Chapter 20

TRIGONOMETRY AND DUALITY

After we have found the equations [The Laws of Cosines and Sines for a Hyperbolic Plane] which represent the dependence of the angles and sides of a triangle; when, finally, we have given general expressions for elements of lines, areas and volumes of solids, all else in the [Hyperbolic] Geometry is a matter of analytics, where calculations must necessarily agree with each other, and we cannot discover anything new that is not included in these first equations from which must be taken all relations of geometric magnitudes, one to another. ... We note however, that these equations become equations of spherical Trigonometry as soon as, instead of the sides a, b, c we put ... a $\sqrt{-1}$,b $\sqrt{-1}$,c $\sqrt{-1}$ …— N. *Lobachevsky, quoted in [***HY***: Greenberg]*

In this chapter, we will first derive, geometrically, expressions for the circumference of a circle on a sphere, the Law of Cosines on the plane, and its analog on a sphere. Then we will talk about duality on a sphere. On a sphere, duality will enable us to derive other laws that will help our two-dimensional bug to compute sides and/or angles of a triangle given ASA, RLH, SSS, or AAA. Finally, we will look at duality on the plane.

PROBLEM 20.1 CIRCUMFERENCE OF A CIRCLE

a. *Find a simple formula for the circumference of a circle on a sphere in terms of its intrinsic radius and make the formula as intrinsic as possible.*

We suggest that you make an extrinsic drawing (like Figure 20.1) of the circle, its intrinsic radius, its extrinsic radius, and the center of the sphere. You may well find it convenient to use trigonometric functions to express your answer. Note that the existence of trigonometric functions for right triangles follows from the properties of similar triangles that were proved in Problem **13.4**.

In Figure 20.1, rotating the segment of *length r´* (***the extrinsic radius***) through a whole revolution produces the same circumference as rotating *r*, which is an arc of the great circle as well as **the intrinsic radius** of the circle on the sphere.

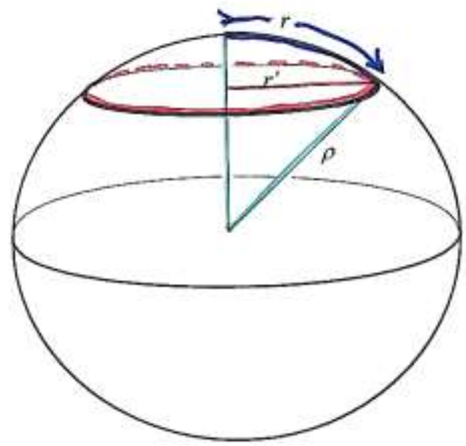


Figure 20.1 Intrinsic radius *r*

Even though the derivation of the formula this way will be extrinsic, it is possible, in the end, to express the circumference only in terms of intrinsic quantities. Thus, also think of the following problem:

b. *How could our 2-dimensional bug derive this formula?*

By looking at very small circles, the bug could certainly find uses for the trigonometric functions they give rise to. Then the bug could discover that the geodesics are actually (intrinsic) circles, but circles that do not have the same trigonometric properties as very small circles. Then what?

Using the expressions of trigonometric and hyperbolic functions in terms of infinite series, it is proved (in [**HY**: Greenberg], p. 337) that

THEOREM 20.1. *In a hyperbolic plane of radius* 1, *a circle with intrinsic radius r has circumference c equal to*

$$c = 2\,\pi\,\sinh(r).$$

c. *Use the theorem to show that on a hyperbolic plane of radius ρ, a circle with intrinsic radius r has circumference c equal to*

$$c = 2\,\pi\,\rho\,\sinh(r/\rho).$$

When going from a hyperbolic plane of radius 1 to a hyperbolic plane of radius *ρ*, all lengths scale by a factor of *ρ*. *Why?*

The formula in part **c** should look very much like your formula for part **a** (possibly with some algebraic manipulations). This is precisely what Lobachevsky was talking about in the quote at the beginning of this chapter. Check this out with part **d**.

d. *Show that, if you replace ρ by iρ in the formula of part* **c**, *then you will get the formula in part* ***a***.

Look up the definition of *sinh* (hyperbolic sine) and express it as a Taylor series and then compare with the Taylor series of sine.

PROBLEM 20.2 LAW OF COSINES

If we know two sides and the included angle of a (small) triangle, then according to SAS the third side is determined. If we know the lengths of the two sides and the measure of the included angle, how can we find the length of the third side? The various formulas that give this length are called the ***Law of Cosines***.

a. *Prove the Law of Cosines for triangles in the plane* (see Figure 20.2):

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

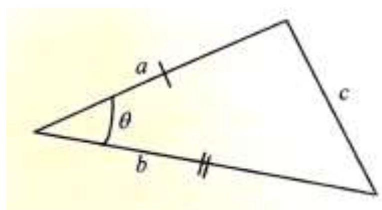


Figure 20.2 Law of Cosines

For a geometric proof of this "law," look at the pictures in Figure 20.3. We first saw the idea for these pictures in the marvelous book [**EG**: Valens]. These pictures show the squares as rigid with hinges at all the points marked with ○. Note that in the middle picture θ is greater than π/2. You must draw a different picture for θ less than π/2. Prove the Law of Cosines on the plane using the pictures in Figure 20.3, or in any other way you wish. Note the close relationship with the Pythagorean Theorem.

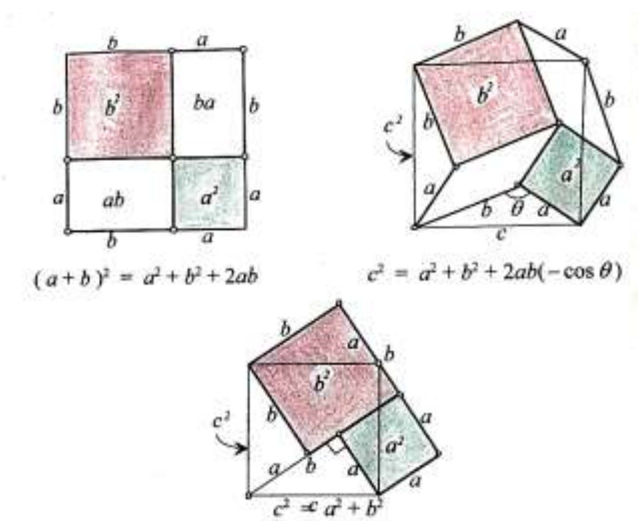


Figure 20.3 Three related geometric proofs

b. *Prove the law of cosines for small triangles on a sphere with radius ρ:*

$$\cos c/\rho = \cos a/\rho \cos b/\rho + \sin a/\rho \sin b/\rho \cos \theta.$$

Hint: One approach that will work is to project the triangle by a gnomic projection onto the plane tangent to the sphere at the vertex of the given angle. This projection will preserve the size of the given angle (*Why?*) and, even though it will not preserve the lengths of the sides of the triangle, you can determine what effect it has on these lengths. Now apply the planar Law of Cosines to this projected triangle and turn the algebra crank. It is very helpful to draw a 3-D picture of this projection.

c. *Derive a formula for the distance between two points on the earth in terms of the latitude and longitude at the two points.*

It is sometimes convenient to measure lengths of great circle arcs on the sphere in terms of the radian measure:

radian measure of the arc = (length of the arc)/ ρ,

where ρ is the radius of the sphere. For example, the radian measure of one quarter of a great circle would be $\pi/2$ and the radian measure of half a great circle would be π. In Figure 20.4, the segment a is subtended by the angle α at the pole and by the same angle α at the center of the sphere. The radian measure of a is the radian measure of α. Most other texts and articles either use radian measure for lengths or assume that the radius ρ is equal to 1. We will most of the time keep the ρ explicit so the connection with the radius will be clearly seen.

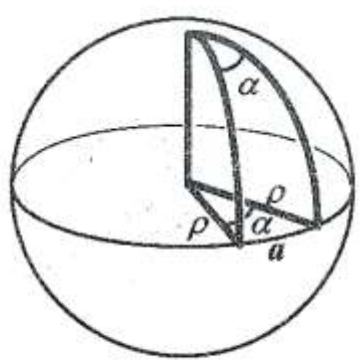


Figure 20.4 Radian measure of lengths

If we measure lengths in radians, then one possible formula for the spherical triangle of radius ρ in part **b** is

$$\cos c = \cos a \cos b + \sin a \sin b \cos \theta.$$

THEOREM 20.2a. A Law of Cosines for right triangles on a sphere with radius ρ is

$$\cos c/\rho = \cos a/\rho \cos b/\rho,$$

or, in radian measure of lengths,

$$\cos c = \cos a \cos b,$$

which can be considered as the spherical equivalent of the Pythagorean Theorem.

THEOREM 20.2b. A Law of Cosines for triangles on a hyperbolic plane with radius ρ is

$$\cosh c/\rho = \cosh a/\rho \cosh b/\rho + \sinh a/\rho \sinh b/\rho \cos \theta .$$

This theorem is proved in [**HY**: Stahl], p. 125, using analytic techniques, and in [**HY**: Greenberg] using infinite series representations. This is one of the equations that Lobachevsky is talking about in the quote at the beginning of the chapter. It is important to realize that a study of hyperbolic trigonometric functions was started by V. Riccati (1707–1775, Italy) and others, and continued by J. H. Lambert (1728–1777, Germany) in 1761–68 — well before their use in hyperbolic geometry by Lobachevsky. For details on the history of hyperbolic functions, see the informative 2004 paper [**HI**: Barnett].

PROBLEM 20.3 LAW OF SINES

Closely related to the Law of Cosines is the ***Law of Sines***. (Figure 20.5)

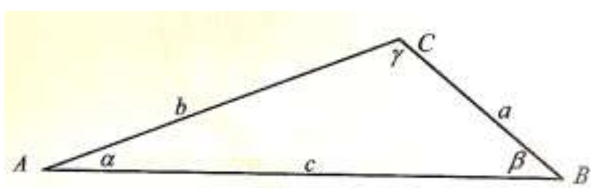


Figure 20.5 Law of Sines

a. *If ΔABC is a planar triangle with sides, a, b, c, and corresponding opposite angles, α, β, γ, then*

$$a/\sin \alpha = b/\sin \beta = c/\sin \gamma .$$

The standard proof for the Law of Sines is to drop a perpendicular from the vertex C to the side c and then to express the length of this perpendicular as both ($b \sin \alpha$) and ($a \sin \beta$). See Figure 20.6. From this the result easily follows. Thus, on the plane the Law of Sines follows from an expression for the sine of an angle in a right triangle.

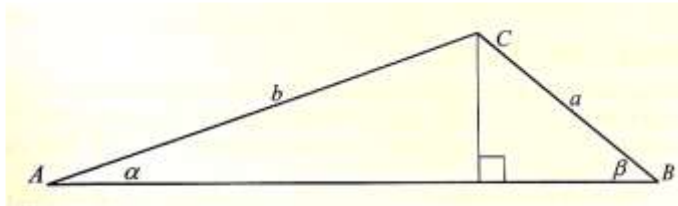


Figure 20.6 Standard proof of Law of Sines on plane

For triangles on the sphere we can find a very similar result.

b. *Show that for a small triangle on a sphere with radius ρ,*

$$(\sin a/\rho)/\sin\alpha = (\sin b/\rho)/\sin\beta = (\sin c/\rho)/\sin\gamma .$$

If ΔACD is a triangle on the sphere with the angle at D being a right angle, then use gnomic projection to project ΔACD onto the plane that is tangent to the sphere at A. Because the plane is tangent to the sphere at A, the size of the angle α is preserved under the projection. In general, angles on the sphere not at A will not be projected to angles of the same size, but in this case the right angle at D will be projected to a right angle. (Be sure you see why this is the case. A good drawing and the use of symmetry and similar triangles will help.) Now express the sine of α in terms of the sides of this projected triangle.

When we measure the sides in radians, on the sphere the Law of Sines becomes

$$(\sin a)/\sin\alpha = (\sin b)/\sin\beta = (\sin c)/\sin\gamma .$$

For a right triangle this becomes (Figure 20.7) $\sin\alpha = (\sin a)/(\sin c)$.

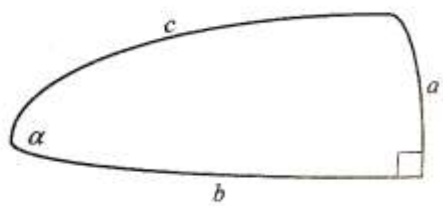


Figure 20.7 Law of Sines for right triangles on a sphere

DUALITY ON A SPHERE

We can now ask, Is it possible to find expressions corresponding to the other triangle congruence theorems that we proved in Chapters 6 and 9? Let us see how we can be helped by a certain concept of duality that we will now develop. When we were looking at *SAS* and *ASA*, we noticed a certain duality between points and lines (geodesics). *SAS* was true on the plane or open hemisphere because two points determine a unique line segment and *ASA* was true on the plane or open hemisphere because two (intersecting) lines determine a unique point. In this section we will make this notion of duality broader and deeper and look at it in such a way that it applies to both the plane and the sphere. On the whole sphere, two distinct points determine a unique straight line (great circle) unless the points are antipodal. In addition, two distinct great circles determine a unique pair of antipodal points. Also, a circle on the sphere has two centers that are antipodal. Remember also that in most of the triangle congruence theorems, we had trouble with triangles that contained antipodal points. So, our first step is to consider not points on the sphere but rather point-pairs, pairs of antipodal points. With this definition in mind, check the following:

- *Two distinct point-pairs determine a unique great circle (geodesic).*
- *Two distinct great circles determine a unique point-pair.*
- *The intrinsic center of a circle is a single point-pair.*
- *SAS, ASA, SSS, AAA are true for all triangles not containing any point-pairs.*

Now we can make the duality more definite.

- *The dual of a great circle is its* ***poles*** (the point-pair that is the intrinsic center of the great circle). Some books use the term ***"polar"*** in place of ***"dual"***.
- *The dual of a point-pair is its* ***equator*** (the great circle whose center is the point-pair).

Note: *If the point-pair P is on the great circle l, then the dual of l is on the dual of P*. See Figure 20.8.

Figure 20.8 *P* is on *l* implies the dual of *l* is on the dual of *P*

If *k* is another great circle through *P*, then notice that the dual of *k* is also on the dual of *P*. Because an angle can be viewed as a collection of lines (great circles) emanating from a point, the dual of this angle is a collection of point-pairs lying on the dual of the angle's vertex. And, vice versa, the dual of the points on a segment of a great circle are great circles emanating from a point-pair that is the dual of the original great circle. Before going on, be sure to understand this relationship between an angle and its dual. Draw pictures. Make models.

PROBLEM 20.4 THE DUAL OF A SMALL TRIANGLE ON A SPHERE

The dual of the small triangle ΔABC is the small triangle $\Delta A^*B^*C^*$, where A^* is that pole of the great circle of BC that is on the same side of BC as the vertex A, similarly for B^*, C^*. See Figure 20.9.

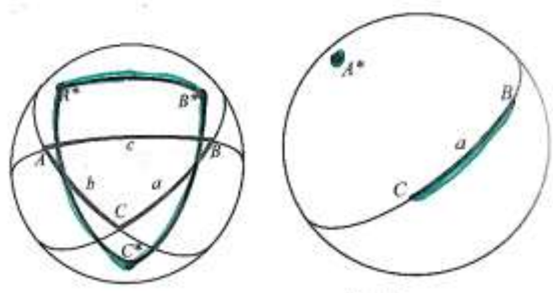


Figure 20.9 The dual of a small triangle

a. *Find the relationship between the sizes of the angles and sides of a triangle and the corresponding sides and angles of its dual.*

b. *Is there a triangle that is its own dual?*

PROBLEM 20.5 TRIGONOMETRY WITH CONGRUENCES

a. *Find a dual of the Law of Cosines on the sphere. That is, find a statement that results from replacing in the Law of Cosines every angle and every side by its dual.*

There is an analogous dual Law of Cosines for a hyperbolic plane, proved in the same books cited with Theorem 20.2, the hyperbolic Law of Cosines.

b. *For each of ASA, RLH, SSS, AAA, if you know the measures of the given sides and angles, how can you find the measures of the sides and angles that are not given? Do this for both spheres and hyperbolic planes.*

Use part **a** and the formulas from Problems **20.1** and **20.2**.

DUALITY ON THE PROJECTIVE PLANE

The gnomic projection, ***g*** (Problem **14.2**), allows us to transfer the above duality on the sphere to a duality on the plane. If P is a point on the plane, then there is a point Q on the sphere such that $g(Q) = P$. The dual of Q is a great circle l on the sphere. If Q is not the south pole, then half of l is in the southern hemisphere and its projection onto the plane, ***g***(l), is a line we can call the dual of P. This defines a dual for every point on the plane except for the point where the south pole of the sphere rests. See Figure 20.10. It is convenient to call this point the origin, O, of the plane.

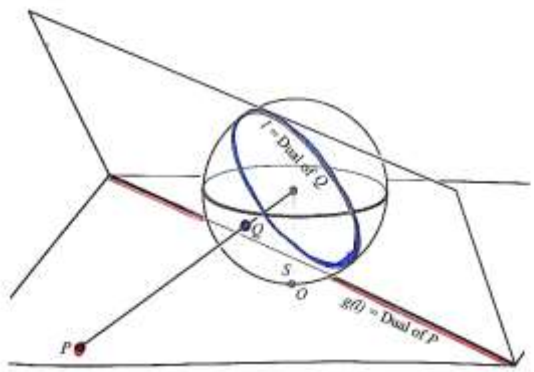


Figure 20.10 Duality on the projective plane

Note that O is the image of S, the south pole, and that the dual of S is the equator, which is projected by G to infinity on the plane. Thus, we define the dual of O to be the ***line at infinity***. If l is any line in the plane, then it is the image of a great circle on the sphere that intersects the equator in a point-pair. The image of this point-pair is considered to be a single point at infinity at the "end" of the line l, the same point at both ends. The plane with the line at infinity attached is called the ***projective plane***. This projective plane is the same as the *real projective plane* that we investigated in Problem **18.3b**, the only difference being that here we are focusing on the gnomic projection onto the plane and in Chapter 18 we were considering it as a spherical 2-manifold.

PROBLEM 20.6 PROPERTIES ON THE PROJECTIVE PLANE

a. *Check that the following hold on the projective plane:*

- *Two points determine a unique line.*
- *Two parallel lines share the same point at infinity.*
- *Two lines determine a unique point.*
- *If a point is on a line, then the dual of the line is a point that is on the dual of the original point.*

b. *If γ is the circle with center at the origin and with radius the same as the radius of the sphere and if (P,Q) is an inversive pair with respect to γ, then show that the dual of P is the line perpendicular to OP at the point $-Q$, the point in the plane that is opposite (with respect to O) of Q.* See Figure 20.11.

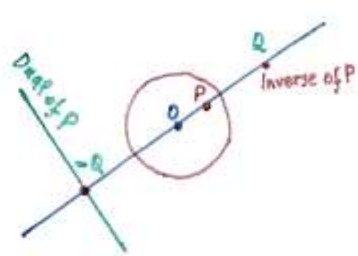


Figure 20.11 Duals/inversions

c. *Show that the dual of a point P on γ is a line tangent to γ at the diametrically opposite point $-P$.*

DUAL VIEWS OF OUR EXPERIENCE

Our usual point of view for perceiving our world is from the origin (because in this view we are the center of our world). We look out in all directions to observe what is outside and we strive to look out as close toward infinity as we can. The dual of this is the point of view where we are the line at infinity (the dual of the origin), we view the whole world (which is within us), and we strain to look in as close toward the center as we can.

PERSPECTIVE DRAWING AND HISTORY

Present-day theories of projective geometry got their start from Euclid's *Optics* [**AT**: Euclid *Optics*] and later from the theories of perspective developed during the Renaissance by artists who studied the geometry inherent in perspective drawings. The first mathematical rule for getting the correct perspective was invented by Filippo Brunelleschi (1377–1455) in or shortly before 1413. In the Renaissance there was no such profession as artist. Some of the most famous artists of 14th and 15th centuries were trained as goldsmiths and were taught some mathematics from books such as Leonardo of Pisa's (c.1170–c.1240) book *Libro d'abaco*, mostly known for its use of Arabic numerals and origin of Fibonacci numbers, but there is much more mathematics in it. Brunelleschi's work included painting and sculpture as well as the design of stage machinery and buildings. He invented his method of "artificial perspective" for drawing different buildings in Rome in a way that appeared realistic. But no written descriptions survive from Brunelleschi.

Left: Masaccio, *Holy Trinity*, fresco; right: perspective diagram of *Holy Trinity*

The earliest works to show parallel lines that converge to a point in a more or less convincing manner are not paintings, but relief panels by a friend of Brunelleschi's, the sculptor Donatello (1386–1466). Another younger friend of Brunelleschi's who was also an early user of perspective was the painter Masaccio (1401–c.1428). Masaccio's fresco *Trinity* (c.1426) in Santa Maria Novella (Florence) provides the best opportunity for studying Masaccio's mathematics and hence making some educated guesses about Brunelleschi's method.

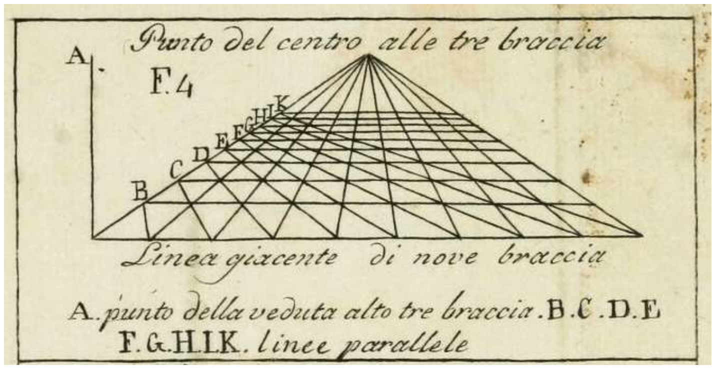
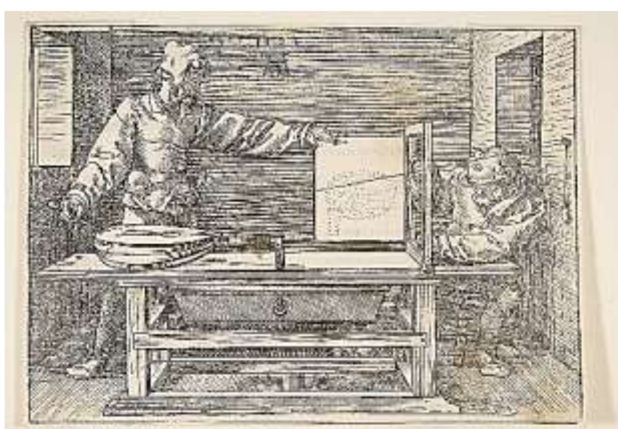


Diagram by Leon Battista Alberti of perspective lines leading to a vanishing point from his treatise Della Pittura

The first surviving written account of a method of perspective construction is by Leon Batista Alberti (1404–1472) in the beginning of a short treatise *On Painting* in 1435. The problem was how to construct the perspective image of a square checkerboard pavement. It is clear from later literary evidence, mainly dating from 16th century, that Alberti's construction was not the only one current in 15th century. Among the first mathematical treatises on perspective we should mention also Pierro della Francesca (1412–1492), *De prospectiva pingendi*. Pierro della Francesca is acknowledged as one of the most important painters of the 15th century, but he had an independent reputation as a highly competent mathematician. In Albrecht Dürer's (1471–1528) *Treatise on measurement with compasses and straightedge* (Nuremberg, 1525) is shown an instrument being used to draw a perspective image of a lute.



Man drawing a lute by Albrecht Dürer

Another mechanical aid described by Dürer goes back to the early 15th century and was described also by Alberti and Leonardo da Vinci. It was a veil of coarse netting, which was used to provide something rather like a system of coordinate lines. Such squaring systems were familiar to artists also as a method of scaling drawings up for transfer to the painting surface. For more discussion on perspective and mathematics in the Renaissance, see [**AD**: Field].

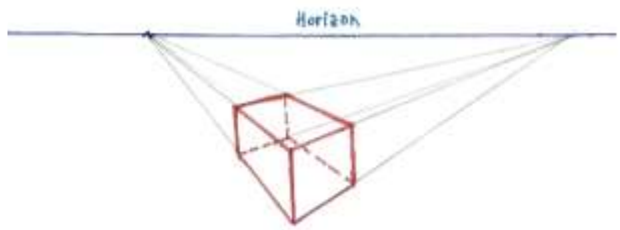


Figure 20.12 Two-point perspective

The "vanishing point" where the lines that are parallel on the box intersect on the horizon of the drawing is an image on the drawing of the point at infinity on these parallel lines. One way to visualize this is to imagine yourself at the center of a transparent sphere looking out at the world. If you look at two parallel straight lines and trace these lines on the sphere, you will be tracing segments of two great circles. If you followed this tracing indefinitely to the ends of the straight lines, you would have a tracing of two half great circles on the sphere intersecting in their endpoints. These endpoints of the semicircles are the images of the point at infinity that is the common intersection of the two parallel lines. If you now use a gnomic projection to project this onto a plane (for example, the artist's canvas), then you will obtain two straight line segments intersecting at one of their endpoints as in the Figure 20.12.

If you are interested in studying more about perspective through connections between mathematics and art, see the book *Perspective and Projective Geometry* by Annalisa Crannell, Mark Frantz, and Fumiko Futamura (Princeton University Press, 2019).

Jacques Villon (Gaston Duchamp) *Abstract Composition*