SOLUTIONS

Chapter 2

Extrinsic Curves

PROBLEM 2.1. Give Examples of F.O.V.'s

a.

The following are a few of the examples that have been brought up in a class:

1. Consider successively a globe of the world, a map of the whole USA, a map of New York State, a map of the Ithaca Area, a map of the Cornell campus, a floor plan of White Hall, my ordinary experience of my office, ... Going from one level to the next the field of view shrinks but more details are discernible. Ithaca is indistinguishable from a point in the f.o.v. of New York State.
2. Wild flowers at a distance are a blur of color but up close you see individual petals.
3. When reading a book, one f.o.v. is the Table of Contents which differs from the f.o.v. of reading the text on an individual page.
4. "You can't see the forest for the trees."
5. A student may view her/his studies from several f.o.v.'s: As part of a career, The degree program, A particular course, What is studied in class one day. This was caused by the fact that the printer could distinguish 1/300 inches from 0 inches whereas to the word processor 1/300 inches = 0.00 inches.
7. One can ask: Is there reality independent of a field of view? See, for example, Eaves & Eaves, *Powers of Ten*, which is both a video and a book (Freeman and Sons). I can also remember a children's book which I have now lost entitled *A View From the Oak*. In that book it was described how the different creatures that lived in and around an oak tree had very differing realities because they viewed the world within different f.o.v.'s.

b.

You can easily see that in any ordinary (linear) f.o.v. you are not able to distinguish \$0, \$1, and the national debt. The only f.o.v. that seems to work is a logarithmic f.o.v. in which \$N is represented by log(N+1). The decibel scale (for sound) and the Richter scale (for earth quakes) are examples of logarithmic f.o.v.'s.

There is no f.o.v. in which [the national debt] is distinguishable from [the national debt]+\$1, because the national debt is not that precisely defined. See also Part **e**, below.

The situation with a truck load of sand is the same as with the national debt.

Compare the fact that [the national debt] is indistinguishable from [the national debt]+\$1 and that [truck load of sand] is indistinguishable from [truck load of sand]+[grain of sand] with the usual mathematical definition:

A set X is infinite if it is in one-to-one correspondence with a proper subset of itself.

c.

No, {1.4, 1.49, 1.499, 1.4999, 1.49999, ...} converges to 1.5, but [1.49...9]=1 and [1.5]=2.

As soon as $(p_n - q)$ is less than 0.5, then $[p_n - q]=0$.

In this f.o.v. it is true that, whenever $|x_n - y| < 1$, then $|[x_n] - [y]| \le 1$. Thus, with the radius $\rho = 5000$, if we set the tolerance $\tau = 1/4999$, then, as soon as $|x_n - y| < 1$, we will have $[x_n]$ indistinguishable from y. In this f.o.v. what one observes is $\{x_n\}$ and $\{y\}$ but not $\{x_n - y\}$.

***d.**

For each n, if we look at all 4-digit floating point numbers of the form abcd $\times 10^n$, then the analysis is exactly the same as in Part **c**, except the radius is 5000×10^n. But, if we try to include floating point numbers with all n, then we get a type of logarithmic f.o.v. in which the tolerance is not constant across the field of view.

***e.**

Normally when we say "one kilometer" we are considering a f.o.v. in which 1000 meters and 1001 meters are not distinguishable. However, when we say "1000 meters" we normally are considering a f.o.v. in which 999 meters and 1001 meters are both distinguishable from 1000 meters.

Possible additional questions:

- ♦ Investigate the differences between standard pixel graphics, with and without dithering.
- ♦ In the standard theory of the reals the infinite repeating decimal 0.999... and the number 1 are considered to be equal. Is there a field of view in which 0.999... and 1 are distinguishable? Explain.
- ♦ Consider the graphs of real-valued functions of one real variable in various f.o.v.'s. When does a graph satisfy the vertical line test in a field of view? Can you find an example of a function (defined for all reals) that in a particular f.o.v. both satisfies the vertical line test and has a jump discontinuity (in the f.o.v.)?

Problem 2.2. Smoothness and Tangent Directions

a.

$y = \sqrt{10^{-30} + x^2}$ is continuously differentiable but in the normal f.o.v. is indistinguishable from the non-differentiable $y = |x|$. Only in f.o.v.'s with radius about 10^{-15} would you see the differentiability at the origin.

$y = \dfrac{x}{\sqrt{10^{-30} + x^2}}$ is continuously differentiable, but in the normal f.o.v. is indistinguishable from the non-continuous step function $y = \dfrac{x}{|x|}$. There is no f.o.v. in which one could see the whole "step" as smooth, but in f.o.v.'s with radius about 10^{-15} you could see the smoothness locally.

$y = |x|$ will look the same in all f.o.v.'s.

$y = x + 10^{-15}\dfrac{x}{|x|}$ is not continuous nor everywhere differentiable, but in the normal f.o.v. is indistinguishable from the straight line $y = x$.

If one wishes to further investigate the relationships between f.o.v.'s and smooth, then one can look at

- *Prove that in any f.o.v. if there is a circle big enough that the center is distinguishable from each point on the circumference then the circle is smooth in the f.o.v. at every point.*
- *Prove the graph of $y = a\,x^2$ is smooth at every point in any field of view where the intersection of the graph with the vertical line $x = 0$ is indistinguishable from a point.*

b.

If $t(x) = f(p) + f'(p)(x-p)$ is the equation of the line tangent to the curve $(x,f(x))$ at the point $(p, f(p))$, then $f(x) - t(x) = \{ [f(x) - f(p)]/(x-p) - f'(p) \}(x-p)$. By definition, if f is differentiable at p then, for every ε there is a δ such that

$$0 < |x-p| < \delta \Rightarrow \left| \frac{f(x)-f(p)}{x-p} - f'(p) \right| < \varepsilon .$$

Thus multiplying by $|x-p|$ we get

$$0 < |x-p| < \delta \Rightarrow |f(x)-t(x)| = \left| \left(\frac{f(x)-f(p)}{x-p} - f'(p) \right)(x-p) \right| < \varepsilon\delta.$$

But this exactly says that, for given tolerance ε there is a radius δ such that in the f.o.v. centered at p with radius δ and tolerance ε the curve and the tangent line are indistinguishable; and thus the curve is infinitesimally straight at p.

c.

Express the tangent line as $t(x) = f(p) + r(x-p)$, where r is a real number. Then the graph is infinitesimally straight if for every tolerance τ there is a radius ρ such that, if x is in the f.o.v. centered at p of radius ρ [$|(x,f(x))-(p,f(p))| < \rho$], then $(x,f(x))$ is within $\tau\rho$ of some point in the tangent line. Since we can pick any f.o.v. with radius < ρ, pick one in which the radius is equal to $|x-p|$. Now look at the geometry in Figure 2.A.

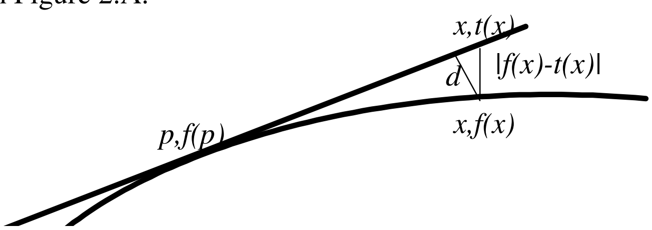

Figure 2.A. Distance between surface and tangent plane.

From the little triangle we see that (letting d be the distance from $(x,f(x))$ to the tangent line)

$$\frac{\sqrt{|f(x)-t(x)|^2 - d^2}}{d} = r \text{ or } |f(x)-t(x)| = d\sqrt{r^2+1} .$$

Thus, since $d < \tau\rho = \tau|x-p|$, $|f(x)-t(x)| = d\sqrt{r^2+1} < \tau|x-p|\sqrt{r^2+1}$. Since r is fixed (and finite), given an $\varepsilon > 0$, we can pick

$$\tau = \frac{\varepsilon}{\sqrt{r^2+1}}$$

and then find the δ (= ρ, above) so that, for $|x-p| < \delta$, $|f(x)-t(x)| < \varepsilon|x-p| < \varepsilon\delta$. We can conclude that f is differentiable. If r is infinite, then there is a vertical tangent and f can not be differentiable there.

***d.**

For $x \neq 0$, we can calculate $f'(x) = 2x\sin(1/x) - \cos(1/x)$, which is continuous for $x \neq 0$, but has no limit as x approaches 0. But the graph lies between the graphs of $y = x^2$ and $y = -x^2$ which are both tangent to the x-axis at $x = 0$. Thus, the graph of the function has a tangent line at $x = 0$ and (by Part **c**) is differentiable everywhere. Because of the $\cos(1/x)$ term there are points on the graph of f arbitrarily close to 0 which have tangent lines with slope close to 1. Clearly any f.o.v. in which the graph is indistinguishable from the tangent line at these points can not contain $x = 0$. Thus the graph can not be uniformly straight in any neighborhood of 0.

***e.**

The function f is differentiable and continuously differentiable in a neighborhood of p if and only if for every tolerance τ there is a radius ρ such that $|p-q|<\rho \Rightarrow \left|\frac{f(q)-f(p)}{q-p}-f'(p)\right|<\varepsilon$ and $|f'(p)-f'(q)|<\varepsilon$, which implies that $|f(q)-f(p)-f'(p)(q-p)|<\varepsilon|q-p|$ and

$$\left|\left\{\frac{f(p)-f(q)}{p-q}-f'(q)\right\}(p-q)\right|=\left|\left\{(f'(p)-f'(q))+\left[\frac{f(q)-f(p)}{q-p}-f'(p)\right]\right\}(p-q)\right|<2\varepsilon|q-p|.$$

Thus continuously differentiable implies smooth (uniformly infinitesimally straight).

If the curve is smooth, then for every tolerance τ there is a radius ρ such that, if $|p-q|<\rho$, then in a f.o.v. of radius $|p-q|$ centered at p we have that q is indistinguishable from a point on the tangent line at p. Thus, the (finite) slope of the tangent line is indistinguishable from the slope of the secant joining p to q. The argument applies also to the tangent line at q. The slopes of the tangent lines at p and q are within $[\varepsilon\tau]/\tau = \varepsilon$ of each other. Thus, f is continuously differentiable.

f.

It is important for you to actually try walking along a path with a sharp corner and to notice what their velocity does at the corner. This exercise works best if the path is icy! Eventually you will come to see that it is possible to have a differentiable (even continuously differentiable) parametrization of a non-smooth curve. However, if the velocity is non-zero p, then it follows (by an argument essentially the same as Part **b**) that the curve is infinitesimally straight at p. And by an argument very similar to that in Part **e**, we can show that, if the parametrization is continuously differentiable with non-zero velocity vectors then the curve must be smooth.

Thus, we can prove:

A constant speed curve is smooth if and only if the velocity vector exists and is continuous at each point.

PROBLEM 2.3. Curvature of a Curve in Space

a.

The important point of this part is for you to think of curvature in concrete intuitive terms. As examples, you can use circles, helixes, and physical curves that you make out of wire. Even though you may remember the classical definition of curvature given in Part **b**, you may will feel comfortable trying out other definitions and methods.

b.

What you do here will, of course, depend on what you proposed in Part **a**. Again it is important for you to explore the intuitive meaning of the classical definition.

c.

You may want to argue something similar to: If the curvature vector had a component in the tangential direction then that component would not contribute to the curving of the curve. But eventually you should come to something like:

The derivative of a unit vector (a geometric direction) is always in a direction perpendicular to the original direction.

If the unit vector were to change in a direction that is not perpendicular to its own direction, then the length of the vector would change. It is important at this point to be able to see the truth of this statement geometrically. But it can also be useful to understand the linear algebra proof that goes like this:

If $\mathbf{V}(s)$ is a vector-valued function of the real variable s such that $|\mathbf{V}(s)|$ = constant, then writing

$$|\mathbf{V}(s)|^2 = \mathbf{V}(s)\cdot\mathbf{V}(s)$$

and differentiating with respect to s and using the product rule, we obtain

$$\mathbf{V}'(s)\cdot\mathbf{V}(s) + \mathbf{V}(s)\cdot\mathbf{V}'(s) = 0$$

and thus, $\mathbf{V}'(s)$ is perpendicular to $\mathbf{V}(s)$, since the dot product commutes.

PROBLEM 2.4. Osculating Circle

a.

What needs to be checked here is that the tangent vector and thus its derivative the curvature vector must lie in the plane of the circle. In that plane a parametrization of the circle with respect to arclength is

$$\gamma(s) = (r\cos(s/r), r\sin(s/r))$$

and we can compute:

$$\boldsymbol{T}(s) = \gamma'(s) = (-\sin(\tfrac{s}{r}), \cos(\tfrac{s}{r})) \text{ and } \boldsymbol{\kappa}(s) = \tfrac{d}{ds}\boldsymbol{T}(s) = \tfrac{1}{r}(-\cos(\tfrac{s}{r}), -\sin(\tfrac{s}{r}))$$

Thus, $|\boldsymbol{\kappa}| = (1/r)$ and the circle is in the plane of $\mathbf{T}(s)$ and $\boldsymbol{\kappa}(s)$.

b.

Since $\mathbf{T}$ is a unit vector, we can write $\mathbf{T}(s) = (\cos\theta(s), \sin\theta(s))$. Then

$$|\kappa| = \left|\frac{d\mathbf{T}}{ds}\right| = |(-\theta'(s)\sin\theta(s), \theta'(s)\cos\theta(s))| = \left|\frac{d\theta}{ds}\right|.$$

Since the normal vector is perpendicular to the unit tangent vector, then (in the plane) it is determined by the tangent vector up to sign. Thus, the normal vector must change at exactly the same scalar rate as the unit tangent vector. You can also write the normal vector in terms of θ. Thus: $|\boldsymbol{\kappa}| = \left|\frac{d\boldsymbol{T}}{ds}\right| = \left|\frac{d\boldsymbol{N}}{ds}\right|$.

Calculate that the helix is such a counterexample. If the curve were not planar, then there could be a component of the rate of change of the $\mathbf{N}$ in a direction perpendicular to $\mathbf{T}$. See Problem **2.6.d**.

***c.**

From Figure 2.B we see that

$$\frac{l_h}{(1/\kappa) - h} = \frac{l}{(1/\kappa)} \text{ or } l_h = l(1 - \kappa h).$$

Remembering that $\kappa = |\boldsymbol{\kappa}|$ we have

$$\tfrac{d}{dh} l_h \equiv \lim_{h\to 0} \frac{l_h - l}{h} = \lim_{h\to 0} \frac{[l(1 - |\boldsymbol{\kappa}|h)] - l}{h} = -|\boldsymbol{\kappa}|l \quad .$$

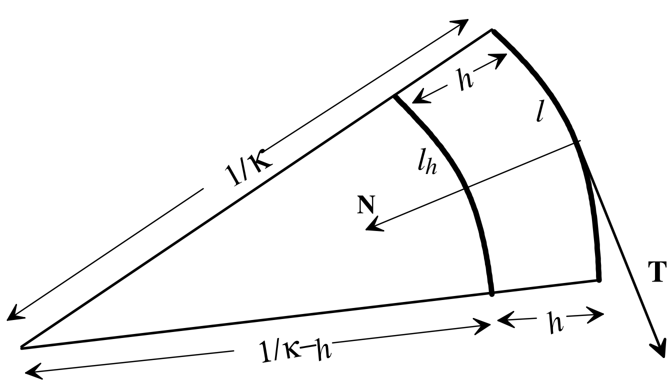

Figure 2.B. Rate of change of arclength.

PROBLEM 2.5. Strakes

a.

Consider the osculating circle to the helix. The radial segments of the strake will (at least in the f.o.v. in which the osculating circle and the helix are indistinguishable) extend the radii of this osculating circle. Thus, we want the inner radius of the osculating circle and the radius of curvature of the helix to be equal.

b.

The helix parametrized by arclength is

$$\boldsymbol{p}(s) = \left(\frac{hs}{\sqrt{h^2+(2\pi r)^2}},\ r\cos\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}},\ r\sin\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}} \right).$$

Thus,

$$\boldsymbol{T}(s) = \tfrac{d}{ds}\boldsymbol{p}(s) = \left(\frac{h}{\sqrt{h^2+(2\pi r)^2}},\ -\frac{2\pi r}{\sqrt{h^2+(2\pi r)^2}}\sin\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}},\ \frac{2\pi r}{\sqrt{h^2+(2\pi r)^2}}\cos\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}} \right)$$

and

$$\boldsymbol{\kappa}(s) = \tfrac{d}{ds}\boldsymbol{T}(s) = \left(0,\ \frac{-4\pi^2 r}{h^2+(2\pi r)^2}\cos\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}},\ \frac{-4\pi^2 r}{h^2+(2\pi r)^2}\sin\frac{2\pi s}{\sqrt{h^2+(2\pi r)^2}} \right).$$

Then the inner radius of curvature is $R_i = \frac{1}{|\boldsymbol{\kappa}|} = \frac{h^2+(2\pi r)^2}{4\pi^2 r} = r + \frac{h^2}{4\pi^2 r}$, which, for h = 10 m and r = 1 m, gives R_i = 3.533 m.

c.

We first compute the lengths of the inner and outer helixes. The inner helix has length

$$l_i = \sqrt{h^2+(2\pi r)^2} = \sqrt{100+4\pi^2} = 11.810 \text{ (meters)}.$$

This is also the same as the length of the inner edge of the annulus. Now the outer edge of the strake is

$$l_o = \sqrt{(10)^2+(2\pi(1.2))^2} = 12.524 \text{ (meters)},$$

but the outer edge of the annulus will be

$$l_i\frac{3.533+0.2}{3.533} = 12.479 \text{ (meters)}.$$

Thus, the annulus can not fit isometrically onto the strake and will have to be stretched by 0.045 meters.

We can also see that the annulus and the strake are not the same by calculating the outer radius of curvature of the strake

$$R_o = \frac{h^2+(2\pi r)^2}{4\pi^2 r} = r + \frac{h^2}{4\pi^2 r} = 1.2 + \frac{100}{4\pi^2(1.2)} = 3.311 \text{ (meters)},$$

which is a *smaller* radius than the inner radius! This compares with 3.733 meters as the outer radius of the annulus.

Since these corresponding lengths are different then the strake cannot be made from the annulus without stretching. But, of course, the metal can easily be stretched a little. However if we wanted to make the strake very wide compared to the cylinder, such as in an auger, then the stretching might be too much.

PROBLEM 2.6. When a Curve Does Not Lie in a Plane

a.

This problem is mainly an issue of seeing. It helps some people to make a model using two sheets of paper and tape. Notice that, by construction, the two approximate osculating planes intersect on the cord from **p–** to **p+**. By construction, each of the centers of curvature, **a**, **b**, lie on a perpendicular bisector to this cord. In the limit as **p–** and **p+** converge to **p**, a unit vector determined by the cord will converge to the unit tangent vector $\mathbf{T_p}$ and the plane containing the centers of curvature will converge to the plane through **p** which is perpendicular to $\mathbf{T_p}$. Thus, we see that, as the osculating plane changes, the center of curvature will change and the osculating planes will pivot infinitesimally about the tangent vector.

b.

Planar $\Rightarrow \tau = 0$. If the curve lies in a plane, then the unit tangent vectors and unit normal vectors also lie in this plane. Since we assume that the curvature vector is nonzero, then the normal vector is defined at each point and thus, so is the binormal $\mathbf{B_p} = \mathbf{T_p} \times \mathbf{N_p}$. Since the binormal is perpendicular to both the tangent vector and the normal vector it must be perpendicular to the plane in which they lie, thus, there are two possibilities for the binormal. But since the curvature is well-defined and nonzero the normal vector is also well-defined and must not switch sides of the tangent vector. Thus, the binormal is constant.

$\tau = 0 \Rightarrow$ *Planar.* Since the torsion is zero the binormal must be constant (since it is well-defined) and thus, the osculating planes (the plane of $\mathbf{T_p}$ and $\mathbf{N_p}$) must always be perpendicular to the constant binormal and thus, parallel to each other. There are now (at least) two ways to argue that the osculating planes are constant. First, let $\gamma(s)$ be a parametrization of the curve with respect to arclength. Then $f(s) = \gamma(s)\cdot\mathbf{B}$ is a real-valued function and if $\gamma(a)$ and $\gamma(b)$ have different osculating planes then $f(a)$ is not equal to $f(b)$. By the Mean Value Theorem, $f(b) - f(a) = f'(c)\,(b - a)$, for some c between a and b. But

$$f'(c) = \gamma'(c)\cdot\mathbf{B} = (|\gamma'(c)|)(\mathbf{T}\cdot\mathbf{B}) = 0,$$

and thus, the osculating planes at a and b must be the same. Second, one may argue infinitesimally using **2.6.a** that if the osculating planes changes, then, since it changes by pivoting around the tangent vector, the binormal must change.

***c.**

By definition of κ the first equation is true (see Problem **2.3**). The picture in Figure 2.12 of the text shows that the binormal changes only in a direction parallel to the normal, and thus the third equation follows. The second equation follows from **2.3.b** and the picture in Figure 2.12. Or, one may differentiate the cross product $\mathbf{N} = \mathbf{B} \times \mathbf{T}$. to get $\mathbf{N}'(s) = \mathbf{B} \times \mathbf{T}' + \mathbf{B}' \times \mathbf{T} = \kappa(s)(\mathbf{B} \times \mathbf{N}) - \tau(s)(\mathbf{N} \times \mathbf{T})$, but from the right-hand rule we can conclude that $\mathbf{N} \times \mathbf{T} = -\mathbf{B}$ and $\mathbf{B} \times \mathbf{N} = -\mathbf{T}$ from which the second equation follows.

d.

The helix parametrized by arclength is

$$\boldsymbol{p}(s) = \left(\frac{hs}{\sqrt{h^2 + (2\pi r)^2}},\ r\cos\frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}},\ r\sin\frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}} \right).$$

And thus

$$\boldsymbol{T}(s) = \tfrac{d}{ds}\boldsymbol{p}(s) = \left(\frac{h}{\sqrt{h^2 + (2\pi r)^2}},\ -\frac{2\pi r}{\sqrt{h^2 + (2\pi r)^2}} \sin \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}}, \frac{2\pi r}{\sqrt{h^2 + (2\pi r)^2}} \cos \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}} \right)$$

and

$$\boldsymbol{\kappa}(s) = \tfrac{d}{ds}\boldsymbol{T}(s) = \left(0,\ \frac{-4\pi^2 r}{h^2 + (2\pi r)^2} \cos \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}}, \frac{-4^2 r}{h^2 + (2\pi r)^2} \sin \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}} \right)$$

and

$$\boldsymbol{N}(s) = \frac{\boldsymbol{\kappa}(s)}{|\boldsymbol{\kappa}(s)|} = \left(0, -\cos \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}}, -\sin \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}} \right).$$

Thus

$$\boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s) = \left(\frac{2\pi r}{\sqrt{h^2 + (2\pi r)^2}}, \frac{h}{\sqrt{h^2 + (2\pi r)^2}} \sin \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}}, \frac{-h}{\sqrt{h^2 + (2\pi r)^2}} \cos \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}} \right),$$

and

$$\boldsymbol{\tau}(s) = \boldsymbol{B}'(s) = \left(0, \frac{2\pi h}{h^2 + (2\pi r)^2} \cos \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}}, \frac{2\pi h}{h^2 + (2\pi r)^2} \sin \frac{2\pi s}{\sqrt{h^2 + (2\pi r)^2}} \right).$$

Then we calculate the scalar torsion

$$\tau(s) = -[\boldsymbol{B}'(s) \cdot \boldsymbol{N}(s)] = \frac{2\pi h}{h^2 + (2\pi r)^2},$$

which is a constant. It is interesting to compare this with the scalar curvature (see **2.5**):

$$\kappa(s) = \frac{4\pi^2 r}{h^2 + (2\pi r)^2}.$$

The helix is the canonical curve with constant nonzero torsion and constant nonzero curvature. The circle has constant nonzero curvature but zero torsion and the straight line has zero curvature and, thus, undefined torsion.