

SOLUTIONS

Chapter 5

Area, Parallel Transport and Intrinsic Curvature

PROBLEM 5.1. The Area of a Triangle on a Sphere

a.

Consider $\triangle ABC$, a triangle that does not self-intersect and that has no collinear vertices. Let A' , B' and C' be the points on the sphere opposite A , B and C , respectively. We can see that the corresponding angles of these triangles are congruent by using the Vertical Angle Theorem twice, and by the fact that a lune has congruent angles. We can conclude that $\triangle ABC \cong \triangle A'B'C'$, since the triangles satisfy the conditions of AAA, or because the sphere has central symmetry.

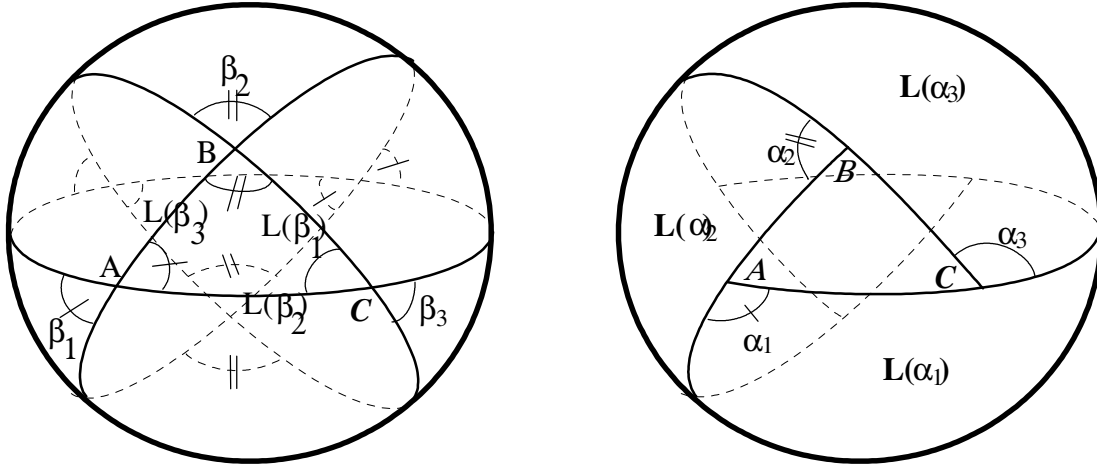


Figure 5.A. Lunes and triangles.

Furthermore, given that $\triangle ABC$ can be seen as the intersection of three lunes namely, $L(\beta_1)$, $L(\beta_2)$, $L(\beta_3)$, $\triangle A'B'C'$ can also be seen as the intersection of the opposite lunes. (See Figure 5.A.) Note that the sphere is covered by these six lunes, which are disjoint except for the fact that both $\triangle ABC$ and $\triangle A'B'C'$ are each covered three times. Now, if we let $A(\Delta)$ denote the area of the congruent triangles, then the whole area of the sphere A can be expressed in the following way: $A = 2A(L(\beta_1)) + 2A(L(\beta_2)) + 2A(L(\beta_3)) - 4A(\Delta)$. Since the area of a lune is proportional to the area of the whole sphere in direct relation to its angle, we know: $A(L(\theta)) = (\theta/2\pi) A$. Then, combining these equations, we get the following intrinsic expression: $A(\Delta) = [\sum \beta_i - \pi] (A/4\pi)$.

Now, focusing on the exterior angles (see Figure 5.B), we can also express the sphere as the union of three lunes (with the α_i as angles) plus the two triangles. Thus, we get $A = A(L(\alpha_1)) + A(L(\alpha_2)) + A(L(\alpha_3)) + 2A(\Delta)$.

$A(L(\alpha_i)) + 2A(\Delta)$, and thus, $A(\Delta) = A/4\pi[2\pi - \sum \alpha_i]$. We know that the area of the whole sphere is $4\pi R^2$, where R is the (extrinsic) radius of the sphere. With this additional information we can rewrite the formula of Problem 5.1.a: $\text{Area}(\Delta) = [\sum \beta_i - \pi] R^2 = [2\pi - \sum \alpha_i] R^2$.

b.

The triangle in the plane is determined by three points. Imagine a large sphere resting on these three points. Then the three points determine a geodesic triangle of the sphere. Now keeping the three points fixed let the radius of the sphere goes to infinity, then the area and angles of the geodesic triangle approaches the area and angles of the original plane triangle. For a small triangle on a large sphere, $\text{Area}(\Delta)/R^2$ is very small and thus $\sum \beta_i$ is very close to π and $\sum \alpha_i$ is very close to 2π . Since the $\text{Area}(\Delta)$ is finite and approaches the area of the plane triangle for a planar triangle, we can conclude that for the plane triangle $\sum \beta_i$ is equal to π and $\sum \alpha_i$ is equal to 2π .

c.

On the plane the sum of the interior angles of a triangle is always equal to π , but on the sphere the sum of the interior angles of a triangle is always greater than π .

PROBLEM 5.2. Dissection of Polygons into Triangles

In the literature there are many incorrect descriptions of dissections of polygons, several by well-known mathematicians. For a discussion of these errors see the article: Chung-Wu Ho, Decomposition of a Polygon into Triangles, *Mathematical Gazette*, vol. 60 (1976), 132-134. You may want to point this out to the students at an appropriate time.

Dissecting convex polygons: A polygon is **convex** if any two points on it can be joined by a geodesic segment lying entirely in its interior. To dissect a convex polygon, pick any vertex and join it to all the others that are not adjacent. This will dissect the convex polygon into triangles. On a sphere if these triangles are not small then dissect them further.

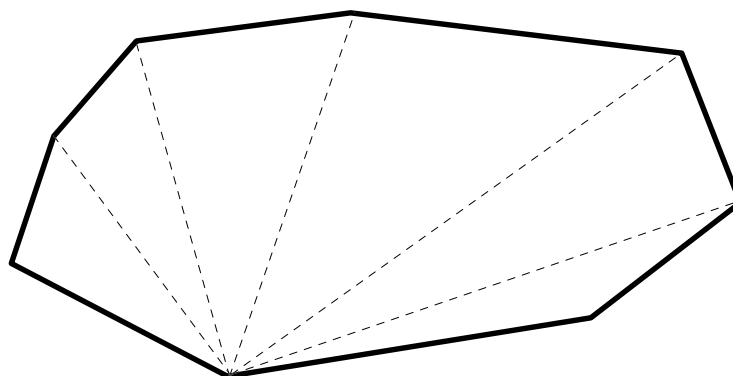


Figure 5.B. Dissecting convex polygons.

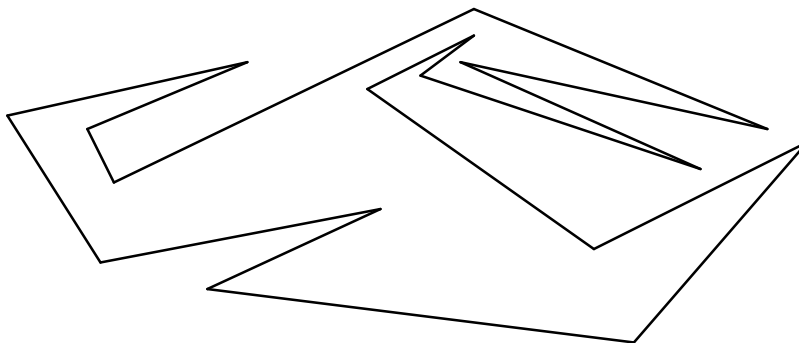


Figure 5.C. Concave polygon.

Dissecting concave vertices: If the polygon is not convex, then it has at least one concave vertex. We say that a vertex is **concave** if its (interior) angle is greater than a straight angle. To dissect a polygon that has concave vertices, at each concave vertex cut along a segment which divides the interior angle into two convex (less than or equal to a straight angle) angles. Now the polygon will be dissected into a finite number of convex polygons, and then each of the convex polygons can be dissected into (small) triangles.

Parallel cuts: Pick any line (great circle) l . For each vertex of the polygon draw a line (great circle) through the vertex and perpendicular to l . If you now cut along the intersection of these lines with the interior of the polygon, the polygon will be dissected into triangles and quadrilaterals, each of which can be dissected into two triangles.

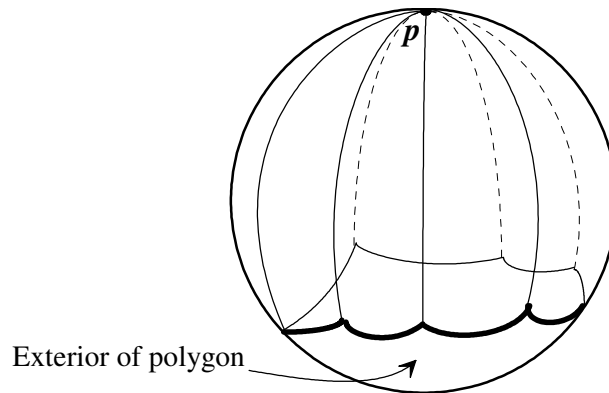


Figure 5.D. Large polygon on sphere.

Dissecting into triangles without adding any new vertices: On a sphere, if there are no convex vertices then the exterior of the polygon is convex. In this case, pick a point p that is the opposite pole of some point in the exterior of the polygon and cut along all the short great circle segments joining p to the vertices of the polygon. See Figure 5.D.

If there is a convex vertex v , then let l be a geodesic joining the two adjacent vertices. In the diagram below, v' and v'' are adjacent to v . If the segment of l between the two vertices lies totally inside the polygon, then make a cut along it. If not, then parallel transport l toward v along vv' until the portion of l within $\Delta vv'v''$ intersects only interior portions of the area of the polygon except for vertices, such as w , which lies on the polygon's perimeter. Now, l must contain a vertex w of the polygon (lying in the interior of $\Delta vv'v''$). Cut along the segment vw . See Figure 5.E.

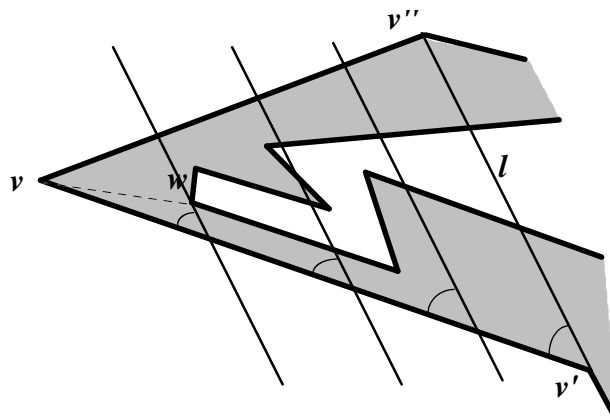


Figure 5.E. Finding vertex to join to v .

In both cases, we have cut the polygon into two polygons each with fewer vertices. Continue until only triangles are left.

You can check that this method dissects the polygon into $n-2$ triangles where n is the number of sides of the polygon. Then, one can use Problem 5.2 to show that:

On the plane, the sum of the angles of a polygon with n sides is $(n-2)\pi$.

PROBLEM 5.3. Gauss-Bonnet for Polygons on a Sphere

Divide the polygon into small triangles, Δ_i . It is possible to do this by constructing geodesic spanning segments in the interior of the polygon (see Problem 5.2). Now we proceed in two steps:

First, check directly that $\mathcal{H}(\Gamma) = 2\pi - \sum \alpha_i$ by parallel transporting a vector tangent to the initial edge of Γ (as depicted in Figure 5.2 of the text, in the case Γ is a triangle, and then extending to arbitrary polygons as in Figure 5.5). Where at each vertex the amount of the exterior angle is added to the angle between the transported vector and the tangent vector of the polygon.

Second, we will show that $\mathcal{H}(\Gamma) = \sum_i \mathcal{H}(\Delta_i)$ and thus $\mathcal{H}(\Gamma) = \sum_i \mathcal{H}(\Delta_i) = \sum_i A(\Delta_i)4\pi/A = A(\Gamma)4\pi/A$. If there is only one triangle then it follows from Problem 5.1. Consider one of the geodesic spanning segments, γ , constructing using Problem 5.2. This segment γ separates the interior of Γ into two pieces bounded by polygons Γ_1 and Γ_2 , where the edges of Γ_i are γ plus some of the edges of Γ . (See Figure 5.F.)

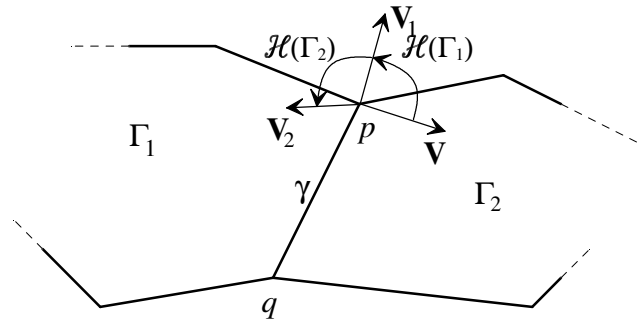


Figure 5.F. Holonomy adds.

Now, we determine the holonomies by parallel transporting a vector around the polygons. Remember that it does not matter which vertex or which vector we start with. First we parallel transport a vector \mathbf{V} around Γ_1 in a counterclockwise direction starting at p , one of the endpoints of γ as indicated in the picture. Call the result of this parallel transport, \mathbf{V}_1 . Next parallel transport \mathbf{V}_1 counterclockwise around Γ_2 starting again at p . Call the resulting vector \mathbf{V}_2 . Now, if we parallel transport \mathbf{V} counterclockwise around Γ except at q (the opposite end of γ from p) we detour along γ to p and then immediately back along γ to q before continue to parallel transport around Γ . Clearly, this detour does not affect the final parallel transport and, thus, we conclude that $\mathcal{H}(\Gamma)$ is also the parallel transport of \mathbf{V} around Γ . Thus, we conclude that $\mathcal{H}(\Gamma) = \mathcal{H}(\Gamma_1) + \mathcal{H}(\Gamma_2)$. Proceed likewise with the other geodesic spanning segments until we have divided the polygon into triangles and there are no geodesic spanning segments left. We will then have the conclusion that $\mathcal{H}(\Gamma) = \sum_i \mathcal{H}(\Delta_i)$.

PROBLEM 5.4. Parallel Fields and Intrinsic Curvature

a.

i. \Rightarrow ii. Note that the derivative $\frac{d}{ds}\mathbf{V}(s)$ is the same as the directional derivative

$$(\gamma'(s))\mathbf{V} = \lim_{h \rightarrow 0} \frac{1}{h} [\mathbf{V}(s+h) - \mathbf{V}(s)].$$

At the corners of a piecewise geodesic curve use one sided derivatives:

$$\frac{d}{ds+} \mathbf{V}(s) = \lim_{h \rightarrow 0+} \frac{1}{h} [\mathbf{V}(s+h) - \mathbf{V}(s)] \quad \text{and} \quad \frac{d}{ds-} \mathbf{V}(s) = \lim_{h \rightarrow 0-} \frac{1}{h} [\mathbf{V}(s+h) - \mathbf{V}(s)] .$$

On a geodesic the angle between the transported vector and the velocity vector is constant.

If $\mathbf{V}(s)$ is parallel to $\gamma'(s)$ then their derivatives are parallel. But the derivative of $\gamma'(s)$ with respect to arclength is exactly the (extrinsic) curvature vector, which is perpendicular to tangent plane at $\gamma(s)$ because γ is a geodesic.

If $\mathbf{V}(s)$ is not parallel to $\gamma'(s)$, then $\langle \mathbf{V}(s), \gamma'(s) \rangle$ is a constant along each geodesics segment and therefore, $0 = \frac{d}{dt} \langle \mathbf{V}(s), \gamma'(s) \rangle = \left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle + \left\langle \mathbf{V}(s), \frac{d}{ds} \gamma'(s) \right\rangle = \left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle + 0$ because $\mathbf{V}(s)$, being in the tangent space, is perpendicular to the curvature vector $\frac{d}{ds} \gamma'(s)$. Thus $\frac{d}{ds} \mathbf{V}(s)$ is perpendicular to $\gamma'(s)$, and it is also perpendicular to $\mathbf{V}(s)$ since the length of $\mathbf{V}(s)$ is constant. But then $\frac{d}{ds} \mathbf{V}(s)$ is perpendicular to the tangent space $T_{\gamma(s)}M$ which is the span of $\mathbf{V}(s)$ and $\gamma'(s)$.

ii. \Rightarrow i. If $\frac{d}{ds} \mathbf{V}(s)$ is perpendicular to the tangent space $T_{\gamma(s)}M$, then along each geodesic segment

$$\frac{d}{ds} \langle \mathbf{V}(s), \gamma'(s) \rangle = \left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle + \left\langle \mathbf{V}(s), \frac{d}{ds} \gamma'(s) \right\rangle = 0,$$

and, thus, there is a constant angle between $\mathbf{V}(s)$ and $\gamma'(s)$.

b.

First we differentiate:

$$\begin{aligned} \frac{d}{ds} \langle \mathbf{V}(s), \gamma'(s) \rangle &= \left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle + \left\langle \mathbf{V}(s), \gamma''(s) \right\rangle = |\mathbf{V}(s)| \frac{d}{ds} \cos \theta = |\mathbf{V}(s)| (-\sin \theta) \frac{d}{ds} \theta(s) \\ |\mathbf{V}(s)| (-\sin \theta) \frac{d}{ds} \theta(s) &= \left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle + \left\langle \mathbf{V}(s), \gamma''(s) \right\rangle = \left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle + \left\langle \mathbf{V}(s), \kappa_g(s) \right\rangle, \end{aligned}$$

since $\gamma''(s) = \kappa(s) = \kappa_g(s) + \kappa_n(s)$ and $\langle \mathbf{V}(s), \kappa_n(s) \rangle = 0$. From Figure 5.7 in the text, we see that $\langle \mathbf{V}(s), \kappa_g(s) \rangle = \pm(-\sin \theta) |\kappa_g|$, positive if γ is turning counterclockwise.

i. \Rightarrow ii. If $\frac{d}{ds} \mathbf{V}(s)$ is perpendicular to the tangent plane at $\gamma(s)$, then $\left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle = 0$ and, thus, $\frac{d}{ds} \theta(s) = \pm |\kappa_g(s)|$, positive if γ is turning counterclockwise. Also $\frac{d}{ds} \theta$ is positive if γ is turning counterclockwise and, thus, $\frac{d}{ds} \theta(s) = \kappa_g(s)$.

ii. \Rightarrow i. If $\frac{d}{ds} \theta(s) = \kappa_g(s)$, then $\left\langle \frac{d}{ds} \mathbf{V}(s), \gamma'(s) \right\rangle = 0$. But also, since the magnitude of $\mathbf{V}(s)$ is constant, $\frac{d}{ds} \mathbf{V}(s)$ must be perpendicular to $\mathbf{V}(s)$. If $\mathbf{V}(s)$ and $\gamma'(s)$ are *not* parallel, then they span the tangent plane and, thus, it follows that $\frac{d}{ds} \mathbf{V}(s)$ must be perpendicular to that tangent plane. If $\mathbf{V}(s)$ and $\gamma'(s)$ are parallel in an interval, then $0 = \frac{d}{ds} \theta(s) = \kappa_g(s)$ and in that interval γ is a geodesic and $\frac{d}{ds} \mathbf{V}(s) = \gamma''(s) = \kappa(s)$ which is perpendicular to the tangent plane. Because $\frac{d}{ds} \mathbf{V}(s)$ is continuous, we conclude that it must always be perpendicular to the tangent plane at $\gamma(s)$.

c.

Let $\mathbf{V}(s) \equiv \lim \mathbf{V}_i(s)$. Then, by Part **a**, each $\mathbf{V}_i'(s)$ is perpendicular to the tangent plane at $\gamma_i(s)$ and, thus, (since the tangent planes vary continuously) $\frac{d}{ds} \mathbf{V}(s)$ is perpendicular to the tangent plane \mathbf{T}_p . If $\mathbf{W}(s)$ is the parallel transport of \mathbf{V} along γ then, from Part **b**, we know that both $\mathbf{W}(s)$ and $\mathbf{V}(s)$ must have the same angle with $\gamma'(s)$ and thus they are equal.

d.

i. If γ is geodesic, then $0 = \kappa_g(s) = \frac{d}{ds} \theta(s)$ and, thus, if we parallel transport $\gamma'(a)$ along γ , then at every point the transported vector will remain $\gamma'(s)$ and, thus, $\gamma'(s)$ is a parallel vector field along γ . Conversely, if $\gamma'(s)$ is a parallel vector field along γ , then $0 = \frac{d}{ds} \theta(s) = \kappa_g(s)$ and, thus, γ is a geodesic.

- ii. Start with a tangent vector V at p on γ and parallel transport V around γ . Along the smooth segments the angle $\theta(s)$ between $\gamma'(s)$ and $V(s)$ will change by $\int \kappa_g(s) ds$, where the integral is along the smooth segment. At the vertices (junctures between successive geodesic segments) the angle between $\gamma'(s)$ and $V(s)$ will change by an amount equal to the exterior angle at that vertex. This leads to the formula.

We can now extend the notion of holonomy and the Gauss-Bonnet formula (Problem 5.3) on the sphere to piecewise smooth curves.

PROBLEM 5.5. Holonomy on Surfaces

a.

Since such geodesic triangle can be flattened into plane triangles, their holonomy's must be zero.

b.

Since there are no exterior angles the holonomy is the integral of the geodesic curvature which is $1/r$, where r is the intrinsic radius of the circle. Thus, using 5.4.d.ii, $\mathcal{H}(\gamma) = 2\pi - \int \kappa_g(s) ds = 2\pi - (1/r)(r\alpha) = 2\pi - \alpha$. When $\alpha < 2\pi$ then the holonomy is positive and when $\alpha > 2\pi$ then the holonomy is negative. When $\alpha = 2\pi$, then the holonomy is equal to zero as expected because then the cone is the plane.

c.

The curves α , δ , and γ are extrinsic circles with their (extrinsic) curvature vectors pointing perpendicular to the surface, thus, they are geodesics and therefore the only contribution to the holonomy is from β . The extrinsic curvature of β is $1/r$ and, thus, so is its geodesic curvature. Therefore the integral of the geodesic curvature along half of β is $(1/r)(\pi r) = \pi$. In the case of the region B this is negative (because β is turning clockwise when we go counterclockwise around B) and for the region A is positive (because β is turning counterclockwise when we go around A counterclockwise). All four vertices have exterior angles $\pi/2$. Thus, $\mathcal{H}(A) = 2\pi - [4(\pi/2) + \pi] = -\pi$, and $\mathcal{H}(B) = 2\pi - [4(\pi/2) - \pi] = \pi$.

d.

Note that the straight horizontal coordinate curves are geodesics, and that the helical coordinate curves have their extrinsic curvatures equal to their intrinsic curvatures, since their extrinsic curvatures are parallel to the surface. Since the four exterior angles are each $\pi/2$, the holonomy of this region on the strake is equal to the negative of the integral of the geodesic (and, thus, extrinsic) curvature. From Problem 2.5.b, we have that the curvature of the helix is $\frac{4\pi^2 r}{h^2 + (2\pi r)^2}$, and the arclength s for a change in θ of Δ is given by $\Delta = \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}}$ or $s = \frac{\Delta}{2\pi} \sqrt{h^2 + (2\pi r)^2}$. In going counterclockwise around the region the outer helix is curving counterclockwise and, thus, its κ_g is positive and, thus, it contributes negatively to the holonomy; the reverse situation happens on the inner helix, thus, for this region on the strake the holonomy is

$$2\pi\Delta \left[\frac{\frac{4\pi^2 r}{h^2 + (2\pi r)^2} \frac{\Delta}{2\pi} \sqrt{h^2 + (2\pi r)^2}}{\sqrt{h^2 + (2\pi r)^2}} - \frac{\frac{4\pi^2 (r+\Delta)}{h^2 + (2\pi(r+\Delta))^2} \frac{\Delta}{2\pi} \sqrt{h^2 + (2\pi(r+\Delta))^2}}{\sqrt{h^2 + (2\pi(r+\Delta))^2}} \right] = 2\pi\Delta \left[\frac{1}{\sqrt{(h/r)^2 + (2\pi)^2}} - \frac{1}{\sqrt{(h/(r+\Delta))^2 + (2\pi)^2}} \right] < 0.$$

Thus, the strake has negative curvature and cannot possibly be locally isomorphic to the plane.

PROBLEM 5.6. Holonomy Explains Foucault's Pendulum

a.

The only significant force acting on the pendulum is perpendicular to the surface and, thus, any effect of this force to change the direction of the swing of the pendulum will be perpendicular to the surface. Thus, the derivative of the pendulum's direction is perpendicular to the surface and the swing direction of the pendulum is a parallel vector field. Therefore the counterclockwise angle between the

starting and ending positions of the swing plane after 24 hours is the holonomy of the latitude circle, but the observed rotation (relative to the latitude circle) is the holonomy minus 2π .

b.

The holonomy is $2\pi - (\text{circumference})(\text{intrinsic curvature}) = 2\pi - (2\pi R \cos \phi)/(R \cot \phi) = 2\pi(1 - \sin \phi)$, and, thus, the observed rotation (relative to the latitude circle) is $-2\pi \sin \phi$.

c.

Since the Pendulum rotates (with respect to the latitude circle) $-2\pi \sin \phi$ radians in 24 hours, the period is $\frac{2\pi}{2\pi \sin \phi} 24 = \frac{24}{\sin \phi}$ hours.

d.

At the North Pole the period will be 24 hours and the pendulum will act exactly like the hand of a 24 hour clock. At the Equator, the swing plane of the pendulum will not change relative to the equator.

e.

Since $\mathcal{H}(\gamma) = A(\gamma)/R^2$, the area on the earth above the latitude ϕ is $(6360)^2(2\pi)(1 - \sin \phi)$.

PROBLEM 5.7. Intrinsic Curvature of a Surface

a.

The intrinsic curvature of the cylinder is zero, because the holonomy is always zero.

On the cone the intrinsic curvature is also zero for the same reason except at the cone point. At the cone point we can calculate the intrinsic curvature using circles with centers at the cone point. From Problem 5.5.b the holonomy of these circles is $2\pi - \int_0^\alpha (1/r)r d\theta = 2\pi - \alpha$, where α is the cone angle. The area of these circles is $(\alpha/2\pi)(\pi r^2)$. Thus, the intrinsic curvature is $\lim_{r \rightarrow 0} \frac{(2\pi - \alpha)}{\frac{\alpha}{2\pi}(\pi r^2)} = \pm\infty$, unless $\alpha = 2\pi$ in which case we have the plane with curvature zero.

b.

We use the results of Problems 5.6.b and 4.5.b (noting carefully that ϕ in Problem 5.6 measures from the Equator and ϕ in Problem 4.5 measures from the North Pole) and measure ϕ from the North Pole. Then the intrinsic curvature of the sphere is $\lim_{\phi \rightarrow \pi/2} \frac{2\pi(1 - \cos \phi)}{R^2 2\pi(1 - \cos \phi)} = \frac{1}{R^2}$.

***c.**

Use Problems 4.5.d and 5.5.d, remembering that $k = h/(2\pi)$. Instead of the closed form expression for the area we will leave it (at least partially) in its integral form and then evaluate the limit by using L'Hôpital's Rule.

$$R(x(\theta, r)) = \lim_{\Delta \rightarrow 0} \frac{2\pi\Delta \left[\frac{1}{\sqrt{(h/r)^2 + (2\pi)^2}} - \frac{1}{\sqrt{(h/(r+\Delta))^2 + (2\pi)^2}} \right]}{\left(\int_\theta^{\theta+\Delta} d\theta \right) \left(\int_r^{r+\Delta} \frac{dr}{\sqrt{r^2 + k^2}} \right)} = \lim_{\Delta \rightarrow 0} \frac{\frac{1}{\sqrt{(h/2\pi r)^2 + 1}} - \frac{1}{\sqrt{(h/(2\pi(r+\Delta))^2 + 1}}}{\int_r^{r+\Delta} \frac{dr}{\sqrt{r^2 + k^2}}}.$$

Now apply L'Hôpital's Rule to get

$$R(p) = \lim_{\Delta \rightarrow 0} \frac{1}{\sqrt{(r+\Delta)^2 + k^2}} \left(-\left(\frac{-1}{2}\right) \frac{k^2(-2)(r+\Delta)^{-3}}{[(k^2)(r^2+\Delta)^{-2} + 1]^{3/2}} \right) = \frac{1}{\sqrt{r^2 + k^2}} \left(\left(\frac{1}{2}\right) \frac{k^2(-2)(r)^{-3}}{((k^2)(r)^{-2} + 1)^{3/2}} \right) = \frac{-k^2(1/r)}{(r^2 + k^2)^2} < 0.$$

d.

Note that since the surface is constructed the same everywhere (as $\delta \rightarrow 0$) it is **homogeneous** (that is, intrinsically and geometrically every point has a neighborhood isometric to a neighborhood of any other point). Thus, the intrinsic curvature is constant. There are (at least) two solution methods:

1. Using 4.5e and 5.4d. See Figure 5.10 in text. By 1.8c the annular curve marked '?' has length $c \exp(-d/r)$ thus (by 4.5e) the area is $A = cr - [c \exp(-d/r)]r$. The geodesic curvature κ_g of the annular

curves is $1/r$ because the extrinsic curvature is tangent to the surface. Then, by **5.4d**, the holonomy is $\mathcal{H} = (1/r)[c - (c \exp(-d/r))]$. We conclude that the intrinsic curvature is $\mathcal{H}/A = -1/r^2$.

2. Using annular definition. Pick a region that crosses the circular edge between two strips and then let $\delta \rightarrow 0$. See the Figure 5.11 in the text. Note that the inner and outer bounding arcs on this region both have radii equal to $r + (\delta/2)$. In calculating the holonomy the exterior angles add up to 2π and, thus, the holonomy is determined by the geodesic curvatures on the two bounding arcs. But the bottom arc is shorter and contributes positively to the holonomy and the upper arc is longer and contributes negatively to the holonomy. Therefore the holonomy (and intrinsic curvature) is negative.

Note that $\frac{m}{l} = \frac{r+\delta}{r+\delta/2}$. Then, the area of the region is the sum of the areas of two annular sectors:

$$\begin{aligned} A &= \frac{l}{2\pi(r+\delta/2)} [\pi(r+\delta)^2 - \pi(r+\delta/2)^2] + \frac{m}{2\pi r} [\pi(r+\delta/2)^2 - \pi r^2] = \\ &= \frac{l}{2(r+\delta/2)} [((r+\delta)^2 - (r+\delta/2)^2) + \frac{r+\delta}{r} ((r+\delta/2)^2 - r^2)] = \\ &= \frac{l}{2(r+\delta/2)} [r^2 + 2r\delta + \delta^2 - (r^2 + r\delta + (\delta/2)^2) + \frac{r+\delta}{r} (r^2 + r\delta + (\delta/2)^2 - r^2)] = \\ &= \frac{l}{2(r+\delta/2)} [r\delta + \frac{3}{4}\delta^2 + \frac{r+\delta}{r} (r\delta + (\delta/2)^2)] = \frac{\delta l}{2(r+\delta/2)} [r + \frac{3}{4}\delta + \frac{r+\delta}{r} (r + \delta/4)] . \end{aligned}$$

The holonomy is (note that $n = (l \frac{r+\delta}{r})$)

$$\mathcal{H} = 2\pi - 4(\pi/2) - \left[\left(l \frac{r+\delta}{r} \right) \left(\frac{1}{r+\delta/2} \right) - l \frac{1}{r+\delta/2} \right] = l \left(1 - \frac{r+\delta}{r} \right) \left(\frac{1}{r+\delta/2} \right) = \frac{-l\delta}{r(r+\delta/2)} .$$

Then

$$\mathcal{H}/A = \frac{-2}{r} [r + \frac{3}{4}\delta + \frac{r+\delta}{r} (r + \delta/4)]^{-1} ;$$

and, thus, the intrinsic curvature is

$$\lim_{\delta \rightarrow 0} \frac{-2}{r} [r + \frac{3}{4}\delta + \frac{r+\delta}{r} (r + \delta/4)]^{-1} = \frac{-1}{r^2} .$$