Chapter 8

Intrinsic Local Descriptions and Manifolds

PROBLEM 8.1. Covariant Derivative and Connection

a.

Using Problem **5.4** we calculate

$$\mathbf{Xf}-\nabla_{\mathbf{X}}\mathbf{f} = \lim_{\delta\to 0}\tfrac{1}{\delta}[\mathbf{f}(\alpha(\delta)) - \mathbf{f}(p)] - \lim_{\delta\to 0}\tfrac{1}{\delta}[\mathbf{f}(\alpha(\delta)) - P(\alpha,p,\alpha(\delta))\mathbf{f}(p)] =$$
$$= \lim_{\delta\to 0}\tfrac{1}{\delta}[P(\alpha,p,\alpha(\delta))\mathbf{f}(p) - \mathbf{f}(p)] = \tfrac{d}{ds}P(\alpha,p,\alpha(s))\mathbf{f}(p)_{s=0} = c\mathbf{n}\ ,$$

this is in the normal direction by Problem **5.4** and since $\nabla_{\mathbf{X}}\mathbf{f}$ is a tangent vector $c\mathbf{n}$ must be the normal component of $\mathbf{Xf}$, which normal component is $\langle \mathbf{Xf}, \mathbf{n}(p)\rangle\mathbf{n}$.

b.

From the definition of geodesic and normal curvatures (and using Part **a**)

$$\boldsymbol{\kappa}_g(0) = \boldsymbol{\kappa}(0) - \boldsymbol{\kappa}_{\boldsymbol{n}}(0) = \mathbf{T}\gamma'_{s=0} - \langle \mathbf{T}\gamma'_{s=0}, \mathbf{n}(0)\rangle\mathbf{n}(0) = \nabla_{\mathbf{T}}\gamma'_{s=0}\,.$$

c.

That this intrinsic derivative is zero implies (using part **a**) that the directional derivative $\gamma'(s)\boldsymbol{V}$ is in the normal direction, which implies (by Problem **5.4**) that the vector field is parallel.

d.

Using part **a** we calculate

$$\nabla_{\mathbf{X+Y}}\mathbf{f} = (\mathbf{X+Y})\mathbf{f} - \langle(\mathbf{X+Y})\mathbf{f},\mathbf{n}\rangle\mathbf{n} = \mathbf{Xf} + \mathbf{Yf} - \langle \mathbf{Xf} + \mathbf{Yf}, \mathbf{n}\rangle\mathbf{n} = \mathbf{Xf} + \mathbf{Yf} - \langle \mathbf{Xf}, \mathbf{n}\rangle\mathbf{n} - \langle \mathbf{Yf}, \mathbf{n}\rangle\mathbf{n} = \nabla_{X}\mathbf{f} + \nabla_{\mathbf{Y}}\mathbf{f}$$
$$\nabla_{a\mathbf{X}}\mathbf{f} = (a\mathbf{X})\mathbf{f} - \langle(a\mathbf{X})\mathbf{f},\mathbf{n}\rangle\mathbf{n} = a(\mathbf{Xf}) - \langle a(\mathbf{Xf}),\mathbf{n}\rangle\mathbf{n} = a[\mathbf{Xf} - \langle \mathbf{Xf},\mathbf{n}\rangle\mathbf{n}] = a\nabla_{\mathbf{X}}\mathbf{f}.$$

e.

Using part **a** and Problem **4.8** we calculate

$$\nabla_{\mathbf{X}} r\mathbf{Y} = \mathbf{X}(r\mathbf{Y}) - \langle \mathbf{X}(r\mathbf{Y}),\mathbf{n}\rangle\mathbf{n} = r(\mathbf{XY}) - \langle r(\mathbf{XY}),\mathbf{n}\rangle\mathbf{n} = r[\mathbf{XY} - \langle \mathbf{XY},\mathbf{n}\rangle\mathbf{n}] = r\nabla_{\mathbf{X}}\mathbf{Y}\ ,$$

and, using the fact that $\mathbf{X}f$ is a scalar and that $\mathbf{Y}$ is perpendicular to $\mathbf{n}$,

$$\nabla_{\mathbf{X}} f\,\mathbf{Y} = \mathbf{X}(f\,\mathbf{Y}) - \langle \mathbf{X}(f\,\mathbf{Y}),\mathbf{n}\rangle\mathbf{n} = [(\mathbf{X}f)\mathbf{Y} + f(\mathbf{XY})] - \langle(\mathbf{X}f)\mathbf{Y} + f(\mathbf{XY}),\mathbf{n}\rangle\mathbf{n} =$$
$$= [(\mathbf{X}f)\mathbf{Y} + f(\mathbf{XY})] - \langle f(\mathbf{XY}),\mathbf{n}\rangle\mathbf{n} = (\mathbf{X}f)\mathbf{Y} + f(\mathbf{XY}) - \langle f(\mathbf{XY}),\mathbf{n}\rangle\mathbf{n} = (\mathbf{X}f)\mathbf{Y} + f\,\nabla_{\mathbf{X}}\mathbf{Y}.$$

**Problem* 8.2. Manifolds — Intrinsic and Extrinsic*

***a.**

Outline of a proof of Problem 8.2.a:

i. First we prove this in the case that the chart is a Monge patch $\mathbf{y}$. The inverse $\mathbf{y}^{-1}|M$ is just the orthogonal projection of M onto $\mathbf{R}^2$. If f is C^k then $\mathbf{y}^{-1}\circ f$ is C^k because it is just the projection onto the first two coordinates. On the other hand, if $\mathbf{y}^{-1}\circ f$ is C^k then so is $\mathbf{y}\circ(\mathbf{y}^{-1}\circ f) = f$.

ii. Now we look at $\mathbf{x}^{-1}\circ\mathbf{y}$. This is one-to-one because it is the composition of one-to-one functions. If $\mathbf{x}$ is defined on U and $\mathbf{y}$ is defined on V then $\mathbf{x}^{-1}\circ\mathbf{y}$ is defined on $\mathbf{y}^{-1}(\mathbf{x}(U)\cap\mathbf{y}(V))$ and maps it to $\mathbf{x}^{-1}(\mathbf{x}(U)\cap\mathbf{y}(V))$. These are both open sets because they are the inverse image of open sets under a continuous map.

iii. By step **i**, the function $\mathbf{y}^{-1}\circ\mathbf{x}$ (the inverse of $\mathbf{x}^{-1}\circ\mathbf{y}$) is C^k. Since $\mathbf{y}^{-1}\circ\mathbf{x}$ is one-to-one and onto an open set its differential $d(\mathbf{y}^{-1}\circ\mathbf{x})$ (which is represented by a matrix for given basis in $\mathbf{R}^2$) is invertible. We can then find the inverse of the matrix and since the entries of $d(\mathbf{y}^{-1}\circ\mathbf{x})$ are C^k the entries of its inverse $d(\mathbf{x}^{-1}\circ\mathbf{y})$ are also C^k. [You can use the Inverse Function Theorem (see Appendix **B.3**) but this is overkill in this case because the hard part of the Inverse Function Theorem is to prove that the function and its inverse are one-to-one and onto.]

b.

Let $\mathbf{x}$ be a local chart whose image contains a neighborhood of p. Look at the projection π which takes a neighborhood of p onto the tangent space at p. Then $\pi\circ\mathbf{x}$ is a C^k function from $\mathbf{R}^n$ to $\mathbf{R}^n$. By the Inverse Function Theorem (Appendix **B.3**), $\pi\circ\mathbf{x}$ has a local C^k inverse g. Then $\mathbf{x}\circ g$ is a map from the tangent space onto a neighborhood of p in M such that $\pi\circ(\mathbf{x}\circ g)$ is the identity. Thus, $\mathbf{x}\circ g$ is a Monge patch.

c.

By part **b**, M has a Monge patch $\mathbf{y}$. The inverse $\mathbf{y}^{-1}|M$ is just the orthogonal projection of M onto $\mathbf{R}^2$. If $\mathbf{x}$ is another chart then $\mathbf{y}^{-1}\circ\mathbf{x}$ is C^k because it is just the projection onto the first two coordinates. Now we look at $\mathbf{x}^{-1}\circ\mathbf{y}$. This is one-to-one because it is the composition of one-to-one functions. If $\mathbf{x}$ is defined on U and $\mathbf{y}$ is defined on V then $\mathbf{x}^{-1}\circ\mathbf{y}$ is defined on $\mathbf{y}^{-1}(\mathbf{x}(U)\cap\mathbf{y}(V))$ and maps it to $\mathbf{x}^{-1}(\mathbf{x}(U)\cap\mathbf{y}(V))$. These are both open sets because they are the inverse image of open sets under a continuous map. Since $\mathbf{y}^{-1}\circ\mathbf{x}$ is one-to-one and onto an open set its differential $d(\mathbf{y}^{-1}\circ\mathbf{x})$ (which is represented by a matrix for given basis in $\mathbf{R}^2$) is invertible. We can then find the inverse of the matrix and since the entries of $d(\mathbf{y}^{-1}\circ\mathbf{x})$ are C^k the entries of its inverse $d(\mathbf{x}^{-1}\circ\mathbf{y})$ are also C^k. [You can use the Inverse Function Theorem (see Appendix **B.3**) but this is overkill in this case because the hard part of the Inverse Function Theorem is to prove that the function and its inverse are one-to-one and onto.]

Now, let $\mathbf{z}$ be any other of the extrinsic charts for M. Then, by the previous argument, $\mathbf{z}$ is compatible with $\mathbf{y}$. Then $\mathbf{x}^{-1}\circ\mathbf{z} = (\mathbf{x}^{-1}\circ\mathbf{y})\circ(\mathbf{y}^{-1}\circ\mathbf{z})$ is the composition of C^k functions and is, thus, C^k. Therefore, the collection of extrinsic charts is an atlas for M.

d.

That a surface with a single chart is a C^k manifold follows immediately from the definition. We now check that the two charts, $\mathbf{x}$ and $\mathbf{z}$, defined for the annular hyperbolic plane (with $r = 1$) in Problem **1.8** are compatible. The compositions $\mathbf{x}^{-1}\circ\mathbf{z}(x,y) = \mathbf{x}^{-1}\circ\mathbf{x}(x,\ln(y)) = (x,\ln(y))$ and $\mathbf{z}^{-1}\circ\mathbf{x}(x,y) = (x,\exp(y)$ are both C^k, and thus the charts are compatible.

e.

If γ and λ are two curves containing the point p in M and $\mathbf{x}$ and $\mathbf{y}$ are two charts containing p, then we can explicitly calculate:

$$(\mathbf{x}^{-1}\circ\gamma)'(0) = (\mathbf{x}^{-1}\circ\lambda)'(0) \Leftrightarrow (\mathbf{y}^{-1}\circ\gamma)'(0) = d(\mathbf{y}^{-1}\circ\mathbf{x})[(\mathbf{x}^{-1}\circ\gamma)'(0)] = d(\mathbf{y}^{-1}\circ\mathbf{x})[(\mathbf{x}^{-1}\circ\lambda)'(0)] = (\mathbf{y}^{-1}\circ\lambda)'(0).$$

Thus, the definition does not depend on which chart containing p you choose.

We now show, for each chart $\mathbf{y}$ (containing p), that the function from the tangent space of $\mathbf{R}^n$ at $q = \mathbf{y}^{-1}(p)$ to T_pM defined by $d\mathbf{y}(\mathbf{X}_q) = [t \to \mathbf{y}(q+t\mathbf{X}_q)]$ is one-to-one and onto. Let $\mathbf{X}_q$ and $\mathbf{Y}_q$ be two tangent vectors at q in $\mathbf{R}^n$. Suppose that $d\mathbf{y}(\mathbf{X}_q) = [t \to \mathbf{y}(q+t\mathbf{X}_q)] = [t \to \mathbf{y}(q+t\mathbf{Y}_q)] = d\mathbf{y}(\mathbf{Y}_q)$, then, by definition, $\mathbf{X}_q = (\mathbf{y}^{-1} \circ \mathbf{y}(q+t\mathbf{X}_q))'(0) = (\mathbf{y}^{-1} \circ \mathbf{y}(q+t\mathbf{Y}_q))'(0) = \mathbf{Y}_q$. This correspondence is onto because, if γ is any curve in M with $\gamma(0) = p$, then $(\mathbf{y}^{-1} \circ \gamma)'(0) = \mathbf{Y}_q$ is a tangent vector at q in $\mathbf{R}^n$, and thus, $[\gamma] = d\mathbf{y}(\mathbf{Y}_q)$.

Use the above one-to-one, onto correspondence (dependent on the chart $\mathbf{y}$) define a vector space structure on T_pM. If $\mathbf{x}$ is any other chart containing p then $d(\mathbf{x}^{-1}\circ\mathbf{y})$ is a linear isomorphism from the tangent of $\mathbf{R}^n$ at $\mathbf{y}^{-1}(p)$ to the tangent space of $\mathbf{R}^n$ at $\mathbf{x}^{-1}(p)$ and $d\mathbf{x}(d(\mathbf{x}^{-1}\circ\mathbf{y})(\mathbf{X}_q)) = (d\mathbf{x}\circ d(\mathbf{x}^{-1}\circ\mathbf{y}))(\mathbf{X}_q) = d(\mathbf{x}\circ(\mathbf{x}^{-1}\circ\mathbf{y}))(\mathbf{X}_q) = d\mathbf{y}(\mathbf{X}_q)$. Thus, the vector space structure defined by $d\mathbf{y}$ will be the same as the structure defined by $d\mathbf{x}$.

PROBLEM 8.3. Christoffel Symbols, Intrinsic Descriptions

a.

1. We calculate $\langle \mathbf{x}_{ij}, \mathbf{x}_l \rangle = \langle \mathbf{x}_i\mathbf{x}_j, \mathbf{x}_l \rangle = \langle (\nabla_{\mathbf{x}_i}\mathbf{x}_j + \langle \mathbf{x}_i\mathbf{x}_j, \mathbf{n} \rangle\mathbf{n}), \mathbf{x}_l \rangle = \langle \nabla_{\mathbf{x}_i}\mathbf{x}_j, \mathbf{x}_l \rangle + \langle \langle \mathbf{x}_i\mathbf{x}_j, \mathbf{n} \rangle\mathbf{n}, \mathbf{x}_l \rangle$, this last term is equal to zero because $\mathbf{n}$ is perpendicular to $\mathbf{x}_l$. Thus,

$$\langle \mathbf{x}_{ij}, \mathbf{x}_l \rangle = \langle \nabla_{\mathbf{x}_i}\mathbf{x}_j, \mathbf{x}_l \rangle = \left\langle \sum_k \Gamma_{ij}^k \mathbf{x}_k, \mathbf{x}_l \right\rangle = \sum_k \Gamma_{ij}^k \langle \mathbf{x}_k, \mathbf{x}_l \rangle = \sum_k \Gamma_{ij}^k g_{kl}.$$

2. The matrix (g^{lk}) is the inverse of the matrix (g_{lk}), which means that

$$\sum_l g_{kl}g^{lm} = 1, \text{ when } k = m, \text{ and } \sum_l g_{kl}g^{lm} = 0, \text{ when } k \neq m.$$

Thus, we show that $\sum_l \langle \mathbf{x}_{ij}, \mathbf{x}_l \rangle g^{lm} = \sum_l \left(\sum_k \Gamma_{ij}^k g_{kl}\right) g^{lm} = \sum_k \sum_l \left(\Gamma_{ij}^k g_{kl} g^{lm}\right) = \sum_k \left(\Gamma_{ij}^k \sum_l g_{kl} g^{lm}\right) = \Gamma_{ij}^m$.

b.

If $\mathbf{Y} = \sum Y^j \mathbf{x}_j$ is a (tangent) vector field (note that the Y^j are real valued functions), then (using Problem **8.1.e**)

$$\begin{aligned} \nabla_{\mathbf{x}_i}\mathbf{Y} &= \sum_j \nabla_{\mathbf{x}_i}(Y^j\mathbf{x}_j) = \sum_j \left[(\mathbf{x}_i Y^j)\mathbf{x}_j + Y^j(\nabla_{\mathbf{x}_i}\mathbf{x}_j)\right] = \\ &= \sum_j \left[(\mathbf{x}_i Y^j)\mathbf{x}_j + Y^j\left(\sum_k \Gamma_{ij}^k \mathbf{x}_k\right)\right] = \sum_j (\mathbf{x}_i Y^j)\mathbf{x}_j + \sum_j Y^j\left(\sum_k \Gamma_{ij}^k \mathbf{x}_k\right) = \sum_k (\mathbf{x}_i Y^k)\mathbf{x}_k + \sum_k \left(\sum_j Y^j \Gamma_{ij}^k \mathbf{x}_k\right) = \\ &= \sum_k \left(\mathbf{x}_i Y^k + \sum_j \Gamma_{ij}^k Y^j\right)\mathbf{x}_k . \end{aligned}$$

c.

We calculate, using properties of the directional derivative from Chapter 4:

$$\mathbf{x}_i\langle \mathbf{x}_j, \mathbf{x}_k \rangle = \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle + \langle \mathbf{x}_j, \mathbf{x}_{ik} \rangle \text{ and, thus, } \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle = \mathbf{x}_i\langle \mathbf{x}_j, \mathbf{x}_k \rangle - \langle \mathbf{x}_j, \mathbf{x}_{ik} \rangle.$$

Applying this three times with different indices we get

$$\begin{aligned} \langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle &= \mathbf{x}_i\langle \mathbf{x}_j, \mathbf{x}_k \rangle - \langle \mathbf{x}_j, \mathbf{x}_{ik} \rangle = \\ &= \mathbf{x}_i\langle \mathbf{x}_j, \mathbf{x}_k \rangle - (\mathbf{x}_k\langle \mathbf{x}_j, \mathbf{x}_i \rangle - \langle \mathbf{x}_{kj}, \mathbf{x}_i \rangle) = \\ &= \mathbf{x}_i\langle \mathbf{x}_j, \mathbf{x}_k \rangle - \mathbf{x}_k\langle \mathbf{x}_j, \mathbf{x}_i \rangle + \mathbf{x}_j\langle \mathbf{x}_k, \mathbf{x}_i \rangle - \langle \mathbf{x}_k, \mathbf{x}_{ji} \rangle. \end{aligned}$$

Thus, $\langle \mathbf{x}_{ij}, \mathbf{x}_k \rangle = \frac{1}{2}\left[\mathbf{x}_i\langle \mathbf{x}_j, \mathbf{x}_k \rangle - \mathbf{x}_k\langle \mathbf{x}_j, \mathbf{x}_i \rangle + \mathbf{x}_j\langle \mathbf{x}_k, \mathbf{x}_i \rangle\right] = \frac{1}{2}\left[\mathbf{x}_i g_{jk} - \mathbf{x}_k g_{ji} + \mathbf{x}_j g_{ki}\right]$.

d.

For a surface with geodesic rectangular (or polar) coordinates, we have

$$(g_{ij}) = \begin{pmatrix} h^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } (g^{ij}) = \begin{pmatrix} h^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, we can calculate

$$\Gamma_{11}^1 = \tfrac{1}{2}\sum_l g^{1l}[\mathbf{x}_1 g_{1l} - \mathbf{x}_l g_{11} + \mathbf{x}_1 g_{l1}] = \tfrac{1}{2}g^{11}[\mathbf{x}_1 g_{11} - \mathbf{x}_1 g_{11} + \mathbf{x}_1 g_{11}] = \tfrac{1}{2}h^{-2}\mathbf{x}_1(h^2) = \tfrac{1}{2}h^{-2}(2hh_1) = h_1/h,$$
$$\Gamma_{11}^2 = \tfrac{1}{2}\sum_l g^{2l}[\mathbf{x}_1 g_{1l} - \mathbf{x}_l g_{11} + \mathbf{x}_1 g_{l1}] = \tfrac{1}{2}g^{22}[\mathbf{x}_1 g_{12} - \mathbf{x}_2 g_{11} + \mathbf{x}_1 g_{21}] = \tfrac{-1}{2}\mathbf{x}_2(h^2) = -hh_2,$$
$$\Gamma_{12}^1 = \Gamma_{21}^1 = \tfrac{1}{2}\sum_l g^{1l}[\mathbf{x}_2 g_{1l} - \mathbf{x}_l g_{12} + \mathbf{x}_1 g_{l2}] = \tfrac{1}{2}g^{11}[\mathbf{x}_2 g_{11} - \mathbf{x}_1 g_{12} + \mathbf{x}_1 g_{12}] = \tfrac{1}{2}h^{-2}(\mathbf{x}_2(h^2)) = h_2/h \quad ,$$
$$\Gamma_{12}^2 = \Gamma_{21}^2 = \tfrac{1}{2}\sum_l g^{2l}[\mathbf{x}_1 g_{2l} - \mathbf{x}_l g_{21} + \mathbf{x}_2 g_{l1}] = \tfrac{1}{2}g^{22}[\mathbf{x}_1 g_{22} - \mathbf{x}_2 g_{21} + \mathbf{x}_2 g_{21}] = 0 \quad ,$$
$$\Gamma_{22}^k = \tfrac{1}{2}\sum_l g^{kl}[\mathbf{x}_2 g_{2l} - \mathbf{x}_l g_{22} + \mathbf{x}_2 g_{l2}] = \tfrac{1}{2}[g^{k1}(\mathbf{x}_2 g_{21} - \mathbf{x}_1 g_{22} + \mathbf{x}_2 g_{12}) + g^{k2}[\mathbf{x}_2 g_{22} - \mathbf{x}_2 g_{22} + \mathbf{x}_2 g_{22}]] = 0 \quad .$$

As derived in the solution to Problem **7.2.a**, for the sphere,

$$h(u^1,u^2) = \cos\tfrac{u^2}{R}, \text{ thus } h_1 = 0 \text{ and } h_2(u^1,u^2) = -\tfrac{1}{R}\sin\tfrac{u^2}{R}.$$

Thus, we calculate

$$\Gamma_{11}^1 = h_1/h = 0,\ \Gamma_{11}^2 = -hh_2 = \tfrac{1}{R}\cos\tfrac{u^2}{R}\sin\tfrac{u^2}{R},\ \Gamma_{12}^1 = \Gamma_{21}^1 = h_2/h = \tfrac{-1}{R}\tan\tfrac{u^2}{R}, \text{ all others zero.}$$

Problem 8.4. Intrinsic Curvature and Geodesics

a.

We calculate using the fact that, for any real-valued function $f(s)$, $\nabla_{\gamma'(a)} f(s)|_{s=a} = f'(a)$,

$$\begin{aligned}\boldsymbol{\kappa}_g(a) &= \nabla_{\gamma'(a)}\gamma' = \nabla_{\gamma'(a)}\textstyle\sum_j(\gamma^j)'\mathbf{x}_j = \sum_j\left[(\gamma'(a)(\gamma^j)')\mathbf{x}_j + (\gamma^j)'\nabla_{\gamma'(a)}\mathbf{x}_j\right] = \\ &= \textstyle\sum_j\left[((\gamma^j)''_a)\mathbf{x}_j + (\gamma^j)'_a\nabla_{\sum_i(\gamma^i)'_a\mathbf{x}_i}\mathbf{x}_j\right] = \sum_j\left[((\gamma^j)''_a)\mathbf{x}_j + (\gamma^j)'_a\sum_i(\gamma^i)'_a\nabla_{\mathbf{x}_i}\mathbf{x}_j\right] = \\ &= \textstyle\sum_j((\gamma^j)''_a)\mathbf{x}_j + \sum_{i,j}(\gamma^j)'_a(\gamma^i)'_a\sum_k\Gamma_{ij}^k(\gamma(a))\mathbf{x}_k = \sum_k((\gamma^k)''_a)\mathbf{x}_k + \sum_k\sum_{i,j}(\gamma^j)'_a(\gamma^i)'_a\Gamma_{ij}^k(\gamma(a))\mathbf{x}_k = \\ &= \sum_k\left[(\gamma^k)''_a + \sum_{i,j}\Gamma_{ij}^k(\gamma(a))(\gamma^i)'_a(\gamma^j)'_a\right]\mathbf{x}_k \quad .\end{aligned}$$

b.

This follows immediately from part **a** because a curve is a geodesic if and only if $\boldsymbol{\kappa}_g = 0$ at every point along the curve.

c.

For geodesic coordinates **x** we have expressions for the Christoffel symbols from Problem **8.3.d**. Thus, we can say that, for a curve $\boldsymbol{\gamma}(s) = \mathbf{x}(\gamma^1(s),\gamma^2(s))$,

$$\begin{aligned}\kappa_g(a) &= \sum_k\left[(\gamma^k)''_a + \sum_{i,j}\Gamma_{ij}^k(\gamma(a))(\gamma^i)'_a(\gamma^j)'_a\right]\mathbf{x}_k = \\ &\left[(\gamma^1)''_a + \tfrac{h_1(\gamma(a))}{h(\gamma(a))}((\gamma^1)'_a)^2 + 2\tfrac{h_2(\gamma(a))}{h(\gamma(a))}(\gamma^1)'_a(\gamma^2)'_a\right]\mathbf{x}_1 + \left[(\gamma^2)''_a - h(\gamma(a))h_2(\gamma(a))((\gamma^1)'_a)^2\right]\mathbf{x}_2 \quad ,\end{aligned}$$

and $\boldsymbol{\gamma}$ is a geodesic if and only if

$$\left[(\gamma^1)''_a + \tfrac{h_1(\gamma(a))}{h(\gamma(a))}((\gamma^1)'_a)^2 + 2\tfrac{h_2(\gamma(a))}{h(\gamma(a))}(\gamma^1)'_a(\gamma^2)'_a\right] = 0 = \left[(\gamma^2)''_a - h(\gamma(a))h_2(\gamma(a))((\gamma^1)'_a)^2\right],$$

for each a.

Problem 8.5. Lie Brackets and Coordinate Vector Fields

a.

From Problems **8.2.a** or **8.2.c** it is clear that $\Gamma_{ij}^k = \Gamma_{ji}^k$, thus, $\nabla_{\mathbf{x}_i}\mathbf{x}_j = \sum_k\Gamma_{ij}^k\mathbf{x}_k = \sum_k\Gamma_{ji}^k\mathbf{x}_k = \nabla_{\mathbf{x}_j}\mathbf{x}_i$.

b.

Let $a(x,y) = y$ then, since **B** is constant, $\nabla_{\mathbf{A}(0,0)}\mathbf{B} = \mathbf{A}(0,0)\mathbf{B} = \mathbf{0} \neq \mathbf{e}_2 = \frac{\partial}{\partial y}\mathbf{A}(x,y) = \mathbf{B}(0,0)\mathbf{A} = \nabla_{\mathbf{B}(0,0)}\mathbf{A}$.

c.

In any geodesic coordinates **x** where the second coordinate curves are not extrinsically straight we have $\mathbf{x}_2\mathbf{x}_2 = \mathbf{x}_{22}$ equal to the extrinsic curvature which is perpendicular to the surface because the curves are geodesics. However, expressing the tangent vectors in local coordinates **x** and using Problem **8.1** and linearity, we can calculate

$$\sum_i (X^i\mathbf{x}_i)_p \sum_j (Y^j\mathbf{x}_j) = \sum_i \left(X_p^i\right) \sum_j \mathbf{x}_i(Y^j\mathbf{x}_j) = \sum_i \left(X_p^i\right) \sum_j \left[(\mathbf{x}_i Y^j)\mathbf{x}_j + Y^j(\mathbf{x}_i\mathbf{x}_j)\right]$$
$$= \sum_{i,j} X_p^i(\mathbf{x}_i Y^j)\mathbf{x}_j + \sum_{i,j} X_p^i Y_p^j(\mathbf{x}_{ij}).$$

In this last expression the first term is a tangent vector and the last term is symmetric in i and j, thus,

$$\mathbf{X}_p\mathbf{Y} - \mathbf{Y}_p\mathbf{X} = \Sigma_{i,j} X_p^i(\mathbf{x}_i Y^j)\mathbf{x}_j - \Sigma_{i,j} Y_p^i(\mathbf{x}_i X^j)\mathbf{x}_j = \Sigma_{i,j}\left[X_p^i(\mathbf{x}_i Y^j) - Y_p^i(\mathbf{x}_i X^j)\right]\mathbf{x}_j$$

is a tangent vector. Then we can calculate

$$[\mathbf{X},\mathbf{Y}]_p \equiv \nabla\mathbf{X}(p)\mathbf{Y} - \nabla\mathbf{Y}(p)\mathbf{X} = \mathbf{X}_p\mathbf{Y} - \langle\mathbf{X}_p\mathbf{Y},\mathbf{n}\rangle\mathbf{n} - \mathbf{Y}_p\mathbf{X} + \langle\mathbf{Y}_p\mathbf{X},\mathbf{n}\rangle\mathbf{n} =$$
$$\mathbf{X}_p\mathbf{Y} - \mathbf{Y}_p\mathbf{X} + \langle\mathbf{Y}_p\mathbf{X},\mathbf{n}\rangle\mathbf{n} - \langle\mathbf{X}_p\mathbf{Y},\mathbf{n}\rangle\mathbf{n} = (\mathbf{X}_p\mathbf{Y} - \mathbf{Y}_p\mathbf{X}) - \langle(\mathbf{X}_p\mathbf{Y} - \mathbf{Y}_p\mathbf{X}),\mathbf{n}\rangle\mathbf{n} = \mathbf{X}_p\mathbf{Y} - \mathbf{Y}_p\mathbf{X},$$

where the last equality is because $\mathbf{X}_p\mathbf{Y} - \mathbf{Y}_p\mathbf{X}$ is a tangent vector, and thus has no projection onto the normal **n**.

***d.**

Outline of a proof: This outline assumes that the reader has a familiarity with flows defined by vector fields and with the theorem from analysis that a C^1 vector field always has a unique flow. For a discussion of these results the interested reader can consult [**An**: Strichartz], Chapter 11, or [**DG**: Dodson/Poston], VII.6 and VII.7. In the latter, the details of this outline are filled in.

1. Given a C^1 vector field **V** defined and nonzero in a neighborhood of p in M then there is a coordinate chart **x** such that $\mathbf{V} = \mathbf{x}_1$.

2. If **V** and **W** are two C^1 vector fields on M with flows ϕ_s and ψ_s then the flows commute

$$\phi_a \circ \psi_b = \psi_b \circ \phi_a, \text{ wherever defined}$$

if and only if $[\mathbf{V},\mathbf{W}]_p = \mathbf{0}$, for all p.

3. Use the flows to define the coordinate chart **x**.

PROBLEM 8.6. Riemann Curvature Tensors

a.

Outline of a proof:

Let $p = \mathbf{x}(0,0)$. Since the covariant derivative and the intrinsic curvature can both be defined in terms of parallel transport, we look at parallel transport along the coordinate curves and use the abbreviations:

$$P_1(\delta,a) = P(t \to \mathbf{x}(t,a), \mathbf{x}(0,a), \mathbf{x}(\delta,a)),\ P_2(a,\delta) = P(t \to \mathbf{x}(a,t),\ \mathbf{x}(a,0), \mathbf{x}(a,\delta)).$$

Since $P_1(\varepsilon,\delta)[P_2(0,\delta)V(p)]$ is the parallel transport $\mathbf{V}(p)$ along the second coordinate curve to the point $\mathbf{x}(0,\delta)$ and then along the first coordinate curve to the point $\mathbf{x}(\varepsilon,\delta)$ and $P_2(\varepsilon,\delta)[P_1(\varepsilon,0)V(p)]$ is the parallel transport of $\mathbf{V}(p)$ along the first coordinate curve to $\mathbf{x}(\varepsilon,0)$ and then along the second coordinate curve to

$\mathbf{x}(\varepsilon,\delta)$, then the angle θ between these two parallel transports is the holonomy of the region R bounded by the coordinate curves with "corners" $p=\mathbf{x}(0,0)$, $\mathbf{x}(\varepsilon,0)$, $\mathbf{x}(\varepsilon,\delta)$, $\mathbf{x}(0,\delta)$. (See Figure 8.4 in the text.)

Then, denote

$$\mathbf{P}(\varepsilon,\delta)=P_1(\varepsilon,\delta)[P_2(0,\delta)\mathbf{V}(p)]-P_2(\varepsilon,\delta)[P_1(\varepsilon,0)\mathbf{V}(p)].$$

Then note that $\pm|\mathbf{P}(\varepsilon,\delta)|/|\mathbf{V}| = 2\sin(\theta/2)$, where we assign $\pm|\mathbf{P}|$ the same sign as θ (positive, if counter-clockwise). We can calculate

$$K(p) = \lim_{R\to 0}(\mathcal{H}(R)/A(R)) = \lim_{R\to 0}(\theta/A(R)) = \lim_{\varepsilon,\delta\to 0}\frac{\theta}{2\sin(\theta/2)}\left(\frac{\pm|\mathbf{P}(\varepsilon,\delta)|/|\mathbf{V}|}{\varepsilon\delta}\right)\frac{\varepsilon\delta}{A(R)} \quad .$$

Since this limit exists it is equal to the product

$$\lim_{\varepsilon,\delta\to 0}\frac{\theta}{2\sin(\theta/2)}\left(\lim_{\varepsilon,\delta\to 0}\frac{\pm|\mathbf{P}(\varepsilon,\delta)|/|\mathbf{V}|}{\varepsilon\delta}\right)\lim_{\varepsilon,\delta\to 0}\frac{\varepsilon\delta}{A(R)}$$

as long as two of these three limits exists. As the region gets smaller it becomes closer and closer to a planar region and, thus, the angle θ goes to zero and the first limit exists and is equal to 1. We look at the inverse of the third limit

$$\begin{aligned}\lim_{\varepsilon,\delta\to 0}\tfrac{1}{\varepsilon\delta}A(R) &= \lim_{\varepsilon,\delta\to 0}\tfrac{1}{\varepsilon\delta}\int_0^\delta\int_0^\varepsilon \sqrt{g_{ij}(u^1,u^2)}\ du^1du^2 = \lim_{\varepsilon,\delta\to 0}\tfrac{1}{\varepsilon\delta}\int_0^\delta\int_0^\varepsilon|\mathbf{x}_1(u^1,u^2)||\mathbf{x}_2(u^1,u^2)|\ du^1du^2 = \\ &= \lim_{\varepsilon,\delta\to 0}\left(\tfrac{1}{\varepsilon}\int_0^\varepsilon|\mathbf{x}_1(u^1,u^2)|\ du^1\right)\left(\tfrac{1}{\delta}\int_0^\delta|\mathbf{x}_2(u^1,u^2)|\ du^2\right) = |\mathbf{x}_1(0,0)||\mathbf{x}_2(0,0)| \quad .\end{aligned}$$

Thus, $|\mathbf{V}||\mathbf{x}_1||\mathbf{x}_2|K(p) = \lim_{\varepsilon,\delta\to 0}\frac{\pm|\mathbf{P}(\varepsilon,\delta)|}{\varepsilon\delta}$.

Now, denoting $\mathbf{V}(\mathbf{x}(a,b)) = \mathbf{V}(a,b)$, we use the limit definition of covariant derivative, the fact that $\nabla_{\mathbf{x}_1}$ is continuous, and the fact that parallel transport is a linear isometry to conclude

$$\begin{aligned}\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{V} &= \nabla_{\mathbf{x}_1}\lim_{\delta\to 0}\tfrac{1}{\delta}[\mathbf{V}(a,\delta)-P_2(a,\delta)\mathbf{V}(a,0)] = \lim_{\delta\to 0}\nabla_{\mathbf{x}_1}\left(\tfrac{1}{\delta}[\mathbf{V}(a,\delta)-P_2(a,\delta)\mathbf{V}(a,0)]\right) = \\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}\{\tfrac{1}{\delta}[\mathbf{V}(\varepsilon,\delta)-P_2(\varepsilon,\delta)\mathbf{V}(\varepsilon,0)] - P_1(\varepsilon,\delta)\tfrac{1}{\delta}[\mathbf{V}(0,\delta)-P_2(0,\delta)\mathbf{V}(0,0)]\} = \\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}\tfrac{1}{\delta}\{\mathbf{V}(\varepsilon,\delta)-P_2(\varepsilon,\delta)\mathbf{V}(\varepsilon,0)-P_1(\varepsilon,\delta)\mathbf{V}(0,\delta)+P_1(\varepsilon,\delta)[P_2(0,\delta)\mathbf{V}(0,0)]\}\end{aligned}$$

Thus,

$$\begin{aligned}&\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{V}-\lim_{\varepsilon,\delta\to 0}\frac{\mathbf{P}(\varepsilon,\delta)}{\varepsilon\delta} = \\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}\tfrac{1}{\delta}\{\mathbf{V}(\varepsilon,\delta)-P_2(\varepsilon,\delta)\mathbf{V}(\varepsilon,0)-P_1(\varepsilon,\delta)\mathbf{V}(0,\delta)+P_1(\varepsilon,\delta)[P_2(0,\delta)\mathbf{V}(0,0)]-\mathbf{P}(\varepsilon,\delta)\} = \\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\tfrac{1}{\varepsilon}\tfrac{1}{\delta}\{\mathbf{V}(\varepsilon,\delta)-P_2(\varepsilon,\delta)\mathbf{V}(\varepsilon,0)-P_1(\varepsilon,\delta)\mathbf{V}(0,\delta)+P_2(\varepsilon,\delta)[P_1(\varepsilon,0)\mathbf{V}(0,0)]\} = \\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\tfrac{1}{\delta}\{\tfrac{1}{\varepsilon}[\mathbf{V}(\varepsilon,\delta)-P_1(\varepsilon,\delta)\mathbf{V}(0,\delta)]-\tfrac{1}{\varepsilon}[P_2(\varepsilon,\delta)\mathbf{V}(\varepsilon,0)-P_2(\varepsilon,\delta)[P_1(\varepsilon,0)\mathbf{V}(0,0)]]\} = \\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\tfrac{1}{\delta}\{\tfrac{1}{\varepsilon}[\mathbf{V}(\varepsilon,\delta)-P_1(\varepsilon,\delta)\mathbf{V}(0,\delta)]-P_2(\varepsilon,\delta)(\tfrac{1}{\varepsilon}[\mathbf{V}(\varepsilon,0)-P_1(\varepsilon,0)\mathbf{V}(0,0)])\} = \\ &= \lim_{\delta\to 0}\tfrac{1}{\delta}\{(\nabla_{\mathbf{x}_1}\mathbf{V})(0,\delta)-P_2(\varepsilon,\delta)[(\nabla_{\mathbf{x}_1}\mathbf{V})(0,0)]\} = \nabla_{\mathbf{x}_2}((\nabla_{\mathbf{x}_1}\mathbf{V})(0,0)) \ .\end{aligned}$$

Therefore,

$$|\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{V}-\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_1}\mathbf{V}| = \left|\lim_{\varepsilon,\delta\to 0}\frac{\mathbf{P}(\varepsilon,\delta)}{\varepsilon\delta}\right| = \lim_{\varepsilon,\delta\to 0}\left|\frac{\mathbf{P}(\varepsilon,\delta)}{\varepsilon\delta}\right| = |\mathbf{V}||\mathbf{x}_1||\mathbf{x}_2||K(p)| \quad .$$

b.

In part **a** we can set $\mathbf{V}$ equal to $\mathbf{x}_1$ and then after parallel transport around the vector $\mathbf{x}_1(0,0)$ will change only in the $\mathbf{x}_2$ direction, because the length of $\mathbf{x}_1$ does not change and the change must be in the tangent plane and, thus, be parallel to the $\mathbf{x}_2$ direction, and thus,

$$\langle(\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{x}_1-\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_1}\mathbf{x}_1),\mathbf{x}_2\rangle_p = \langle\mathbf{x}_1,\mathbf{x}_1\rangle\langle\mathbf{x}_2,\mathbf{x}_2\rangle K(p)$$

because $\pm\langle(\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{x}_1-\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_1}\mathbf{x}_1),\mathbf{x}_2\rangle_p = |\mathbf{x}_1||\mathbf{x}_1||\mathbf{x}_2||K(p)||\mathbf{x}_2| = \pm\langle\mathbf{x}_1,\mathbf{x}_1\rangle\langle\mathbf{x}_2,\mathbf{x}_2\rangle K(p)$, where the left $\pm$ is positive when the change after parallel transport in the positive $\mathbf{x}_2$-direction and in this case the angle of change (which is the holonomy) is positive, and thus $K(p)$ is positive.

c.

If $\mathbf{F}(\mathbf{X})$ is a vector field that depends linearly on another vector field $\mathbf{X}$, then there is a trick that works to check whether $\mathbf{F}_p(\mathbf{X})$ depends only on $\mathbf{X}_p$. Let k be any real-valued function defined in a neighborhood of p such that $k(p) = 1$, then $\mathbf{F}_p(\mathbf{X})$ depends only on $\mathbf{X}_p$ if and only if $\mathbf{F}_p(k\mathbf{X}) = k(p)\mathbf{F}_p(\mathbf{X}) = \mathbf{F}_p(\mathbf{X})$. Note that, in this case and because $\mathbf{F}$ is linear, if $\mathbf{X} = \Sigma X^i\mathbf{x}_i$ then $\mathbf{F}_p(\mathbf{X}) = \Sigma X^i(p)\mathbf{F}_p(\mathbf{x}_i)$. So we now calculate, $k(p)$ is as above,

$$\nabla_{\mathbf{X}_p}\nabla_{\mathbf{Y}}(k\mathbf{Z}) = \nabla_{\mathbf{X}_p}[(\mathbf{Y}k)\mathbf{Z} + k(p)\nabla_{\mathbf{Y}}\mathbf{Z}] = (\mathbf{X}_p\mathbf{Y}k)\mathbf{Z} + (\mathbf{X}_pk)\nabla_{\mathbf{Y}_p}\mathbf{Z} + (\mathbf{Y}_pk)\nabla_{\mathbf{X}_p}\mathbf{Z} + k(p)\nabla_{\mathbf{X}_p}\nabla_{\mathbf{Y}}\mathbf{Z},$$
$$\nabla_{Y_p}\nabla_{X}(k\mathbf{Z}) = \nabla_{\mathbf{Y}_p}[(\mathbf{X}k)\mathbf{Z} + k(p)\nabla_{\mathbf{X}}\mathbf{Z}] = (\mathbf{Y}_p\mathbf{X}k)\mathbf{Z} + (\mathbf{Y}_pk)\nabla_{\mathbf{X}_p}\mathbf{Z} + (\mathbf{X}_pk)\nabla_{\mathbf{Y}_p}\mathbf{Z} + k(p)\nabla_{\mathbf{Y}_p}\nabla_{\mathbf{X}}\mathbf{Z},$$
$$\nabla_{[\mathbf{X},\mathbf{Y}]_p}(k\mathbf{Z}) = (\mathbf{Y}_p\mathbf{X}k)\mathbf{Z} - (\mathbf{X}_p\mathbf{Y}k)\mathbf{Z} + k(p)\nabla_{[\mathbf{X},\mathbf{Y}]_p}\mathbf{Z}.$$

Thus, we can cancel terms and get

$$\mathbf{R}_p(\mathbf{X},\mathbf{Y})(k\mathbf{Z}) \equiv \nabla_{\mathbf{X}_p}\nabla_{\mathbf{Y}}(k\mathbf{Z}) - \nabla_{\mathbf{Y}_p}\nabla_{\mathbf{X}}(k\mathbf{Z}) - \nabla_{[\mathbf{X},\mathbf{Y}]_p}(k\mathbf{Z}) =$$
$$= k(p)\nabla_{\mathbf{X}_p}\nabla_{\mathbf{Y}}\mathbf{Z} - k(p)\nabla_{\mathbf{Y}_p}\nabla_{\mathbf{X}}\mathbf{Z} - k(p)\nabla_{[\mathbf{X},\mathbf{Y}]_p}\mathbf{Z} = k(p)\mathbf{R}_p(\mathbf{X},\mathbf{Y})\mathbf{Z} = \mathbf{R}_p(\mathbf{X},\mathbf{Y})\mathbf{Z}.$$

Thus, we have established that $\mathbf{R}_p$ depends on $\mathbf{Z}_p$ and not on the rest of the field $\mathbf{Z}$. Now we look at whether it depends on $\mathbf{Y}_p$. We calculate

$$\nabla_{\mathbf{X}_p}\nabla_{k\mathbf{Y}}\mathbf{Z} - \nabla_{k\mathbf{Y}_p}\nabla_{\mathbf{X}}\mathbf{Z} = (\mathbf{X}_pk)\nabla_{\mathbf{Y}}\mathbf{Z} + k(p)\nabla_{\mathbf{X}_p}\nabla_{\mathbf{Y}}\mathbf{Z} - k(p)\nabla_{\mathbf{Y}_p}\nabla_{\mathbf{X}}\mathbf{Z},$$
$$\nabla_{[\mathbf{X},k\mathbf{Y}]_p}\mathbf{Z} = \nabla_{\mathbf{X}_p(k\mathbf{Y})}\mathbf{Z} - \nabla_{k\mathbf{Y}_p\mathbf{X}}\mathbf{Z} = \nabla_{(\mathbf{X}_pk)\mathbf{Y}_p}\mathbf{Z} + \nabla_{k(p)\mathbf{X}_p\mathbf{Y}}\mathbf{Z} - k(p)\nabla_{\mathbf{Y}_p\mathbf{X}}\mathbf{Z} = (\mathbf{X}_pk)\nabla_{\mathbf{Y}_p}\mathbf{Z} + k(p)\left[\nabla_{\mathbf{X}_p\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}_p\mathbf{X}}\mathbf{Z}\right] \quad .$$

Thus, $\mathbf{R}_p(\mathbf{X},k\mathbf{Y})\mathbf{Z} = \nabla_{\mathbf{X}_p}\nabla_{k\mathbf{Y}}\mathbf{Z} - \nabla_{k\mathbf{Y}_p}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},k\mathbf{Y}]_p}\mathbf{Z} = k(p)\mathbf{R}_p(\mathbf{X},\mathbf{Y})\mathbf{Z}$, and by symmetry we have that $\mathbf{R}_p$ depends only on $\mathbf{Y}_p$ and on $\mathbf{X}_p$.

d.
We calculate $K(\mathbf{X}\wedge\mathbf{Y}) = \mathbf{R}_p(\mathbf{X},\mathbf{Y},\mathbf{X},\mathbf{Y}) = \mathbf{R}_p(\Sigma_i X^i\mathbf{x}_i, \Sigma_j Y^j\mathbf{y}_j, \Sigma_k X^k\mathbf{x}_k, \Sigma_h Y^h\mathbf{y}_h) = \sum_i\sum_j\sum_k\sum_h R_{ijkh}X^iY^jX^kY^h$.

Problem 8.7. Intrinsic Calculations in Examples

a. *the cylinder,*

In geodesic rectangular coordinates $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Gamma^k_{ij} = 0$, $R^h_{ijk} = 0 = R_{ijkh}$, $K(\mathbf{x}_1 \wedge \mathbf{x}_2) = 0$.

We also know all of these because the cylinder is locally isometric to the plane.

b. *the sphere,*

In geodesic rectangular coordinates

$$(g_{ij}(\theta,\phi)) = \begin{pmatrix} \sin^2(\phi/r) & 0 \\ 0 & 1 \end{pmatrix},$$
$$\Gamma^1_{11} = \tfrac{h_1}{h} = 0,\ \Gamma^2_{11} = -hh_2 = \tfrac{-\sin(\phi/r)\cos(\phi/r)}{r},\ \Gamma^1_{12} = \Gamma^1_{21} = \tfrac{h_2}{h} = \tfrac{\cot(\phi/r)}{r},\ \text{others zero}$$
$$R^1_{212} = K = \tfrac{1}{r^2},\ R^2_{121} = h^2K = \tfrac{\sin^2(\phi/r)}{r^2},\ R_{2121} = R^1_{212}g_{11} = \tfrac{\sin^2(\phi/r)}{r^2} = R^2_{121}g_{22} = R_{1212}$$
$$K(\mathbf{x}_1 \wedge \mathbf{x}_2) = \tfrac{R_{1212}}{g_{11}g_{22}} = \tfrac{R_{2121}}{g_{11}g_{22}} = \tfrac{1}{r^2} \quad .$$

c. *the torus* ($\mathbf{S}^1 \times \mathbf{S}^1$) *in* $\mathbf{R}^4$ *with coordinates* $\mathbf{x}(u^1,u^2) = (\cos u^1, \sin u^1, \cos u^2, \sin u^2)$.

We compute

$$\mathbf{x}_1(u^1,u^2) = (-\sin u^1, \cos u^1, 0, 0),\ \ \mathbf{x}_2(u^1,u^2) = (0, 0, -\sin u^2, \cos u^2)$$
$$g_{11} = g_{22} = 1,\ g_{12} = g_{21} = 0,\ \Gamma^k_{ij} = 0,\ R^h_{ijk} = 0 = R_{ijkh},\ K(\mathbf{x}_1 \wedge \mathbf{x}_2) = 0.$$

The name *flat torus* is appropriate because it is locally isometric to the plane.

d. *the annular hyperbolic plane with respect to its natural geodesic rectangular coordinate system.* (See Problem **1.8**.)

In rectangular geodesic coordinates

$$(g_{ij}(x,y)) = \begin{pmatrix} \exp^2(-y/r) & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Gamma^1_{11} = \tfrac{h_1}{h} = 0,\ \Gamma^2_{11} = -hh_2 = \tfrac{\exp^2(-y/r)}{r},\ \Gamma^1_{12} = \Gamma^1_{21} = \tfrac{h_2}{h} = \tfrac{-1}{r},\ \text{others zero}$$

$$R^1_{212} = K = \tfrac{-1}{r^2},\ R^2_{121} = h^2 K = \tfrac{-\exp^2(-y/r)}{r^2},\ R_{2121} = R^1_{212} g_{11} = \tfrac{-\exp^2(-y/r)}{r^2} = R^2_{121} g_{22} = R_{1212}$$

$$K(\mathbf{x}_1 \wedge \mathbf{x}_2) = \tfrac{R_{1212}}{g_{11}g_{22}} = \tfrac{R_{2121}}{g_{11}g_{22}} = \tfrac{-1}{r^2} \quad .$$

e. *the 3-manifold* $\mathbf{S}^2\times\mathbf{R} \subset \mathbf{R}^4$, *that is the set of those points* $\{(x,y,z,w)\in \mathbf{R}^4 \mid (x,y,z) \in \mathbf{S}^2 \subset \mathbf{R}^3\}$.
In local coordinates, $\mathbf{x}(\theta,\phi,w) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi, w)$, we compute

$$\mathbf{x}_1(\theta,\phi,w) = (-\sin\theta\sin\phi, \cos\theta, \phi, 0, 0),\ \mathbf{x}_2(\theta,\phi,w) = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi, 0),\ \mathbf{x}_3 = (0,0,0,1)$$

$$(g_{ij}) = \begin{pmatrix} \sin^2\phi & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma^k_{ij} = \tfrac{1}{2}\sum_l g^{kl}\left[\mathbf{x}_j g_{il} - \mathbf{x}_l g_{ij} + \mathbf{x}_i g_{lj}\right] = \tfrac{1}{2} g^{kk}\left[\mathbf{x}_j g_{ik} - \mathbf{x}_k g_{ij} + \mathbf{x}_i g_{kj}\right] \quad ,$$

and thus $\Gamma^k_{ij} = 0$ unless two of i, j, k are equal to 1. We compute

$$\Gamma^1_{11} = \tfrac{\mathbf{x}_1 g_{11}}{2\sin^2\phi} = 0,\ \Gamma^2_{11} = \tfrac{\mathbf{x}_1 g_{12} - \mathbf{x}_2 g_{11} + \mathbf{x}_1 g_{21}}{2} = -\sin\phi\cos\phi,\ \Gamma^3_{11} = \tfrac{\mathbf{x}_1 g_{13} - \mathbf{x}_3 g_{11} + \mathbf{x}_1 g_{31}}{2} = 0$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \tfrac{\mathbf{x}_2 g_{11} - \mathbf{x}_1 g_{12} + \mathbf{x}_1 g_{12}}{2\sin^2\phi} = \cot\phi,\ \Gamma^1_{13} = \Gamma^1_{31} = \tfrac{\mathbf{x}_3 g_{11} - \mathbf{x}_1 g_{13} + \mathbf{x}_1 g_{13}}{2\sin^2\phi} = 0.$$

Then we compute, using

$$R^h_{ijk} = \mathbf{x}_j \Gamma^h_{ik} + \textstyle\sum_l \Gamma^l_{ik}\Gamma^h_{jl} - \mathbf{x}_i \Gamma^h_{jk} - \sum_l \Gamma^l_{jk}\Gamma^h_{il},$$

that

$$R^1_{212} = \mathbf{x}_1\Gamma^1_{22} + \textstyle\sum_l \Gamma^l_{22}\Gamma^1_{1l} - \mathbf{x}_2\Gamma^1_{12} - \sum_l \Gamma^l_{12}\Gamma^1_{2l} = 0 + 0 - \mathbf{x}_2\cot\phi - \cot^2\phi =$$

$$= -\mathbf{x}_2\cot\phi - \cot^2\phi = \tfrac{1}{\sin^2\phi} - \tfrac{\cos^2\phi}{\sin^2\phi} = 1 \quad ,$$

$$R^2_{121} = \mathbf{x}_2\Gamma^2_{11} + \textstyle\sum_l \Gamma^l_{11}\Gamma^2_{2l} - \mathbf{x}_1\Gamma^2_{21} - \sum_l \Gamma^l_{21}\Gamma^2_{1l} = \mathbf{x}_2(-\sin\phi\cos\phi) + 0 - 0 - \cot\phi(-\sin\phi\cos\phi) =$$

$$= (-\cos\phi\cos\phi + \sin\phi\sin\phi) + \cos^2\phi = \sin^2\phi,$$

and thus

$$R_{2121} = R^1_{212} g_{11} = \sin^2\phi = R^2_{121} g_{22} = R_{1212}$$

$$K(\mathbf{x}_1 \wedge \mathbf{x}_2) = \tfrac{R_{1212}}{g_{11}g_{22}} = \tfrac{R_{2121}}{g_{11}g_{22}} = 1.$$

Then we compute

$$R^1_{313} = \mathbf{x}_1\Gamma^1_{33} + \textstyle\sum_l \Gamma^l_{33}\Gamma^1_{1l} - \mathbf{x}_3\Gamma^1_{13} - \sum_l \Gamma^l_{13}\Gamma^1_{3l} = 0 + 0 - 0 - 0 = 0$$

$$R^3_{131} = \mathbf{x}_3\Gamma^3_{11} + \textstyle\sum_l \Gamma^l_{11}\Gamma^3_{3l} - \mathbf{x}_1\Gamma^3_{31} - \sum_l \Gamma^l_{31}\Gamma^3_{1l} = 0 + 0 - 0 - 0 = 0$$

$$R_{3131} = R_{1313} = K(\mathbf{x}_1 \wedge \mathbf{x}_3) = 0 \quad ,$$

and

$$R^2_{323} = \mathbf{x}_2\Gamma^2_{33} + \textstyle\sum_l \Gamma^l_{33}\Gamma^2_{2l} - \mathbf{x}_3\Gamma^2_{23} - \sum_l \Gamma^l_{23}\Gamma^2_{3l} = 0 + 0 - 0 - 0 = 0$$

$$R^3_{232} = \mathbf{x}_3\Gamma^3_{22} + \textstyle\sum_l \Gamma^l_{22}\Gamma^3_{3l} - \mathbf{x}_2\Gamma^3_{32} - \sum_l \Gamma^l_{32}\Gamma^3_{2l} = 0 + 0 - 0 - 0 = 0$$

$$R_{3232} = R_{2323} = K(\mathbf{x}_2 \wedge \mathbf{x}_3) = 0 \quad .$$

This makes sense since the surface in the $\mathbf{x}_1$,$\mathbf{x}_2$ directions is a sphere of radius one and the surfaces in the $\mathbf{x}_1$,$\mathbf{x}_3$ and $\mathbf{x}_2$,$\mathbf{x}_3$ directions are cylinders, and thus have curvature zero.