with a healthy emphasis on topology, algebraic and differential geometry, number theory, and differential equations. Out of such a treatment would emerge the fact that the subject has great power in applications through the simplicity of its fundamental theorems and its remarkable formulas.

This modest appendix sacrifices the broad view to get at some facts about complex analysis that provide useful tools in real analysis. Accordingly it merely touches on the topological/geometric aspect of the subject and does not get into number theory or differential equations at all.

Notation and the first definitions appear in Sections A4 and A5 of Appendix A, and Section A7 is relevant, too. Additional notation and definitions appear in Sections III.11 and III.12, and it is assumed that the reader is familiar with all of this material.

One point deserves emphasis here, namely the correspondence between linear functions and matrices and how it impacts the relationship between the set $\mathbb{C}$ of complex numbers and the set $\mathbb{R}$ of real numbers. The vector space $\mathbb{R}^n$ of n-dimensional *column* vectors is denoted by $\mathbb{R}^n$. The linear functions $T : \mathbb{R}^n \to \mathbb{R}^m$ correspond to the m-by-n real matrices once we fix the standard bases $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ and $\{u_1, \dots, u_m\}$ of $\mathbb{R}^m$, as follows. The matrix A corresponding to T is given conveniently in terms of the dot product by $A_{ij} = T(e_j) \cdot u_i$.

The passage back and forth between complex numbers and their real and imaginary parts is fundamental. This passage allows us to identify for some purposes the set $\mathbb{C}$ of complex numbers with $\mathbb{R}^2$, the vector space of two-dimensional column vectors with real entries. In making this identification, we ordinarily single out $\{1, i\}$ as an ordered basis of $\mathbb{C}$ over $\mathbb{R}$, and then $a + bi$ in $\mathbb{C}$ gets identified with the column vector $\binom{a}{b}$ in $\mathbb{R}^2$. Multiplication by a fixed complex number $a + bi$ is a complex linear function of $\mathbb{C}$ into itself, and under the identification $\mathbb{C} \cong \mathbb{R}^2$, it yields a real linear function of $\mathbb{R}^2$ into itself. The matrix that corresponds to this linear function by the above prescription has first column the expression of $(a + bi)1$ in the basis $\{1, i\}$, namely $\binom{a}{b}$; the second column contains the expression of $(a + bi)i$ in the basis $\{1, i\}$, namely $\begin{pmatrix} -b \\ a \end{pmatrix}$. Thus the matrix corresponding to multiplication by $a + bi$ is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Let us call this 2-by-2 matrix $M(a+bi)$, and let us write $M(\mathbb{C})$ for the set of all such real matrices.

The emphasis in this appendix will be on functions carrying a subset of $\mathbb{C}$, usually an open set, into $\mathbb{C}$. Such a function is often written as f or as $z \mapsto f(z)$. Traditionally one refers to the function simply as $f(z)$, and we shall sometimes follow this tradition. In terms of real and imaginary parts, we typically write $z = x+iy$ and $f(z) = u(x,y) + iv(x,y)$. Thus we may regard f as a certain kind of function from an open subset of $\mathbb{R}^2$ into $\mathbb{R}^2$. We shall introduce complex differentiation of such functions and interpret the notion of complex derivative in the light of the above discussion.

Often we make use of certain geometric shapes in $\mathbb{C}$. The **open disk** of radius r about z_0 is the set of all z with $|z-z_0| < r$, and the corresponding **closed disk** is the set with $|z-z_0| \le r$. The edge of the closed disk, namely the set where $|z-z_0| = r$, is a **circle**.[1] The circle has an **inside**, namely the set where $|z-z_0| < r$, and an **outside**, namely the set where $|z-z_0| > r$. The **unit disk**, open or closed, is the disk of center 0 and radius 1, and the **unit circle** is its edge.

A **polygon** is the union of finitely many line segments L_j such that the endpoint of L_{j-1} matches the initial point of L_j and the endpoint of the last line segment matches the initial point of the first. A **triangle** T is a three-sided polygon. The **inside** of T is the bounded component of $\mathbb{C} - T$. The corresponding **filled triangle** is the union of T and its inside. Similarly a **rectangle** is a four-sided polygon with two sets of parallel sides and with right angles between consecutive sides. The **inside** of R is the bounded component of $\mathbb{C} - R$. The corresponding **filled rectangle** is the union of R and its inside.

Just as in Chapter I we made occasional use of limits of the form $\lim_{x\to x_0} g(x)$, where g is a real-valued function on a subset of $\mathbb{R}$ containing an open interval centered at x_0 but possibly not containing x_0 itself, now we shall make occasional use of limits of the form $\lim_{z\to z_0} h(z)$, where h is a complex-valued function defined on a subset of $\mathbb{C}$ containing an open disk about z_0 but possibly not containing z_0 itself. This **limit** will be said to exist and equal c if for each $\epsilon > 0$, there is some $\delta > 0$ such that $|f(z) - c| < \epsilon$ whenever $0 < |z-z_0| < \delta$. As in the real case, we omit the details about why such limits respect addition, subtraction, and multiplication and about how they can be reformulated in terms of limits of sequences. Occasionally we shall make use of limits written $\lim_{z\to\infty} f(z)$; this means nothing more than $\lim_{w\to 0} f(1/w)$.

Let $f : U \to \mathbb{C}$ be a function defined on an open set. If z_0 is in the open set, we say that f has the complex number $f'(z_0)$ as **complex derivative** at z_0 if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

[1] In Section B2 we shall consider parametrizations of circles, but we do not do so now.

In view of the discussion in the previous paragraph, the condition is that for any $\epsilon > 0$, there is a $\delta > 0$ such that whenever $|z - z_0| < \delta$ and $z \neq z_0$, then $\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$. The first proposition relates this notion to the notion of differentiability in Chapter III.

Proposition B.1. Let $f = u + iv$ be a complex-valued function defined on an open set in $\mathbb{C}$ containing the point $z_0 = x_0 + iy_0$. Then f has a complex derivative $f'(z_0)$ at z_0 if and only if the function $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is differentiable at (x_0, y_0) with a Jacobian matrix in $M(\mathbb{C})$, and in this case the Jacobian matrix of $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is $M(f'(z_0))$.

We give the proof at the end of this section. Let us observe the following consequence.

Corollary B.2 (Cauchy–Riemann equations). Let $f = u + iv$ be a complex-valued function defined on an open set in $\mathbb{C}$ containing the point $z_0 = x_0 + iy_0$. If f has a complex derivative at z_0, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at } (x_0, y_0),$$

and $f'(z_0)$ is given by

$$f'(z_0) = \tfrac{\partial u}{\partial x}(x_0, y_0) + i\tfrac{\partial v}{\partial x}(x_0, y_0) = -i\tfrac{\partial u}{\partial y}(x_0, y_0) + \tfrac{\partial v}{\partial y}(x_0, y_0).$$

Conversely if the first partial derivatives of u and v exist in a neighborhood of (x_0, y_0) and if the first partial derivatives are continuous at (x_0, y_0) and satisfy $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at (x_0, y_0), then f has a complex derivative at z_0.

PROOF. If $f = u + iv$ has a complex derivative at z_0, then Proposition B.1 shows that the Jacobian matrix of $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ exists and is of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $f'(z_0) = a + bi$. Since the Jacobian matrix has the form

$$\begin{pmatrix} \frac{\partial u}{\partial x}(x_0, y_0) & \frac{\partial u}{\partial y}(x_0, y_0) \\ \frac{\partial v}{\partial x}(x_0, y_0) & \frac{\partial v}{\partial y}(x_0, y_0) \end{pmatrix},$$

the first partial derivatives have to satisfy the indicated equations. The asserted formulas for $f'(z_0)$ follow. For the converse Theorem 3.7 says that the existence of the first partial derivatives in a neighborhood of (x_0, y_0) and the continuity of them at (x_0, y_0) implies that $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$ is differentiable at (x_0, y_0). The differentiability at (x_0, y_0) and the equations satisfied by the partial derivatives together imply that f has a complex derivative at z_0, by Proposition B.1. $\square$

For a complex-valued function $f = u + iv$, let us understand $\frac{\partial f}{\partial x}(z_0)$ to mean $\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$. Similarly $\frac{\partial f}{\partial y}(z_0)$ means $\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)$. In this notation we can rephrase Corollary B.2 as follows.

Corollary B.2′ (Cauchy–Riemann equations, complex form). Let $f = u+iv$ be a complex-valued function defined on an open set in $\mathbb{C}$ containing the point $z_0 = x_0 + iy_0$. If f has a complex derivative at z_0, then

$$\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0),$$

and each of these equals $f'(z_0)$. Conversely if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist in a neighborhood of z_0 and are continuous at z_0 and satisfy $\frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$, then f has a complex derivative at z_0, and it equals both of these quantities.

Our interest will be in functions that have a complex derivative at every point of an open set. It is customary to work with open sets that are connected. Taking a cue from Section III.12, we use the term **region** to refer to any nonempty connected open subset of $\mathbb{C}$. Any two points in a region can be connected by a piecewise C^1 curve, according to Lemma 3.46.

If U is a region in $\mathbb{C}$, we say that $f : U \to \mathbb{C}$ is an **analytic function** if f has a complex derivative at every point of U. The term "regular function" was in use formerly, and the term "holomorphic function" is often used by people who also have in mind functions of several complex variables. A function that is analytic in the region $\mathbb{C}$ is said to be **entire**.

EXAMPLES.

(1) Any complex linear combination of analytic functions is analytic, and so is the product of two analytic functions. The quotient of two analytic functions is analytic on the (open) subset where the denominator is nonzero. The proofs are the same as in calculus, and the usual product and quotient rules remain valid for complex differentiation.

(2) Any constant function is analytic on all of $\mathbb{C}$, the complex derivative being the 0 function, and $z \mapsto z^n$ is entire for any integer $n \geq 0$, the complex derivative being nz^{n-1}. The proofs are the same as in calculus. Similarly when n is a negative integer, $z \mapsto z^n$ is analytic on $\mathbb{C} - \{0\}$ with complex derivative nz^{n-1}.

(3) If f and g are analytic functions and the domain of g contains the image of f, then the composition $g \circ f$ is analytic on the domain of f, and $(g \circ f)'(z) = g'(f(z))f'(z)$. The proof is the same as in calculus.

(4) From Theorem 1.37 we know that if a power series $\sum_{n=0}^{\infty} c_n z^n$ converges in $\mathbb{C}$ for some z_0 with $|z_0| = R$, then it converges absolutely for $|z| < R$. In this

situation the function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ turns out to be analytic for $|z| < R$, and its complex derivative is $\sum_{n=0}^{\infty} (n+1)c_{n+1}z^n$. With some effort one can work to adapt the proof of Theorem 1.23 to handle this situation. But it is much easier to derive this fact by using complex line integrals, and we therefore postpone the proof to the next section.

PROOF OF PROPOSITION B.1. Write $z = x + iy$, let a candidate for $f'(z_0)$ be $a + ib$, and temporarily put $z - z_0 = h + ik$. The expression that is to tend to 0 in the definition of complex derivative is

$$\begin{aligned}
&|z|^{-1}\big(f(z) - f(z_0) - (z - z_0)f'(z_0)\big)\\
&= |z|^{-1}\big(f(z) - f(z_0) - (a + ib)(h + ik)\big)\\
&= |x + iy|^{-1}\big(u(x, y) - u(x_0, y_0) + iv(x, y) - iv(x_0, y_0) - (a + ib)(h + ik)\big)\\
&= |x + iy|^{-1}\Big(u(x, y) - u(x_0, y_0) - \begin{pmatrix} a & -b \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\Big)\\
&\quad + |x + iy|^{-1}i\Big(v(x, y) - v(x_0, y_0) - \begin{pmatrix} b & a \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\Big),
\end{aligned}$$

and this tends to 0 in $\mathbb{C}$ if and only if

$$|(x, y)|^{-1}\Big(\begin{pmatrix} u(x,y)-u(x_0,y_0) \\ v(x,y)-v(x_0,y_0) \end{pmatrix} - \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}\Big)$$

tends to 0 in $\mathbb{R}^2$. This latter expression is what is to tend to 0 in the definition of differentiability with Jacobian matrix in $M(\mathbb{C})$. □

B2. Complex Line Integrals

This section introduces complex line integrals. These amount to ordinary line integrals as in Section III.12 but with a change of notation. The parametrically defined curve γ is now viewed as taking its values in $\mathbb{C}$, rather than in $\mathbb{R}^2$, and the continuous vector field F that is defined at least on the image of γ and takes values in $\mathbb{R}^2$ is replaced by a continuous function f that is defined at least on the image of γ and takes values in $\mathbb{C}$. With this change in notation, Theorem 3.44 becomes the following statement.

Proposition B.3. If $\gamma : [a, b] \to \mathbb{R}^2$ is a rectifiable simple arc and f is a continuous complex-valued function on the image of γ, then there exists a unique number, denoted $\int_\gamma f(z)\,dz$, with the following property. For any $\epsilon > 0$, there exists a $\delta > 0$ such that any partition $P = \{t_j\}_{j=0}^m$ of $[a, b]$ with $\mu(P) < \delta$ has

$$\Big|\int_\gamma f(z)\,dz - \sum_{j=1}^m f(\gamma(t_{j-1}))\big(\gamma(t_j) - \gamma(t_{j-1})\big)\Big| < \epsilon.$$

REMARKS.

(1) This result is just a restatement of Theorem 3.44 for dimension 2 in slightly different terminology, and no further proof will be needed. The number $\int_\gamma f(z)\,dz$ is called the **complex line integral of f over γ**.

(2) A subtle mixing of terms is involved in the passage from Theorem 3.44 to Proposition B.3. To understand it, write F for the complex-valued function in Theorem 3.44, and write F in terms of its real and imaginary parts as $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$. Decompose f in Proposition B.3 into its real and imaginary parts as $u + iv$, and view $\gamma(t)$ in the respective cases as $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and $x(t) + iy(t)$. The sum over the partition in Theorem 3.44 involves a dot product, namely

$$F_1(x, y)(x(t_j) - x(t_{j-1})) + F_2(x, y)(y(t_j) - y(t_{j-1})),$$

while the sum in Proposition B.3 involves a product of two complex numbers, namely

$$(u(x, y) + iv(x, y))\big(((x(t_j) - x(t_{j-1})) + i(y(t_j) - y(t_{j-1}))\big)$$

To match the two we want to have $F_1 = u + iv$ and $F_2 = i(u + iv)$. That is, the complex line integral $\int_\gamma f(z)\,dz$ equals the ordinary line integral over γ of the complex-valued vector field $\begin{pmatrix} f \\ if \end{pmatrix}$. Theorem 3.44 implies Proposition B.3 because Theorem 3.44 was actually valid for complex-valued vector fields.

(3) For our purposes the virtue of formulating complex line integrals as a limit of a sum of this kind is that we can see by inspection that the answer is independent of the parametrization as long as a reparametrization is orientation-preserving. The reasoning is the same as in Section III.12.

(4) The definition of complex line integral immediately extends to piecewise C^1 curves in $\mathbb{C}$, as they were defined in Section III.12. If $\gamma : [a, b] \to \mathbb{R}^2$ is the curve, if the intervals on which it is a tamely behaved simple arc are those relative to a partition $\{c_j\}_{j=0}^m$, and if f is a continuous complex-valued function on the image on γ, then the definition of the **complex line integral of f over γ** extends to this situation by the formula

$$\int_\gamma f(z)\,dz = \sum_{j=1}^m \int_{\gamma_{[c_{j-1},c_j]}} f(z)\,dz.$$

(5) Of particular interest to us will be the case of the **standard circle** in $\mathbb{C}$ of radius $r > 0$ and center a. By this we mean the piecewise C^1 curve $z(t) = a + re^{it}$ for $0 \le t \le 2\pi$. This circle is traversed counterclockwise when $\mathbb{C}$ is viewed in the usual way as a plane with the real axis horizontal pointing to the right and the imaginary axis vertical pointing up.[2] Integration over the standard circle of radius r and center a is often indicated by the notation $\int_{|z-a|=r}$. The circle is not a simple arc, being equal at the two ends, but the theory applies to it. It can be viewed, for example, as built from two tamely behaved simple arcs. However it is viewed, complex line integrals over it give the same result independently of what pieces are used and what parametrization is used, as long as the pieces and the pieces of the standard circle are related by an orientation-preserving reparametrization.

Proposition B.4. If $\gamma : [a, b] \to \mathbb{C}$ is a tamely behaved simple arc and if f is a continuous complex-valued function on the image of γ, then the complex line integral of f over γ, which exists by Proposition B.3, is given by

$$\int_\gamma f(z)\,dz = \lim_{\substack{a'\downarrow a,\ b'\uparrow b,\\ a<a'<b'<b}} \int_{a'}^{b'} f(\gamma(t))\gamma'(t)\,dt.$$

REMARKS.
(1) This follows from Theorem 3.44.
(2) As usual, we abbreviate the right side of this formula as

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt,$$

ignoring the fact that the integrand may be unbounded and that the integral is not strictly a Riemann integral. It is, however, a Lebesgue integral in the sense of Chapter V, and it presents no difficulty.
(3) In analogy with what happened for ordinary line integrals in Section III.12, the Schwarz inequality gives

$$\left| \int_a^b f(\gamma(t))\gamma'(t)\,dt \right| \le \int_a^b |f(\gamma(t))\gamma'(t)|\,dt,$$

and this translates into a way of estimating a complex line integral in terms of an integral with respect to arc length:

$$\left| \int_\gamma f(z)\,dz \right| \le \int_\gamma |f|\,ds.$$

[2]However, if perversely one wants to view the imaginary axis as pointing down, then standard circles are traversed clockwise.

(4) The formula of the proposition, with notation simplified as in Remark 2, immediately extends to complex line integrals over curves that are piecewise C^1 in the sense[3] of Section III.12. The formula in Remark 4 with Proposition B.3 is used for the whole curve, and the formula of the present proposition applies to each constituent simple arc.

Proposition B.5. If f is a continuous complex-valued function on a region U in $\mathbb{C}$, then f is the complex derivative of some analytic function on U if and only if the value of the complex line integral $\int_\gamma f(z)\,dz$ over each piecewise C^1 curve $\gamma : [a,b] \to U$ depends only on the endpoints $\gamma(a)$ and $\gamma(b)$ and not on the values of $\gamma(t)$ for $a < t < b$. In this case, any analytic function F whose complex derivative is f satisfies $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$.

PROOF. Remark 2 with Proposition B.3 shows that the complex-valued vector field on U that corresponds to f is $\begin{pmatrix} f \\ if \end{pmatrix}$. Proposition 3.47 shows that the ordinary line integrals of this vector field are independent of the path, depending only on the endpoints, if and only if $\begin{pmatrix} f \\ if \end{pmatrix}$ is the gradient of some (complex-valued) C^1 function F on U. That condition means that $f = \frac{\partial F}{\partial x}$ and $if = \frac{\partial F}{\partial y}$.

If such an F exists, then $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$ on U, and Corollary B.2′ shows that F is analytic on U with $F'(z) = f(z)$ on U. Conversely if F is any analytic function on U with $F' = f$, then Corollary B.2′ says that $f = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$. Hence $if = i(-i\frac{\partial F}{\partial y}) = \frac{\partial F}{\partial y}$.

The formula $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$ comes out of the proof of Proposition 3.47. □

Corollary B.6. If U is a region in $\mathbb{C}$, then a continuous function $f : U \to \mathbb{C}$ is the complex derivative of an analytic function on U if and only if $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 closed curve γ in U.

PROOF. If f is the complex derivative of some F, then $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 closed curve in U as a consequence of Proposition B.5 with $\gamma(b) = \gamma(a)$. Conversely suppose $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 closed curve in U, and let p and q be in U. If γ_1 and γ_2 are two piecewise C^1 curves from p to q, then $\gamma_1 - \gamma_2$ is a piecewise C^1 closed curve from p to itself. By hypothesis, $\int_{\gamma_1-\gamma_2} f(z)\,dz = 0$. Hence $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$. By Proposition B.5, f is the complex derivative of some F. □

[3] *Warning.* The definition of piecewise C^1 in Section III.12 requires less at the endpoints of each constituent interval than the standard condition given in Section A2 of Appendix A. Thus any reader who skipped Section III.12 might do well to read the definition in Section III.12 now.

For each integer $n \geq 0$, the power z^n is the complex derivative of $z^{n+1}/(n+1)$ on $\mathbb{C}$. Therefore $\int_\gamma z^n\,dz = 0$ for every piecewise C^1 closed curve γ in $\mathbb{C}$ when $n \geq 0$. Combining this fact with Corollary B.6, we can complete Example 4 of analytic functions, given in the previous section, as follows.

Corollary B.7. Suppose that the power series $\sum_{n=0}^{\infty} c_n z_0^n$ is convergent for some z_0 with $|z_0| = R$. Then the series $\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent for $|z| < R$, and the sum $f(z)$ is an analytic function for $|z| < R$ whose complex derivative is given by term-by-term complex differentiation of the series for $f(z)$.

PROOF. The absolute convergence was proved in Theorem 1.37. That theorem showed that if R' is any positive number with $R' < R$, then the series $\sum_{n=0}^{\infty} |c_n z^n|$ is uniformly convergent for $|z| \leq R'$, and so is the series $\sum_{n=0}^{\infty} |(n+1)c_{n+1}z^n|$. Introduce $f_N(z) = \sum_{n=0}^{N} c_n z^n$ and $g_N(z) = \sum_{n=0}^{N-1} (n+1)c_{n+1}z^n$, and define $g(z) = \sum_{n=0}^{\infty} (n+1)c_{n+1}z^n$. Observe that $f_N'(z) = g_N(z)$ for all z and that $f(0) = f_N(0) = c_0$ for all N.

Let R' be any positive number with $R' < R$, and let γ be any piecewise C^1 closed curve from 0 to z_0 with image in the open disk $|z| < R'$. The straight line segment from 0 to z_0 is one such curve. By Proposition B.5, $f_N(z_0) = c_0 + \int_\gamma g_N(z)\,dz$ for every N. On the image of γ, $g_N(z)$ tends uniformly to $g(z)$, and also $\lim_N f_N(z_0) = f(z_0)$. By Theorem 1.31, $\lim_N \int_\gamma g_N(z)\,dz = \int_\gamma g(z)$. Therefore $f(z_0) = c_0 + \int_\gamma g(z)\,dz$, and $\int_\gamma g(z)\,dz$ is exhibited as depending only on the endpoints of γ. Since z_0 is arbitrary with $|z_0| < R'$, Proposition B.5 shows that $g(z)$ is the complex derivative of an analytic function F for $|z| < R'$ and that function $F(z)$ has $F(z) - F(0) = \int_\gamma g(z)\,dz$. From $F(z) - F(0) = f(z) - c_0$, we see that $f(z)$ is analytic and that $f'(z) = g(z)$. $\square$

EXAMPLE. It follows immediately that the functions

$$e^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!},$$

which were defined and analyzed in Section I.7, are all analytic functions on $\mathbb{C}$.

For each integer $n \le -2$ on $\mathbb{C}-\{0\}$, z^n is the complex derivative of $z^{n+1}/(n+1)$. Therefore Corollary B.6 shows that $\int_\gamma z^n\,dz = 0$ for all integers $n \le -2$ for every piecewise C^1 closed curve in $\mathbb{C}-\{0\}$. However, there exist piecewise C^1 closed curves in $\mathbb{C}-\{0\}$ over which z^{-1} does not have integral 0. In fact, if γ is any standard circle centered at 0, say the one parametrized as $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$, then

$$\int_\gamma z^{-1}\,dz = \int_0^{2\pi} (re^{it})^{-1}\gamma'(t)\,dt = \int_0^{2\pi} r^{-1}e^{-it}ire^{it}\,dt = 2\pi i \ne 0.$$

More generally $\int_\gamma (z-a)^{-1}\,dz = 2\pi i$ whenever γ is a standard circle centered at a.

Just as in Section III.13, we can introduce the notion of a **piecewise C^1 chain**. This is a formal sum of piecewise C^1 curves, say $\gamma = \gamma_1 + \cdots + \gamma_r$, and a complex line integral over γ is defined as the corresponding sum:

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{r} \int_{\gamma_k} f(z)\,dz.$$

Two such chains are equal if all line integrals defined on both are equal. There is no need to dwell on the formal properties of chains at this time, but we mention that in this notation, $-\gamma$ will denote the reverse of γ and a chain involving terms γ_j and $-\gamma_j$ equals the chain with those two terms dropped. We return to consider chains in more detail in Section B12.

B3. Goursat's Lemma and the Cauchy Integral Theorem

A.–L. Cauchy was the person who discovered and almost single-handedly developed the foundations of complex variable theory. The key elementary fact about analytic functions is that the complex derivative of an analytic function is again an analytic function. The gateway toward getting at this fact is to examine the result of Corollary B.6, that certain functions f have $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 closed curve γ in the region U. It is not true that this conclusion is valid for every analytic function f and every region U: toward the end of Section B2, we saw that a standard circle γ about 0 in $U = \mathbb{C}-\{0\}$ always has $\int_\gamma z^{-1}\,dz \ne 0$.

However, Cauchy proved for certain kinds of regions that $\int_\gamma f(z)\,dz$ is always 0 for f analytic when the image of γ lies in the region. This result is called the **Cauchy Integral Theorem**. In retrospect we can see instances of this equality from Green's Theorem as in Theorem 3.48.[4] For a region as in Theorem 3.48, the fact that $\int_\gamma f(z)\,dz = 0$ for $f = u + iv$ analytic on an open set containing the region and its boundary is an immediate consequence of the Cauchy–Riemann equations, provided the partial derivatives of u and v are continuous. See Problem 3 at the end of this appendix for the case of a filled rectangle. This kind of argument was what Cauchy used, but then the treatment of analytic functions had to work with continuous complex derivatives and it was necessary to prove that the complex derivative of such a function is again such a function.

Much later E. Goursat found a way around the assumption of continuity of f', and the resulting proofs of the foundational results are not much more difficult than the ones they replaced. In this section we shall give Goursat's proof of a key lemma that will be used to prove the Cauchy Integral Theorem for a disk with no assumption that f' is continuous. This version of the Cauchy Integral Theorem can be viewed as the local form of a global theorem that will be addressed later. The local form is good enough to get at the infinite differentiability of analytic functions, a step we shall carry out in the next section after establishing the Cauchy Integral Formula.

Theorem B.8 (Goursat's Lemma). If f is analytic in a region containing the filled rectangle

$$R = \{z = x + iy \mid a \le x \le b \text{ and } c \le y \le d\},$$

and if ∂R is its boundary, parametrized as a piecewise C^1 closed curve so as to be traversed counterclockwise, then $\int_{\partial R} f(z)\,dz = 0$.

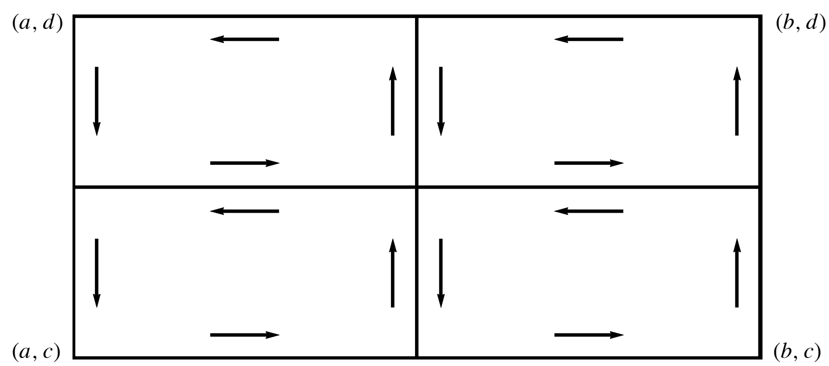

FIGURE B.1. First bisection of the rectangle R in Goursat's Lemma.

PROOF. We use a method of bisection to isolate the worst possible behavior within R, and then we examine the resulting situation. For any filled rectangle

[4] The Cauchy Integral Theorem predates Green's Theorem.

R' contained in R, define

$$\eta(R') = \int_{\partial R'} f(z)\,dz.$$

Define $R^{(0)} = R$. Write R as the union of four nonoverlapping congruent rectangles R_1, R_2, R_3, R_4 of half the base and half the height. The orientation of the boundary of each is to be counterclockwise, and then

$$\partial R = \partial R_1 + \partial R_2 + \partial R_3 + \partial R_4 \qquad (*)$$

in the sense of equality of piecewise C^1 chains as at the end off the previous section. See Figure B.1. The reason for the equality $(*)$ is that each interior edge of $R_1, \dots, R_4$ that appears on the right side is traversed twice, once in each direction. Consequently at least one of $R_1, \dots, R_4$, say R_j, has $|\eta(R_j)| \geq \frac{1}{4}|\eta(R)|$. Choose one such R_j, and call it $R^{(1)}$. Then we have

$$|\eta(R^{(1)})| \geq \tfrac{1}{4}|\eta(R^{(0)})|.$$

We repeat this process with $R^{(1)}$, writing it as the union of four nonoverlapping congruent rectangles, etc., and we select one of the four, which we call $R^{(2)}$, in such a way that $|\eta(R^{(2)})| \geq \frac{1}{4}|\eta(R^{(1)})|$. Proceeding inductively, we obtain a nested sequence of filled rectangles $\{R^{(k)}\}_{k=0}^{\infty}$ with diagonals tending to 0 such that

$$|\eta(R^{(k)})| \geq \tfrac{1}{4}|\eta(R^{(k-1)})|$$

for all $k \geq 1$. Hence

$$|\eta(R^{(k)})| \geq 4^{-k}|\eta(R)| \qquad (**)$$

for $k \geq 0$. By the Heine–Borel Theorem (Corollary 2.37 and Proposition 2.35), $\bigcap_{k=0}^{\infty} R^{(k)}$ is nonempty. Let z_0 be in the intersection.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ small enough so that the disk of radius δ about z_0 is contained in the region on which f is analytic and so that $0 < |z - z_0| < \delta$ implies

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \leq \epsilon.$$

Then

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \epsilon|z - z_0|$$

for $|z - z_0| < \delta$. We know from the comments before Corollary B.7 that 1 and z have integral 0 over $\partial R^{(k)}$ for each k. Therefore

$$\eta(R^{(k)}) = \int_{\partial R^{(k)}} \big(f(z) - f(z_0) - (z - z_0)f'(z_0)\big)\,dz.$$

Since every point z of $\partial R^{(k)}$ has $|z - z_0| < \delta$, Remark 3 with Proposition B.4 shows that

$$\begin{aligned}|\eta(R^{(k)})| &= \Big| \int_{\partial R^{(k)}} \big(f(z) - f(z_0) - (z - z_0) f'(z_0)\big)\, dz \Big| \\ &\le \int_{\partial R^{(k)}} \epsilon |z - z_0|\, ds \le \epsilon \text{ diameter}(\partial R^{(k)}) \text{ perimeter}(R^{(k)}).\end{aligned}$$

Combining this inequality with $(**)$ gives

$$4^{-k}|\eta(R)| \le \eta(R^{(k)}) \le \epsilon\, 2^{-k}\text{diameter}(\partial R)\, 2^{-k}\text{perimeter}(\partial R)$$

and therefore

$$|\eta(R)| \le \epsilon \text{ diameter}(\partial R) \text{ perimeter}(\partial R). \qquad \square$$

In order to apply Theorem B.8 easily in certain situations, we shall make its statement at once more general and more ugly, as follows.

Theorem B.8′. If $f(z)$ is analytic in a region containing a filled rectangle R except for finitely many interior points z_j and if

$$\lim_{z \to z_j} (z - z_j) f(z) = 0$$

for each of the exceptional points, then $\int_{\partial R} f(z)\, dz = 0$.

PROOF. It is sufficient to consider the case of a single exceptional point z_0, since R can be subdivided into finitely many nonoverlapping rectangles, each containing at most one exceptional point. When the resulting complex line integrals over boundaries of rectangles are added, the contributions from the edges of these rectangles that are interior to R will cancel in pairs and the result will follow.

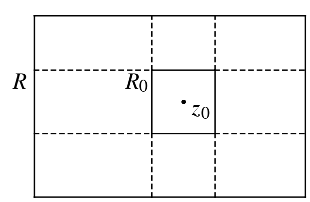

FIGURE B.2. Handling an exceptional point in Goursat's Lemma.

Since the single exceptional point z_0 is an interior point, we can choose a small square R_0 centered at z_0 as in Figure B.2, and continuation of its edges until they meet the edges of R will determine a total of nine constituent rectangles of R. Again the sum of the complex line integrals over all the boundaries of all nine rectangles will equal the integral over the boundary of R. We apply Theorem B.8 to the eight constituent rectangles other than R_0 and we get 0 for each. Thus the complex line integral over ∂R equals the complex line integral over ∂R_0.

Let $\epsilon > 0$ be given. By hypothesis we can choose R_0 small enough so that $|f(z)| \leq \epsilon |z - z_0|^{-1}$ on ∂R_0. Then

$$\left|\int_{\partial R} f(z)\,dz\right| = \left|\int_{\partial R_0} f(z)\,dz\right| \leq \Big(\max_{z \in \partial R_0} |f(z)|\Big)\,\text{perimeter}(\partial R_0)$$
$$\leq \epsilon \Big(\max_{z \in \partial R_0} |z - z_0|^{-1}\Big)\,\text{perimeter}(\partial R_0).$$

If each side of R_0 has length r_0, then the expression $|z - z_0|^{-1}$ is as large as possible when z is at the center of one of the sides of R_0. There it is $2r_0^{-1}$. The perimeter of R_0 is $4r_0$, and the estimate above becomes $|\int_{\partial R} f(z)\,dz| \leq 8\epsilon$. $\square$

Theorem B.9 (Cauchy Integral Theorem, local form). If f is analytic in an open disk D, then

$$\int_\gamma f(z)\,dz = 0$$

for every piecewise C^1 closed curve in D.

PROOF. Let z_0 be the center of D. For $z \in D$, define $F(z) = \int_\gamma f(\zeta)\,d\zeta$, where γ is a polygonal path from z_0 to z whose constituent line segments are each horizontal or vertical. Let us argue by induction on the number of line segments in the polygonal path that the definition of $F(z)$ is independent of the path. If two consecutive segments are horizontal or if they are both vertical, we can combine them into a single segment and reduce the number of segments. Thus we may assume that the segments alternate in type, horizontal and vertical. If there are at least three, we show how to reduce their number by means of Theorem B.8. We may assume by symmetry that three consecutive segments are horizontal, then

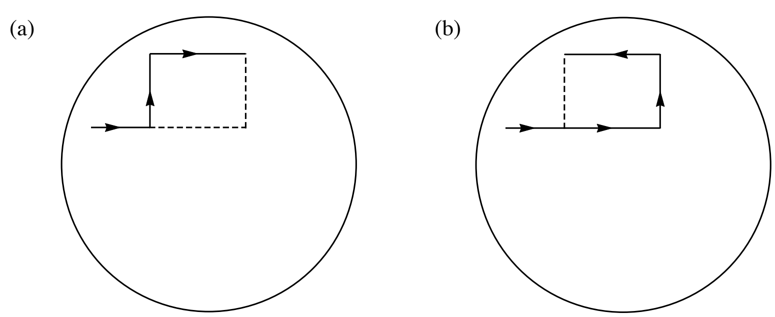

FIGURE B.3. Reduction steps in the proof of Theorem B.9.

vertical, then horizontal. If the two horizontal segments go in the same direction, left or right, then the starting point and the final point lie on the diagonal of a rectangle, and one of the other two vertices of that rectangle lies[5] in the given disk D. Say that the vertex on the continuation of the first horizontal segment lies in D, as in Figure B.3a. Then the vertical and the second horizontal can be replaced by the other two sides of a rectangle—the continued horizontal segment and a new vertical segment—in D because the complex line integral around the rectangle is 0. Consequently the three given segments—first horizontal, first vertical, second horizontal—can be replaced by the first horizontal, its continuation, and the new vertical. The two horizontal segments—the first horizontal and its continuation— can be combined into one, reducing the number of segments. A similar argument works if the vertex lying in D is the other vertex of the rectangle; the three segments get replaced by a vertical one followed by a horizontal one.

If among the three segments the two horizontal segments go in opposite directions, as in Figure B.3b, then the vertical and the shorter horizontal span a rectangle that lies in D, and the complex line integral around that rectangle is 0 by Theorem B.8. The three segments get replaced by two, one horizontal and one vertical.

Thus the value of the integral over a polygonal path is the same as the integral over a polygonal path with at most two segments. Suppose there are exactly two segments. The nontrivial possibilities for two segments are horizontal then vertical or else vertical then horizontal. The four segments in question are the sides of a rectangle, around which the complex line integral gives 0. Thus the two possibilities give the same definition for $F(z)$. The only remaining possibilities are that there is only one segment, and then the path is unique, or else there are zero segments, in which case z is the center of the disk and the value of $F(z)$ is 0. Thus $F(z)$ is well defined.

The function $F(z)$ is certainly continuous, as a change from z_1 to z_2 produces a change in the integral of at most the maximum value of $|f(\zeta)|$ on a polygonal path from z_1 to z_2, times the length of the polygonal path. Parametrizing horizontal and vertical segments, we compute the partial derivatives of F. If the last segment of a path is taken to be horizontal, we see that $\frac{\partial F}{\partial x}(z) = f(z)$. If it is taken to be vertical, we see that $\frac{\partial F}{\partial y}(z) = if(z)$. Both partial derivatives are continuous, and Corollary B.2′ implies that F has a complex derivative at each point, namely the value of $\frac{\partial F}{\partial x}$, which is f. In other words, f is the complex derivative of an analytic function. Corollary B.6 therefore shows that $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 closed curve in D. □

[5]If $|(x_1, y_1) - (a,b)| < r$ and $|(x_2, y_2) - (a,b)| < r$, then either $|(x_1, y_2) - (a,b)| < r$ or $|(x_2, y_1) - (a,b)| < r$ because $|(x_1, y_1) - (a,b)|^2 + |(x_2, y_2) - (a,b)|^2 = |(x_1, y_2) - (a,b)|^2 + (x_2, y_1) - (a,b)|^2$.

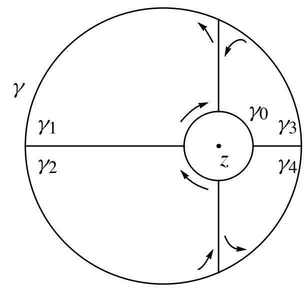

FIGURE B.4. Computation of $\int_\gamma (\zeta - z)^{-1}\, d\zeta$ over a standard circle when z is not the center.

EXAMPLE. If γ is any standard circle containing the point z inside it,[6] then

$$\int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi i.$$

To see this equality, form a standard circle γ_0 inside γ that is centered at z. We shall show that

$$\int_\gamma \tfrac{d\zeta}{\zeta - z} = \int_{\gamma_0} \tfrac{d\zeta}{\zeta - z}. \tag{$*$}$$

To do so, we adjoin integrations over four canceling pairs of line segments as in Figure B.4, thereby introducing four closed curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that tracing out γ amounts to tracing out $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, and γ_0. (Two of the line segments lie on the line of centers for the two circles, and the other two lie on the line through z perpendicular to the line through the centers.) Then

$$\int_\gamma \tfrac{d\zeta}{\zeta - z} = \sum_{j=1}^{4} \int_{\gamma_j} \tfrac{d\zeta}{\zeta - z} + \int_{\gamma_0} \tfrac{d\zeta}{\zeta - z}. \tag{$**$}$$

Each of $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ is a closed curve lying in a disk within $\mathbb{C}$ on which $(\zeta - z)^{-1}$ is analytic, and Theorem B.9 shows that $\int_{\gamma_j} \frac{d\zeta}{\zeta - z} = 0$ for $j = 1, 2, 3, 4$. Thus $(**)$ implies $(*)$. The right side of $(*)$ is $2\pi i$ by a computation near the end of Section B2, and consequently $\int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi i$.

Just as we did with Theorem B.8, we shall modify the statement of Theorem B.9 to make it at once more general and more ugly, as follows. The modified result will be easier to apply in certain situations.

[6] As usual, "inside it" means that the distance from z to the center of γ is less than the radius of γ.

Theorem B.9′. If f is analytic in an open disk D except for finitely many interior points z_j and if $\lim_{z\to z_j}(z - z_j)f(z) = 0$ for each of the exceptional points, then

$$\int_\gamma f(z)\,dz = 0$$

for every piecewise C^1 closed curve in D that does not pass through any of the points z_j.

PROOF. Instead of considering polygonal paths starting from the center of D, we consider those starting from another point chosen so that its x coordinate does not match the x coordinate of any vertical line segment and the y coordinate does not match the y coordinate of of any horizontal line segment.

We then imitate as much as possible of the argument for Theorem B.9 except that we consider only polygonal paths that do not pass through any z_j and we invoke Theorem B.8′ repeatedly instead of Theorem B.8. The inductive argument reduces the number of segments in the polygonal path, and the reduction is again to two segments if neither coordinate of z matches a coordinate of some z_j, but the reduction can be only to three segments if one coordinate of z matches a coordinate of some z_j and it can be only to four segments if each coordinate of z matches a coordinate of some z_j.

The reduced case can be handled by Theorem B.8′ in a noncanonical way that we shall not write out. Then the rest of the argument, using Corollary B.2′ and Corollary B.6, applies to the region obtained by deleting the points z_j from D, and the proof is complete. □

B4. Cauchy Integral Formula

Theorem B.9′ readily implies that an analytic function is given locally by an integral formula. Then a manageable interchange of limits will show that the complex derivative of the analytic function is given by differentiating under the integral sign, and consequently the complex derivative is an analytic function. Therefore analytic functions have complex derivatives of all orders, and each of them satisfies an estimate for its size. We shall carry out these steps in this section. In the next section we shall use the estimates to prove that the infinite Taylor series expansion of an analytic function about the center of an open disk converges everywhere in the disk to the analytic function.

Theorem B.10 (Cauchy Integral Formula, local form). Let f be analytic in an open disk D, and let γ be any standard circle in D. If z is any point inside γ, then

$$\boxed{f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}.}$$

REMARKS.

(1) Say that γ has center a and radius r. Recall that the condition that γ be a "standard circle" means that γ it is given by $t \mapsto a + re^{it}$ with $0 \le t \le 2\pi$. It is enough that the circle be piecewise C^1 and that each of its pieces is an orientation-preserving reparametrization of the corresponding piece of $t \mapsto a + re^{it}$.

(2) Once again the condition that z be "inside γ" means that $|z - a| < r$.

PROOF. We apply the Cauchy Integral Theorem (Theorem B.9′) to the function

$$g(\zeta) = \frac{f(\zeta) - f(z)}{\zeta - z}$$

on the disk D, counting z as a single exceptional point. Since $f'(z)$ exists, $\lim_{\zeta\to z}(\zeta - z)g(\zeta) = 0$. Theorem B.9′ applies and gives $\int_\gamma g(\zeta)\,d\zeta = 0$. Therefore

$$\int_\gamma \tfrac{f(\zeta)\,d\zeta}{\zeta - z} = f(z)\int_\gamma \tfrac{d\zeta}{\zeta - z}. \qquad (*)$$

Since $\int_\gamma \frac{d\zeta}{\zeta - z} = 2\pi i$ by the example following the proof of Theorem B.9, the theorem follows. □

The Cauchy Integral Formula gives an explicit integral for the values of $f(z)$ inside the circle in terms of the values on the circle and is the first suggestion that the values of $f(z)$ on rather thin sets determine the values of $f(z)$ everywhere.

This formula is just asking to be differentiated in z so as to give an integral formula for $f'(z)$ in terms of the values of f. Justifying on the basis of general theorems the interchange of any kind of derivative and a limit such as the one defining an integral is normally hard. Theorem 1.23 shows what is involved in the case of real functions of a real variable. Here it is better to proceed directly, taking advantage of properties of the known function $(\zeta - z)^{-1}$. Once we have succeeded, we can attempt to iterate the process and obtain formulas for complex derivatives of all orders. The result of the iteration is as follows.

Theorem B.11. The complex derivative of an analytic function is analytic. More specifically if C denotes a standard circle and if $f(z)$ is analytic in an open disk containing C and its inside, then f has complex derivatives of all orders inside C, and they are given by

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}$$

at all points z inside C.

Before coming to the proof, let us draw one important inference. That inference is that once we can justify an interchange of complex derivative and complex line integral in this one case, then we can handle future cases more simply. Namely we can replace the complex differentiation by a complex line integral, and we are confronting two integrals. The two integrals can normally be interchanged by Fubini's Theorem (Corollary 3.33), and the result is that the interchange of a complex derivative and a complex line integral normally is easily justified. An example will be given in Corollary B.15.

The theorem will follow immediately from two lemmas.

Lemma B.12. Let γ be a piecewise C^1 closed curve in $\mathbb{C}$, and let U be an open subset of $\mathbb{C}$. If g is a continuous complex-valued function on $\text{image}(\gamma) \times U$, then the function $z \mapsto \int_\gamma g(\zeta, z)\, d\zeta$ is continuous on U.

REMARK. The idea of the proof here is more important than the specific statement. Variants of the lemma in which the proof needs only a small adjustment are used in problems in Chapters VI and VIII.[7]

PROOF. Fix z_0 in U, and let N be a closed disk centered at z_0 and lying in U. Since the set $\text{image}(\gamma) \times N$ is compact, the restriction of g to it is uniformly continuous. Given $\epsilon > 0$, choose $\delta > 0$ less the the radius of N such that any two points (ζ_1, z_1) and (ζ_2, z_2) in $\text{image}(\gamma) \times N$ at distance $< \delta$ have $|g(\zeta_1, z_1) - g(\zeta_2, z_2)| < \epsilon$. Then $|g(\zeta, z) - g(\zeta, z_0)| < \epsilon$ for all ζ in $\text{image}(\gamma)$ as long as $|z - z_0| < \delta$. Consequently

$$\begin{aligned}\left| \int_\gamma g(\zeta, z)\, d\zeta - \int_\gamma g(\zeta, z_0)\, d\zeta \right| &= \left| \int_\gamma [g(\zeta, z) - g(\zeta, z_0)]\, d\zeta \right| \\ &\le \int_\gamma |g(\zeta, z) - g(\zeta, z_0)|\, ds \le \epsilon \ell(\gamma),\end{aligned}$$

where $\ell(\gamma)$ is the length of γ. □

Lemma B.13. Let γ be a piecewise C^1 curve in $\mathbb{C}$, and suppose that φ is a continuous complex-valued function on the image of γ. Then for $n \ge 1$, the continuous function

$$F_n(z) = \int_\gamma \frac{\varphi(\zeta)\, d\zeta}{(\zeta - z)^n}$$

is analytic in each of the regions making up the complement of the image of γ, and its complex derivative is given by $F_n'(z) = nF_{n+1}(z)$.

REMARKS. The continuity of $F_n(z)$ is a special case of Lemma B.12. Next, the image of γ is a compact set in $\mathbb{C}$, and its complement is open in $\mathbb{C}$. Any open

[7] In the variants the integral is a Lebesgue integral, $\text{image}(\gamma)$ is replaced by some more general compact metric space, and the integral is taken relative to a specific instance of what Chapter XI calls a finite Borel measure.

set in $\mathbb{C}$ is the union of its connected components, each of which is open and is therefore a region. The first conclusion of the lemma is that $F_n(z)$ is analytic on each component.

PROOF. We begin with the case $n = 1$. Let z_0 be a point not in the image of γ, and let z be a point not in the image of γ that is close enough to z_0 to meet a condition to be specified. Then

$$F_1(z) - F_1(z_0) = \int_\gamma \big(\tfrac{1}{\zeta - z} - \tfrac{1}{\zeta - z_0}\big)\varphi(\zeta)\,d\zeta = (z - z_0)\int_\gamma \tfrac{\varphi(\zeta)\,d\zeta}{(\zeta - z)(\zeta - z_0)}.$$

Hence

$$\begin{aligned} \tfrac{F_1(z)-F_1(z_0)}{z-z_0} - \int_\gamma \tfrac{\varphi(\zeta)\,d\zeta}{(\zeta-z_0)^2} &= \int_\gamma \big(\tfrac{1}{(\zeta-z)(\zeta-z_0)} - \tfrac{1}{(\zeta-z_0)^2}\big)\varphi(\zeta)\,dz \\ &= \int_\gamma \big(\tfrac{1}{\zeta-z} - \tfrac{1}{\zeta-z_0}\big)\tfrac{\varphi(\zeta)\,d\zeta}{\zeta-z_0} \\ &= (z-z_0)\int_\gamma \tfrac{\varphi(\zeta)\,d\zeta}{(\zeta-z)(\zeta-z_0)^2}. \qquad (*) \end{aligned}$$

To estimate $(*)$, let M be the maximum value of $|\varphi(\zeta)|$ and choose δ small enough so that the disk of radius δ and center z_0 does not meet the image of γ. Then $|\zeta - z_0| \geq \delta$ for ζ in the image of γ. If z is in the disk of radius $\delta/2$ about z_0, then every ζ in the image of γ has

$$|\zeta - z| \geq |\zeta - z_0| - |z - z_0| \geq \delta - \delta/2 = \delta/2$$

by the triangle inequality. Taking the absolute value of both sides of $(*)$ gives

$$\Big|\tfrac{F_1(z)-F_1(z_0)}{z-z_0} - \int_\gamma \tfrac{\varphi(\zeta)\,d\zeta}{(\zeta-z)^2}\Big| \leq |z - z_0|\,2\delta^{-3}M\ell(\gamma).$$

This tends to 0 as z tends to z_0. Therefore $F_1(z)$ has a complex derivative at z_0, and $F_1'(z_0) = F_2(z_0)$ is as asserted. This completes the argument for $n = 1$.

We turn to the inductive step. To indicate that F_n depends on φ, let us write $F_n^{(\varphi)}$ in place of F_n. Inductively for $n \geq 2$, suppose that $F_{n-1}^{(\varphi)}{}' = (n-1)F_n^{(\varphi)}$ for every φ. We start with the identity

$$\begin{aligned} \tfrac{1}{(\zeta-z)^n} - \tfrac{1}{(\zeta-z_0)^n} &= \tfrac{1}{(\zeta-z)^{n-1}(\zeta-z_0)}\big(1 + \tfrac{z-z_0}{\zeta-z}\big) - \tfrac{1}{(\zeta-z_0)^n} \\ &= \big(\tfrac{1}{(\zeta-z)^{n-1}(\zeta-z_0)} - \tfrac{1}{(\zeta-z_0)^n}\big) + \tfrac{z-z_0}{(\zeta-z)^n(\zeta-z_0)}, \end{aligned}$$

insert $\varphi(\zeta)\,d\zeta$ throughout, integrate over γ, and divide by $z - z_0$ to get

$$\tfrac{F_n^{(\varphi)}(z)-F_n^{(\varphi)}(z_0)}{z-z_0} = (z-z_0)^{-1}\Big(\int_\gamma \tfrac{\varphi(\zeta)\,d\zeta}{(\zeta-z)^{n-1}(\zeta-z_0)} - \int_\gamma \tfrac{\varphi(\zeta)\,d\zeta}{(\zeta-z_0)^n}\Big) + \int_\gamma \tfrac{\varphi(\zeta)\,d\zeta}{(\zeta-z)^n(\zeta-z_0)}.$$

If we define $\psi(\zeta)$ to be the continuous function $\frac{\varphi(\zeta)}{\zeta - z_0}$, then we recognize this identity as saying that

$$\tfrac{F_n^{(\varphi)}(z)-F_n^{(\varphi)}(z_0)}{z-z_0} = \tfrac{F_{n-1}^{(\psi)}(z)-F_{n-1}^{(\psi)}(z_0)}{z-z_0} + F_n^{(\psi)}(z).$$

As z tends to z_0, the inductive hypothesis implies that the first term on the right side tends to $F_{n-1}^{(\psi)}{}'(z_0)$. Meanwhile, Lemma B.12 implies that the second term on the right side tends to $F_n^{(\psi)}(z_0)$. Thus the right side tends to $F_{n-1}^{(\psi)}{}'(z_0)+F_n^{(\psi)}(z_0)$, which by inductive hypothesis equals $(n-1)F_n^{(\psi)}(z_0) + F_n^{(\psi)}(z_0) = nF_n^{(\psi)}(z_0)$, and we conclude that $F_n^{(\varphi)}{}'(z_0)$ exists and equals $nF_n^{(\psi)}(z_0) = nF_{n+1}^{(\varphi)}(z_0)$. This completes the induction and the proof. □

PROOF OF THEOREM B.11. The case $n = 1$ of the theorem is immediate from the case $n = 1$ of Lemma B.13 if we take $\varphi(\zeta) = f(\zeta)$ and apply the Cauchy Integral Formula (Theorem B.10). For general n, we proceed inductively, starting from the formula for $n = 1$ in the theorem and making use of the general case of Lemma B.13 with $\varphi(\zeta) = f(\zeta)$ to handle the inductive step. □

With Theorem B.11 now completely proved, we take note of some consequences.

Continuity and differentiability are local properties. Thus if they hold in a disk about each point, they hold everywhere. The first corollary takes advantage of this fact.

Corollary B.14 (Morera's Theorem). If U is a region and $f : U \to \mathbb{C}$ is a continuous function for which $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 curve in U, then f is analytic in U.

PROOF. Corollary B.6 shows that f is the complex derivative of an analytic function. By Theorem B.11, f is analytic. □

Let us see a situation in which Morera's Theorem and Theorem B.11 can be combined with Fubini's Theorem to justify the insertion of an operation of complex differentiation inside an integral sign.

Corollary B.15. If U is a region in $\mathbb{C}$ and $\varphi(z, t)$ is a continuous complex-valued function on $U \times [a, b]$ that is analytic as a function of z for each fixed t, then the function

$$F(z) = \int_a^b \varphi(z, t)\,dt$$

is analytic for z in U and satisfies

$$F'(z) = \int_a^b \frac{\partial\varphi(z, t)}{\partial z}\,dt.$$

Here $\frac{\partial\varphi(z,t)}{\partial z}$ is the complex derivative of the function $z \mapsto \varphi(z, t)$ with t fixed.

REMARKS. The same comment as with Lemma B.12 applies to variants of this result. Observe that Corollary B.15 generalizes Lemma B.13.

PROOF. We observe that $F(z)$ is continuous in z by Lemma B.12. Let C be a circle in U whose inside lies in U. If γ is any piecewise C^1 closed curve in the inside of C, then

$$\int_\gamma F(z)\,dz = \int_\gamma \Big[\int_a^b \varphi(z,t)\,dt\Big]\,dz = \int_a^b \Big[\int_\gamma \varphi(z,t)\,dz\Big]\,dt = 0$$

by Fubini's Theorem (Corollary 3.33) and the Cauchy Integral Theorem (Theorem B.9). By Morera's Theorem (Corollary B14), $F(z)$ is analytic on the inside of C. Since C is arbitrary, $F(z)$ is analytic in U. For the verification of the formula for $F'(z)$, formula Theorem B.11 tells us that we can compute $F'(z)$ as a complex line integral. Let z be in U, and choose a circle C in U whose inside lies in U and contains z. Corollary 3.33 and two applications of Theorem B.11 combine to give

$$\begin{aligned} F'(z) = \tfrac{1}{2\pi i}\int_C \tfrac{F(\zeta)\,d\zeta}{(\zeta-z)^2} &= \tfrac{1}{2\pi i}\int_C \Big[\int_a^b \tfrac{\varphi(\zeta,t)\,dt}{(\zeta-z)^2}\Big]\,d\zeta \\ &= \int_a^b \tfrac{1}{2\pi i}\Big[\int_C \tfrac{\varphi(\zeta,t)\,d\zeta}{(\zeta-z)^2}\Big]\,dt = \int_a^b \tfrac{\partial\varphi(z,t)}{\partial z}\,dt. \end{aligned}$$ □

Corollary B.16 (Cauchy's estimate). If f is analytic on an open disk containing a circle C of radius r and center a and if $|f| \le M$ on C, then

$$|f^{(n)}(a)| \le Mn!\,r^{-n}.$$

PROOF. We put $z = a$ in the formula of Theorem B.11 and take the absolute value of both sides. Then

$$|f^{(n)}(a)| \le \tfrac{n!}{2\pi} M r^{-(n+1)} \ell(C) = n! M r^{-n},$$

since $\ell(C) = 2\pi r$. □

Corollary B.17 (Liouville's Theorem). The only bounded entire functions are the constant functions.

PROOF. If f is everywhere analytic and bounded on $\mathbb{C}$ and if $|f(z)| \le M$ everywhere, then Corollary B.16 with $n = 1$ says that $|f'(a)| \le Mr^{-n}$ for all a and all r. Thus f' is the 0 function. If $f = u + iv$, then u and v have first partial derivatives identically 0 and are hence constant. Therefore f is constant. □

Corollary B.18 (Fundamental Theorem of Algebra). If P is a polynomial with complex coefficients and with degree ≥ 1, then P has a root in $\mathbb{C}$, i.e., $P(z_0) = 0$ for some z_0 in $\mathbb{C}$.

PROOF. Arguing by contradiction, suppose that $P(z)$ is nowhere 0. Then $1/P(z)$ is analytic in all of $\mathbb{C}$. If P has degree n and leading coefficient $c_n \ne 0$, then $\lim_{z\to\infty} |P(z)|/|z|^n = |c_n| \ne 0$. Therefore $|1/P(z)|$ is bounded for $|z|$ sufficiently large, say for $|z| \ge R$. But also $|1/P(z)|$ is continuous for $|z| \le R$ and has to assume its maximum value. Consequently $1/P(z)$ is a bounded function analytic on all of $\mathbb{C}$ and must be constant, by Corollary B.17. □

B5. Taylor's Theorem

This section establishes the complex-variable form of Taylor's Theorem with remainder, and it goes on to show that the infinite Taylor series of an analytic function about $z = a$ converges to the function in the largest open disk centered at a in which the function is analytic.

By way of preliminaries we need to identify what is happening at the "exceptional points" that were allowed in Theorem B.8′ and Theorem B.9′. Those points are called **removable singularities**. The preliminary result is that the analytic function can be defined at such points in a way that makes those points no longer exceptional.

Proposition B.19 (Removable Singularity Theorem). Let U be a region in $\mathbb{C}$, let a be a point in U, and let $U' = U - \{a\}$. If f is analytic in U' and $\lim_{z\to a}(z-a)f(z) = 0$, then there exists a unique extension of f to an analytic function on U. The value of the extension at a is $\lim_{z\to a} f(z)$.

PROOF. Let C be a standard circle such that it and its inside are in U and such that a is in the inside. We rederive the Cauchy Integral Formula (Theorem B.10) using Theorem B.9′ with the function $g(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z}$ and two exceptional points, rather than one. One exceptional point is the point z where f is evaluated, and the other is the point a. The hypothesis for the exceptional point z is that $\lim_{\zeta\to z}(\zeta - z)g(\zeta) = 0$ and that is satisfied because $g'(z)$ exists. The hypothesis for the exceptional point a is that $\lim_{\zeta\to a}(\zeta - a)g(\zeta) = 0$, which follows since $z \neq a$ and $\lim_{\zeta\to a}(\zeta - a)f(\zeta) = 0$. The derivation yields

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad (*)$$

for all points $z \neq a$ inside C. Lemma B.13 shows that the right side of $(*)$ is an analytic function in the complement of C in $\mathbb{C}$. The function on U that equals the right side of $(*)$ inside C and that equals $f(z)$ elsewhere on $U - C$ is the required extension of f from U' to U. □

Theorem B.20 (Taylor's Theorem, first form). If f is an analytic function in a region U containing a, then there exists an analytic function $f_n(z)$ in U such that

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n.$$

The function $f_n(z)$ has $f_n(a) = \frac{1}{n!} f^{(n)}(a)$. Moreover, if C is a circle such that C and its inside are contained in U and if a is inside C, then $f_n(z)$ is given inside C by

$$f_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - a)^n (\zeta - z)}.$$

REMARK. One can solve the first formula of the theorem for $f_n(z)$ when $z \neq a$, and it follows that f_n is uniquely determined. The second formula of the theorem tells what $f_n(z)$ actually is.

PROOF. We begin by using Proposition B.19 to construct inductively certain analytic functions $f_1, \dots, f_n$ in U. To define f_1, we observe that the function defined by $g_1(z) = \frac{f(z) - f(a)}{z - a}$ for $z \neq a$ is analytic and that $\lim_{z \to a} g_1(z) = f'(a)$ exists. Thus g_1 satisfies $\lim_{z \to a}(z - a) g_1(z) = 0$. By Proposition B.19, g_1 extends to an analytic function f_1 defined on all of U.

Inductively suppose that we have constructed $f_1, \dots, f_{k-1}$. Then the function $g_k(z) = \frac{f_{k-1}(z) - f_{k-1}(a)}{z - a}$ for $z \neq a$ is analytic, and $\lim_{z \to a} g_k(z) = f^{(k)}(a)$ exists. Thus g_k satisfies $\lim_{z \to a}(z - a) g_k(z) = 0$. By Proposition B.19, g_k extends to an analytic function f_k defined on all of U. The induction is complete, and the result is that

$$\begin{aligned} f(z) = f(a) + (z - a) f_1(a) + (z - a)^2 f_2(a) \\ + \cdots + (z - a)^{n-1} f_{n-1}(a) + (z - a)^n f_n(z). \end{aligned}$$

Differentiating k times and setting $z = a$, we obtain $f^{(k)}(a) = k!\, f_k(a)$. The first formula of the theorem follows.

If C is as in the statement of the theorem, then the Cauchy Integral Formula gives

$$f_n(z) = \tfrac{1}{2\pi i} \int_C \tfrac{f_n(\zeta)\, d\zeta}{\zeta - z}. \qquad (*)$$

We put $z = \zeta$ in the first formula of the theorem, solve for $f_n(\zeta)$, and substitute into the right side of $(*)$. Then we get

$$f_n(z) = \tfrac{1}{2\pi i} \int_C \tfrac{f(\zeta)\, d\zeta}{(\zeta - a)^n (\zeta - z)} - \tfrac{1}{2\pi i} \sum_{k=0}^{n-1} \tfrac{1}{2\pi i} \tfrac{f^{(k)}(a)}{k!} \int_C \tfrac{d\zeta}{(\zeta - a)^{n-k} (\zeta - z)}. \qquad (**)$$

We shall show that

$$\int_C \frac{d\zeta}{(\zeta - a)^l (\zeta - z)} = 0 \qquad (\dagger)$$

for every integer $l \geq 1$. Then all the terms in the expression $\sum_{k=1}^{n-1}$ in $(**)$ will be 0, and $(**)$ will reduce to the second formula of the theorem. For $l = 1$, the left side of $(\dagger)$ is

$$\int_C \tfrac{d\zeta}{(\zeta - a)(\zeta - z)} = \tfrac{1}{z - a} \int_C \big(\tfrac{1}{\zeta - z} - \tfrac{1}{\zeta - a}\big)\, d\zeta,$$

and the part of the argument in the proof of Theorem B.10 that referred to Figure B.4 shows that this is 0. With z fixed, let us take the complex derivative of the identity

$$0 = \int_C \frac{d\zeta}{(\zeta-a)(\zeta-z)}$$

m times in the variable a, with z constant. Lemma B.13 allows us to put the complex derivatives underneath the integral sign. From the differentiation formula $\big(\frac{d}{da}\big)^m(\zeta-a)^{-1} = m!\,(\zeta-a)^{-(m+1)}$, we obtain

$$0 = \Big(\frac{d}{da}\Big)^m \int_C \frac{d\zeta}{(\zeta-a)(\zeta-z)} = \int_C \Big(\Big(\frac{d}{da}\Big)^m(\zeta-a)^{-1}\Big)\frac{d\zeta}{\zeta-z} = m!\int_C \frac{d\zeta}{(\zeta-a)^{m+1}(\zeta-z)}.$$

This proves (†) and completes the proof of the theorem. □

Theorem B.21 (Taylor's Theorem, second form). If f is an analytic function in a region U containing a, then the infinite series expansion

$$f(z) = f(a) + \sum_{k=1}^{\infty} \frac{f^{(k)}(a)}{k!}\,(z-a)^k$$

is valid everywhere in each open disk centered at a that is contained in U.

PROOF. Let C be a standard circle of radius R and center a that lies completely in U, and let M be the maximum value of $|f(\zeta)|$ on C. Fix z inside C. For any n, Theorem B.20 gives

$$f(z) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}\,(z-a)^k + (z-a)^{n+1}\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}(\zeta-z)}.$$

For $\zeta \in C$, we have $|\zeta-a| = R$ and $|\zeta-z| \geq R-|z-a|$. Thus the remainder term has

$$\begin{aligned}\Big|(z-a)^{n+1}\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n}(\zeta-z)}\Big| &\leq \frac{1}{2\pi}|z-a|^{n+1}MR^{-(n+1)}(R-|z-a|)^{-1}2\pi R\\ &= \frac{M|z-a|}{R-|z-a|}\big(|z-a|/R\big)^n\end{aligned}$$

Since $|z-a| < R$, the remainder term has limit 0 as n tends to infinity. □

B6. Local Properties of Analytic Functions

This section examines the zeros and poles of analytic functions. The point of departure is Taylor's Theorem (Theorems B.20 and B.21).

Proposition B.22. If f is analytic in a region U, if z_0 is a point in U, and if $f^{(n)}(z_0) = 0$ for all $n \geq 0$, then f is identically 0 on U.

PROOF. Let E be the subset of points a of U where $f^{(n)}(a) = 0$ for all n. This set is nonempty, since z_0 is in it. Theorem B.21 shows for each member a of E that there is a disk centered at a such that $f(z) = 0$ for all z in the disk, and therefore E is open. Since each $f^{(n)}$ is continuous, the set where any particular $f^{(n)}$ equals 0 is relatively closed in U. Taking the intersection over n of these sets, we see that E is relatively closed in U. Thus E is nonempty, open, and closed in the connected metric space U, and it must be all of U. □

Proposition B.23. If f is analytic in a region U and is not identically 0, then the zeros of f are isolated, i.e., for each z_0 for which $f(z_0) = 0$, there is a neighborhood of z_0 such that $f(z)$ is nonzero at all points of that neighborhood other than z_0. Consequently If f_1 and f_2 are two analytic functions in the region U and if $f_1(z_0) = f_2(z_0)$ at a subset of points z_0 in U with a limit point in U, then f_1 is identically equal to f_2 on U.

REMARKS. The second conclusion of the proposition is sometimes called the **Identity Theorem**. It is an immediate consequence of the Identity Theorem that various trigonometric identities that are valid for real variables remain valid for complex variables as well. For example, it follows from the identity $\sin^2 x + \cos^2 x = 1$ for a real variable x, which was proved in Section I.7, that $\sin^2 z + \cos^2 z = 1$ for a complex variable z.

PROOF. For the first conclusion suppose that $f(z_0) = 0$. If f is not identically 0, Proposition B.22 shows that $f^{(n)}(z_0) \neq 0$ for some n. Let h be the smallest integer for which $f^{(h)}(z_0) \neq 0$. Theorem B.20 shows that $f(z) = f_h(z)(z - z_0)^h$ with f_h analytic in U and with $f_h(z_0) = \frac{1}{n!} f^{(h)}(a)$. By assumption on h, $f_h(z)$ is a continuous function that is nonzero at z_0. It is therefore nonzero in a neighborhood of z_0, and the neighborhood in question is the neighborhood whose existence is asserted by the first conclusion of the proposition. The second conclusion follows by applying the first conclusion to $f = f_1 - f_2$. □

EXAMPLE. Suppose that $f(z)$ is known to be an entire function with $f(i/n) = e^{-1/n^2}$. What is $f(1)$? Normally there is no formula for extending f from the known points to other points, but here we can argue as follows. Let $g(z) = e^{z^2}$. Then $g(i/n) = e^{-1/n^2}$, and Proposition B.23 says that $f(z) = e^{z^2}$. Therefore $f(1) = e$.

If an analytic function f in a region U has $f(z_0) = 0$ but f is not the zero function, then the integer h in the proof of Proposition B.23, i.e., the smallest integer for which $f^{(h)}(z_0) \neq 0$, is called the **order** of the zero of f at z_0.

Corollary B.24 (Maximum Modulus Theorem). If f is analytic in a region U and if $|f|$ has a local maximum at a point of U, then f is a constant function.

PROOF. Without loss of generality, we may assume that $f(z_0) \geq 0$. Let D be a disk of radius R and center z_0 small enough so that the closure $\overline{D}$ of D is contained in U and so that $|f(z)| \leq |f(z_0)|$ for all z in $\overline{D}$. Let r be any number with $0 < r \leq R$. Parametrizing the circle $|z - z_0| = r$ in the standard way and applying the Cauchy Integral Formula (Theorem B.10), we obtain

$$f(z_0) = \tfrac{1}{2\pi i} \int_0^{2\pi} \tfrac{f(z_0+re^{i\theta})}{re^{i\theta}}\, rie^{i\theta}\, d\theta = \tfrac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \qquad (*)$$

Writing $f = u + iv$, extracting the real part of $(*)$, and taking into account that $f(z_0)$ is real, we obtain

$$f(z_0) = \tfrac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Hence

$$\tfrac{1}{2\pi} \int_0^{2\pi} [f(z_0) - u(z_0 + re^{i\theta})]\, d\theta = 0. \qquad (**)$$

By assumption the integrand on the left side of $(**)$ is everywhere ≥ 0, and it is continuous. Therefore it is identically 0, and $u(z_0 + re^{i\theta}) = f(z_0)$ for all θ. Since $|f(z_0)| \geq |f(z_0 + re^{i\theta})|$, we conclude that $f(z_0 + re^{i\theta}) = f(z_0)$ for all θ. This being true for all r with $0 < r \leq R$, f is constantly equal to $f(z_0)$ in a neighborhood of z_0. By Proposition B.23 applied to the function $f(z) - f(z_0)$, f is a constant function on U. □

We turn our attention to functions f that are analytic in an open set U containing z_0 but maybe not at z_0 itself. In this case, z_0 is said to be an **isolated singularity** of f. The case of a removable singularity was treated in Proposition B.19. If z_0 is a removable singularity, then $f(z_0)$ can be defined in such a way that the extended function is analytic on all of U.

The next case of interest is that $\lim_{z\to z_0} f(z) = \infty$, i.e., that $\lim_{z\to z_0} \frac{1}{f(z)} = 0$. In this case, we say that z_0 is a **pole** of f. The limit relation implies that there is a number $\delta > 0$ such that $g(z) = 1/f(z)$ is a bounded complex-valued function for $0 < |z - z_0| < \delta$. Since division respects analyticity, g is analytic for $0 < |z - z_0| < \delta$ and g has an isolated singularity at z_0. The hypothesis is that $\lim_{z\to z_0} g(z) = 0$, and therefore the singularity of g is removable. By Proposition B.19, g extends to be analytic for $|z - z_0| < \delta$ if we define $g(z) = 0$. Then z_0 is a zero of some order h for g, and we can write $g(z) = (z - z_0)^h g_h(z)$ for an analytic function g_h with $g_h(z_0) \neq 0$. Put $f_h(z) = 1/g_h(z)$. This is analytic in some possibly smaller disk $|z - z_0| < \delta'$, and Theorem B.20 allows us to write

$$f_h(z) = b_h + b_{h-1}(z - z_0) + b_{h-2}(z - z_0)^2 + \cdots + b_1(z - z_0)^{h-1} + (z - z_0)^h c(z)$$

with $b_h \neq 0$ and with $c(z)$ analytic for $|z - z_0| < \delta'$. Consequently

$$f(z) = 1/g(z) = (z - z_0)^{-h}/g_h(z) = (z - z_0)^{-h} f_h(z),$$

and we see that $f(z)$ has an expansion

$$f(z) = b_h(z - z_0)^{-h} + b_{h-1}(z - z_0)^{-h+1} + \cdots + b_1(z - z_0)^{-1} + c(z).$$

We say that the pole of f at z_0 has **order** h at z_0. The pole is **simple** if its order is 1. The part of the above expansion that involves the powers $(z-z_0)^{-h}, \ldots, (z-z_0)^{-1}$ is called the **singular part** of f about z_0. If f and φ are two analytic functions in some region $0 < |z| < \delta$ having a pole at z_0 and having the same singular part about z_0, then $f - \varphi$ has a removable singularity at z_0.

A function in a region that is analytic except for poles is said to be a **meromorphic function**. The set of meromorphic functions on a region U is closed under addition, subtraction, multiplication, and division except for division by 0.

An isolated singularity that is not removable and is not a pole is called an **essential singularity**. For example the function $e^{1/z}$ has an essential singularity at $z = 0$. The first fact about such singularities is the following classical result of Weierstrass. We shall have more to say about these singularities in Corollary B.48.

Proposition B.25 (Weierstrass). An analytic function comes arbitrarily close to each complex value in every neighborhood of an essential singularity.

PROOF. Assume the contrary for an essential singularity of $f(z)$ at the point a. Then there we can find a complex number w_0 and a positive number ϵ such that $|f(z) - w_0| \geq \epsilon$ for all z in some set $0 < |z - a| < \delta$. Put $g(z) = 1/(f(z) - w_0)$. Then g is analytic and bounded for $0 < |z - z| < \delta$, and Proposition B.19 shows that g has a removable singularity at z_0. Either $g(z)$ is nonzero at z_0, in which case $f(z) - w_0$ and also $f(z)$ would be analytic in a neighborhood z_0, contradiction, or g has a zero at z_0 of some order $h > 0$, $f(z) - w_0$ has a pole at z_0 of order h, and $f(z)$ has a pole of order h, contradiction. □

We conclude our discussion of local properties of analytic functions by deriving the Inverse Function Theorem for analytic functions of one complex variable from the Inverse Function for C^1 functions of two real variables.

Proposition B.26 (Inverse Function Theorem). Let $f(z)$ be analytic on a region U, let z_0 be in U, and put $w_0 = f(z_0)$. If $f'(z_0) \neq 0$, then there exist open sets U and V in $\mathbb{C}$ such that z_0 is in U, w_0 is in V, f is one-one onto from U onto V, and the inverse function $g : V \to U$ is analytic. In this situation the complex derivatives of f and g are related by $g'(f(z)) = f'(z)^{-1}$ for z in U.

PROOF. Write $z = x + iy$ and $f = u + iv$. We associate to f the function F of two real variables $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$. Proposition B.1 shows that F is differentiable at every point of U, and its Jacobian matrix at each point is $M(f'(z))$. Since f' is analytic, the entries of the Jacobian matrix are continuous. Therefore F is of class C^1. The matrix $M(f'(z_0)) = M(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}) = \begin{pmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{pmatrix}(x_0, y_0)$ is invertible, since $f'(z_0) \neq 0$, with inverse given by $\big((\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2\big)^{-1} M(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x})$. The real-variable Inverse function Theorem (Theorem 3.17) applies and yields a C^1 inverse function to F. Because of Proposition B.1 the form of the Jacobian matrix of the inverse function shows that the inverse function, when viewed as a function of one complex variable, is analytic. □

B7. Logarithms and Winding Numbers

In this section we address the question of an inverse for the exponential function $z \mapsto e^z$, and we introduce the notion of the "winding number" or "index" of a piecewise C^1 closed curve around a point.

From Section I.7 the exponential function satisfies $e^{x+iy} = e^x(\cos y + i \sin y)$. We know that $x \mapsto e^x$ is one-one from $(-\infty, \infty)$ onto $(0, \infty)$ and that $y \mapsto \cos y + i \sin y$ is one-one from $[0, 2\pi)$ onto the unit circle $|z| = 1$ in $\mathbb{C}$. In stating this property of $y \mapsto \cos y + i \sin y$, we know that we can replace $[0, 2\pi)$ by any interval of $\mathbb{R}$ of length 2π that contains one of its endpoints but not the other.

Thus the function $z \mapsto e^z$ carries $\mathbb{C}$ onto $\mathbb{C} - \{0\}$, but it is not one-one, having a periodicity property in the y variable. If w is any point in $\mathbb{C} - \{0\}$, $\log w$ can refer to any of the infinitely many values of z for which $e^z = w$. In that sense, log is not a function, not being single-valued.

To make a function usable in complex analysis, one restricts attention to some open set U of $\mathbb{C} - \{0\}$ such that e^z maps an open set one-one onto U. Since e^z has complex derivative $e^z \neq 0$ everywhere, the Inverse Function Theorem (Proposition B.26) applies everywhere, and the result is a determination of the logarithm as an analytic function on the set U. Such a determination is called a (single-valued) **branch of the logarithm**. Let us observe that any branch of the logarithm has complex derivative $1/z$; in fact, if g is such a branch, then Proposition B.26 gives $g'(e^z) = 1/e^z$, and hence $g'(z) = 1/z$ for all $z \neq 0$. The **principal branch** is the determination that uses $U = \mathbb{C} - (-\infty, 0]$; it is called $\operatorname{Log} z$. Thus $\operatorname{Log} z$ is a one-one analytic function from $\mathbb{C} - (-\infty, 0]$ onto $\{(z \in \mathbb{C} \mid -\pi < \operatorname{Im} z < \pi\}$. Frequently branches of the logarithm, the principal branch being one, have domain the difference of $\mathbb{C}$ and some ray from the origin. But this need not be the case; it is necessary only that the branch of the logarithm be a partial inverse of the exponential function.

The n^{th} root behaves similarly. A complex number $z \neq 0$ has n n^{th} roots, differing by n^{th} roots of 1, and $z^{1/n}$ can refer to any of them. To make a function out of n^{th} root that is usable in complex analysis, one restricts attention to some open set U of $\mathbb{C}-\{0\}$ such that z^n maps an open set one-one onto U. Since z^n has complex derivative nz^{n-1} everywhere, the Inverse Function Theorem (Proposition B.26) applies everywhere on $\mathbb{C}-\{0\}$, and the result is a determination of $z^{1/n}$ as an analytic function on the set U. Let us check what the complex derivative is for a branch g of n^{th} root. We have $g'(z^n) = 1/(nz^{n-1}) = z/(nz^n)$; putting $w = z^n$ and $z = g(w)$, we obtain $g'(w) = g(w)/(nw)$. This formula is valid for every branch of n^{th} root. The **principal branch** of n^{th} root is the determination that uses $U = \mathbb{C} - (-\infty, 0]$; it carries U onto $\{z = re^{i\theta} \in \mathbb{C} \mid r > 0 \text{ and } -\frac{\pi}{n} < \theta < \frac{\pi}{n}\}$.

Branches of n^{th} root can alternatively be defined in terms of the logarithm. The formal side calculation starts from the desired formula $\log(z^{1/n}) = \frac{1}{n}\log z$ and exponentiates to get $z^{1/n} = e^{\frac{1}{n}\log z}$. Any branch of the logarithm then leads to the definition of a branch of n^{th} root. To compute the complex derivative, we use the usual rules: $\frac{d}{dz}(z^{1/n}) = \frac{d}{dz}(e^{\frac{1}{n}\log z}) = e^{\frac{1}{n}\log z}\frac{d}{dz}(\frac{1}{n}\log z) = z^{1/n}\frac{1}{nz}$, the same as in the previous paragraph.

Since either definition is available for n^{th} root, let us use the one in terms of logarithm, which will extend to a definition of z^p for any real number p: $z^p = e^{p\log z}$. Again any branch of the logarithm leads to a branch[8] of z^p. By the same computation as for n^{th} root, any branch of z^p has complex derivative $pz^{1/p}/z$.

Similar considerations apply to the arcsine and arctangent functions. We carry out the details for the arcsine function in the next paragraph and leave the details for the arctangent function to Problem 34 at the end of this appendix.

The sine function has $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$. We can solve $w = \frac{1}{2i}(e^{iz} - e^{-iz})$ for z in terms of w by first solving a quadratic equation for e^{iz} and then taking a logarithm and dividing by i. The result is that $z = -i\log\big(iw + \sqrt{1-w^2}\big)$ for branches of the logarithm and the square root. We readily check that the complex derivative of this expression with respect to w is $1/\sqrt{1-w^2}$ consistently with the case that w is a real number in $(-1, 1)$, known from Corollary 1.46a. To decide what branches of logarithm and square root we can use, let us try for the principal branch of the logarithm. Then the expression whose logarithm is being taken, namely $iw + \sqrt{1-w^2}$, must not be real and ≤ 0. The exceptional case is that $iw + \sqrt{1-w^2} = r \leq 0$. Squaring shows that the exceptional case must

[8]The definition given above for a branch of the logarithm or of the n^{th} root was in terms of an inverse function, but that definition is inadequate in the case of z^p. Let us simply take **branch** to mean a consistent determination of an analytic function on a region of $\mathbb{C}$ that satisfies appropriate properties for that function. The formal definition is in terms of "global analytic functions," but we shall not pursue the matter.

have $(r - iw)^2 = 1 - w^2$, hence must have $r^2 - 2iw = 1$. The exceptional case thus has w imaginary. Say $w = iv$. Then $i(iv) + \sqrt{1-(iv)^2} = r \le 0$, i.e., $-v + \sqrt{1+v^2} = r \le 0$. This never happens for v real. Thus we can use the principal branch of the logarithm in all circumstances, and we have only to make sense of the square root. The principal branch of the square root asks that $1 - w^2$ not be real ≤ 0, thus that w not be real with $|w| \ge 1$. If we exclude this set, then $\arcsin w$ is well defined as $-i \operatorname{Log}\big(iw + \sqrt{1-w^2}\big)$, with the understanding that the principal branch of the square root is to be used. The values of the square root are thus in the open right half plane.

The fact that there is no single-valued analytic function $\log z$ on $\mathbb{C} - \{0\}$ is intimately related to the fact that $\int_C z^{-1}\,dz \ne 0$ when C is the unit circle. Indeed, any branch of the logarithm has complex derivative $1/z$ on its domain. If there were an analytic function with complex derivative $1/z$ on all of $\mathbb{C} - \{0\}$, then Corollary B.6 would say that $\int_C z^{-1}\,dz$ is indeed 0 over every piecewise C^1 closed curve that does not pass through the origin. Let us take advantage of this fact to introduce the notion of the "winding number" or "index" of a closed curve about a point.

Proposition B.27. If the piecewise C^1 closed curve γ in $\mathbb{C}$ does not pass through the point z_0, then the value of the complex line integral

$$\int_\gamma \frac{dz}{z - z_0}$$

is a multiple of $2\pi i$.

PROOF. Let γ be given parametrically by $\gamma(t)$ for $a \le t \le b$, so that the value of the given integral is $h(b)$, where $h : [a, b] \to \mathbb{C}$ is the function

$$h(t) = \textstyle\int_a^t \frac{\gamma'(u)}{\gamma(u)-z_0}\,du.$$

The function h is continuous on $[a, b]$, and on each open interval where $\gamma(t)$ is C^1, h is differentiable with derivative

$$h'(t) = \gamma'(t)/\big(\gamma(t) - z_0\big). \tag{$*$}$$

Thus the complex-valued function

$$e^{-h(t)}(\gamma(t) - z_0) \tag{$**$}$$

is continuous on $[a, b]$, and on each open interval where $\gamma(t)$ is C^1, $(**)$ is differentiable; in view of $(*)$, the derivative of $(**)$ is

$$e^{-h(t)}h'(t)(\gamma(t) - z_0) + e^{-h(t)}\gamma'(t) = 0.$$

Accordingly $(**)$ is constant on each closed interval between whose ends $\gamma(t)$ is C^1, and it follows that $(**)$ is constant on $[a, b]$. Comparing the values at the endpoints gives

$$\gamma(a)-z_0 = e^{-h(a)}(\gamma(a)-z_0) = e^{-h(b)}(\gamma(b)-z_0) = \exp\left(-\int_\gamma \tfrac{dz}{z-z_0}\right)(\gamma(a)-z_0).$$

Since $\gamma(a) \neq z_0$, we conclude that $\exp\left(-\int_\gamma \frac{dz}{z-z_0}\right) = 1$. Hence $\int_\gamma \frac{dz}{z-z_0}$ is in $2\pi i\mathbb{Z}$. $\square$

If γ is a piecewise C^1 closed curve that does not pass through z_0, let $n(\gamma, z_0)$ be the integer

$$n(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}.$$

This is the **winding number** or **index** of γ about z_0.

Some intuition about winding numbers may be helpful. The relevant quantity to discuss is the **argument** of a nonzero complex number. Any nonzero complex number z can be decomposed as $z = re^{i\theta}$ with $r = |z| > 0$. Here r and $e^{i\theta}$ are unique, but θ is not unique, being determined only up to an additive multiple of 2π. The nonunique number θ is called the **argument** of z, written $\arg z$. It is nothing more than the imaginary part of $\log z$. Let the above closed curve γ have domain $[a, b]$. Since γ is closed, $\gamma(a) = \gamma(b)$. Fix t in $[a, b]$. Since γ does not pass through z_0, the argument of $\gamma(t) - z_0$ is well defined up to an additive multiple of 2π. Although there is an ambiguity in the definition of argument, it is intuitively plausible (and we shall it rigorously at the end of Section B12) that the argument of $\gamma(t) - z_0$ can be chosen to vary continuously in t once a choice is made for the argument of $\gamma(a) - z_0$. With this fact in mind, it follows from the equality $\gamma(a) = \gamma(b)$ that the difference of the values of the argument at $t = b$ and $t = a$ must be a multiple of 2π. As will be shown in Section B12, *this multiple is the winding number of* γ *about* z_0.

Proposition B.28. If γ is a piecewise C^1 closed curve in $\mathbb{C}$, then the function $z_0 \mapsto n(\gamma, z_0)$ is constant on each connected component of the open complement of image(γ) in $\mathbb{C}$.

PROOF. The function $z_0 \mapsto n(\gamma, z_0)$ is integer-valued, according to Proposition B.27. An integer-valued continuous function has to be constant on connected components, and thus is enough to show that $z_0 \mapsto n(\gamma, z_0)$ is continuous. For each $\delta > 0$, consider points at a distance $> \delta$ from image(γ). If z_1 and z_0 are two such points, then

$$n(\gamma, z_1) - n(\gamma, z_0) = \tfrac{1}{2\pi i}\int_\gamma \left[\tfrac{1}{z-z_1} - \tfrac{1}{z-z_0}\right] dz = \tfrac{z_1-z_0}{2\pi i}\int_\gamma \tfrac{dz}{(z-z_1)(z-z_0)},$$

and

$$|n(\gamma, z_1) - n(\gamma, z_0)| \le \tfrac{|z_1 - z_0|}{2\pi}\,\delta^{-2}\ell(\gamma). \qquad (*)$$

If z_0 is given, let δ be twice the distance of z_0 to image(γ). All points z_1 sufficiently close to z_0 are at distance $> \delta$ from image(γ), and the estimate $(*)$ applies. Thus the function $z_0 \mapsto n(\gamma, z_0)$ is continuous at z_0. □

Since image(γ) is compact, it lies in the closed disk of radius R centered at 0, for some sufficiently large R. The complement of the disk is connected, and thus the complement of image(γ) contains exactly one unbounded component.

Proposition B.29. If γ is a piecewise C^1 closed curve in $\mathbb{C}$, then $n(\gamma, z_0) = 0$ on the unbounded component of the complement of image(γ).

PROOF. Suppose that image(γ) is contained in the closed disk of radius R centered at 0. If z_0 is outside this disk, then

$$|n(\gamma, z_0)| = \tfrac{1}{2\pi}\Big|\int_\gamma \tfrac{dz}{z - z_0}\Big| \le \tfrac{1}{2\pi}\tfrac{1}{|z_0| - R}\,\ell(\gamma),$$

and this tends to 0 as $|z_0|$ tends to infinity. Since according to Proposition B.28, $n(\gamma, z_0)$ is constant on the unbounded component, its value must be 0. □

Let us take note of some simple cases where we can compute winding numbers easily:

(i) If k is a positive integer and σ is a piecewise C^1 curve that traverses k times over the image of γ, then $n(\sigma, z_0) = kn(\gamma, z_0)$.

(ii) $n(-\gamma, z_0) = -n(\gamma, z_0)$.

(iii) If C is a standard circle with center a, the $n(C, z_0) = 1$ for every point z_0 inside C. In fact, we saw by direct computation near the end of Section B2 that $n(C, a) = 1$. From Proposition B.28 it follows that $n(C, z_0) = 1$ for every point z_0 inside C, since a and z_0 lie in the same component of the complement of image(γ). (Actually we effectively observed the equality $n(\gamma, z_0) = n(\gamma, a)$ earlier in an example following Theorem B.9; at that time we proved it by introducing a small standard circle centered at z_0 and canceling line segments between the two circles, as in Figure B.4.)

In each case we observe that the asserted winding number matches the value from the intuitive definition given before Proposition B.28.

Using winding numbers, we can generalize the setting of the Cauchy Integral Formula so that any piecewise C^1 closed curve is allowed. For now, the region will still be restricted to a disk, but in Sections B12 and B13 we shall see how to allow more general regions.

Theorem B.30 (Cauchy Integral Formula, general form for a disk). Let f be analytic in an open disk D, and let γ be any piecewise C^1 closed curve in D. If z is any point of D not on image(γ), then

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

PROOF. We apply the Cauchy Integral Theorem (Theorem B.9) to the function

$$g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \text{for } \zeta \in D - \{z\} \\ f'(z) & \text{for } \zeta = z. \end{cases}$$

on the disk D. This is analytic except possibly at z. However, z is a removable singularity, and thus g is analytic in all of D. Theorem B.9 applies and gives $\int_\gamma g(\zeta)\, d\zeta = 0$. Therefore

$$\tfrac{1}{2\pi i} \int_\gamma \tfrac{f(\zeta)\, d\zeta}{\zeta - z} = \left(\tfrac{1}{2\pi i} \int_\gamma \tfrac{d\zeta}{\zeta - z}\right) f(z) = n(\gamma, z) f(z). \qquad \square$$

B8. Operations on Taylor Series

This section concerns the computation of Taylor series coefficients. For a few functions an appeal to the definition is as good a method as any for finding Taylor series coefficients. Among these are the series for exp, sin, and cos, which were already noted in Section B2.

Another series that can be computed directly from the definition is the **binomial series**, the series for $(1-z)^{-p}$. Let us pay particular attention to p real.[9] We are defining general real powers by means of the logarithm, and we use the principal branch of the logarithm here. The principal branch makes $(1-z)^{-p}$ meaningful except when z is real and ≥ 1. Thus $(1-z)^{-p}$ is analytic for $|z| < 1$, and Taylor's Theorem (Theorem B.21) says that the series will be convergent there.[10] The series is

$$(1-z)^{-p} = 1 + \sum_{n=1}^{\infty} \tfrac{p(p-1)\cdots(p-n+1)}{n!} z^n.$$

In Section I.7 we observed the convergence of this expansion by elementary means, but the fact that the series converged to the function required additional steps, Theorem B.21 not being available at the time. Instead we showed that the

[9] We build a minus sign into the exponent because the most familiar case is the case of $(1-z)^{-1}$, which gives the geometric series $\sum_{n=0}^{\infty} z^n$.

[10] It will of course also be convergent for *all* z when p is a nonpositive integer.

sum of the series satisfied a differential equation, and we solved the differential equation.

In principle as many coefficients as desired can be computed from the definition for any other Taylor series, but in practice there are often easier methods. Any method that yields a power series converging to the function in question is actually finding the Taylor series, since the sum of the series determines the coefficients, according to Theorem B.20. Actually to find the Taylor coefficients for a function through the z^n term, it is not necessary to seek the entire Taylor series nor even to know the radius of convergence. Theorem B.20 tells us that if f is analytic near 0, then $f(z)$ can be written in the form

$$a_0 + a_1 z + \cdots + a_n z^n + [z^{n+1}],$$

where $[z^{n+1}]$ is an analytic function that has a zero at least of order $n+1$ at 0; moreover, in any such expansion each a_k is the Taylor coefficient $f^{(k)}/k!$.

Let us see how this approach can work in the context of an example. Consider

$$f(z) = \frac{z}{e^z - 1} = \frac{z}{\sum_{n=1}^{\infty} \frac{z^n}{n!}} = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{z^n}{(n+1)!}} = \frac{1}{1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + [z^5]}.$$

The sum formula for a geometric series tells us that $\frac{1}{1+w} = \sum_{n=0}^{\infty} (-1)^n w^n = 1 - w + w^2 - w^3 + w^4 + [w^5]$. Substituting $w = \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + [z^5]$, we obtain

$$\begin{aligned} f(z) = 1 &- (\tfrac{z}{2} + \tfrac{z^2}{6} + \tfrac{z^3}{24} + \tfrac{z^4}{120} + [z^5]) + (\tfrac{z}{2} + \tfrac{z^2}{6} + \tfrac{z^3}{24} + \tfrac{z^4}{120} + [z^5])^2 \\ &- (\tfrac{z}{2} + \tfrac{z^2}{6} + \tfrac{z^3}{24} + \tfrac{z^4}{120} + [z^5])^3 + (\tfrac{z}{2} + \tfrac{z^2}{6} + \tfrac{z^3}{24} + \tfrac{z^4}{120} + [z^5])^4 + [z^5], \end{aligned}$$

the last step using that $[w^5]$ expands out as $[z^5]$. Expanding the second, third, and fourth powers and then lumping all powers of z higher than 4 into $[z^5]$ shows that

$$\begin{aligned} f(z) &= 1 - (\tfrac{z}{2} + \tfrac{z^2}{6} + \tfrac{z^3}{24} + \tfrac{z^4}{120}) + (\tfrac{z^2}{4} + \tfrac{z^3}{6} + \tfrac{5z^4}{72}) - (\tfrac{z^3}{8} + \tfrac{z^4}{8}) + \tfrac{z^4}{16} + [z^5] \\ &= 1 - \tfrac{z}{2} + \tfrac{z^2}{12} - \tfrac{z^4}{720} + [z^5]. \end{aligned}$$

An alternative way of making this computation without using the geometric series is to write

$$(1 + \tfrac{z}{2} + \tfrac{z^2}{6} + \tfrac{z^3}{24} + \tfrac{z^4}{120} + [z^5])(1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + [z]^5) = 1,$$

expand everything out, equate coefficients of like powers of z, and solve recursively for the coefficients a_1, a_2, a_3, a_4 of the power series expansion of $f(z)$.

Complex differentiation is handled by term-by-term differentiation of a series, as is seen directly from the formula for Taylor coefficients. In view of Proposition B.5, if $f(z)$ is a given analytic function, integration in a disk amounts to a complex line integral of the form $\int_\gamma f(\zeta)\,d\zeta$, where γ goes from z_0 to z within a disk; the complex line integral is independent of the path, which can therefore be ignored. The effect on Taylor coefficients is the opposite of the effect for differentiation. Thus the power series expansion of $\log(1-z)$ is well defined for $|z|<1$ and is given by integrating the series for $(1-z)^{-1}$ term by term.[11]

The most interesting operation is composition. If f is analytic for $|z|<r$ and g is analytic in a disk containing $f(\{z \mid |z|<r\})$, then $z \mapsto g(f(z))$ is analytic for $|z|<r$, and the series expansion for the composition is obtained by composing the two series. The computation can be unpleasant unless $f(0)=0$, a condition that tends to be satisfied in practice. Composition with $f(0)=0$ is already interesting when f is a polynomial. For example, e^w has a known series expansion. If we substitute the polynomial $w=z^2$, we obtain the expansion for e^{z^2} as

$$e^{z^2}=\sum_{n=0}^{\infty}\tfrac{z^{2n}}{n!},$$

and again this has to be the Taylor series expansion about 0. This formula is not at all obvious by computing the Taylor coefficients of e^{z^2} from the definition.

Similarly the binomial series $(1-w)^{-1/2}=1+\sum\limits_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2^n n!}\,w^n$ leads to the expansion

$$(1-z^2)^{-1/2}=1+\sum_{n=1}^{\infty}\tfrac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2^n n!}\,z^{2n},$$

and this can be integrated term by term to give an expansion for $\arcsin z$, since we know from Section B7 that $\frac{d}{dz}\arcsin z=(1-z^2)^{-1/2}$.

When the inside function in a composition is not a polynomial, the computation is messier, but the principles are the same, at least if $f(0)=0$. Suppose that $f(z)=\sum_{n=1}^{\infty}a_n z^n$ and $g(w)=\sum_{n=0}^{\infty}b_n w^n$. Substitution gives

$$g(f(z))=b_0+b_1\Big(\sum_{n=1}^{\infty}a_n z_n\Big)+b_2\Big(\sum_{n=1}^{\infty}a_n z_n\Big)^2+\dots.$$

When the left side is written out in powers of z, the contribution from $b_k\Big(\sum\limits_{l=1}^{\infty}a_l z_l\Big)^k$ starts with $b_k a_1^k z^k$. Thus only the coefficients $b_0,\dots,b_n$ can contribute to the coefficient of z^n.

[11] … and taking the constant term to be 0 so that the value of the series at $z=0$ matches the value of the function there.

EXAMPLE. It will be shown by complex-variable theory that

$$\exp\left(z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \dots\right) = 1/(1-z) \qquad \text{for all } |z| < 1.$$

This identity was established in a complicated way using real-variable methods in Section I.10 and Problems 30–35 at the end of Chapter I. This example will establish this identity easily by complex-variable methods. In fact, we know either from this section or from properties of geometric series that $\sum_{n=0}^\infty z^n = 1/(1-z)$ for $|z| < 1$. For $|z| < 1$, $1/(1-z)$ is in the right half plane, and therefore the principal branch of the logarithm, Log, of it is analytic on $1/(1-z)$ for $|z| < 1$. Hence its Taylor series about $z = 0$ converges to it for $|z| < 1$. Moreover, $\frac{d}{dz}\text{Log}(1/(1-z)) = 1/(1-z) = \sum_{n=0}^\infty z^n$, and consequently $\text{Log}(1/(1-z)) = \sum_{n=0}^\infty \frac{1}{n+1} z^{n+1} = \sum_{n=1}^\infty \frac{1}{n} z^n$. Since Log and exp are inverse functions, we can exponentiate both sides to get $1/(1-z) = \exp\left(\sum_{n=1}^\infty \frac{1}{n} z^n\right)$.

Finally we make some remarks about the Taylor series expansions of inverse functions. Let us assume that the analytic function f has $f(0) = 0$ and $f'(z) \neq 0$. Write $f(z) = \sum_{k=1}^\infty a_k z^k$ in a disk centered at 0. The Inverse Function Theorem (Proposition B.26) applies and gives us an analytic function $g(w)$ with $g(0) = 0$ and $g(f(z)) = z$. Let $g(w) = \sum_{n=1}^\infty b_n w^n$. We do not readily get an estimate for a radius of convergence of the series for g, but we know that it is positive. We can write out the composed series as

$$z = \sum_{n=1}^\infty b_n \Big(\sum_{k=1}^\infty a_k z^k\Big)^n$$

and solve recursively for the unknown coefficients b_n. Specifically the low order terms are

$$z = b_1(a_1 z + a_2 z^2 + [z^3]) + b_2((a_1 z + a_2 z^2 + [z^3])^2 + [z^3].$$

The coefficient of z on both sides gives us the equation $1 = b_1 a_1$, which is solvable since $a_1 = f'(0) \neq 0$. From z^2, we get $0 = b_1 a_2 + b_2 a_1^2 = 0$, which is solvable for b_1 again since $a_1 \neq 0$. In general when $n > 1$, the equation from z^n is

$$0 = b_1(\cdots) + b_2(\cdots) + \cdots + b_{n-1}(\cdots) + b_n a_1^n,$$

and this can be solved for b_n in terms of the earlier coefficients because $a_1 \neq 0$. In this way we are able to obtain all the coefficients recursively.

B9. Argument Principle

This section examines the local behavior of an analytic function beyond the analysis of zeros and poles that appeared in Section B6. The new ingredient is a consideration of the effect of applying an analytic function to a piecewise C^1 curve.

Lemma B.31. If γ is a piecewise C^1 curve in $\mathbb{C}$ and f is an analytic function defined on a region containing image(γ), then $f \circ \gamma$ is a piecewise C^1 curve. Moreover, if γ is parametrized by $t \mapsto z(t)$ for $t \in [a, b]$, then

$$\frac{d}{dt} f(z(t)) = f'(z(t))z'(t)$$

for all t for which $z'(t)$ exists, where f' denotes the complex derivative of f.

PROOF. The interval $[a, b]$ is the union of finitely many nonoverlapping intervals $[\alpha, \beta]$ such that γ is continuous on $[\alpha, \beta]$, is of class C^1 on (α, β), and has each component of $\gamma'(t)$ bounded above or bounded below near α and β. If we regard f as a C^1 function from an open subset of $\mathbb{R}^2$ containing image(γ) into $\mathbb{R}^2$, then the composition $f \circ \gamma$ has the same properties on each interval $[\alpha, \beta]$ that γ does. Hence $f \circ \gamma$ is a piecewise C^1 curve.

Write $z(t) = x(t) + iy(t)$ and $f(z) = u(x, y) + iv(x, y)$. The chain rule in the calculus of two real variables gives

$$\begin{aligned} \tfrac{d}{dt} f(z(t)) &= \left(\tfrac{\partial u}{\partial x}(x(t), y(t))\tfrac{dx}{dt} + \tfrac{\partial u}{\partial y}(x(t), y(t))\tfrac{dy}{dt}\right) \\ &\quad + i\left(\tfrac{\partial v}{\partial x}(x(t), y(t))\tfrac{dx}{dt} + \tfrac{\partial v}{\partial y}(x(t), y(t))\tfrac{dy}{dt}\right) \\ &= \tfrac{\partial f}{\partial x}(x(t), y(t))x'(t) + \tfrac{\partial f}{\partial y}(x(t), y(t))y'(t). \end{aligned}$$

By the Cauchy–Riemann equations (Corollary 8.2′), $\frac{\partial f}{\partial x}(z) = -i\frac{\partial f}{\partial y}(z) = f'(z)$. Thus the above expression is

$$= f'(z(t))x'(t) + if'(z(t))y'(t) = f'(z(t))z'(t). \qquad \square$$

Lemma B.32. Suppose that γ is a piecewise C^1 closed curve in $\mathbb{C}$ and that f is an analytic function defined on a region containing image(γ) and nowhere equal to 0 on image(γ). Let Γ be the piecewise C^1 closed curve $\Gamma = f \circ \gamma$. Then

$$n(\Gamma, 0) = \int_\gamma \frac{f'(z)}{f(z)}\, dz.$$

PROOF. Parametrize γ as $z(t)$ for $a \le t \le b$. Then Lemma B.31 gives

$$n(\Gamma, 0) = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_a^b \frac{\frac{d}{dt}(f(z(t))\,dt}{f(z(t))} = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))z'(t)\,dt}{f(z(t))} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

$\square$

Proposition B.33 (Argument Principle, local form). Let $\{a_j\}$ be the set of distinct zeros and $\{b_l\}$ be the set of distinct poles of a meromorphic function $f(z) \not\equiv 0$ defined in a disk U, and suppose that a_j has order h_j and b_l has order k_l. For every piecewise C^1 closed curve in U not passing through a zero or pole,

$$\sum_j h_j n(\gamma, a_j) - \sum_l k_l n(\gamma, b_l) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

Consequently if $\Gamma = f \circ \gamma$, then

$$n(\Gamma, 0) = \sum_j h_j n(\gamma, a_j) - \sum_l k_l n(\gamma, b_l).$$

REMARKS.

(1) Although we are allowing the sets $\{a_i\}$ and $\{b_j\}$ to be infinite, each of the sums in the result has only finitely many terms, as will be observed in the proof.

(2) The name "Argument Principle" comes from the second of the formulas, which computes the amount by which the argument of $\Gamma(t)$, the angle made with the origin, changes as one traverses the image curve Γ. Think of $f'(z)/f(z)$ as the complex derivative of the multivalued function $\log f(z)$. The failure of this expression to represent a single-valued function when traversing a closed curve comes from the imaginary part, since the real part of $\log re^{i\theta} = \log r + i\theta$ comes back to itself along a closed curve while the imaginary part, the argument, may not.

PROOF. Let U have center z_0 and radius R. The open disks $\{z \mid |z - z_0| < r\}$ for $r < R$ form an open cover of the compact set image(γ), and there must be a finite subcover. Hence there are finitely many such disks whose union contains image(γ). These are all contained in one of them, and there thus exists some number $r < R$ such that image$(\gamma) \subseteq \{z \mid |z - z_0| < r\}$. If infinitely many points a_j have $|a_j - z_0| \leq r$, then Theorem 2.36 shows that they have a limit point a, necessarily in U. The Identity Theorem (Proposition B.23) shows that the existence of such a limit point within U forces f to be identically 0, and we are assuming that that is not the case. Similarly if infinitely many points b_l have $|b_l - z_0| \leq r$, then Theorem 2.36 shows that they have a limit point b, necessarily in U. The existence of such a limit point within U contradicts the assumption that f is meromorphic in U.

Let $U_0 = \{z \mid |z - z_0| < r\}$. All points a_j or b_l outside U_0 lie in the unbounded component of the complement of image(γ), and thus $n(\gamma, a_j) = n(\gamma, b_l) = 0$ for them, by Proposition B.29. In other words, there are only finitely many points a_j and b_l in U_0, and we can disregard all points a_j and b_l that lie outside U_0.

For the pole b_1, $f(z)(z - b_1)^{k_1}$ has a removable singularity at b_1 and is nonzero there, and we can regard the product as analytic there. Applying the same reasoning to $b_2, b_3, \dots$, we see that $w(z) = f(z)\prod_l (z - b_l)^{k_l}$ is analytic in U_0 and has the same zeros and orders of zeros as $f(z)$.

If we apply Taylor's Theorem (Theorem B.20) to $w(z)$ about $z = a_1$, we can write $w(z) = (z - a_1)^{h_1} w_1(z)$. Applying the same reasoning in turn to $a_2, a_3, \dots$, we see that $w(z) = \prod_j (z - a_j)^{h_j} g(z)$ is analytic and nonvanishing in U_0. Therefore

$$f(z) = \prod_j (z - a_j)^{h_j} \prod_l (z - b_l)^{-k_l} g(z).$$

in U_0, with $g(z)$ analytic and nonvanishing in U_0. Taking the logarithmic complex derivative yields

$$\frac{f'(z)}{f(z)} = \sum_j \frac{h_j}{z-a_j} - \sum_l \frac{k_l}{z-b_l} + \frac{g'(z)}{g(z)}.$$

Since γ does not pass through any a_j or b_l, we can integrate both sides over γ and obtain

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} = \sum_j h_j n(\gamma, a_j) - \sum_l k_l n(\gamma, b_l) + \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz.$$

The last term on the right is 0, by the Cauchy Integral Theorem for the disk (Theorem B.9), since $g(z)$ is nowhere 0, and the first formula of the proposition follows. The second formula follows by combining this conclusion with Lemma B.32. □

In the simplest applications of Proposition B.33, one supposes that no poles are involved, and one chooses γ so that $n(\gamma, a_j)$ equals 0 or 1 for each zero. For example, γ might be a circle or a rectangle. Then Proposition B.33 counts the total number of zeros, with their multiplicities, inside the circle or rectangle.

Proposition B.33 can be used in computer calculations in checking whether a particular analytic function f has a zero in a specific region (once we know that the result applies to certain kinds of regions other than disks). The reason is that the left side in the first formula of the proposition is an integer, and one does not need an exact calculation of an integral to compute the left side, only a calculation with an error of less than $1/2$.

A more theoretical application of the Argument Principle to analytic functions is to proving Rouché's Theorem in Problem 40 at the end of the appendix.

We shall concentrate here on still another theoretical application, which gives a finer analysis of the behavior of an analytic function near a zero. A consequence, as we shall see, is that every nonconstant analytic function is an open mapping, i.e., carries open sets onto open sets.

Proposition B.34. Suppose that $f(z)$ is analytic in a disk about z_0, that $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. If $\epsilon > 0$ is sufficiently small, then there exists a number $\delta > 0$ such that for every c with $|c - w_0| < \delta$, the equation $f(z) = c$ has exactly n roots in the disk $\{z \mid |z - z_0| < \epsilon\}$.

PROOF. Choose a radius $\epsilon > 0$ so small that $f(z) - w_0$ is nowhere 0 on the closed disk $\{z \bigm| |z - z_0| \le \epsilon\}$ other than at z_0 itself and so that $f'(z)$ is nowhere 0 on the same disk other than possibly at z_0 itself. We can do so because the zeros of $f(z) - w_0$ and the zeros of $f'(z)$ are isolated. Let γ be the standard circle of radius ϵ about z_0, and define

$$\delta = \tfrac{1}{2} \min_{|z-z_0|=\epsilon} |f(z) - w_0|.$$

If $|c - w_0| < \delta$ and $|z - z_0| = \epsilon$, then

$$|f(z) - c| \ge |f(z) - w_0| - |w_0 - c| \ge 2\delta - \delta > 0.$$

For any such c, we apply Proposition B.33 to our standard circle γ and to the function $f(z) - c$. The zeros of $f(z) - c$ are the roots of the equation $f(z) = c$, and we write $z_j(c)$ for them and $h_j(c)$ for their orders. The winding number $n(\gamma, z_j(c))$ equals 1 if $|z_j(c) - z_0| < \epsilon$ and equals 0 otherwise. Since $f(z)$ is nowhere c on γ, Proposition B.33 gives

$$\sum_{|z_j(c)-z_0|<\epsilon} h_j(c) = \tfrac{1}{2\pi i} \int_\gamma \tfrac{f'(z)}{f(z)-c}\, dz$$

and
$$n(\Gamma, c) = \sum_{|z_j(c)-z_0|<\epsilon} h_j(c).$$

Let c' be on the line segment in $\mathbb{C}$ from w_0 to c. Then $|f(z) - c'| \ge \delta > 0$, and c' is not on $\text{image}(f \circ \gamma) = \text{image}(\Gamma)$. Thus w_0 and c lie in the same component of the complement of $\text{image}(\Gamma)$, and Proposition B.28 shows that $n(\Gamma, w_0) = n(\Gamma, c)$. Since w_0 satisfies the same hypotheses as c,

$$n(\Gamma, w_0) = \sum_{|z_j(w_0)-z_0|<\epsilon} h_j(w_0).$$

Because of the choice of ϵ, the only point z in the closed disk of radius ϵ about z_0 where $f(z) = w_0$ is the point $z = z_0$. Thus there is only one term on the right side, and it is n. Also when $c \ne w_0$ and $|z_j(c) - z_0| < \epsilon$, the multiplicity of the root $z_j(c)$ of $f(z) - c$ is $h_j(c) = 1$, since $f'(z) \ne 0$ for $0 < |z - z_0| \le \epsilon$. Putting these facts together with the equality $n(\Gamma, w_0) = n(\Gamma, c)$, we conclude that

$$\#\{\text{roots of } f(z) - c \text{ for } |z - z_0| < \epsilon\} = \sum_{|z_j(c)-z_0|<\epsilon} 1 = \sum_{|z_j(c)-z_0|<\epsilon} h_j(c) = n,$$

as asserted. $\square$

Corollary B.35. Every nonconstant analytic function carries open sets onto open sets.

PROOF. In the notation of Proposition B.34, if f is analytic, then with one proviso, f carries any sufficiently small disk $\{z \,\big|\, |z - z_0| < \epsilon\}$ to a superset of the set $\{w \,\big|\, |w - f(z_0)| < \delta\}$. The proviso is that ϵ is so small that $f(z) - f(z_0)$ and $f'(z)$ can vanish on $\{z \,\big|\, |z - z_0| < \epsilon\}$ only at z_0 itself. Such a positive ϵ exists as soon as f is nonconstant. □

Corollary B.36. Suppose that $f(z)$ is analytic in a disk about z_0, that $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. Then there exists an open disk $\{z \,\big|\, |z - z_0| < \epsilon\}$ inside which $f(z) - w_0$ can be written as a composition of an analytic function with a zero at z_0 followed by the function $\zeta \mapsto \zeta^n$ about $\zeta = 0$. In formulas,

$$\begin{aligned} f(z) - w_0 &= \zeta(z)^n \\ \zeta(z) &= (z - z_0)h(z) \end{aligned}$$

with $\zeta(z)$ and $h(z)$ analytic.

PROOF. Because $f(z)-w_0$ has a zero of order n at z_0, we can write $f(z)-w_0 = (z-z_0)^n g(z)$ with $g(z)$ analytic and $g(z_0) \neq 0$. Choose $\epsilon > 0$ so that $|z-z_0| < \epsilon$ implies $|g(z) - g(z_0)| < |g(z_0)|$. This inequality says that $|1 - g(z)/g(z_0)| < 1$.

Put $w = g(z)/g(z_0)$. The set of w with $|1 - w| < 1$ excludes the negative real axis, and the principal value of the n^{th} root of w is defined there. Define $h_1(z)$ to be the principal value of the n^{th} root of $g(z)/g(z_0)$. This is an analytic function with $h_1(z)^n = g(z)/g(z_0)$ such that $h_1(z) = re^{i\theta}$ has $-\frac{\pi}{n} < \theta < \frac{\pi}{n}$. Fix an n^{th} root $g(z_0)^{1/n} = \rho e^{i\varphi}$ of $g(z_0)$, and define $h(z) = g(z_0)^{1/n} h_1(z)$. This is an analytic function having $h(z) = \rho r e^{i\psi}$ with $-\frac{\pi}{n} + \varphi < \psi < \frac{\pi}{n} + \varphi$.

To complete the proof, we simply compute

$$\begin{aligned} ((z - z_0)h(z))^n &= (z - z_0)^n h(z)^n = (z - z_0)^n (g(z_0)^{1/n} h_1(z))^n \\ &= (z - z_0)^n (g(z_0)^{1/n})^n (h_1(z))^n = (z - z_0)^n g(z_0)(g(z)/g(z_0)) \\ &= (z - z_0)^n g(z) = f(z) - w_0. \end{aligned}$$ □

B10. Residue Theorem

The Residue Theorem is an important tool of complex analysis for calculating ordinary definite integrals, both proper and improper. We shall state and prove the Residue Theorem for a disk in this section, and in Section B11 we shall give some applications to the calculation of definite integrals. A version of the theorem for regions other than disks will be obtained in Section B12, and the need for such a theorem will be apparent from Example 6 at the end of Section B11.

We suppose that $f(z)$ is a function analytic in a disk except for isolated singularities, and for the moment we suppose that those singularities are all poles and there are only finitely many of them. Let γ be a piecewise C^1 closed curve in the disk. The question is to evaluate $\int_\gamma f(z)\,dz$.

If there is only one pole, say at z_0, we can proceed as follows. Let the pole have order h. Following the prescription in Section B6, we expand $f(z)$ as the sum of its singular part and the rest:

$$f(z) = b_h(z-z_0)^{-h} + b_{h-1}(z-z_0)^{-h+1} + \cdots + b_1(z-z_0)^{-1} + c(z).$$

Here $c(z)$ is analytic in the disk, and its integral is 0 by the Cauchy Integral Theorem (Theorem B.9). Each of the powers $(z-z_0)^{-k}$ with $k < -1$ is a complex derivative in $\mathbb{C} - \{z_0\}$, and its integral is 0 by Corollary B.6. Thus

$$\int_\gamma f(z)\,dz = b_1 \int_\gamma (z-z_0)^{-1}\,dz = 2\pi i b_1 n(\gamma, z_0),$$

where $n(\gamma, z_0)$ is the winding number studied in Section B7.

The coefficient b_1 is defined to be the **residue** of $f(z)$ at z_0, denoted by $\operatorname{Res}_f(z_0)$, and the above computation shows that the value of the integral is $2\pi i$ times the product of the residue and the winding number. When the order of the pole is 1, the residue is easy to compute; it is $\operatorname{Res}_f(z_0) = \lim_{z\to z_0}(z-z_0)f(z)$.

The computation is only a little harder when the order h is greater than 1. The residue is just

$$\operatorname{Res}_f(z_0) = \frac{1}{(h-1)!}\lim_{z\to z_0}\Big(\frac{d}{dz}\Big)^{h-1}\big((z-z_0)^h f(z)\big),$$

a fact that we readily check by substituting for $f(z)$ the expression above for the sum of the singular part and the rest.

Theorem B.37 (Residue Theorem). Let $f(z)$ be a function analytic in a disk except for poles at points $\{z_j\}$. If γ is a piecewise C^1 closed curve in the disk not passing through any of the poles, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \operatorname{Res}_f(z_j) n(\gamma, z_j),$$

only finitely many of the terms on the right side being nonzero.

REMARK. The formula of the theorem remains valid if some of the z_j are essential singularities, but the proof breaks down. Also the formula for computing the residues is no longer meaningful once order infinity is allowed. We return to this matter when we take up Laurent series in Section B13. Fortunately applications of the Residue Theorem normally do not involve essential singularities.

PROOF. The same reasoning as at the beginning of the proof of Proposition B.33 shows that we can shrink the disk slightly and assume that there are only finitely many poles, say $z_1, \dots, z_m$. Poles outside the smaller disk will not contribute to the formula for $\int_\gamma f(z)\,dz$. Let the respective singular parts be $s(z_j, z)$ and the respective residues be $\mathrm{Res}_f(z_j)$. Then

$$f(z) - \sum_{j=1}^{m} s(z_j, z) = a(z)$$

is analytic in the shrunk disk, and each term $s(z_j, z) - \frac{\mathrm{Res}_f(z_j)}{z - z_j}$ is the complex derivative of an analytic function in $\mathbb{C} - \{z_j\}$. Write

$$\int_\gamma f(z)\,dz = \int_\gamma a(z)\,dz + \sum_{j=1}^{m} \int_\gamma \big(s(z_j, z) - \tfrac{\mathrm{Res}_f(z_j)}{z - z_j}\big)\,dz + \sum_{j=1}^{m} \int_\gamma \tfrac{\mathrm{Res}_f(z_j)\,dz}{z - z_j}.$$

The first term on the right side is 0 by the Cauchy Integral Theorem (Theorem B.9), the second term is 0 by Corollary B.6, and the third term equals $2\pi i \sum_{j=1}^{m} \mathrm{Res}_f(z_j) n(\gamma, z_j)$ by definition of $n(\gamma, z_j)$. The theorem follows. □

EXAMPLE. Find the residues of $f(z) = \frac{e^z}{(z-a)(z-b)}$ if $a \neq b$. The poles are both simple. At a, we have

$$\mathrm{Res}_f(a) = \lim_{z \to a} (z - a) f(z) = \lim_{z \to a} \tfrac{e^z}{z - b} = \tfrac{e^a}{a - b},$$

and at b, we have

$$\mathrm{Res}_f(b) = \lim_{z \to b} (z - b) f(z) = \lim_{z \to b} \tfrac{e^z}{z - a} = \tfrac{e^b}{b - a}.$$

B11. Evaluation of Definite Integrals

Ordinarily the Residue Theorem is applied in situations where γ is a simple closed curve oriented so that the inside of the curve is on the left. Then the winding number is 1 for γ about the poles inside the closed curve and is 0 for γ about the poles outside the closed curve. The statement of the theorem in this case is that $\int_\gamma f(z)\,dz$ equals $2\pi i$ times the sum of the residues at the poles inside the curve. After the theorem is applied, some passage to the limit is involved so that the initial curve matches the desired interval of integration in the limit.

The definite integrals to be evaluated with the help of the Residue Theorem are typically of the form $\int_{-\infty}^{+\infty}$ or $\int_0^{2\pi}$, but other intervals are possible in more complicated cases. The integrals may or may not be absolutely convergent.

EXAMPLE 1. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$. This is a simple example of an absolutely convergent integral of a rational function from $-\infty$ to $+\infty$. Such integrals can always be evaluated directly as a limit of the integral $\int_{-R}^{R}$, with $\int_{-R}^{R}$ handled exactly by the method of partial fractions of calculus. For this particular case the integrand is the derivative of $\arctan x$, and thus the integral equals

$$\lim_{R\to\infty} (\arctan R - \arctan(-R)) = \pi/2 - (-\pi/2) = \pi.$$

In most cases using the Residue Theorem tends to be easier than using partial fractions. Let us see what happens here. The curve γ is taken as the line segment from $-R$ to R on the real axis, followed by the large semicircle in the upper half plane that closes the curve. The semicircle may be parametrized as $t \mapsto Re^{it}$ with $0 \le t \le \pi$. We have

$$\int_{\text{line segment}} (z^2+1)^{-1}\,dz + \int_{\text{semicircle}} (z^2+1)^{-1}\,dz$$
$$= \int_\gamma (z^2+1)^{-1}\,dz = 2\pi i \sum_{\substack{\text{poles inside}\\ \text{semicircle}}} \operatorname{Res}_{(z^2+1)^{-1}}(z_j).$$

The expression after the first equals sign is a complex line integral that on the one hand equals the sum of the two integrals on the left and on the other hand equals the expression on the right obtained from the Residue Theorem.

The first integral on the left is just $\int_{-R}^{R}(x^2+1)\,dx$ and tends to $\int_{-\infty}^{\infty}(x^2+1)\,dx$ in the limit as R tends to infinity. The second integral on the left is equal to $\int_0^\pi ((Re^{it})^2+1)^{-1} Rie^{it}\,dt$. The absolute value of the integrand in this case is $\le R/(R^2-1)$ if $R > 1$, and the length of the interval is π. Thus the absolute value of the second integral on the left is of the order of π/R and tends to 0 as R tends to infinity. In other words, the limit as R tends to ∞ of the left side of the displayed equation is the integral $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$.

The only poles of $(z^2+1)^{-1}$ in $\mathbb{C}$ are at $\pm i$. The pole at $-i$ is never inside the semicircle in question, and the pole at $+i$ is inside the semicircle when $R > 1$. The poles are simple and the residue at $+i$ is $\lim_{z\to i}(z-i)(z^2+1)^{-1} = \lim_{z\to i}(z+i)^{-1} = 1/(2i)$. Taking into account the factor $2\pi i$, we obtain the result $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$.

Let us step back from Example 1 to see what made it work. We started with an integral of the form $\int_{\infty}^{\infty} \frac{P(x)\,dx}{Q(x)}$, where P and Q are polynomials. Implicitly we assumed that $Q(x)$ was nowhere 0 on the real axis. We replaced $P(x)$ and $Q(x)$ by $P(z)$ and $Q(z)$ and considered a complex line integral $\int_{\gamma} \frac{P(z)\,dz}{Q(z)}$, where γ consisted of a line segment on the real axis, followed by a semicircle in the upper half plane. It was relevant that the integral over the semicircle involved an extra factor of Rie^{it}. For the integral over the semicircle, we estimated $P(Re^{it})/Q(e^{it})$, and the main consideration was $\deg P - \deg Q$. If this was -2 or less, then the product $RP(Re^{it})/Q(e^{it})$ would still have tended to 0. If $\deg P - \deg Q$ had been -1, this would not have happened. Once we could handle the integral over the semicircle, all we needed was knowledge of the residues in the upper half plane. For that knowledge, an exact factorization of Q was handy; an approximation would have been good enough to get an approximate answer if all poles were simple. The result was that if $\deg P - \deg Q \le -2$, then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \times \begin{cases} \text{sum of residues of } P/Q \\ \text{in upper half plane} \end{cases}.$$

EXAMPLE 2. $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$, another absolutely convergent integral. It is tempting to proceed exactly as in Example 1, using $\frac{\cos z}{z^2+1}$, but this approach does not work because $\cos z$ gets quite large in the imaginary direction. Instead one uses e^{iz} in place of $\cos z$, since $e^{iz} = e^{ix}e^{-y}$ is small in the positive imaginary direction. Just as in Example 1, the integral over the semicircle tends to 0, and the result is that

$$\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+1} = 2\pi i \times \begin{cases} \text{sum of residues of } e^{ix}/(x^2+1) \\ \text{in upper half plane} \end{cases}.$$

The only relevant pole is from $+i$, and the residue is $\lim_{z\to i}(z-i)e^{iz}(z^2+1)^{-1} = \lim_{z\to i} e^{iz}(z+i)^{-1} = e^{-1}/(2i)$. Thus the integral of $e^{ix}/(x^2+1)$ is πe^{-1}. Taking the real part shows that the integral of $\cos x/(x^2+1)$ is πe^{-1}.

If we had considered $e^{-iz}/(z^2+1)$ instead of $e^{iz}/(z^2+1)$, then the same technique would have worked if γ had consisted of the line segment on the real axis followed by a semicircle in the *lower* half plane. In this case we would have to take into account that the winding number of γ about $-i$ is -1.

Adjusting the integrand in Example 2 slightly, we see that we could have handled $\int_{-\infty}^{\infty} e^{-2\pi iux}(x^2+1)^{-1}\,dx$. The choice of semicircle would have depended on the sign of u, with either choice working when $u = 0$. Readers who have peeked at Chapter VIII will know that the function $u \mapsto \int_{\infty}^{\infty} e^{-2\pi iux}(x^2+1)^{-1}\,dx$ is the Fourier transform of $(x^2+1)^{-1}$. Fourier transforms are of great importance in real analysis and electrical engineering.

Putting the techniques of Examples 1 and 2 together, we see that we can compute $\int_{-\infty}^{\infty} e^{-2\pi i u x}(P(x)/Q(x))\,dx$ as long as $P(x)$ and $Q(x)$ are polynomials with $\deg P - \deg Q \le -2$ and $Q(x)$ is nowhere 0 on the real axis.

EXAMPLE 3. $\int_{-\infty}^{\infty} e^{ix}\frac{x}{x^2+1}\,dx$. Here the difference of degrees of the numerator and denominator of $x/(x^2+1)$ is -1, and the above condition is not satisfied. This time the integral is not absolutely convergent. However, it will still be true that $\lim_{X_1,X_2\to\infty}\int_{-X_1}^{X_2} e^{ix}\frac{x}{x^2+1}\,dx$ exists. The curve to use is the boundary γ of the filled rectangle with $-X_1 \le x \le X_2$ and $0 \le y \le Y$, and γ is to be oriented so as to be traversed counterclockwise as usual. We shall assume that $Y > 1$.

The contribution to the complex line integral from the right side of the rectangle is an integral from 0 to Y of an integrand in y that in absolute value equals

$$\begin{aligned} e^{-y}|X_2+iy||(X_2+iy)^2+1|^{-1} &= e^{-y}|X_2+iy||X_2+iy+i|^{-1}|X_2+iy-i|^{-1} \\ &\le e^{-y}|X_2+iy-i|^{-1} \le X_2^{-1}e^{-y}, \end{aligned}$$

and $\int_0^Y X_2^{-1}e^{-y} \le X_2^{-1}$. So the contribution from the right side of the rectangle is $\le X_2^{-1}$. Similarly the contribution from the left side of the rectangle is $\le X_1^{-1}$.

The contribution to the complex line integral from the top side of the rectangle is

$$\int_{-X_1}^{X_2} e^{i(x+iY)}(x+iY)((x+iY)^2+1)^{-1}\,dx,$$

and the absolute value of the integrand is $\le e^{-Y}|x+iY-i|^{-1} \le e^{-Y}/(Y-1)$. Thus the contribution from the top side of the rectangle is $\le e^{-Y}(X_1+X_2)/(Y-1)$. For fixed X_1 and X_2, this tends to 0 as Y tends to infinity.

Consequently

$$\Big|\int_{-X_1}^{X_2} e^{ix}\frac{x}{x^2+1}\,dx - 2\pi i\sum_{\text{poles inside }\gamma}(\text{residue at pole})\Big| \le X_2^{-1}+X_1^{-1}.$$

The only pole inside γ is at i, and the residue there is $\lim_{z\to i}\big(e^{iz}z/(z+i)\big) = -e^{-1}i/(2i)$. Therefore $\lim_{X_1,X_2\to\infty}\int_{-X_1}^{X_2} e^{ix}\frac{x}{x^2+1}\,dx = -\pi i e^{-1}$.

If we had considered $e^{-ix}x/(x^2+1)$ instead of $e^{ix}x/(x^2+1)$, then the same technique would have worked if γ had consisted of the line segment on the real axis followed by the other three sides of a rectangle in the *lower* half plane.

Adjusting the integrand in Example 3 slightly, we see that we can handle $\int_{-\infty}^{\infty} e^{-2\pi i u x}x(x^2+1)^{-1}\,dx$. The choice of rectangle depends on the sign of u, with neither choice working when $u = 0$.

Adjusting the integrand even more, we see that we can handle any integral of the form $\int_{-\infty}^{\infty} e^{-2\pi i u x}P(x)/Q(x)\,dx$ whenever P and Q are polynomials with $Q(x)$ nonvanishing for x real and with $\deg P - \deg Q = -1$.

EXAMPLE 4. $\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$, an integral that is not absolutely convergent at infinity or at 0. Because of the failure of absolute convergence at infinity, the proper approach is to consider a limit of $\int_{-X_1}^{X_2}$ as X_1 and X_2 tend to infinity, rather than a limit of $\int_{-R}^{R}$ as R tends to infinity. Thus we use a rectangle as in Example 3 rather than a semicircle as in Examples 1 and 2. In addition, there is a pole on the real axis, and it requires special treatment. We therefore adjust the curve γ of Example 3 slightly, to include the pole at $z = 0$ within the curve. Fix positive numbers X_1, X_2, Y, and ϵ with ϵ small. The integral proceeds on the real axis from $-X_1$ to $-\epsilon$, around a small semicircle ϵe^{it} with $\pi \le t \le 2\pi$, and along the real axis from ϵ to X_2, and it closes up by going up the right side, across the top, and down the left side as in Example 3. The same reasoning as in Example 3 therefore gives us

$$\Big| \int_{-X_1}^{-\epsilon} e^{ix}/x\,dx + \int_{\epsilon}^{X_2} e^{ix}/x\,dx + \int_{\text{semicircle}} e^{iz}/z\,dz - 2\pi i \operatorname{Res}_{e^{iz}/z}(0)\Big| \le c_2 X_2^{-1} + c_1 X_1^{-1}.$$

for suitable constants c_1 and c_2. Letting X_1 and X_2 tend to infinity gives

$$\int_{|x|\ge\epsilon} e^{ix}/x\,dx = 2\pi i \operatorname{Res}_{e^{iz}/z}(0) - \int_{\text{semicircle}} e^{iz}/z\,dz$$

The pole at $z = 0$ is simple, and the residue is $\lim_{z\to 0} z e^{iz}/z = 1$. To estimate the integral over the semicircle, we can write $e^{iz}/z = 1/z + a(z)$ with $a(z)$ analytic near $z = 0$. Then

$$\begin{aligned}\int_{\text{semicircle}} e^{iz}/z\,dz &= \int_{\text{semicircle}} 1/z\,dz + \int_{\text{semicircle}} a(z)\,dz \\ &= \int_{\pi}^{2\pi} 1/(\epsilon e^{it})\, i\epsilon e^{it}\,dt + \int_{\text{semicircle}} a(z)\,dz.\end{aligned}$$

On the right side the first term equals πi, and the second term is $\le$ the product of the length of the semicircle, namely $\epsilon\pi$, by the supremum of $|a(z)|$ on the semicircle. Thus the second term tends to 0 as ϵ tends to 0. We obtain

$$\int_{|x|\ge\epsilon} (e^{ix}/x)\,dx = 2\pi i - \pi i + (\text{error term})$$

with $\lim_{\epsilon\to 0}(\text{error term}) = 0$. Consequently

$$\lim_{\epsilon\to 0} \int_{|x|\ge\epsilon} (e^{ix}/x)\,dx = \pi i.$$

The left side is known as the **Cauchy principal value** of the integral of e^{ix}/x, and one writes

$$\text{PV} \int_{-\infty}^{\infty} e^{ix}/x\,dx = \pi i.$$

EXAMPLE 4′. Variants. If we had started with e^{-ix}/x instead of e^{ix}/x and if we had argued with a rectangle in the *lower* half plane and a small semicircle in the *upper* half plane, we would have found that PV $\int_{-\infty}^{\infty} e^{-ix}/x\,dx = -\pi i$. Subtracting the two results, and dividing by $2i$, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi.$$

The PV is not needed, since $\frac{\sin x}{x}$ has no singularity at $x = 0$. Since $\frac{\sin x}{x}$ is a even function, it is customary to write this result as

$$\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

If we go over the reasoning in Examples 4 and 4′ carefully, we see that we can handle an arbitrary finite number of *simple* poles on the real axis in an integral $\int_{\infty}^{\infty} \frac{e^{ix}P(x)}{Q(x)}\,dx$, as well as finitely many poles in the upper half plane, as long as $\deg P - \deg Q \le -1$. The integral will be $2\pi i$ times the sum of the residues at the poles in the open upper half plane it plus πi times the sum of the residues at the simple poles on the real axis. The integral will involve a Cauchy principal value at each pole on the real axis.

It is instructive to see what happens with a double pole on the real axis. The integral $\int_{-\infty}^{\infty} e^{ix}x^{-2}\,dx$ is absolutely convergent at infinity. The above method gives us

$$\begin{aligned}
\int_{|x|\ge\epsilon} e^{iz}z^{-2}\,dx &= 2\pi i\,\mathrm{Res}_{e^{iz}z^{-2}}(0) + \int_{\text{semicircle}} e^{iz}z^{-2}\,dz \\
&= -2\pi + \int_{\text{semicircle}} z^{-2}(1 + iz + a(z))\,dz \quad \text{with } z^{-2}a(z) \text{ analytic} \\
&= -2\pi + \int_{\pi}^{2\pi} \epsilon^{-2}e^{-2it}i\epsilon e^{it}\,dt + \int_{\pi}^{2\pi} i\epsilon^{-1}e^{-it}i\epsilon e^{it}\,dt + (\text{error}) \\
&= -2\pi + i\epsilon^{-1}\int_{\pi}^{2\pi} e^{-it}\,dt - \pi + (\text{error}) \\
&= -2\pi - 2\epsilon^{-1} - \pi + (\text{error})
\end{aligned}$$

with "(error)" referring to a term that tends to 0 as ϵ tends to 0. What we see is that we do not get a finite limit on the right side as ϵ tends to 0.

EXAMPLE 5. $I = \int_0^{2\pi} R(\sin\theta, \cos\theta)\,d\theta$, where R is a rational function everywhere finite for $0 \le \theta \le 2\pi$. Let us substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}\,d\theta = iz\,d\theta$, so that $d\theta = \frac{-i\,dz}{z}$. Then I becomes an integral around the standard unit circle C:

$$I = \int_C R\Big(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\Big)(-i)z^{-1}\,dz$$

Expanding out the integrand reveals a meromorphic function with no poles on C. Thus the value of I is just $2\pi i$ times the sum of the residues of the integrand inside the unit circle. For a particular case, we use this approach to evaluate

$$\int_0^{2\pi} \frac{\cos\theta\,d\theta}{5+4\cos\theta} = -i \int_C \frac{\frac{1}{2}(z+z^{-1})}{5+4(\frac{1}{2}(z+z^{-1}))}\,\frac{dz}{z}$$
$$= -i \int_C \frac{z^2+1}{z(4z^2+10z+4)}\,dz.$$

The integrand has poles at 0, $-\frac{1}{2}$, and -2. The value of the integral (without the coefficient $-i$) is therefore $2\pi i$ times the sum of the residues at 0 and $-\frac{1}{2}$:

$$\int_0^{2\pi} \frac{\cos\theta\,d\theta}{5+4\cos\theta} = 2\pi i(-i)(\tfrac{1}{4} - \tfrac{5}{12}) = -\tfrac{\pi}{3}.$$

EXAMPLE 6. $I = \int_{-\infty}^{\infty} \frac{\log|x|}{(1+x^2)^2}\,dx$, an absolutely convergent integral. This looks like a candidate for the kind of analysis we have done in this section, but we have to make some preliminary adjustments. The integrand is not the restriction to the real axis of an analytic function, but the integrand has compensations. For one thing the integrand is even; in addition, the value of the natural analytic function $\frac{\log z}{(1+z^2)^2}$ on the negative real axis is related to the value of the same analytic function on the positive real axis. Consequently handling the integral of $\frac{\log z}{(1+z^2)^2}$ may well lead to a value for I. To do so, we choose a branch of the logarithm that is analytic in a region that includes the upper half plane and both halves of the real axis, such as the one that excludes the negative imaginary axis. Then there will be a pole at $z = i$ and some other singularity at $z = 0$. We can use a small semicircle around 0, we can make our estimates, and we can write down the result of applying the Residue Theorem. There is just one difficulty. We proved the Residue Theorem for a disk, not for a region like the plane with the negative imaginary axis omitted. To obtain a better version of the Residue Theorem, we need a better version of the Cauchy Integral Theorem, one that takes global properties of the region into account. We shall obtain the improved version of the Cauchy Integral Theorem in the next section, and later we shall come back to the integral in this example.

B12. Global Theorems in Simply Connected Regions

We saw an example at the end of the previous section showing that the scope of the Cauchy Integral Theorem as stated in Theorem B.9 is not broad enough for some purposes. Theorem B.9 was stated for a disk of any radius, but we needed a version that applied to another kind of region.

Following the approach in Ahlfors's *Complex Analysis*, we define a bounded region of $\mathbb{C}$ to be **simply connected** if its complement in $\mathbb{C}$ is connected.[12] By way of example, the inside of a circle (an open disk) is simply connected. It is fairly clear intuitively from the definition that a simply connected region cannot have any holes or punctures in it. Indeed, any hole or puncture would constitute a component of the complement of the region. This definition of "simply connected" is easy to check and relatively easy to apply, but it is not the standard one. The standard one, roughly speaking, is that a region is simply connected if every loop (closed path) based at a point can be continuously deformed to a point within the region without moving the base point; we shall be more precise at the end of this section.

The question of addressing the equivalence of the present definition and the standard one goes by way of Lemma B.39. Lemma B.39 will reformulate the present definition of simply connected in terms of winding number, and what needs proof is that the winding-number definition is equivalent with the standard definition. This matter will be taken up at the end of this section.

Theorem B.40 below will say that the Cauchy Integral Theorem extends to be valid for all bounded simply connected regions, not just for disks. The first step toward a proof will be to formulate the notion "simply connected" in terms of winding numbers.

By way of preliminaries we reintroduce the notion of a **piecewise C^1 chain** that was mentioned at the end of Section B2 and was used also in Section III.13. This is a formal sum of piecewise C^1 curves, say $\gamma = \gamma_1 + \cdots + \gamma_r$, without regard to the order of the terms. We regard two chains as **equal** if they can be obtained from each other by a sequence of operations of the form

(i) subdivision of an arc,
(ii) fusion of subarcs into a single arc,
(iii) reparametrization of an arc,
(iv) cancellation of a pair of opposite arcs,
(v) insertion of a pair of opposite arcs,
(vi) dropping a one-point arc (with domain of the form $[a, a]$), and
(vii) insertion of a one-point arc.

In analogy with what happened with ordinary line integrals in Section III.13, we define a complex line integral over a piecewise C^1 chain γ as the corresponding

[12]To make this definition apply to an unbounded region, one must first adjoin a point at infinity to $\mathbb{C}$ and regard $\mathbb{C}$ as a subset of a 2-dimensional sphere, as in Problems 5–8 at the end of this appendix. Then a region in $\mathbb{C}$ can be defined to be simply connected if its complement in the sphere is connected. Preferring to avoid unenlightening complications, we shall not use the term "simply connected" in dealing with unbounded regions.

sum of complex line integrals over the constituent piecewise C^1 curves:

$$\int_\gamma f\,dz = \sum_{k=1}^{r} \int_{\gamma_k} f\,dz.$$

If two such chains are equal, then all complex line integrals defined on both are equal.

As in Section III.13, we denote the reverse of γ by $-\gamma$. If $\gamma = \gamma_1 + \cdots + \gamma_r$ and $\sigma = \sigma_1 + \cdots + \sigma_s$ are chains, let $\gamma + \sigma = \gamma_1 + \cdots + \gamma_r + \sigma_1 + \ldots \sigma_s$. Then $\int_{\gamma+\sigma} f\,dz = \int_\gamma f\,dz + \int_\sigma f\,dz$. We shall write $n\gamma$ for $\gamma + \cdots + \gamma$ (n times) and $-n(\gamma) = n(-\gamma)$ and $0(\gamma) =$ (empty arc). Then every chain can be written as $\gamma = a_1\gamma_1 + \cdots + a_n\gamma_k$ with the a_j positive integers and the γ_j distinct, and if we allow some coefficients to be 0, then any two chains can be expressed as sums of the same γ_j's.

The new ingredient in all this formalism, not present in Section III.13 or Section B2, is the notion of a "cycle." A chain is a **cycle** if it can be represented as the sum of (piecewise C^1) closed curves.

Lemma B.38. A chain is a cycle if and only if in any representation of the chain as the sum of piecewise C^1 closed curves, the initial point and endpoints are identical in pairs.

PROOF. If the condition is satisfied for $\gamma_1 + \cdots + \gamma_n$, take the piecewise C^1 curve γ_1 and if γ_1 is not closed, use the hypothesis to find γ_k whose initial point matches the endpoint of γ_1. Then $\gamma_1 + \gamma_k$ is a piecewise C^1 curve, and either it is closed, or we can join another constituent curve to it. Continuing in this way, we eventually express the chain as the sum of closed curves.

Conversely if we have a chain expressed as the sum of closed curves, we need only check that operations (i) through (vii) preserve the condition in the statement of the lemma. We omit the verification. □

The **winding number** or **index** of a cycle γ about a point not in the image of any of the constituent curves of γ is $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$. It is a well defined integer, and it satisfies

$$n(c_1\gamma_1 + c_2\gamma_2, a) = c_1 n(\gamma_1, a) + c_2 n(\gamma_2, a)$$

whenever c_1 and c_2 are integers.

Lemma B.39. A bounded region U in $\mathbb{C}$ is simply connected if and only if $n(\gamma, a) = 0$ for all piecewise C^1 cycles γ in U and points a in $\mathbb{C} - U$.

PROOF OF NECESSITY. Suppose U is bounded and simply connected, so that $\mathbb{C}-U$ is connected. Let γ be a piecewise C^1 cycle. In view of Lemma B.38, it is enough to prove that $n(\gamma, a) = 0$ for all piecewise C^1 closed curves γ in U and points a in $\mathbb{C}-U$. Since γ is in U, $\mathbb{C}-U \subseteq \mathbb{C}-\text{image}(\gamma)$. As a connected set, $\mathbb{C}-U$ must be contained in one of components of $\mathbb{C}-\text{image}(\gamma)$. The set $\mathbb{C}-U$ contains the exterior of any sufficiently large disk centered at the origin, and the component in question must be the unbounded component. By Proposition B.29, $n(\gamma, a) = 0$ for all a in the unbounded component, hence for all a in $\mathbb{C}-U$. □

PROOF OF SUFFICIENCY. With U bounded, suppose that $n(\gamma, a) = 0$ for all piecewise C^1 cycles γ in U and points a in $\mathbb{C}-U$. We are to prove that U is simply connected. Arguing by contradiction, suppose $\mathbb{C}-U$ fails to be connected. Then we can write $\mathbb{C}-U = X \cup Y$ with X and Y nonempty, disjoint, closed, and relatively open. Since the exterior of a sufficiently large disk is contained in one of the components, exactly one of the connected components of $\mathbb{C}-U$, say E, is unbounded. Since $E \subseteq X \cup Y$ and E is connected, E is contained in either X or Y. Say that $E \subseteq X$. Then Y is bounded, as well as closed; hence Y is compact (Corollary 2.37). Fix a point a of Y. We shall produce a cycle γ in U for which $n(\gamma, a) = 1$, and then the proof will be complete.

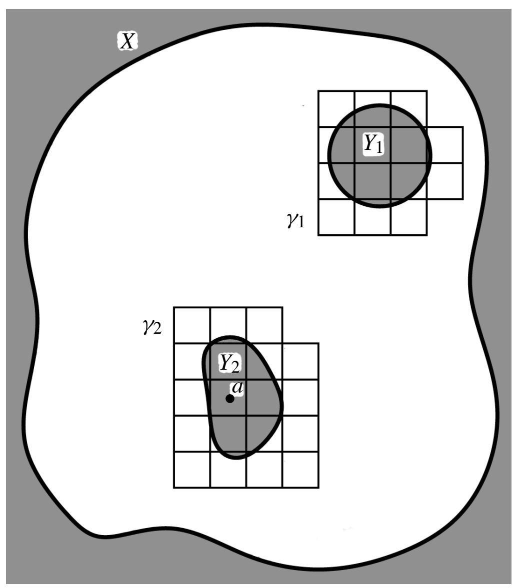

FIGURE B.5. Construction of a cycle γ and a point a with $n(\gamma, a) = 1$. Here $Y = Y_1 \cup Y_2$, a lies in Y_2, and $\gamma = \gamma_1 + \gamma_2$.

Since X is closed, the distance $D(x, X)$ of a point x to X is a continuous function (Proposition 2.16) on $\mathbb{C}$ that vanishes on X and only on X (Proposition

2.19). Corollary 2.39 shows that it attains a minimum value $m > 0$ on the compact set Y. Cover $\mathbb{C}$ with a grid of nonoverlapping filled (closed) squares of a fixed side $< m/\sqrt{2}$, and position the squares so that the chosen point a is at the center of one of them. The size of the squares is arranged so that no square meets both X and Y.

If Q is any of the squares, we let ∂Q be its boundary, oriented so as to be traversed counterclockwise. Each ∂Q, being a closed piecewise C^1 curve, is a cycle. Define $\gamma = \sum_i \partial Q_i$, where the sum is taken over all squares Q_i such that $Q_i \cap Y \neq \varnothing$. The sum is finite because Y is compact and the squares are nonoverlapping. Thus γ is a cycle. Since a is in Y and lies at the center of a square, there is exactly one square Q_0 for which $n(\partial Q_0, a) = 1$. Thus $n(\gamma, a) = 1$. See Figure B.5.

The cycle γ does not meet X because no square of the grid meets both X and Y. Let us see that when cancellations of sides of squares are taken into account, γ does not meet Y. Thus let Q be a square that contributes to γ. It has $Q \cap Y \neq \varnothing$. If one of Q's sides meets Y, Q shares that side with exactly one other square Q' of the grid, and then $Q' \cap Y \neq \varnothing$. So the side appears in the expression for γ with one orientation from ∂Q and with the opposite orientation from $\partial Q'$, and the two cancel. We conclude that γ does not meet Y. Thus γ lies in the complement of $X \cup Y = \mathbb{C} - U$, hence in U. □

Theorem B.40 (Cauchy Integral Theorem, global form). If f is analytic in a bounded simply connected region U in $\mathbb{C}$, then

$$\int_\gamma f(z)\, dz = 0$$

for every piecewise C^1 closed curve in U.

REMARKS. By Lemma B.38 it follows that $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 cycle in U. In the statement of the theorem the hypothesis "bounded" is unnecessary, but we shall not make the effort to drop it.

PROOF. Fix z_0 in U, and define $F(z) = \int_\sigma f(\zeta)\,d\zeta$ for z in U, where σ is a polygonal path from z_0 to z in U with sides parallel to the axes. The main step is to prove that $F(z)$ is well defined. Let us see how the theorem then follows.

The function $F(z)$ is certainly continuous, as a change from z_1 to z_2 produces a change in the integral of at most the maximum value of $|f(\zeta)|$ on a polygonal path from z_1 to z_2, times the length of the polygonal path. Parametrizing horizontal and vertical segments, we compute the partial derivatives of F. If the last segment of a path is taken to be horizontal, we see that $\frac{\partial F}{\partial x}(z) = f(z)$. If it is taken to be vertical, we see that $\frac{\partial F}{\partial y}(z) = if(z)$. Both partial derivatives are continuous, and Corollary B.2′ implies that F has a complex derivative at each point, namely the value of $\frac{\partial F}{\partial x}$, which is f. Thus f is the complex derivative of an analytic function, and Corollary B.6 shows that $\int_\gamma f(z)\,dz = 0$ for every piecewise C^1 closed curve in U. So the theorem follows if F is well defined.

If σ_1 and σ_2 are two polygonal paths from z_0 to z in U with sides parallel to the axes, then the difference $\gamma = \sigma_1 - \sigma_2$ is a closed polygonal path in U with sides parallel to the axes. Consequently it is enough to show that $\int_\gamma f(z)\,dz = 0$ for every closed polygonal path γ in U with sides parallel to the axes. We may assume that γ contains at least one horizontal segment and one vertical segment, and we regard the straight-line segments in γ as fixed for the remainder of the proof. In particular, the set image(γ) is a meaningful subset of U.

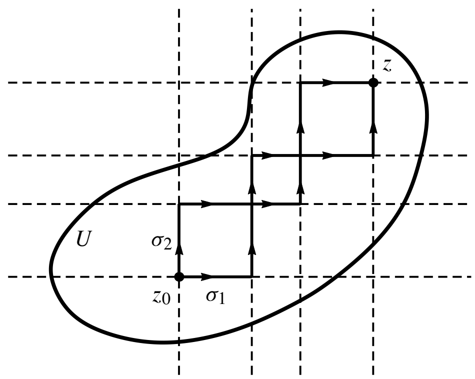

FIGURE B.6. Two polygonal paths σ_1 and σ_2 from z_0 to z in the region U.

Figure B.6 illustrates the situation. As indicated in that figure, we introduce some auxiliary lines by extending in both directions to an infinite line each horizontal or vertical segment of γ. The configuration of horizontal and vertical lines makes a finite grid on $\mathbb{C}$ of finite and infinite filled closed rectangles, and the assumption that γ contains at least one horizontal segment and one vertical segment means that the grid contains at least one finite rectangle.

Denote by R_i a typical finite filled rectangle. For each R_i, let a_i be a point in the inside of R_i, and let ∂R_i be the boundary of R_i, viewed as a piecewise C^1 closed curve oriented so as to be traversed counterclockwise. Let γ_0 be the cycle

$$\gamma_0 = \sum_i n(\gamma, a_i)\partial R_i. \qquad (*)$$

In addition, denote by R'_j a typical infinite rectangle. For each such R'_j, let a'_j be a point on the inside of R'_j. Let us check that

$$n(\gamma - \gamma_0, a) = 0 \qquad (**)$$

whenever a is one of the points a_i or a'_j. In fact, $n(\partial R_i, a_k)$ equals 1 if $k = i$ and equals 0 otherwise, and so

$$n(\gamma_0, a_k) = \sum_i n(\gamma, a_i) n(\partial R_i, a_k) = n(\gamma, a_k),$$

which proves $(**)$ for the points a_i. Since $n(\partial R_i, a'_j) = 0$ for all i and j, we have

$$n(\gamma_0, a'_j) = \sum_i n(\gamma, a_i) n(\partial R_i, a'_j) = 0,$$

and this proves $(**)$ for the points a'_j.

From $(**)$ we wish to prove that the cycles γ and γ_0 are equal. Express $\gamma - \gamma_0$ as an integer combination of sides; this is possible since the sides of γ were what were used to form the grid. Each side that can appear in γ or γ_0 as a finite side is a side of exactly two rectangles, which are adjacent. Since the two rectangles have this side in common, either they are both finite or else one is finite and the other is infinite.

Suppose that R_i and R_k have a side σ in common. To fix the signs, let us orient σ so that ∂R_i, which is traversed counterclockwise, contains σ as one of its four sides while ∂R_k contains $-\sigma$. Suppose that the expression of $\gamma - \gamma_0$ contains the integer multiple $c\sigma$ of σ. Then the cycle $\gamma - \gamma_0 - c\partial R_i$ does not contain σ. A straight line segment from a_i to a_k therefore does not meet the cycle $\gamma - \gamma_0 - c\partial R_i$, and it follows from Proposition B.28 that

$$n(\gamma - \gamma_0 - c\partial R_i, a_i) = n(\gamma - \gamma_0 - c\partial R_i, a_k). \qquad (\dagger)$$

In view of $(**)$, the left side of $(\dagger)$ is $-c$, and the right side is 0. Thus $c = 0$. This proves that no side common to two finite rectangles appears in the expression for $\gamma - \gamma_0$.

If R_i and R'_j have a side σ in common, we argue similarly. Again to fix signs, let us orient σ so that σ is one of the sides of R_i. If the expression of $\gamma - \gamma_0$ contains the integer multiple $c\sigma$ of σ, then just as with $(\dagger)$, we obtain

$$n(\gamma - \gamma_0 - c\partial R_i, a_i) = n(\gamma - \gamma_0 - c\partial R_i, a'_j). \qquad (\dagger\dagger)$$

By $(**)$ the left side is $-c$, and the right side is 0. Thus $c = 0$. This proves that no side common to a finite and an infinite rectangle appears in the expression for $\gamma - \gamma_0$. Combining this result with $(*)$ gives us

$$\gamma = \sum_i n(\gamma, a_i) \partial R_i. \qquad (\ddagger)$$

Let us now prove that every R_i in $(\ddagger)$ for which the coefficient $n(\gamma, a_i)$ is nonzero lies completely in U, i.e., the inside of R_i lies in U. It is time to use the hypothesis that U is simply connected. The function $a \mapsto n(\gamma, a)$ is constant on the inside of R_i since the inside is connected and lies in the complement of γ, and a_i is in the inside. Thus $n(\gamma, a)$ is nonzero everywhere on the inside of R_i. Because U is simply connected, Lemma B.39 tells us that $n(\gamma, a) = 0$ for all piecewise C^1 cycles in U and points a in $\mathbb{C} - U$. Thus no point of the inside of R_i lies in $\mathbb{C} - U$, i.e, the inside of R_i lies completely in U. By Goursat's Lemma, $\int_{\partial R_i} f(z)\, dz = 0$. Forming the integer combination of such integrals indicated by (‡), we obtain $\int_\gamma f(z)\, dz = 0$. □

In short order we can derive global forms of the Cauchy Integral Formula, the Argument Principle, and the Residue Theorem. In addition, we shall see how to make $\log z$ into a function on any bounded simply connected region. The results are as follows.

Corollary B.41 (Cauchy Integral Formula, global form). Let f be analytic in a simply connected region U, and let γ be any piecewise C^1 cycle in U. If z is any point of U not on image(γ), then

$$n(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z}.$$

REMARK. The condition that z not be on image(γ) means that there is some expression for γ as a combination of piecewise C^1 curves such that z is not on the image of any of the curves.

PROOF. We apply the Cauchy Integral Theorem (Theorem B.40) to the function

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{for } \zeta \in U - \{z\} \\ f'(z) & \text{for } \zeta = z. \end{cases}$$

on the simply connected region U. This is analytic except possibly at z. However, z is a removable singularity, and thus g is analytic in all of D. Theorem B.40 applies and gives $\int_\gamma g(\zeta)\, d\zeta = 0$. Therefore

$$\tfrac{1}{2\pi i} \int_\gamma \tfrac{f(\zeta)\, d\zeta}{\zeta - z} = \big(\tfrac{1}{2\pi i} \int_\gamma \tfrac{d\zeta}{\zeta - z}\big) f(z) = n(\gamma, z) f(z).$$ □

Corollary B.42. If $f(z)$ is analytic and nowhere vanishing in a bounded simply connected region U, then it is possible to define a (single-valued) analytic branch of $\log z$ in U.

PROOF. Fix z_0 in U. By Theorem B.40, $\int_\gamma \frac{f'(z)}{f(z)}\,dz$ is 0 over every piecewise C^1 closed curve in U. Hence we can define an indefinite integral of $f'(z)/f(z)$ by putting $F(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta$ for any piecewise C^1 curve from z_0 to z. The function $F(z)$ is analytic is U, and its complex derivative is $f'(z)/f(z)$. Consequently

$$\tfrac{d}{dz}\big(f(z)e^{-F(z)}\big) = \big(f'(z) - f(z)F'(z)\big)e^{-F(z)} = 0$$

on U, and $f(z)e^{-F(z)}$ is constant. That is, $f(z) = ce^{F(z)}$ for some nonzero constant c. Putting $z = z_0$ shows that $f(z_0) = ce^{F(z_0)} = c$. Thus $f(z) = f(z_0)e^{F(z)}$. Making an arbitrary choice for the value $\log f(z_0)$ of the logarithm at z_0 allows us to rewrite this equation as $f(z) = e^{F(z)+\log f(z_0)}$. Consequently we can define $\log f(z) = F(z) + \log f(z_0)$. □

Corollary B.43 (Argument Principle, global form). Let $\{a_j\}$ be the set of distinct zeros and $\{b_l\}$ be the set of distinct poles of a meromorphic function $f(z) \not\equiv 0$ defined in a simply connected region U, and suppose that a_j has order h_j and b_l has order k_l. For every piecewise C^1 cycle γ in U not passing through a zero or pole,

$$\sum_j h_j n(\gamma, a_j) - \sum_l k_l n(\gamma, b_l) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.$$

Consequently if $\Gamma = f \circ \gamma$, then

$$n(\Gamma, 0) = \sum_j h_j n(\gamma, a_j) - \sum_l k_l n(\gamma, b_l).$$

REMARKS. The condition that γ does not pass through a zero or pole means that there is some expression for γ as a combination of piecewise C^1 curves with no zero or pole on the image of any of the curves. The result is derived from the Cauchy Integral Theorem (Theorem B.40) in the same way that Theorem B.33 was derived from Theorem B.9.

Corollary B.44 (Residue Theorem). Let $f(z)$ be a function analytic in a simply connected region U except for poles at points $\{z_j\}$. If γ is a piecewise C^1 cycle in U not passing through any of the poles, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \operatorname{Res}_f(z_j)n(\gamma, z_j),$$

only finitely many of the terms on the right side being nonzero.

REMARKS. The result is derived from the Cauchy Integral Theorem (Theorem B.40) in the same way that Theorem B.37 was derived from Theorem B.9.

We now return to the consideration of examples of applications of the Residue Theorem that we suspended near the end of Section B11.

EXAMPLE 6 (CONTINUED). $I = \int_{-\infty}^{\infty} \frac{\log|x|}{(1+x^2)^2}\, dx$, an absolutely convergent integral. If we replace $\log|x|$ by $\log z$, this looks like a candidate for the kind of analysis we have did in Section B11. The complex line integral to consider will use a large semicircle in the upper half plane and line segments on the real axis. In addition, we shall use a small semicircle in the upper half plane about the singularity 0. The piecewise C^1 curve thus goes from $(-R, 0)$ to $(-\epsilon, 0)$ on the real axis, around the small semicircle in the upper half plane given by $t \mapsto \epsilon e^{it}$ with t going from π down to 0, from $(\epsilon, 0)$ to $(R, 0)$ on the real axis, and then around the large semicircle in the upper half plane $t \mapsto Re^{it}$ with t going from 0 to π. The expression $\log z$ is to be interpreted as the branch of the logarithm that excludes the negative imaginary axis and takes values whose imaginary parts are between $-\pi/2$ and $+3\pi/2$.

A region containing this curve is the difference of the open set set $\operatorname{Im} z > -\epsilon$ and the closed disk $|z| \le \epsilon$. It is simply connected, and Corollary B.44 is applicable. When $z = x+iy$, the value of $\log z$ is $\log|z|+i\theta$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Thus it is $\log|z|$ plus a bounded expression. Let us bound the complex line integrals over the two semicircles. On the large semicircle, where $|z| = R$ and $R \ge 2$, the value of the integrand is $\le CR^{-4}\log R$ for some constant C. The length of the large semicircle is πR, and thus the absolute value of the complex line integral is $\le CR^{-4}(\log R)2\pi R$. This tends to 0. On the small semicircle, where $|z| = \epsilon$ and $\epsilon \le \frac{1}{2}$, the value of the integrand is $\le C\log(\epsilon^{-1})$ for some constant C. The length of the small semicircle is $\pi\epsilon$, and thus the absolute value of the complex line integral is $\le 2\pi C\epsilon\log(\epsilon^{-1})$. This tends to 0.

The only pole of the integrand in the upper half plane is at $z = i$. The residue there is

$$\begin{aligned}\lim_{z\to i}\big(\tfrac{d}{dz}\big)\big((z-i)^2(\log z)(1+z^2)^{-2}\big) &= \lim_{z\to i}\big(\tfrac{d}{dz}\big)\big((z+i)^{-2}(\log z)\big)\\ &= \lim_{z\to i}\big(-2(z+i)^{-3}\log z + (z+i)^{-2}z^{-1}\big)\\ &= \tfrac{\pi}{8} + \tfrac{i}{4}.\end{aligned}$$

Letting $R \to \infty$ and $\epsilon \to 0$ gives

$$\int_{-\infty}^{0} \frac{\log z}{(1+z^2)^2}\, dz + \int_0^{\infty} \frac{\log z}{(1+z^2)^2}\, dz = 2\pi i\big(\tfrac{\pi}{8} + \tfrac{i}{4}\big)$$

For negative real x, we have $\log x = \log(-x) + \pi i$. Thus we can rewrite this equation as

$$2\int_0^{\infty} \frac{\log z}{(1+z^2)^2}\, dz + \pi i \int_0^{\infty} \frac{1}{(1+z^2)^2}\, dz = 2\pi i\big(\tfrac{\pi}{8} + \tfrac{i}{4}\big).$$

We could evaluate the second term on the left side by the method of Example 1 in Section B11, but it is not necessary to do so. All we have to do is to equate the real parts of the two sides of the equation. We obtain

$$I = \int_{-\infty}^{\infty} \tfrac{\log|x|}{(1+x^2)^2}\,dx = 2\int_0^{\infty} \tfrac{\log z}{(1+z^2)^2}\,dz = -\tfrac{\pi}{2}.$$

EXAMPLE 7. $I = \int_0^{\infty} \frac{x^\alpha}{1+x^2}\,dx$ with $0 < \alpha < 1$, an absolutely convergent integral. This example is rather similar to Example 6. The integrand is to be $\frac{z^\alpha}{1+z^2}$, where $z^\alpha = e^{\alpha \log z}$ and $\log z$ is the same branch as in Example 6. That is, the branch of the logarithm excludes the negative imaginary axis and takes values whose imaginary parts are between $-\pi/2$ and $+3\pi/2$. The piecewise C^1 curve over which to integrate is the same as in Example 6, and the simply connected region is the same as in that example.

The integrals over the semicircles are handled more or less as in Example 6. The only pole of the integrand in the upper half plane is at $z = i$, and it is simple. The residue there is

$$\lim_{z\to i} \tfrac{(z-i)z^\alpha}{1+z^2} = \lim_{z\to i} z^\alpha/(z+i) = e^{\alpha \log i}/(2i) = e^{\alpha i\pi/2}/(2i).$$

The Residue Theorem, in the form of Corollary B.44, gives

$$\int_{-\infty}^{0} \tfrac{z^\alpha}{1+z^2}\,dz + \int_0^{\infty} \tfrac{z^\alpha}{1+z^2}\,dz = 2\pi i \operatorname{Res}_{z^\alpha/(1+z^2)}(i) = \pi e^{\alpha i\pi/2}.$$

In the first integral, we change z into $-z$ and use the following formula, valid for positive real z:

$$(-z)^\alpha = e^{\alpha \log(-z)} = e^{\alpha(i\pi + \log z)} = e^{\alpha i\pi} z^\alpha$$

Then we see that

$$\int_{-\infty}^{0} \tfrac{z^\alpha}{1+z^2}\,dz = \int_0^{\infty} \tfrac{e^{\alpha i\pi} z^\alpha}{1+z^2}\,dz = e^{\alpha i\pi} \int_0^{\infty} \tfrac{z^\alpha}{1+z^2}\,dz.$$

Hence

$$(1 + e^{\alpha i\pi}) \int_0^{\infty} \tfrac{x^\alpha}{1+x^2}\,dx = \pi e^{\alpha i\pi/2},$$

and

$$\int_0^{\infty} \tfrac{x^\alpha}{1+x^2}\,dx = \tfrac{\pi e^{\alpha i\pi/2}}{1+e^{\alpha i\pi}} = \tfrac{\pi}{2\cos(\alpha\pi/2)}.$$

Let us return to the question of the equivalence for bounded regions U in $\mathbb{C}$ of the text's definition of the term "simply connected," i.e., that $\mathbb{C} - U$ is connected, and the standard definition. It is time to state the standard definition precisely.

A pathwise connected metric space X such as our bounded region U is **simply connected** in the standard sense if every **loop** $\gamma(t)$ defined for t in a nontrivial closed interval $[a, b]$ and **based** at a point p of X is **homotopic** to a **constant loop**.[13] We need to define the terms "loop," "based," "homotopic," and "constant loop." A **loop** is a path γ with $\gamma(a) = \gamma(b)$. It is **based** at p if $\gamma(a) = p$. A loop γ based at p is **homotopic** to another loop σ based at p if there exists a continuous function $f : [a, b] \times [0, 1] \to X$ such that

$$\begin{aligned} f(t, 0) &= \gamma(t) && \text{for all } t \in [a, b], \\ f(t, 1) &= \sigma(t) && \text{for all } t \in [a, b], \\ f(a, s) &= p && \text{for all } s \in [0, 1], \\ f(b, s) &= p && \text{for all } s \in [0, 1]. \end{aligned}$$

The relation "is homotopic to" is easily seen to be an equivalence relation. The **constant loop** σ based at p is the one having $\sigma(t) = p$ for all $t \in [a, b]$.

The intuition is that γ and σ are homotopic if the one loop can be continuously deformed into the other through loops based at p. The function f gives the deformation, being continuous and having $f(\,\cdot\,, 0) = \gamma$ and $f(\,\cdot\,, 1) = \sigma$. The deformed loops are $f(\,\cdot\,, s)$ for $s \in [0, 1]$. "Simply connected" in this sense means that every loop based at p can be continuously deformed to the constant loop.

The question to be addressed is why for a bounded region U of $\mathbb{C}$ the condition that $\mathbb{C} - U$ be connected holds if and only if every loop based at a point p of U is homotopic to a constant loop. In view of Lemma B.39, it is enough to say why a bounded region U of $\mathbb{C}$ has $n(\gamma, z_0) = 0$ for all piecewise C^1 cycles γ in U and points z_0 in $\mathbb{C} - U$ if and only if every loop in U based at p is homotopic to a constant loop.

For our current purposes the interesting matter is the relationship between winding numbers and loops homotopic to constant loops. We begin with that relationship, treating the remainder of the discussion of equivalent definitions of "simply connected" as a digression and putting it in small type. Fix z_0 in $\mathbb{C} - U$, and consider winding numbers about z_0.

We shall make use of the following notation. For any $z \in \mathbb{C} - \{0\}$, let $\{z\} = z/|z|$, a complex number with $|\{z\}| = 1$. The number $\{z\}$ captures the notion of the argument of z unambiguously: for any value of $\arg z$, we have $\{z\} = e^{i \arg z}$.

Let γ be a loop in U based at $p \in U$, and let $\{\gamma\}$ be the loop based at $\{p - z_0\}$ in the unit circle of $\mathbb{C}$ and defined by $\{\gamma\}(t) = \{\gamma(t) - z_0\}$ for $t \in [a, b]$. The

[13]The choice of p is irrelevant because loops based at two different points are related by a path from the one point to the other.

winding numbers of γ about z_0 and of $\{\gamma\}$ about 0 are given by

$$n(\gamma, z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0} = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)\,dt}{\gamma(t)-z_0}$$

and
$$n(\{\gamma\}, 0) = \frac{1}{2\pi i}\int_{\{\gamma\}} \frac{dz}{z} = \frac{1}{2\pi i}\int_a^b \frac{\{\gamma\}'(t)\,dt}{\{\gamma\}(t)}.$$

These two winding numbers turn out to be equal, but more is true. As in Section III.11, define $\gamma_{[a,s]}$ for $a \le s \le b$ to be the restriction of γ from $[a, b]$ to $[a, s]$. Then actually[14]

$$\int_{\{\gamma\}_{[a,s]}} \frac{dz}{z} = i \operatorname{Im} \int_{\gamma_{[a,s]}} \frac{dz}{z-z_0}$$

for every s in $[a, b]$. The integral $\int_{\gamma_{[a,s]}} \frac{dz}{z-z_0}$ is trying to be $\log(\gamma(s))) - \log(\gamma(a))$, but log is multivalued. We run into trouble when we consider that integral just when $s = b$, but the trouble goes away when we consider the integral as a continuous function of s. As a result we can treat the left side as a well defined continuous function of s, and its value is $i\big(\arg(\gamma(s)) - \arg(\gamma(a))\big)$. In effect, from the values of $\{\gamma\}(s) = e^{i\arg(\gamma(s))}$ in S^1 and a definite choice of $\arg(\gamma(a))$, we have managed to construct a "lift" of the values to $i\big(\arg(\gamma(s)) - \arg(\gamma(a))\big)$ in $i\mathbb{R}$. The winding number of γ about z_0 is 0 if and only if the lifted value $i\big(\arg(\gamma(s)) - \arg(\gamma(a))\big)$ at $s = b$ is 0.

In fact, this line of reasoning does not seriously make use of integration and can be reformulated without it. Thus we can work with any loop γ in U based at p, not necessarily a piecewise smooth closed curve. Starting from such a γ, we form $\{\gamma\}$ as above, and we fix a value for $i\arg(\gamma(a))$. Using the uniform continuity of $\{\gamma\}$, we can form a partition of $[a, b]$, say $a = t_0 < t_1 < \cdots < t_n = b$, such that $\{\gamma\}$ varies by less than a fraction of the diameter of S^1 on each interval $[t_{j-1}, t_j]$ of the partition. Inductively on j, we can choose a unique value for $\arg(\gamma(t_j))$ so that $|\arg(\gamma(t_j)) - \arg(\gamma(t_j))| < \frac{1}{2}$, and the result is that we can lift $\{\gamma\}$ to a continuous function $i\big(\arg(\gamma(s)) - \arg(\gamma(a))\big)$ with values in $i\mathbb{R}$. The value of $\arg(\gamma(b)) - \arg(\gamma(a))$ may be defined to be the winding number of γ. The loop γ will be homotopic to a constant loop only if this winding number is 0.

We digress from the complex analysis to sketch some further details about the equivalence of the definitions of "simply connected" for a region. With γ as above, suppose that the winding number is nonzero. Let us see that γ is not homotopic to a constant loop in $\mathbb{C} - \{z_0\}$, much less in $\mathbb{C} - U$. Arguing by contradiction, suppose that γ is homotopic to a constant loop in $\mathbb{C} - \{z_0\}$. Then $\{\gamma\}$ is homotopic to a constant in S^1. Using topological reasoning analogous to the reasoning in the previous paragraph, we lift the whole homotopy for $\{\gamma\}$ to have values in $i\mathbb{R}$. The constant loop lifts to a constant function, and continuity of the homotopy demands that $\arg(\gamma(b)) - \arg(\gamma(a)) = 0$. So the winding number had to be 0, and we have arrived at a contradiction.

[14]This equality follows from the identity $\frac{d}{dt}|\gamma(t)-z_0| = \frac{1}{2}\frac{d}{dt}\big((\gamma(t)-z_0)(\overline{\gamma(t)-z_0})\big)/|\gamma(t)-z_0|$ and an elementary computation that we omit.

The proof that has just been sketched shows that the existence of a nonzero winding number implies the existence of a loop in $\mathbb{C} - U$ that is not homotopic to a constant loop. The converse is more subtle. In fact, what one shows is that if all the winding numbers are 0, then the bounded region is homeomorphic with the open unit disk, and it is an easy matter to verify that every loop in the open unit disk is homotopic to a constant loop. Actually more is true. If all the winding numbers are 0, then there is an analytic function that carries U one-one onto the unit disk (and necessarily has an analytic inverse). This result is known as the Riemann Mapping Theorem. Its proof is beyond the scope of this appendix, but a proof can be found in the book *Complex Analysis* by Ahlfors.

B13. Global Theorems in General Regions

Built into Lemma B.39 is the statement that in any bounded region that is not simply connected, there is an an analytic function $f(z)$ for which the complex line integral of f over some cycle is not 0. Thus any generalization of the Cauchy Integral Theorem to arbitrary bounded regions must either limit the cycles in some way or limit the analytic functions. In this section we shall obtain a version of the Cauchy Integral Theorem applicable to all analytic functions but only certain cycles.

A piecewise C^1 cycle γ in a bounded region U of $\mathbb{C}$ will be said to be a **boundary cycle** if $n(\gamma, a) = 0$ for every a in $\mathbb{C} - U$. The book *Complex Analysis* by Ahlfors refers to boundary cycles as "cycles that are homologous to 0."

Theorem B.45 (Cauchy Integral Theorem, homology form). If f is analytic in a bounded region U in $\mathbb{C}$, then

$$\int_\gamma f(z)\,dz = 0$$

for every piecewise C^1 boundary cycle γ in U.

REMARK. If U is simply connected, then every piecewise C^1 cycle is a piecewise C^1 boundary cycle by Lemma B.39. Consequently Theorem B.45 reduces to Theorem B.40 in the simply connected case.

Warning. Although Lemma B.38 says that any cycle is a combination of piecewise C^1 closed curves, the constituent piecewise C^1 closed curves of a boundary cycle need not be boundary cycles.

PROOF. First we prove the theorem for the special case that the boundary cycle γ is polygonal with each side parallel to the real or imaginary axis. We may assume that γ contains at least one horizontal segment and one vertical segment, and we regard the straight-line segments as fixed for the reminder of the proof of the special case. Then image(γ) is a meaningful subset of U.

The proof of Theorem B.40 introduced some finite rectangles R_i and some infinite rectangles R'_j, as well as points a_i in R_i and a'_j in R'_j, and we use those constructs again here. Repeating the justification of (‡) in that proof, we find that γ is of the form

$$\gamma = \sum_i n(\gamma, a_i)\partial R_i. \qquad (*)$$

Let us now prove that every R_i in $(*)$ for which the coefficient $n(\gamma, a_i)$ is nonzero lies completely in U, i.e., the inside of R_i lies in U. To do so, we shall use the hypothesis that the cycle γ is a boundary cycle.

The function $a \mapsto n(\gamma, a)$ is constant on the inside of R_i since the inside is connected and lies in the complement of γ, and a_i is in the inside. Since $n(\gamma, a_i)$ is being assumed to be nonzero, $n(\gamma, a)$ is nonzero everywhere on the inside of R_i. Because γ is a boundary cycle, each such a is in U. Consequently each R_i for which $n(\gamma, a_i)$ in $(*)$ is nonzero lies completely in U. By Goursat's Lemma, $\int_{\partial R_i} f(z)\,dz = 0$. Forming the integer combination of such integrals indicated by $(*)$, we obtain $\int_\gamma f(z)\,dz = 0$. This completes the proof in the special case.

In the general case let γ be a piecewise C^1 boundary cycle. Fix a decomposition of γ into a combination of piecewise C^1 curves, so that image(γ) is meaningful. As in the proof of Lemma B.39, there exists a number $m > 0$ such that every point of image(γ) is at distance $> m$ from $\mathbb{C} - U$. Each of the finitely many component curves of γ has a parametrization $t \mapsto z(t)$. By uniform continuity let $\delta > 0$ be chosen so that each such parametrization has $|z(t) - z(t')| < m$ whenever $|t - t'| < \delta$. We divide the respective domains of the curves into subintervals each of length $< \delta$, and we let $\{\gamma_i\}$ be the resulting curves. Because of the way that m was defined, each γ_i is contained in an open disk of radius m that lies entirely in U. The endpoints of γ_i can be joined by a polygonal curve σ_i lying completely in the disk and having each side parallel to the real or imaginary axis. Since the Cauchy Integral Theorem has been proved for a disk (Theorem B.9), we have $\int_{\gamma_i} g(z)\,dz = \int_{\sigma_i} g(z)\,dz$ for every analytic function $g(z)$ in U. Let σ be the polygonal cycle $\sum_i \sigma_i$. Summing on i, we obtain

$$\int_\gamma g(z)\,dz = \int_\sigma g(z)\,dz. \qquad (**)$$

Taking $g(z) = 1/(z - a)$ in $(**)$ for any point $a \in \mathbb{C} - U$ shows that $n(\gamma, a) = n(\sigma, a)$. Since γ is a boundary cycle, so is σ. Taking $g(z)$ in $(**)$ to be our given analytic function $f(z)$ shows that that

$$\int_\gamma f(z)\,dz = \int_\sigma f(z)\,dz. \qquad (\dagger)$$

Since σ is a boundary cycle, $\int_\sigma f(z)\,dz = 0$ by the special case already proved. In view of (†), $\int_\gamma f(z)\,dz = 0$. □

EXAMPLES.

(1) The difference for an annulus (washer) of a closed curve with winding number 1 about the center and a second closed curve with winding number 1. The result is a boundary cycle in the annulus. See Figure B.7a.

(2) In a region with two holes, let closed curves γ_1 and γ_2 wind about the respective holes once, and let a closed curve γ_3 wind about both holes once in the opposite direction. Then $\gamma_1 + \gamma_2 + \gamma_3$ is a boundary cycle. See Figure B.7b, which shows γ_1, γ_2, and γ_3 with dashing.

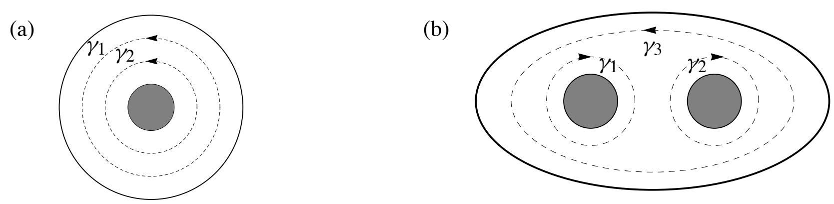

FIGURE B.7. Boundary cycle: (a) as the difference of two cycles, (b) as the sum of three cycles

Corollary B.46 (Residue Theorem). Let $f(z)$ be a function analytic in a region U except for poles at points $\{z_j\}$. If γ is a piecewise C^1 boundary cycle in U not passing through any of the poles, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \operatorname{Res}_f(z_j)\,n(\gamma, z_j),$$

only finitely many of the terms on the right side being nonzero.

REMARKS. The result is derived from the Cauchy Integral Theorem (Theorem B.44) in the same way that Theorem B.37 was derived from Theorem B.9.

B14. Laurent Series

An **annulus** about $z_0 \in \mathbb{C}$ is a set of the form $\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$, where $R_1 \geq 0$ and $R_2 \leq \infty$. In this section, we shall classify the analytic functions in an arbitrary annulus and see that all such functions are uniquely the sum of an analytic function of z for $|z - z_0| < R_2$ and an analytic function of z for $|z - z_0| > R_1$ that vanishes at infinity. Specializing our result to the case that $R_1 = 0$ will reveal the nature of essential singularities.

Theorem B.47. If $f(z)$ is an analytic function in the annulus

$$\{z \in \mathbb{C} \bigm| R_1 < |z - z_0| < R_2\},$$

then there exist unique functions $g_1(z)$ and $g_2(z)$ such that $g_1(z)$ is analytic for $|z - z_0| < R_2$, $g_2(z)$ is analytic for $|z - z_0| > R_1$, $\lim_{z\to\infty} g_2(z) = 0$, and $f(z) = g_1(z) + g_2(z)$ for $R_1 < |z - z_0| < R_2$. If r is any number with $R_1 < r < R_2$ and if C denotes the standard circle of radius r centered at z_0, then g_1 and g_2 are given by

$$g_1(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad \text{for } |z - z_0| < r < R_2,$$

$$g_2(z) = -\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{\zeta - z} \qquad \text{for } R_1 < r < |z - z_0|.$$

Moreover, with the definition

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$

for $-\infty < n < \infty$, the functions g_1 and g_2 are given by

$$g_1(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n \quad \text{for } |z - z_0| < R_2$$

$$g_2(z) = \sum_{n=1}^{\infty} A_{-n} (z - z_0)^{-n} \quad \text{for } |z - z_0| > R_1.$$

REMARKS. The series for g_1 is a power series in $z - z_0$ and converges absolutely in an open disk as usual; the convergence is uniform on any proper closed subdisk. The series for g_2 is a power series in $1/(z - z_0)$, and its convergence follows the usual rules for convergence of power series. Since $f = g_1 + g_2$, we can write

$$f(z) = \sum_{n=-\infty}^{\infty} A_n (z - z_0)^n,$$

and the series on the right converges in an annulus without regard to the order of the terms. It is called the **Laurent series expansion** of f in the given annulus about z_0.

PROOF OF UNIQUENESS. If we have two decompositions of the function f, then subtracting them yields a decomposition $0 = h_1 + h_2$ with h_1 analytic for $|z - z_0| < R_2$ and with h_2 analytic for $|z - z_0| > R_1$. The function defined as h_1 for $|z - z_0| < R_2$ and as $-h_2$ for $|z - z_0| > R_1$ is consistently defined, is analytic in all of $\mathbb{C}$, and tends to 0 as z tends to infinity. By Liouville's Theorem (Corollary B.17), this function is identically 0. Thus h_1 and h_2 are 0. □

PROOF OF EXISTENCE. Let z be in the annulus defined by $R_1 < |z - z_0| < R_2$. If r_1 and r_2 are numbers with $R_1 < r_1 < r_2 < R_2$ and if C_1 and C_2 denote the respective standard circles of radius r_1 and r_2 centered at z_0, then it is immediate from the definitions that $C_2 - C_1$ is a boundary cycle in the annulus. Therefore Corollary B.46 gives

$$\int_{C_2 - C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z} = 2\pi i f(z)\, n(C_2 - C_1, z) \qquad (*)$$

as long as $|z - z_0|$ is not r_1 or r_2.

If $|z - z_0| < r_1$, then $n(C_2 - C_1, z) = 0$, and it follows from $(*)$ that $\int_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \int_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z}$. Consequently g_1 is analytic and well defined independently of the radius r of C as long as the inequality $|z - z_0| < r < R_2$ remains valid. Adapting the radius r suitably to handle a disk about any given z, we see that the formula for g_1 consistently defines an analytic function for $|z - z_0| < R_2$.

Similarly if $|z - z_0| > r_2$, then $n(C_2 - C_1, z) = 0$, and we see that the formula for g_2 consistently defines an analytic function for $R_1 < |z - z_0|$.

If z is in the annulus, choose r_1 and r_2 so that $R_1 < r_1 < |z - z_0| < r_2 < R_2$, and let C_1 and C_2 be the standard circles of radius r_1 and r_2 centered at z_0. This time we have $n(C_2, z) = 1$ and $n(C_1, z) = 0$. Thus $(*)$ gives

$$\int_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \int_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z} = 2\pi i f(z). \qquad (**)$$

From our definitions, $\int_{C_2} \frac{f(\zeta)\,d\zeta}{\zeta - z} = 2\pi i g_1(z)$ and $-\int_{C_1} \frac{f(\zeta)\,d\zeta}{\zeta - z} = 2\pi i g_2(z)$. Therefore $(**)$ says that $g_1(z) + g_2(z) = f(z)$.

Since $|g_2(z)| \le \frac{1}{2\pi} \sup_{\zeta \in C} |f(\zeta)| \left| \int_C \frac{d\zeta}{\zeta - z} \right|$, $\lim_{z \to \infty} g_2(z) = 0$. This completes the proof of existence of the decomposition of f. $\square$

PROOF OF THE EXPANSIONS OF g_1 AND g_2 IN TERMS OF A_n. According to Taylor's Theorem (Theorem B.20) and the complex derivative formula (Theorem B.11), $g_1(z)$ has the Taylor expansion $g_1(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n$, where

$$A_n = \frac{1}{2\pi i} \int\limits_{|\zeta - z| = r} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$

for any $r < R_2$.

Let us now consider g_2. This is an analytic function of $(z - z_0)^{-1} = z'$ for $R_1 < |z - z_0| < \infty$, i.e., for $0 < |z'| < R_1^{-1}$, and it tends to 0 as z' tends to 0. Thus it has a removable singularity at $z' = 0$, and we can write $g_2(z_0 + z'^{-1}) = \sum_{n=1}^{\infty} B_n z'^n$ for $|z'| < R_1^{-1}$.

Let us obtain an explicit formula for the coefficient B_n. For $|z - z_0| > r$, i.e., for $|z'| < r^{-1}$, $g_2(z)$ is given explicitly by $g_2(z) = -\frac{1}{2\pi i}\int_{|\zeta - z_0|=r} \frac{f(\zeta)\,d\zeta}{\zeta - z}$. Hence

$$g_2(z_0 + z'^{-1}) = \frac{1}{2\pi i}\int_{|\zeta - z_0|=r} \frac{f(\zeta)\,d\zeta}{\zeta - z_0 - z'^{-1}}.$$

The formula for the Taylor coefficient B_n is

$$B_n = \frac{1}{2\pi i}\int_{|z'|=r'^{-1}} \frac{g_2(z_0 + z'^{-1})\,dz'}{z'^{n+1}}$$

for any r' with $r'^{-1} < r^{-1}$, i.e., $r' > r$. In the integral for B_n, we make the change of variables $z' = (z - z_0)^{-1}$. Lemma B.31 shows that we are to substitute $dz' = -dz/(z - z_0)^{-2}$ and use the corresponding curve in the z plane, which is the reverse of the standard circle $|z - z_0| = r'$. The minus sign for switching back to the usual orientation for the standard circle $|z - z_0| = r'$ cancels the minus sign in the formula for dz, and we see that

$$\begin{aligned} B_n &= \tfrac{1}{2\pi i}\int_{|z-z_0|=r'} \tfrac{g_2(z)(z-z_0)^{-2}\,dz}{(z-z_0)^{-n-1}} \\ &= \left(\tfrac{1}{2\pi i}\right)^2 \int_{|z-z_0|=r'} \left[\int_{|\zeta - z_0|=r} \tfrac{f(\zeta)\,d\zeta}{\zeta - z}\right] \tfrac{dz}{(z-z_0)^{-n+1}}. \end{aligned}$$

Fubini's Theorem (Corollary 3.33) allows us to interchange the order of integration and obtain

$$B_n = \tfrac{1}{2\pi i}\int_{|\zeta - z_0|=r}\left[\tfrac{1}{2\pi i}\int_{|z-z_0|=r'} \tfrac{dz}{(z-z_0)^{-n+1}(\zeta - z)}\right] f(\zeta)\,d\zeta.$$

In the integration within the brackets, ζ lies inside the circle $|z - z_0| = r'$ because $|\zeta - z_0| = r < r' = |z - z_0|$. Thus the Cauchy Integral Formula (Theorem B.10) shows that the expression in brackets equals $1/(\zeta - z_0)^{-n+1}$, and we obtain

$$B_n = \tfrac{1}{2\pi i}\int_{|\zeta - z_0|=r} \tfrac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{-n+1}} = A_{-n}.$$

Consequently $g_2(z) = \sum_{n=1}^{\infty} B_n z'^n = \sum_{n=1}^{\infty} A_{-n}(z - z_0)^{-n}$, as asserted. □

Corollary B.48. If f is analytic in a region U except possibly for an isolated singularity at the point z_0 of U, then there exist functions g and h such that g is analytic in U, h is analytic in all of $\mathbb{C}$, $h(0) = 0$, and $f(z) = g(z) + h((z - z_0)^{-1})$ on $U - \{z_0\}$.

PROOF. Choose $R_2 > 0$ so that the disk of radius R_2 and center z_0 lies in U. Application of Theorem B.47 to $f(z)$ with $R_1 = 0$ produces functions g_1 and g_2 with g_1 analytic for $|z-z_0| < R_2$, g_2 analytic for $|z-z_0| > 0$, $\lim_{z\to\infty} g_2(z) = 0$, and $f = g_1 + g_2$ for $0 < |z-z_0| < R_2$. Define $h(z') = g_2(z_0 + z'^{-1})$, so that $g_2(z) = h((z-z_0)^{-1})$. Since $h(z')$ is analytic for $z' \neq 0$ and has $\lim_{z'\to 0} h(z') = 0$, h extends to be analytic in all of $\mathbb{C}$ with $h(0) = 0$. The equation $f(z) = g_1(z) + h((z-z_0)^{-1})$ is valid for $0 < |z-z_0| < R_2$, and the terms $f(z)$ and $h((z-z_0)^{-1})$ are meaningful for all z in $U - \{z_0\}$. Thus we can consistently define $g(z)$ to be $g_1(z)$ for $|z-z_0| < R_2$ and to be $f(z) - h((z-z_0)^{-1})$ for $z \in U - \{z_0\}$, and g and h have the required properties. □

In Corollary B.48 if f has a pole at z_0, then h is a polynomial and the function $h((z-z_0)^{-1})$ is the singular part of f about z_0, as defined in Section B6. Corollary B.48 extends the analysis to include essential singularities, and the term $h((z-z_0)^{-1})$ may be regarded as the **singular part** of a general isolated singularity about z_0. The singular part is thus built by forming $h((z-z_0)^{-1})$ from a function $h(z)$ analytic in all of $\mathbb{C}$.

Corollary B.48 allows us to extend any of the various forms of the Residue Theorem to include arbitrary isolated singularities, not just poles. For example, the version in Corollary B.46 becomes the following.

Corollary B.49 (Residue Theorem). Let $f(z)$ be a function analytic in a region U except for isolated singularities at points $\{z_j\}$. If γ is a piecewise C^1 boundary cycle in U not passing through any of the poles, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_j \operatorname{Res}_f(z_j)\, n(\gamma, z_j),$$

only finitely many of the terms on the right side being nonzero.

For the proof we go over the arguments for Theorem B.37 and Corollary B.46 and make suitable small adjustments. Because of Corollary B.48 the singular part about a singularity z_0 is now given by a series in powers of $(z-z_0)^{-1}$. We isolate the term $A_{-1}(z-z_0)^{-1}$ containing the power $(z-z_0)^{-1}$, and sum of the remaining terms is the complex derivative of an analytic function. The term $A_{-1}(z-z_0)^{-1}$ makes the contribution $2\pi i A_{-1} n(\gamma, z_0)$ to the formula for $\int_\gamma f(z)\,dz$.

What breaks down is the formula of Section B11 for the residue in terms of values of complex derivatives. Our analysis leads us to no such formula. Thus Corollary B.49 is really only of theoretical interest and does not play the important role that the Residue Theorem plays, for example, in the evaluation of definite integrals. Another casualty of the failure of the usual formula for residues is that the Argument Principle breaks down. Here is an example.

EXAMPLE. Let $f(z) = e^{1/z}$ and $g(z) = \sin(1/z)$ in $\mathbb{C} - \{0\}$, and let C be the standard circle of radius 1 about the origin. The functions f and g have essential singularities at $z = 0$ and only there. The function f has neither any poles nor any zeros, whereas the function g has no poles but has a sequence of zeros tending to 0.

Let us compute the quantities that occur in the Argument Principle when the only singularities are poles. Since $f'(z) = -z^{-2}e^{1/z}$, we have

$$\int_C \frac{f'(z)}{f(z)}\,dz = \int_C -z^{-2}\,dz = 0.$$

For $g(z)$, we have $g'(z) = -z^{-2}\cos(1/z)$ and $g'(z)/g(z) = -z^{-2}\cot(1/z)$. We compute the initial terms of the Laurent series expansion of $-z^{-2}\cot(1/z)$ by the method of Section B8. We have

$$\begin{aligned}
\frac{g'(z)}{g(z)} &= -\frac{\cos(1/z)}{z^2\sin(1/z)} = -\frac{1 - \frac{1}{2}z^{-2} + [z^{-4}]}{z^2(z^{-1} - \frac{1}{6}z^{-3} + [z^{-5}])} \\
&= z^{-1} \times \frac{1 - \frac{1}{2}z^{-2} + [z^{-4}]}{1 - (\frac{1}{6}z^{-2} + [z^{-4}])} \\
&= z^{-1}(1 - \tfrac{1}{2}z^{-2} + [z^{-4}])(1 + \tfrac{1}{6}z^{-2} + [z^{-4}]) \\
&= z^{-1}(1 - \tfrac{1}{3}z^{-2} + [z^{-4}]) = z^{-1} + \begin{pmatrix}\text{complex derivative of} \\ \text{an analytic function}\end{pmatrix}.
\end{aligned}$$

Thus
$$\int_C \frac{g'(z)}{g(z)}\,dz = \int_C z^{-1}\,dz = 2\pi i.$$

Neither the integral for f nor the integral for g is something one might predict on the basis of the number and nature of the zeros and singularities.

B15. Holomorphic Functions of Several Variables

Holomorphic functions of several variables play a role in the Chapters VII and VIII of *Advanced Real Analysis*, and they will be introduced here. Such functions play no role in Chapter I through XII of *Basic Real Analysis*.

Our concern will be with equivalent definitions of the notion of a "holomorphic function" on an open subset of $\mathbb{C}^n$. The notation $\mathbb{C}^n$ refers to the complex vector space of n-dimensional *column* vectors with entries in $\mathbb{C}$. For economy of presentation, such vectors are often written as n-tuples of complex numbers with entries separated by commas, thus as tuples $c = \{c_j\}_{j=1}^n = (c_1, \dots, c_n)$. The norm of the member c of $\mathbb{C}^n$ is defined as

$$|c| = |(c_1, \dots, c_n)| = \Big(\sum_{j=1}^n |c_j|^2\Big)^{1/2}.$$

The space $\mathbb{C}^n$ becomes a metric space if the distance between two members $c = \{c_j\}_{j=1}^n$ and $d = \{d_j\}_{j=1}^n$ is defined to be $\big(\sum_{j=1}^n |c_j - d_j|^2\big)^{1/2}$.

We can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by

$$(a_1 + ib_1, \dots, a_n + ib_n) \longleftrightarrow (a_1, b_1, \dots, a_n, b_n).$$

This identification respects addition, multiplication by real scalars, and the metric.

If $a = \{a_j\}_{j=1}^n$ is in $\mathbb{C}^n$ and if $r = (r_1, \dots, r_n)$ is an n-tuple of positive numbers, then the **open polydisk** of **polyradius** r about the **center** a is the subset of $\mathbb{C}^n$ of the form

$$\Delta(a, r) = \big\{ z \in \mathbb{C}^n \,\big|\, |z_j - a_j| < r_j \quad \text{for } 1 \le j \le n \big\}.$$

That is, $\Delta(a, r)$ is the set-theoretic product over j from 1 to n of the open disk about a_j of radius r_j. The **closed polydisk** of **polyradius** r about the **center** a is the subset of $\mathbb{C}^n$ of the form

$$\overline{\Delta}(a, r) = \big\{ z \in \mathbb{C}^n \,\big|\, |z_j - a_j| \le r_j \quad \text{for } 1 \le j \le n \big\}.$$

The set $\overline{\Delta}(a, r)$ is the closure of $\Delta(a, r)$.

A function $f : U \to \mathbb{C}$ on an open subset U of $\mathbb{C}^n$ is said to be **holomorphic** on U if it is continuous[15] and has the property that each restriction $z_j \mapsto f(c_1, \dots, c_{j-1}, z_j, c_{j+1}, \dots, c_n)$ to a function of one complex variable is an analytic function on each connected component of its domain. Briefly a function is said to be holomorphic if it is analytic in each variable. Such a function of course satisfies the Cauchy–Riemann equations in each complex variable.

Our interest is in characterizing holomorphic functions in terms of power series expansions, and for this purpose it is handy to use "multi-indices." Define $D_j = \frac{\partial}{\partial z_j}$ for $1 \le j \le n$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of nonnegative integers, we write $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. We call α a **multi-index**, and we call $|\alpha|$ the **total order** of α.

[15]The assumption of continuity can be dropped, as is shown in Hartogs's Theorem, given as Theorem 6 in Section B of Gunning's book.

Theorem B.50 (Osgood's Lemma). Let $f : U \to \mathbb{C}$ be a holomorphic function on an open subset U of $\mathbb{C}^n$, and let $\Delta(a, r)$ be an open polydisk contained in U. Then f has an absolutely convergent power series expansion

$$f(z) = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} c_\alpha (z-a)^\alpha$$

valid on $\Delta(a, r)$. The convergence is uniform on any compact subset of $\Delta(a, r)$, f is a C^∞ function (of $2n$ real variables) on U, and the coefficient c_α is necessarily given by

$$c_\alpha = \frac{(D^\alpha f)(a)}{\alpha!}.$$

Conversely a function $f : U \to \mathbb{C}$ is automatically holomorphic on U if to each p in U corresponds some open polydisk $\Delta(a, r) \subseteq U$ such that p is in $\Delta(a, r)$ and f has an absolutely convergent power series expansion $f(z) = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} c_\alpha (z-a)^\alpha$ valid on $\Delta(a, r)$.

PROOF. Throughout the proof let us write $r' = (r'_1, \dots, r'_n)$ for an n-tuple of positive numbers with $r'_j < r_j$ for all j. If f is holomorphic on $\Delta(a, r)$, then we can apply the Cauchy Integral Formula (Theorem B.10) in each variable to write

$$f(z) = (2\pi i)^{-n} \int_{|\zeta_1 - a_1| = r'_1} (\zeta_1 - z_1)^{-1} \Big[\int_{|\zeta_2 - a_2| = r'_2} (\zeta_2 - z_2)^{-1} \cdots \times \Big[\int_{|\zeta_n - a_n| = r'_n} (\zeta_n - z_n)^{-1} f(\zeta_1, \dots \zeta_n)\, d\zeta_n \Big] \cdots d\zeta_2 \Big] d\zeta_1.$$

Since f is continuous, Fubini's Theorem (Corollary 3.33) allows us to write the iterated integral as a multiple integral. Therefore

$$f(z) = (2\pi i)^{-n} \int_{|\zeta_j - a_j| = r'_j \text{ for all } j} \frac{f(\zeta_1, \dots, \zeta_n)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}.$$

For any fixed z in $\Delta(a, r')$, the geometric series expansion

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{j_1, \dots, j_n = 0}^{\infty} \frac{(z_1 - a_1)^{j_1} \cdots (z_n - a_n)^{j_n}}{(\zeta_1 - a_1)^{j_1+1} \cdots (\zeta_n - a_n)^{j_n+1}}$$

is uniformly convergent on the set of integration, and thus we can interchange the integral and the sum to obtain

$$f(z) = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} c_\alpha (z-a)^\alpha$$

with

$$c_\alpha = (2\pi i)^{-n} \sum_{j_1,\dots,j_n=0}^{\infty} \int_{|\zeta_j - a_j| = r_j' \text{ for all } j} \frac{f(\zeta_1,\dots,\zeta_n)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - a_1)^{j_1+1} \cdots (\zeta_n - a_n)^{j_n+1}}.$$

Taking into account Corollary 3.33 and Cauchy's formula for the complex derivative of an analytic function (Theorem B.11), we see that $c_\alpha = (D^\alpha f)(a)/\alpha!$, as asserted.

The fact that f is C^∞ on U follows from Theorem 1.23. Indeed, application of one of the $2n$ operators $\frac{\partial}{\partial x_j}$ or $\frac{\partial}{\partial y_j}$ term by term to the power series expansion yields a series of the same kind but with coefficients multiplied by α_j and perhaps also a power of i. The differentiated series converges in the same polydisk, and Theorem 1.23 says that the complex derivative and sum can be interchanged. The uniform convergence assures the continuity of the sum of the differentiated series. Iterating this argument shows that any complex derivative of any order of the series exists and is continuous. By Corollary 3.8, f is C^∞ as a function of $2n$ real variables. Corollary B.2$'$ shows that $D^\alpha f$ can be computed as a composition of the corresponding operators $\frac{\partial}{\partial x_j}$, and the formula for c_α follows by evaluating a differentiated series at $z = a$.

Most of the converse has already been proved. Arguments in the previous paragraph show that any convergent sum $f(z) = \sum_{m=0}^{\infty} \sum_{|\alpha|=m} c_\alpha (z-a)^\alpha$ is a C^∞ function and that its partial derivatives can be computed term by term. Making the computation and applying the Cauchy–Riemann equations in the form of Corollary B.2, we see that $f(z)$ is analytic in each complex variable. Since f is continuous and is analytic in each variable, f is holomorphic. $\square$

B16. Problems

1. (a) Can a function $f(z)$ that is defined and continuous for $|z| < 1$ have a complex derivative at $z = 0$ and only there?

   (b) Can a function $f(z)$ that is defined and continuous for $|z| < 1$ have a complex derivative at each point the real axis and only there?

2. Compute $\int_\gamma x\, dz$, where γ is the line segment from 0 to $1 + i$ starting at 0.

3. Suppose R is a filled rectangle in $\mathbb{C}$ with sides parallel to the axes, and let ∂R be its boundary traversed counterclockwise. Suppose that a function is defined to be analytic on a region if it has a *continuous* complex derivative. Without using the Cauchy Integral Theorem but arguing as in the special case of Green's Theorem in Example 1 of Section III.13, prove directly that if f is an analytic function defined on a region containing R, then $\int_{\partial R} f(z)\, dz = 0$.

4. This problem compares definitions of differentiability and complex differentiability for functions of n complex variables. Part (a) uses only material from Section B1, while part (b) uses material from Section B15. Let f be a complex-valued function defined on an open subset U of $\mathbb{C}^n$, and let $z = (z_1, \ldots, z_n)$ be in E. We say that f is **complex differentiable** at z if there exists a complex linear mapping $L : \mathbb{C}^n \to \mathbb{C}$ such that

$$\lim_{\zeta \to 0} \frac{|f(z+\xi) - f(z) - L(\zeta)|}{|\zeta|} = 0.$$

In this case we write $f'(z)$ for it and call $f'(z)$ the **complex derivative** of f at z. Its matrix is called the **complex Jacobian matrix** of f at z. The function f can be regarded as a function $f_{\mathbb{R}}$ of $2n$ real variables, specifically $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$, where x_j and y_j are the real and imaginary parts of z_j, with values in $\mathbb{R}^2 = \{(u, v)\}$. Section III.2 gives a definition of (real) differentiability of $f_{\mathbb{R}}$ and of a real Jacobian matrix of size 2-by-$2n$.

(a) Prove that f is complex differentiable at z if and only if $f_{\mathbb{R}}$ is differentiable at $(x_1, \ldots, y_n)$ and the real 2-by-$2n$ Jacobian matrix J satisfies

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} J = J \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

in block form. Here the entries in the matrix that multiplies J on the left side of the equation are numbers, while the entries in the matrix that multiplies J on the right side of the equation are n-by-n real matrices.)

(b) Use the definitions and results of Section B15 to prove that f is complex differentiable at every point of an open set of $\mathbb{C}^n$ if and only if f is holomorphic on the open set, if and only if $f_{\mathbb{R}}$ is C^∞ on the open set and its (real) Jacobian matrix satisfies the condition in (a) at every point of the open set.

Problems 5–8 introduce the **Riemann sphere** and "stereographic projection." The unit sphere S in $\mathbb{R}^3$ is given as $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. To each point (x_1, x_2, x_3) of $S - \{0, 0, 1)\}$, we associate the complex number $z = \varphi(x_1, x_2, x_3) = (x_1 + i x_2)/(1 - x_3)$.

5. Show that the above function $z = \varphi(x_1, x_2, x_3)$ satisfies $|z|^2 = \frac{1+x_3}{1-x_3}$ and carries $S - \{(0, 0, 1)\}$ one-one onto $\mathbb{C}$ with inverse function φ^{-1} given by $(x_1, x_2, x_3) = \varphi^{-1}(z)$ with

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2}, \qquad x_2 = \frac{z - \bar{z}}{i(1 + |z|^2)}, \qquad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

6. The inverse function φ^{-1} in the previous problem is called the **stereographic projection** of $\mathbb{C}$ to $S - \{(0, 0, 1)\}$. Explain this terminology by showing that if z is written as $x + iy$, then the points $(0, 0, 1)$, (x_1, x_2, x_3), and $(x, y, 0)$ lie on a straight line in $\mathbb{R}^3$.

7. Show that stereographic projection φ^{-1} in Problem 5 is a homeomorphism of $\mathbb{C}$ onto $S - \{(0, 0, 1)\}$.

8. Explain why stereographic projection carries straight lines and circles in $\mathbb{C}$ to circles on S, i.e., subsets of S that are the intersection of $S - \{(0, 0, 1)\}$ with a plane in $\mathbb{R}^3$. Why is every such subset obtained in this way?

Problems 9–35 make use of the Cauchy Integral Theorem in a disk, as well as its immediate consequences and its implications for Taylor series and the Argument Principle. In complex line integrals taken over circles, it is understood that the circle is a standard one, traced out counterclockwise.

9. Evaluate $\int_{|z|=1} \frac{e^z}{z}\, dz$.

10. (a) Let f be an entire function such that $f(z + 1) = f(z)$ for all $z \in \mathbb{C}$. Prove or disprove that f is constant.
    (b) Let f be an entire function such that $f(x + 1) = f(x)$ for all $x \in \mathbb{R}$ and $f(iy + i) = f(iy)$ for all $y \in \mathbb{R}$. Prove or disprove that f is constant.

11. Let f be an entire function. Decide whether each of the following statements is true or false. For those that are true, explain why. For those that are false, give a a counterexample.
    (a) If there exists a sequence $\{z_n\}$ in $\mathbb{C}$ with $\lim_n f(z_n) = 0$, then f is identically zero.
    (b) If $\lim_{r\to\infty} f(re^{i\theta}) = 0$ for some θ in $[0, 2\pi)$, then f is identically zero.
    (c) If $\lim_{r\to\infty} f(re^{i\theta}) = 0$ for $\theta = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$, then f is identically zero.
    (d) If $\lim_{|z|\to\infty} f(z) = 0$, then f is identically zero.

12. Evaluate $\int_\gamma \frac{e^z\, dz}{(1+z)^2}$ if γ is given by $\theta \mapsto 2e^{i\theta}$ for $0 \le \theta \le 2\pi$.

13. Does there exist an entire function $f(z)$ with the property that $f(1/n) = \frac{1}{n(n-1)}$ for every positive integer n? Explain.

14. Does there exist an even entire function $f(z)$ with $f'''(0) = 27$? Explain.

15. Evaluate $\int_{|z|=1} \frac{dz}{(z-a)^m(z-b)^n}$ for all integers $m \ge 1$ and $n \ge 1$ under the assumption that $|a| < 1 < |b|$.

16. Show
    (a) from the Cauchy–Riemann equations and
    (b) by means of Taylor series

    that if $f(z)$ is analytic on the open set U, then $g(z) = \overline{f(\bar{z})}$ is analytic on the open set V of complex conjugates of U.

17. Show that if an entire function f assumes real values on the real and imaginary axes, then $f(-z) = f(z)$ for all z in $\mathbb{C}$.

18. Prove that if the entire function $f(z)$ is real on the real axis and purely imaginary on the imaginary axis, then f is an odd function: $f(-z) = -f(z)$ for all z.

19. By considering $F'(z)/F(z)$, prove that any nowhere vanishing entire function is of the form $F(z) = e^{f(z)}$ with $f(z)$ entire.

20. Let $f(z)$ be analytic for $|z| < 2$, and suppose that $|zf'(z)| \leq 1$ for $|z| < 2$. Prove that $|f(0) - f(1)| \leq \frac{1}{2}$.

21. Suppose that $f(z)$ is an analytic function for $0 < |z| < 1$ such that $|zf(z)| > 1$ everywhere. Suppose also that $f(\frac{1}{2}) = 2$. Prove that $f(z) = 1/z$.

22. Let f be an entire function. Assume that there exist constants $R > 0$ and $\alpha > 0$ such that $|f(z)| \leq A|z|^\alpha$ for all $|z| > R$. Prove that f is a polynomial. Find the maximum degree of such a polynomial.

23. If $f(z)$ is analytic in a region containing 0, show that for some $M > 0$, $|f^{(n)}(0)| \leq M^n n!$ for all $n > 0$.

24. What kind of isolated singularity do the following functions exhibit at the indicated points, and why?
    (a) $\sin \frac{1}{1-z}$ at $z = 1$,
    (b) $\frac{1}{1-e^z}$ at $z = 2\pi i$,
    (c) $\frac{1}{\sin z - \cos z}$ at $z = \pi/4$.

25. Prove that an entire function f has an inverse function only if $f(z) = az + b$ with $a \neq 0$.

26. Let $P(z)$ and $Q(z)$ be polynomials with $Q(z)$ not identically 0, and let $r_1, \ldots, r_k$ be the distinct roots of $Q(z)$. Prove that there exist unique polynomials g and $P_1, \ldots, P_k$ such that

$$\frac{P(z)}{Q(z)} = \sum_{j=1}^{k} P_j\left(\tfrac{1}{z-r_j}\right) + g(z).$$

(This decomposition is called the **partial fractions** decomposition of $\frac{P(z)}{Q(z)}$.)

27. Expand $\dfrac{1}{z(z+1)^2(z+2)^3}$ in partial fractions.

28. If $Q(z)$ is a polynomial with distinct roots $r_1, \dots, r_n$ and if $P(z)$ is a polynomial of degree $< n$, prove that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(r_k)}{Q'(r_k)(z - r_k)}.$$

29. Use the result of the previous problem to write down explicitly a polynomial of degree $< n$ that takes the values $c_1, \dots, c_n$ at n distinct points $r_1, \dots, r_n$. (This polynomial is unique and is called the **Lagrange interpolation polynomial**.)

30. Let f be an analytic function with domain the unit disk such that $f(0) = f'(0) = 0$. Prove that f is not one-one.

31. For which values of the complex parameter λ is the analytic function $f_\lambda(z) = z + \lambda z^2$ one-one on the region where $|z| < 1$?

32. Let U be a region in $\mathbb{C}$, and let $\mathcal{F}$ be the set of all analytic functions $p : U \to \mathbb{C}$ of the form
$$\mathcal{F} = \{p(z) = az + b \mid a \in \mathbb{R} \text{ and } b \in \mathbb{C}\}.$$
Suppose that $f : U \to \mathbb{C}$ is an analytic function such that for each $z \in U$, there exists $p \in \mathcal{F}$ with $f'(z) = p'(z)$. (Here p is allowed to depend on f.) Prove that f is in $\mathcal{F}$.

33. Let f be a continuous function from the closed unit disk $E = \{|z| \le 1\}$ into $\mathbb{C}$, and suppose that $f(z)$ is analytic for $|z| < 1$ and has $f(z)$ purely imaginary for $|z| = 1$. Prove that f is constant.

34. Following the model for how the arcsine function was defined in Section B7, show how the arctangent function can be defined in terms of a suitable branch of the logarithm.

35. Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ be the power series expansion of a function analytic in the disk $\{|z| < r_0\}$, and let $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be its reciprocal: $g(z) = 1/f(z)$.
    (a) Find a recursion formula for the coefficients b_n.
    (b) Find in terms of the coefficients a_n, a lower bound for the radius of convergence of $\sum_{n=1}^{\infty} b_n z^n$.

Problems 36–39 concern **Schwarz's Lemma** in complex analysis, which is the inequality proved in Problem 36. Schwarz's Lemma in complex analysis is not to be confused with the Schwarz inequality in real analysis (Lemma 2.2).

36. By considering the function $f(z)/z$, prove that if $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \le 1$ and $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. Prove in addition that equality holds in the two inequalities only if $f(z) = cz$ with c a constant of absolute value 1.

37. Let f be analytic on an open set containing the closed unit disk and satisfying $f(0) = 0$ and $|f(z)| \le |e^z|$ for $|z| = 1$. How large can $|f(\log 2)|$ be?

38. Suppose that $f(z)$ is a one-one analytic function on the open unit disk $D = \{z \,\big|\, |z| < 1\}$ such that $f(0) = 0$ and $f'(0) = 1$. Let $\alpha = \inf_{w \notin D} |w|$. Use Schwarz's Lemma to prove that $|\alpha| \le 1$.

39. Suppose that $f(z)$ is a function analytic in a region containing the closed unit disk and that $f(z)$ satisfies $f(0) = f'(0) = 0$ and $|f(e^{i\theta})| \le M$ for $0 \le \theta \le 2\pi$. Prove that $|f(z)| \le M|z|^2$ for $|z| \le 1$.

Problems 40–49 concern Rouché's Theorem. Let $f(z)$ and $g(z)$ be analytic functions in an open disk, and suppose that γ is a piecewise C^1 closed curve in the disk such that $f(z)$ and $g(z)$ are nowhere 0 on γ. Suppose further that $|f(z) - g(z)| < |f(z)|$ on the image of γ. **Rouché's Theorem** is the assertion that γ encloses the same number of zeros for f as it does for g in the following sense: if the zeros of $g(z)$ are a_j with order h_j and the zeros of $f(z)$ are b_l with order k_l, then $\sum_j h_j n(\gamma, a_j) = \sum_l k_l n(\gamma, b_l)$.

40. Prove Rouché's Theorem by carrying out the following steps:
    (a) Let $F(z) = g(z)/f(z)$. Observe that $\left|\frac{g(z)}{f(z)} - 1\right| < 1$ on the image of γ, and deduce that the values of $F(z)$ on the image of γ are contained in the disk of radius 1 centered at $w = 1$.
    (b) Put $\Gamma = F \circ \gamma$. Deduce from (a) that $n(\Gamma, 0) = 0$.
    (c) Using the Argument Principle, deduce Rouché's Theorem from (b).

41. Let $g(z) = 10z^8 - z^6 + 3z^3 + 5$. Compute $\int_{|z|=1} \frac{g'(z)}{g(z)}\, dz$, the integration being taken counterclockwise over the standard circle of radius 1 and center 0.

42. Using Rouché's Theorem, decide how many zeros the function $g(z) = z^6 + 4z^5 + z + 1$ has with $|z| < 1$.

43. Using Rouché's Theorem, decide how many zeros $g(z) = 2z^5 - 6z^2 + z + 1$ has in the annulus $1 < |z| < 2$.

44. Prove the Fundamental Theorem of Algebra by means of Rouché's Theorem.

Problems 45–50 concern the Residue Theorem and its applications. In complex line integrals taken over circles, it is understood that the circle is a standard one, traced out counterclockwise.

45. Evaluate the complex line integral $\int_{|z|=2} \frac{dz}{z^2+1}$.

46. Evaluate the complex line integral $\int_{|z|=1} \frac{dz}{2z^2+3z-2}$.

47. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^4+3x^2+2}$ by the method of residues.

48. Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx$.

49. Evaluate $\int_{-\infty}^{\infty} \frac{(1+x)\sin x}{x^2-2x+2}\,dx$.

50. Evaluate $\int_0^{\pi} \frac{dx}{a+b\cos x}$ for $a > b > 0$.

Problems 51–54 concern Laurent series.

51. Let $f(z) = \frac{1}{z(z-1)}$. Expand f in a Laurent series
    (a) for $0 < |z| < 1$.
    (b) $1 < |z| < \infty$.

52. Let $f(z)$ be the function $f(z) = \frac{1}{1-z^2} + \frac{1}{3-z}$.
    (a) How many Laurent expansions of the form $\sum_{n=-\infty}^{\infty} c_n z^n$ does $f(z)$ have? For each such expansion find the maximal region A so that the expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ is valid in A.
    (b) For one such expansion explicitly find the coefficients c_n.

53. Show that the Laurent series for $(e^z - 1)^{-1}$ about $z = 0$ is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}.$$

(The numbers B_k are known as **Bernoulli numbers**. One of their remarkable properties is that $\sum_{n=1}^{\infty} n^{-2k} = \frac{2^{2k-1}}{(2k)!} B_k \pi^{2k}$. Consequently all the B_k are positive.)

54. Express the Laurent series of $\cot z$ about $z = 0$ in terms of the Bernoulli numbers of the previous problem.

Problems 55–60 deal with the interplay between uniform convergence and analytic functions. If U is a region in $\mathbb{C}$, a sequence of functions $\{f_n(z)\}$ is said to **converge uniformly on compact sets** in U to the function $f(z)$ if for each compact subset K of U and each $\epsilon > 0$, there is an N such that $n \geq N$ implies $|f_n(z) - f(z)| < \epsilon$ for all $z \in K$. A set E of analytic functions on a region U is called a **normal family** if it is uniformly bounded on each compact subset of U.

55. If $\{f_n(z)\}$ is a sequence of analytic functions in a region U convergent uniformly on compact sets to a function $f(z)$, prove that $f(z)$ is analytic on U.

56. Let $\{f_n(z)\}$ be a sequence of analytic functions in a region U. Prove that if $\lim_n f_n(z) = 0$ uniformly on every compact subset of U, then $\lim_n f_n'(z) = 0$ uniformly on every compact subset of U.

57. If $\{f_n(z)\}$ is a sequence of nowhere-zero analytic functions in a region U convergent uniformly on compact sets to an analytic function $f(z)$, prove that either $f(z)$ is nowhere 0 or $f(z)$ is identically 0. (This result is known as **Hurwitz's Theorem**.)

58. Prove that if E be a normal family of analytic functions on a region U, then the complex derivatives of the members of E are uniformly bounded on each compact subset of E.

59. Prove that if E is a normal family of analytic functions on a region U, then on each compact subset K of U, E is uniformly equicontinuous in the sense of Section II.10. Follow these steps to do so:
    (a) Fix a number $r > 0$ less than the minimum distance from a point of K to U^c, and let K' be the set of points at distance $\leq r$ from K. Why does it follow that K' is compact and that the closed disk of radius r about any point $z_0 \in K$ lies in K' and hence in U?
    (b) Let M be the maximum value of $|f(\zeta)|$ on K'. Use the Cauchy Integral Formula for a disk of radius r about a point z_0 of K to show that any two points z and z' with distance $\leq r/2$ from the center of the disk have $|f(z) - f(z')| \leq 4M|z - z'|/r$.
    (c) Let $\epsilon > 0$ be given, and choose δ to be the minimum of $r/2$ and $\epsilon r/(4M)$. Show that if z_1 and z_2 are points in K with $|z_1 - z_2| \leq \delta$, then $|f(z_1) - f(z_2)| \leq \epsilon$.

60. Using Ascoli's Theorem, prove that if E is a normal family of analytic functions on a region U, then any sequence $\{f_n(z)\}$ of members of E has a subsequence that is uniformly convergent on each compact subset of U. (The limit function is analytic by Problem 55.)

Problems 61–68 concern linear fractional transformations. If a, b, c, d are complex numbers with $ad - bc \neq 0$, then the analytic function $L(z) = \frac{az+b}{cz+d}$ is called a **linear fractional transformation**. The domain of L is $\mathbb{C}$ if $c = 0$ and is $\mathbb{C} - \{-d/c\}$ if $c \neq 0$. It is often convenient to enlarge $\mathbb{C}$ to a set $\mathbb{C} \cup \{\infty\}$ and to extend the definition of L to a function carrying $S \cup \{\infty\}$ to itself by setting $L(\infty)$ equal to a/c if $c \neq 0$, $L(-d/c) = \infty$ if $c \neq 0$, and $L(\infty) = \infty$ if $c = 0$.

61. Show that the linear fractional transformation $L(z) = \frac{az+b}{cz+d}$, considered as a function defined on $\mathbb{C}$ if $c = 0$ and defined on $\mathbb{C} - \{-d/c\}$ if $c \neq 0$,
    (a) is one-one,
    (b) has image equal to $\mathbb{C}$ if $c = 0$ and equal to $\mathbb{C} - \{a/c\}$ if $c \neq 0$,
    (c) has inverse the linear fractional transformation $w \mapsto \frac{dw-b}{-cw+a}$.

62. Show that the linear fractional transformation L above, when extended to a function from $S \cup \{\infty\}$ to $S \cup \{\infty\}$, is one-one onto.

63. Suppose one extends stereographic projection, as defined in Problem 6, to a function from all of S into $\mathbb{C} \cup \{\infty\}$ by the definition $S(0, 0, 1) = \infty$, and suppose one transfers the definition of the linear fractional transformation $L(z) = \frac{az+b}{cz+d}$ to the sphere S by means of stereographic projection, using $\varphi \circ L \circ \varphi^{-1}$ as the transformation of S corresponding to L. Prove that $\varphi \circ L \circ \varphi^{-1}$ is actually a homeomorphism of S onto itself.

64. Show that if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is associated to the linear fractional transformation $z \mapsto \frac{az+b}{cz+d}$, then a matrix product of two such matrices is associated to the composition of the corresponding linear fractional transformations.

65. (a) Show that the only linear fractional transformation fixing $1, 0$, and ∞ is the identity.
    (b) If z_2, z_3, z_4 are three distinct points in $\mathbb{C}$, show that there exists a linear fractional transformation carrying them into 1, 0, and ∞ in this order. Show also that if one of z_2, z_3, z_4 is ∞, there is still a linear fractional transformation carrying them into $1, 0$, and ∞ in this order.

66. If z_1, z_2, z_3, z_4 are four points in $\mathbb{C}\cup\{\infty\}$ with z_2, z_3, z_4 distinct, their **cross ratio**, denoted (z_1, z_2, z_3, z_4), is the image of z_1 under the unique linear fractional transformation that carries z_2, z_3, z_4 into $1, 0, \infty$. It lies in $\mathbb{C}\cup\{\infty\}$. If T is any linear fractional transformation, prove that $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$ by making use of the linear fractional transformation S with $Sz = (z, z_2, z_3, z_4)$.

67. (a) Show that a linear fractional transformation carries the upper half plane $\{z \mid \operatorname{Im} z > 0\}$ into itself if it corresponds to a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d real and with $ad - bc > 0$.
    (b) Show that any linear fractional transformation that carries the upper half plane to itself coincides with one of the transformations in (a).

68. (a) Show that any linear fractional transformation that carries the open unit disk $\{z \,\big|\, |z| < 1\}$ to itself corresponds to a matrix $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ with $|a|^2 - |\beta|^2 > 0$.
    (b) Show that any linear fractional transformation that carries the open unit disk to itself coincides with one of the transformations in (a).

Problems 69–77 relate harmonic functions in $\mathbb{R}^2$ to complex analysis. The **Laplacian** in $\mathbb{R}^2$ is the differential operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. A real-valued C^2 function $u(x, y)$ on an open set in $\mathbb{R}^2$ is said to be **harmonic** if it satisfies $\Delta u(x, y) = 0$ everywhere on the open set. Such functions were introduced briefly in Problems 14–15 of Chapter III, and harmonic functions in $\mathbb{R}^n$ are studied more extensively in later chapters of the book.

69. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in an open set U of $\mathbb{C}$. Use the Cauchy–Riemann equations to prove that $u(x, y)$ and $v(x, y)$ are harmonic in U.

70. If $u(x, y)$ is a harmonic function in a region U of $\mathbb{C}$ and if $v(x, y)$ is a harmonic function such that $u(x, y)+iv(x, y)$ is analytic in U, then v is called a **conjugate harmonic function** to u.
    (a) Show that a conjugate harmonic function to u in U, if it exists, is unique up to an additive constant.

(b) Show that if v is a conjugate harmonic function to u, then $-u$ is a conjugate harmonic function to v.

(c) Show that $u(x, y) = \log(x^2 + y^2)$ is harmonic in $U = \mathbb{C} - \{(0, 0)\}$ and that it has no conjugate harmonic function in U.

71. (a) On $\mathbb{R}^2$, find all conjugate harmonic functions to $(e^x + e^{-x}) \sin y$.

(b) Prove by using integration that a harmonic function on all of $\mathbb{R}^2$ always has a conjugate harmonic function. Observe that the same argument applies to an open disk, to the inside of a filled rectangle, and to an open half plane. Why does it follow that a harmonic function on any region is necessarily C^∞.

72. By making suitable adjustments to the proof of Theorem B.40, prove that if $u(x, y)$ is harmonic in a bounded simply connected region U of $\mathbb{C}$, then $u(x, y)$ has a well defined conjugate harmonic function $v(x, y)$ on U, i.e., there is a function $v(x, y)$ such that $z = x + iy \mapsto u(x, y) + iv(x, y)$ is analytic in U.

73. Let $u(x, y)$ be harmonic on $\mathbb{R}^2$, and suppose that $A(z)$ is an analytic function on some region U. Prove that $u \circ A$ is harmonic.

74. If $u(x, y)$ is harmonic in a region U of $\mathbb{R}^2$, prove that $u(x, y)$ has an open interval (possibly infinite) as image or else u is constant.

75. If $u(x, y)$ is harmonic in a region U of $\mathbb{R}^2$, prove that $u(x, y)$ does not attain a local maximum value in U unless u is constant.

76. Let $U : \mathbb{R}^2 \to \mathbb{R}$ be an everywhere positive harmonic function. Prove that u is constant.

77. Suppose that $u(x, y)$ is a function that is continuous for $|(x, y)| \leq 1$ and is harmonic for $|(x, y)| < 1$. Prove that $u(0, 0) = \frac{1}{2\pi} \int_0^{2\pi} u(\cos\theta, \sin\theta)\, d\theta$.

Problems 78–81 concern vector-valued "holomorphic" functions of several variables. They make use of Problem 4 and of material from Section B15. Let U be an open set in $\mathbb{C}^n$, and let $f : U \to \mathbb{C}^m$ be a function defined on U. The function f is defined to be **complex differentiable** at $z \in U$ if each of its component functions $f_k : U \to \mathbb{C}$ is complex differentiable at z in the sense of Problem 4. For $z \in U$, define the complex derivative of f to be the linear mapping $L : \mathbb{C}^n \to \mathbb{C}^m$ whose k^{th} component is the complex derivative of f_k at z. The m-by-n complex matrix of L is called the **complex Jacobian matrix** of f at z; we write $J_\mathbb{C}$ for it.. Finally by taking a cue from Problem 4, we can regard f as a function $f_\mathbb{R}$ of $2n$ real variables, specifically $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$, that takes values in $\mathbb{R}^{2m}$ with coordinates $(u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_m)$. If f is complex differentiable at z, then Problem 4 shows that $f_\mathbb{R}$ is differentiable at $(x_1, \ldots, y_n)$ and has an ordinary Jacobian matrix J that is a $2n$-by-$2m$ real matrix.

78. Let $f : U \to \mathbb{C}^m$ be complex differentiable at $z \in U$.
   (a) Show that the Jacobian matrix J of $f_{\mathbb{R}}$ at z satisfies

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} J = J \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

   in block form. Here the entries in the matrix that multiplies J on the left side of the equation are m-by-m real matrices 0, -1, and 1; and the entries in the matrix that multiplies J on the right side of the equation are n-by-n real matrices.
   (b) Decompose the entries of $J_{\mathbb{C}}$ into their real and imaginary parts as $J_{\mathbb{C}} = \operatorname{Re} J_{\mathbb{C}} + i \operatorname{Im} J_{\mathbb{C}}$. Prove that J in block form is given by $J = \begin{pmatrix} \operatorname{Re} J_{\mathbb{C}} & -\operatorname{Im} J_{\mathbb{C}} \\ \operatorname{Im} J_{\mathbb{C}} & \operatorname{Re} J_{\mathbb{C}} \end{pmatrix}$.
79. If U is open in $\mathbb{C}^n$, then a function $f : U \to \mathbb{C}^m$ is said to be **holomorphic** if each component $f_k : U \to \mathbb{C}$ is holomorphic on U in the sense of Section B15. Suppose that $f : U \to \mathbb{C}^m$ is holomorphic and that $g : V \to \mathbb{C}^r$ is holomorphic on an open set V of $\mathbb{C}^m$ that contains $f(U)$. Prove that the composition $g \circ f$ is holomorphic on U and that its complex Jacobian matrix at a point $z \in U$ is the product of the complex Jacobian matrix of g at $f(z)$ and the complex Jacobian matrix of f at z.
80. State and prove a complex-variable version of the real-variable Inverse Function Theorem given in Theorem 3.17.
81. State and prove a complex-variable version of the real-variable Implicit Function Theorem given in Theorem 3.16.