Chapter 8

Intrinsic Local Descriptions and Manifolds

In Chapter 5 we developed geometrically intrinsic descriptions of holonomy, parallel transport, and curvature of surfaces. In Chapter 6 we developed extrinsic descriptions of Gaussian curvature and showed that it was the same as the intrinsic curvature for all C^2 surfaces. In Chapter 7, we found intrinsic local descriptions of Gaussian (intrinsic) curvature with respect to extrinsically defined local coordinates, using (extrinsic) directional derivatives. Now, in this chapter we will develop an intrinsic directional derivative that will allow intrinsic local descriptions of parallel transport. Then we will introduce the notion of manifolds that may have only intrinsically defined local coordinates. We will then put this all together to find for manifolds intrinsic local descriptions of the important intrinsic notions: covariant derivatives, geodesics, parallel transport, holonomy, Gaussian curvature, and others.

PROBLEM 8.1. Covariant Derivative and Connection

If $\mathbf{X}_p$ is a tangent vector at the point p in M, and $\mathbf{f}$ is a vector field (a function that gives a tangent vector at each point) defined near p, then the directional derivative $\mathbf{X}_p\mathbf{f}$ is not in general a tangent vector and, thus, is not intrinsic. But we can define an intrinsic directional derivative by slightly modifying the definition of $\mathbf{X}_p\mathbf{f}$. In particular, if $\alpha(t)$ is a curve in M with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{X}_p$, then

$$\mathbf{X}_p\mathbf{f} = \lim_{\delta\to 0} \tfrac{1}{\delta}[\mathbf{f}(\alpha(\delta)) - \mathbf{f}(p)].$$

This fails to be intrinsic only in the vector subtraction

$$[\mathbf{f}(\alpha(\delta)) - \mathbf{f}(p)].$$

Even in Euclidean space this subtraction does not literally make sense, because $\mathbf{f}(\alpha(\delta))$ is a (free) vector with base at the point $\alpha(\delta)$,and $\mathbf{f}(p)$ is a (free) vector with base at p. So in Euclidean space we perform the subtraction by first parallel translating $\mathbf{f}(\alpha(\delta))$ to a (bound) vector $\mathbf{f}(\alpha(\delta))_p$ based at p. (See Figure 8.1.) We can more correctly define

$$\mathbf{X}_p\mathbf{f} = \lim_{\delta\to 0} \tfrac{1}{\delta}\left[\mathbf{f}(\alpha(\delta))_p - \mathbf{f}(p)_p\right].$$

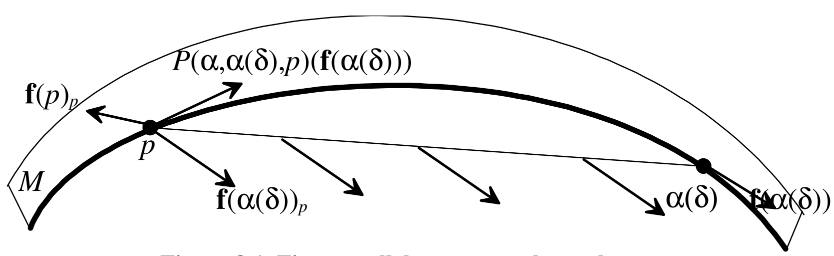

Figure 8.1. First parallel transport, then subtract.

Now this fails to be intrinsic because, even though $\mathbf{f}(\alpha(\delta))_p$ is a vector based at p, it will not in general be a vector that is tangent to M. To correct this situation we parallel transport $\mathbf{f}(\alpha(\delta))$ along α to $p = \alpha(0)$. (See Figure 8.1.) In Problem **5.4** we called this transported vector

$$P(\alpha,\alpha(\delta),p)\mathbf{f}(\alpha(\delta))$$

and it is a tangent vector at p. Now

$$[P(\alpha,\alpha(\delta),p)\mathbf{f}(\alpha(\delta)) - \mathbf{f}(p)]$$

is intrinsic since the subtraction takes place in the tangent space T_pM. It is technically convenient to use, instead of that substraction,

$$[\mathbf{f}(\alpha(\delta)) - P(\alpha,p,\alpha(\delta))\mathbf{f}(p)].$$

This is allowable, because (see Problem **5.4.b**) the change in the angle between the transported vector and the velocity vector of α depends on the geodesic curvature of α and not on which vector is being transported. Thus, $P(\alpha,q,p)$ defines an isometry from the tangent plane T_qM to the tangent plane T_pM. Since the rate of change of the parallel vector field exists (and is perpendicular to the tangent plane), this isometry is continuous and

$$[P(\alpha,\alpha(\delta),p)\mathbf{f}(\alpha(\delta)) - \mathbf{f}(p)] = P(\alpha,\alpha(\delta),p)[\mathbf{f}(\alpha(\delta)) - P(\alpha,p,\alpha(\delta))\mathbf{f}(p)].$$

We then define the ***intrinsic directional derivative*** (often called the ***covariant derivative***) to be:

$$\nabla_{\mathbf{X}}\mathbf{f} = \lim_{\delta\to 0} \frac{1}{\delta}[\mathbf{f}(\alpha(\delta)) - P(\alpha, p, \alpha(\delta))\mathbf{f}(p)].$$

Let **f** *be a (tangent) vector field defined in a neighborhood of the point p on the surface M, then:*

a. *Show that if* **X** *is a tangent vector at p, then the intrinsic derivative in the direction of* **X** *is the projection of the extrinsic directional derivative onto the tangent space to the surface,* T_pM*. That is,*

$$\nabla_{\mathbf{X}}\mathbf{f} = \mathbf{X}\mathbf{f} - \langle \mathbf{X}\mathbf{f}, \mathbf{n}(p)\rangle\mathbf{n}.$$

[Hint: Use Problem **5.4**.]

In many books this is taken as the definition of the covariant derivative.

b. *If* $\gamma(s)$ *is a unit speed smooth curve with tangent vector* $\mathbf{T} = \gamma'(0)$, *then show that* $\nabla_{\mathbf{T}}\gamma'(s)$ *is the intrinsic curvature vector* $\boldsymbol{\kappa}_g$ *of* γ *at* $s = 0$.

[Hint: Use Part **a**.]

c. *Show that* $\mathbf{V}(s)$ *is a parallel vector field along* γ *if and only if*

$$\nabla_{\gamma'(s)}\mathbf{V} = 0, \textit{ for all } s.$$

[Hint: Use Part **a** and Problem **5.4**.]

d. *Show that for fixed* **f**, *the covariant derivative*

$$\mathbf{X} \to \nabla_{\mathbf{X}}\mathbf{f}$$

is a linear operator, that is

$$\nabla_{\mathbf{X}+\mathbf{Y}}\mathbf{f} = \nabla_{\mathbf{X}}\mathbf{f} + \nabla_{\mathbf{Y}}\mathbf{f} \text{ and } \nabla_{a\mathbf{X}}\mathbf{f} = a\nabla_{\mathbf{X}}\mathbf{f}.$$

[Hint: Use Part **a**, and then properties of the extrinsic directional derivative and Riemannian metric.]

e. *For a real number r and a real-valued function f, show that*

$$\nabla_{\mathbf{X}} r\mathbf{Y} = r\nabla_{\mathbf{X}}\mathbf{Y} \text{ and } \nabla_{\mathbf{X}} f\,\mathbf{Y} = (\mathbf{X}f)\mathbf{Y} + f\,\nabla_{\mathbf{X}}\mathbf{Y}.$$

[Hint: Use Part **a** and Problem **4.8**.]

In many treatments of differential geometry any function ∇, for which

- ∇: $(\mathbf{X},\mathbf{Y}) \to \nabla_{\mathbf{X}}\mathbf{Y}$, where $\mathbf{X}$, $\mathbf{Y}$, $\nabla_{\mathbf{X}}\mathbf{Y}$ are vector fields on M,

and

- ∇ satisfies **8.1.d** and **8.1.e,**

is called a ***connection*** on M. There is clearly a close relationship between covariant derivatives and connections. In the literature the terms are sometimes used interchangeably, with the term *connection* used when the abstract properties in **8.1.d** and **8.1.e** are being emphasized, and, the term *covariant derivative* when its role of describing rates of change is emphasized. We use the word 'connection' because a connection allows to connect the tangent vector spaces T_pM and T_qM at two different points. This is done via parallel transport (defined as we have seen in **8.1.c**) along a curve α that joins p to q. Thus any curve from p to q determines a linear transformation

$$P(\alpha,p,q)\colon T_p M \to T_q M,$$

where, for each $\mathbf{V}_p$ in T_pM, $\mathbf{V}(t) = P(\alpha,p,t)\mathbf{V}_p$ is a parallel vector field along α.

**PROBLEM 8.2. Manifolds–Intrinsic and Extrinsic*

In Chapter 1, we have already seen surfaces in 3-space, $\mathbf{R}^3$, with extrinsic local coordinates, and surfaces in $\mathbf{R}^3$ with intrinsic local coordinates. We can say that extrinsic local coordinates, $\mathbf{x}$: $\mathbf{R}^2 \to \mathbf{R}^3$, are differentiable (or C^k) if $\mathbf{x}$ is differentiable (or C^k) as a function from $\mathbf{R}^2$ to $\mathbf{R}^3$. (See, for example, Problems **4.4** and **4.8**). But, for intrinsic local coordinates, no such definition is directly possible since intrinsic local coordinates are not defined in terms of a coordinate system in $\mathbf{R}^3$.

This problem can be seen clearly in the case of the annular hyperbolic plane, H. If we have a function f: $\mathbf{R} \to H$ (such as the parametrization of a curve on H), what would it mean to say that f is differentiable?

Locally we can (see Problem **3.1.f**) embed certain neighborhoods in H as a smooth surface in $\mathbf{R}^3$, but (as explained in **3.1.f**) it is not possible to embed the entire H as a smooth surface. And, even if it were possible to embed H as a smooth surface, its intrinsic description does not include such an embedding. So, is there any way we can say that f: $\mathbf{R} \to H$ is differentiable? There is one coordinate chart $\mathbf{x}$: $\mathbf{R}^2 \to H$ for all of H (see Problem **1.8.b**) that is one-to-one, and so $\mathbf{x}^{-1}$ is defined. Then $\mathbf{x}^{-1}\circ f$ is a function from $\mathbf{R}$to $\mathbf{R}^2$ and we can ask if it is differentiable or not. This leads to a definition:

> If $\mathbf{x}$: $U \subset \mathbf{R}^n \to M$ is a local coordinate chart, and if f: $V \subset \mathbf{R}^m \to M$ is a function, then we say that f ***is differentiable (or*** C^k***) with respect to the chart*** $\mathbf{x}$ if $\mathbf{x}^{-1} \circ f$ is differentiable (or C^k) where it is defined (which is on $f^{-1}[\mathbf{x}(U) \cap f(V)]$).

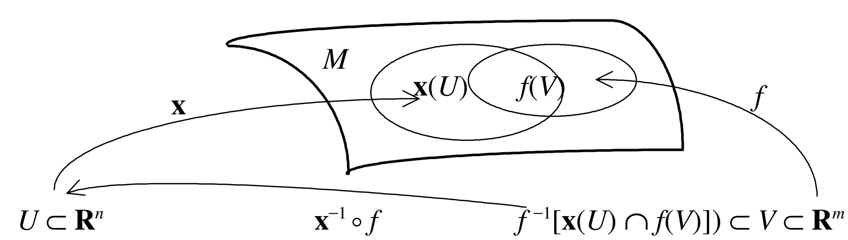

Figure 8.2. Examining functions into a coordinate chart.

Recall from multivariable analysis that a function f from $\mathbf{R}^m$ to $\mathbf{R}^n$ is differentiable at p if there is a linear transformation df (called the ***differential*** of f) from $\mathbf{R}^m$ to $\mathbf{R}^n$ such that, given any error (tolerance) τ there is a radius ρ such that $|x - p| < \rho$ implies

$$|f(x) - [f(p) + df(x-p)]| \leq |x-p|\tau \quad .$$

That df is of maximal rank is equivalent to df taking $\mathbf{R}^m$ linearly onto an m-dimensional subspace of $\mathbf{R}^n$. (See Appendix B.)

We now need to check that this definition is compatible with other definitions:

***a.** *Let M be a smooth surface in $\mathbf{R}^n$ and $\mathbf{x}$: $\mathbf{R}^2 \to M$ be a* C^1 (*or* C^2) *local coordinate chart* (see Problems **4.4** and **4.8**, and note that the differential $d\mathbf{x}$ must have maximal rank) *with image $U = \mathbf{x}(\mathbf{R}^2)$. Then a function f: $\mathbf{R}^m \to U$ is* C^1 (*or* C^2) *with respect to* $\mathbf{x}$ *if and only if* $\mathbf{x}^{-1} \circ f$ *is* C^1 (*or* C^2) *as a function from* $\mathbf{R}^m$ *to* $\mathbf{R}^n$.

Outline of a proof of **8.2.a**:

1. First prove this in the case that the chart is a Monge patch $\mathbf{y}$. (What is the inverse of a Monge patch?) (See Problem **3.1**.)
2. Then look at $\mathbf{x}^{-1} \circ \mathbf{y}$ and argue that this is a one-to-one function from an open subset of $\mathbf{R}^2$ onto an open subset of $\mathbf{R}^2$.
3. The inverse of $\mathbf{x}^{-1} \circ \mathbf{y}$ is C^1 (or C^2), and then it follows that $\mathbf{x}^{-1} \circ \mathbf{y}$ is C^1 (or C^2). [You can use the Inverse Function Theorem (see Appendix B), but this is overkill in this case because the hard part of the Inverse Function Theorem is to prove that the function and its inverse are one-to-one and onto. In this case it is more direct to look at the differential $d(\mathbf{x}^{-1} \circ \mathbf{y})$.]

Now, if the surface M has two local coordinate charts, it is possible that a function will be differentiable with respect to one but not with respect to the other. (For example, let $\mathbf{R}^2$ be a surface with the identity as one chart and a non-differentiable one-to-one function from $\mathbf{R}^2$ to $\mathbf{R}^2$ as another chart.) To take care of this problem, we will require that two coordinate charts, $\mathbf{x}$ and $\mathbf{y}$, for the surface M will be compatible in the sense that

$$\mathbf{x}^{-1} \circ \mathbf{y} \ \text{ and } \ \mathbf{y}^{-1} \circ \mathbf{x}$$

are both differentiable. Then any function that is differentiable with respect to $\mathbf{x}$ will also be differentiable with respect to $\mathbf{y}$. [Be sure you see why this is true.]

We can now use this idea to expand the notion of surface, so that we can work intrinsically and in higher dimensions:

A ***differentiable*** [or C^k] ***n-manifold*** is a metric space M with a collection (called an ***atlas***) $\mathcal{A}$ such that:

- Each member of $\mathcal{A}$ is a chart $\mathbf{x}$: $U \subset \mathbf{R}^n \to M$ such that both U and $\mathbf{x}(U)$ are open, and $\mathbf{x}$ and $\mathbf{x}^{-1}$ are both continuous.
- Each point in M is contained in the image of at least one chart from A.
- If $\mathbf{x}$: $U \subset \mathbf{R}^n \to M$ and $\mathbf{y}$: $V \subset \mathbf{R}^n \to M$ are two charts in $\mathcal{A}$, then

$$\mathbf{x}^{-1} \circ \mathbf{y} \ \text{ and } \ \mathbf{y}^{-1} \circ \mathbf{x}$$

are both differentiable [or C^k] where they are defined.

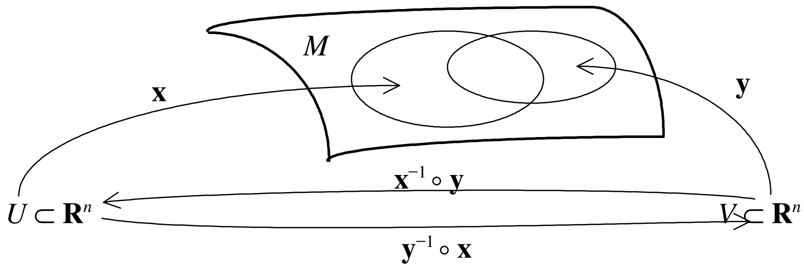

Figure 8.3. Charts in an atlas.

We say that a chart **x** for M is ***compatible*** with the atlas $\mathcal{A}$ if, for every chart **y** in $\mathcal{A}$,

$$\mathbf{x}^{-1}\circ\mathbf{y} \text{ and } \mathbf{y}^{-1}\circ\mathbf{x}$$

are both differentiable [or C^k] where they are defined.

A function $f : (M,\mathcal{A}) \to (N,\mathcal{B})$, between an m-manifold and an n-manifold are said to be [C^k] ***differentiable*** if, for every chart **x** in $\mathcal{A}$ and every chart **y** in $\mathcal{B}$,

$$\mathbf{y}^{-1}\circ f\circ\mathbf{x}$$

is [C^k] differentiable wherever it is defined as a function from a subset of $\mathbf{R}^m$ to a subset of $\mathbf{R}^n$.

Two n-manifolds, $(M,\mathcal{A})$ and $(N,\mathcal{B})$, are said to be the same, or [C^k] ***diffeomorphic***, if there is a one-to-one onto function $f : M \to N$, such that both f and f^{-1} are [C^k] differentiable.

If the n-manifold M is a subset of $\mathbf{R}^m$, and if each chart **x** in the atlas of M is C^k as a function from $\mathbf{R}^n$ to $\mathbf{R}^m$ and has its differential $d\mathbf{x}$ of maximal rank, then we say that M is a ***submanifold of*** $\mathbf{R}^m$.

Problem **8.2.a** shows that every smooth surface in $\mathbf{R}^m$ is a 2-manifold.

These definitions differ from those in some other books. In particular, some texts use charts that are functions from an open subset of M onto an open subset of $\mathbf{R}^n$. Such texts tend to use charts from $\mathbf{R}^n$ to the manifold for manifolds that are described extrinsically as submanifolds in a higher-dimensional Euclidean space, and to use charts from the manifold to $\mathbf{R}^n$ for manifolds that are described intrinsically. In our text, our discussion intertwines the two types of manifolds, and thus, it seems to make sense to use the same direction for the charts. In addition, some texts instead of an *atlas* require a ***maximal atlas*** for M (that is, an atlas $\mathcal{A}$ such that, if **x** is a coordinate chart for M, which is not in $\mathcal{A}$, then there is a coordinate chart **y** in $\mathcal{A}$ such that either (or both) of

$$\mathbf{x}^{-1}\circ\mathbf{y} \text{ and } \mathbf{y}^{-1}\circ\mathbf{x}$$

are both differentiable [or C^k] where they are defined). We will avoid talking about maximal atlases because atlases can be explicitly constructed, whereas maximal atlases can, in general, only be posited by using the following (easily proved) non-constructive result:

LEMMA. *If $\mathcal{A}$ is a C^k atlas for the manifold M, then the set of all charts that are compatible with $\mathcal{A}$ is the unique maximal atlas containing $\mathcal{A}$.*

We shall first look at the case of an extrinsically defined submanifold M in $\mathbf{R}^m$. Clearly the graph of a C^k function f from n-space to m-space is an (extrinsic) n-submanifold in $(n+m)$-space with an atlas consisting of the single chart $\mathbf{x}(p) = (p, f(p))$. We also can prove the following converse:

b. *If M is a C^k n-submanifold of $(n+m)$-space, and p is any point in M, then some neighborhood of p is the graph of a C^k function (that is, p has a C^k Monge patch).*

[Hint: Look at the projection π, which takes a neighborhood of p onto the tangent space at p, and apply the Inverse Function Theorem (Appendix **B.2**).]

c. *If M in $\mathbf{R}^n$ has local (extrinsic) charts that are C^k and maximal rank, then show that M is a C^k manifold with the extrinsic charts as its atlas.*

[Hint: Use the same idea of proof as in Part **a**.]

d. *Show that any surface covered by a single chart is a C^k 2-manifold, for all k. (This in particular applies to the annular hyperbolic plane and any open subset of Euclidean space.) Further, show that the two charts for the annular hyperbolic plane in Problem **1.8** are compatible.*

There is another class of manifolds that are defined by implicit equations. For example, the unit n-sphere is the solution of the equation

$$\sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2 + (x^{n+1})^2} = 1 \ .$$

As a consequence of the Implicit Function Theorem in analysis, we have (see Appendix **B.3**):

Theorem. *Let F: $\mathbf{R}^n \to \mathbf{R}^{n-m}$ be a C^1 function, and suppose $dF(x)$ has maximal rank $n - m$ at every point on a level set*

$$M = \{\ x \mid F(x) = c\ \}.$$

Then M is a C^1 m-submanifold of $\mathbf{R}^n$.

We can now extend to (intrinsic) manifolds the intrinsic notions that we have covered in this text. The trick is merely to use the intrinsically defined forms, which we have found throughout our investigations.

For example, let us look at the notion of ***tangent vector*** and ***tangent space***. If M is an extrinsic manifold in $\mathbf{R}^n$ with C^1 chart $\mathbf{x}$ for a neighborhood of the point p, then (as in Problem **4.1**) every tangent vector at p (which is extrinsic) is the velocity vector of a curve in M. And it is easy to see that, if γ and λ are two C^1 curves in M with $\gamma(0) = \lambda(0) = p$, then they have the same velocity vectors, $\gamma'(0) = \lambda'(0)$, if and only if $\mathbf{x}^{-1}\circ\gamma$ and $\mathbf{x}^{-1}\circ\lambda$ have the same velocity vectors, $(\mathbf{x}^{-1}\circ\gamma)'(0) = (\mathbf{x}^{-1}\circ\lambda)'(0)$, as curves in $\mathbf{R}^n$. Thus we can define the (***intrinsic***) ***tangent space***, T_pM, at p in M to be the equivalence classes of C^1 curves γ in M with $\gamma(0) = p$ with the equivalence relation

$$\gamma \approx \lambda \text{ if and only } (\mathbf{x}^{-1}\circ\gamma)'(0) = (\mathbf{x}^{-1}\circ\lambda)'(0).$$

Each equivalence class of curves in the tangent space is then called an (***intrinsic***) ***tangent vector***, and we will denote the equivalence class of γ by the notation $[\gamma]$.

e. *If the manifold M has an atlas, show that the definition of the intrinsic tangent space at p does not depend on which chart (containing p) you choose from the atlas. Show also that for each chart $\mathbf{y}$ (containing p), the function from the tangent space of $\mathbf{R}^n$ at $q = \mathbf{y}^{-1}(p)$ to T_pM defined by*

$$d\boldsymbol{y}(\mathbf{X}_q) = [t \to \boldsymbol{y}(q + t\mathbf{X}_q)]$$

is one-to-one and onto. Use this to define a vector space structure on $T_p M$ that is independent of which chart (containing p) you use.

Now, how do we visualize this intrinsically? The answer is that we do it naturally all the time! Our three-dimensional physical universe is a 3-manifold, which according to physicists is not Euclidean three space. However, in the very small neighborhood of 3-space in which we physically move our bodies and draw pictures, we have no problems drawing the usual pictures of vectors, and for curves (in our normal physical experience), we have no trouble imagining a tangent vector as a straight line (geodesic) segment with an arrowhead on one end. This makes sense because the space near us is indistinguishable from a region in Euclidean 3-space.

If we assume that each point p in an n-manifold M has a neighborhood that intrinsically is indistinguishable from a region in n-space, then we can use it to define a ***Riemannian metric*** as in Problem **4.3**—this is the case with the annular hyperbolic plane. Or we can posit a symmetric, bilinear, positive definite, real-valued function $\langle \mathbf{X},\mathbf{Y}\rangle_p$ that varies C^k with respect to p and call it a Riemannian metric. Then we can use Riemannian metric to define angles and lengths by setting

$$\langle \mathbf{X},\mathbf{Y}\rangle = |\mathbf{X}|\,|\mathbf{Y}| \cos \theta_{\mathbf{XY}}.$$

As always, we consider a function defined on M or, in this case, on the tangent vectors of M to be C^k if, using a chart $\mathbf{y}$, the corresponding functions on $\mathbf{R}^n$ (or in this case its tangent vectors) are C^k. In fact, we can consider the collection of all tangent vectors on M to be a differentiable manifold (of dimension n^2) called the ***tangent bundle*** TM ***of*** M. The tangent bundle $T\mathbf{R}^n$ of $\mathbf{R}^n$ is the collection of all *bound* vectors on $\mathbf{R}^n$. Since at each point q in $\mathbf{R}^n$ the vectors bound at q form an n-dimensional space and, thus, the tangent bundle has n^2 dimensions. Then the map $d\mathbf{x}$ (defined in Part **e**) maps a vector $\mathbf{X}_q$ bound at q to the tangent vector $d\mathbf{x}(\mathbf{X}_q)$ in $T_{\mathbf{x}(q)}$. Then the atlas for TM consists of a chart $d\mathbf{x}$ for each chart $\mathbf{x}$ in the atlas for M.

Directional derivatives of real-valued functions on an n-manifold M are intrinsic because they are just real numbers, and the definition (after Problem **4.5**)

$$X_p f = \frac{d}{dt} f(\gamma(t))_{t=0} = \lim_{h\to 0} \frac{f(\gamma(h)) - f(\gamma(0))}{h}$$

works unchanged where $\mathbf{X}_p = [\gamma]$.

Directional derivatives of vector valued functions (vector fields) are not intrinsic because $\mathbf{X}_p\mathbf{Y}$ will not in general be a tangent vector, so there is no hope of defining them on an intrinsic manifold; however, we can define the (***intrinsic***) ***covariant derivative*** on any manifold M in the following ways:

If M is an (extrinsic) m-dimensional manifold in n-space, and if $\mathbf{X}$ is a (tangent) vector in the tangent space T_pM at the point p in M, and if $\mathbf{f}$ is a tangent vector *field*, then the ***covariant derivative*** of $\mathbf{f}$ with respect to $\mathbf{X}$ can be extrinsically defined as

$$\nabla_{\mathbf{X}}\mathbf{f} = \{\text{projection of } \mathbf{Xf} \text{ onto the tangent space } T_pM\}.$$

If M is an intrinsic m-manifold, and we have an intrinsic notion of parallel transport (as with the annular hyperbolic plane), then we can define the ***covariant derivative*** as in Problem **8.1**. Or, we can define a ***connection*** on M as any function

$$\nabla: (\mathbf{X},\mathbf{Y}) \to \nabla_{\mathbf{X}}\mathbf{Y}, \text{ where } \mathbf{X}, \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Y} \text{ are vector fields on } M,$$

which satisfies **8.1.d** and **8.1.e**.

None of these ways allows us to intrinsically compute the covariant derivative from the knowledge of a coordinate chart and a Riemannian metric—this is the deficiency we will correct in the next problem.

Problem 8.3. Christoffel Symbols

If $\mathbf{x}$ is a local coordinate system for a neighborhood of p in the manifold M, then the covariant derivative $\nabla_{\mathbf{x}_i}\mathbf{x}_j$ is a tangent vector to M and thus is a linear combination of the basis vectors $\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_m$:

$$\nabla_{\mathbf{x}_i}\mathbf{x}_j = \Gamma_{ij}^1\mathbf{x}_1 + \Gamma_{ij}^2\mathbf{x}_2 + \cdots + \Gamma_{ij}^m\mathbf{x}_m = \sum_k \Gamma_{ij}^k\mathbf{x}_k.$$

The coefficients Γ_{ij}^k are called the ***Christoffel symbols*** and are clearly intrinsic quantities. In this problem we wish to find intrinsic formulae of these symbols and therefore of the intrinsic derivative, but along the way we will need to use the extrinsic description in Problem **8.1.a**. So, until we give a purely intrinsic description at the end of this problem, the manifold M is extrinsic.

a. *Show that*

$$\Gamma_{ij}^k = \sum_l \left\langle \mathbf{x}_{ij}, \mathbf{x}_l \right\rangle g^{lk},$$

where the matrix (g^{lk}) *is the inverse of the matrix* (g_{kl}). Many texts take this latter expression as the definition of the Christoffel symbols, but I believe such a definition hides the geometric meaning.

Outline of solution:

1. Argue that

$$\left\langle \mathbf{x}_{ij}, \mathbf{x}_l \right\rangle = \left\langle \nabla_{\mathbf{x}_i}\mathbf{x}_j, \mathbf{x}_l \right\rangle = \sum_k \Gamma_{ij}^k g_{kl}$$

2. Using the fact that the matrix (g^{lk}) is the inverse of the matrix (g_{kl}), show that

$$\sum_l \left\langle \mathbf{x}_{ij}, \mathbf{x}_l \right\rangle g^{lm} = \sum_l \left(\sum_k \Gamma_{ij}^k g_{kl} \right) g^{lm} = \Gamma_{ij}^m.$$

b. *Explain each step of the following argument:*
If $\mathbf{Y} = \sum Y^j\mathbf{x}_j$ *is a (tangent) vector field (note that the* Y^j *are real valued functions), then*

$$\nabla_{\mathbf{x}_i}\mathbf{Y} = \sum_j \nabla_{\mathbf{x}_i}(Y^j\mathbf{x}_j) = \sum_j \left[(\mathbf{x}_i Y^j)\mathbf{x}_j + Y^j(\nabla_{\mathbf{x}_i}\mathbf{x}_j) \right] =$$
$$= \sum_j \left[(\mathbf{x}_i Y^j)\mathbf{x}_j + Y^j\left(\sum_k \Gamma_{ij}^k \mathbf{x}_k \right) \right] = \sum_k \left(\mathbf{x}_i Y^k + \sum_j \Gamma_{ij}^k Y^j \right)\mathbf{x}_k$$

Some texts use

$$\nabla_{\mathbf{x}_i}\mathbf{Y} = \sum_k \left(\mathbf{x}_i Y^k + \sum_j \Gamma_{ij}^k Y^j \right)\mathbf{x}_k$$

as the definition of the covariant derivative.

c. *Show that*

$$\left\langle \mathbf{x}_{ij}, \mathbf{x}_k \right\rangle = \tfrac{1}{2}\left[\mathbf{x}_i g_{jk} - \mathbf{x}_k g_{ji} + \mathbf{x}_j g_{ki} \right]$$

and thus

$$\Gamma_{ij}^k = \tfrac{1}{2}\sum_l g^{kl}\left[\mathbf{x}_j g_{il} - \mathbf{x}_l g_{ij} + \mathbf{x}_i g_{lj} \right].$$

[Hint:

$$\begin{aligned} \left\langle \mathbf{x}_{ij}, \mathbf{x}_k \right\rangle &= \mathbf{x}_i\left\langle \mathbf{x}_j, \mathbf{x}_k \right\rangle - \left\langle \mathbf{x}_j, \mathbf{x}_{ik} \right\rangle = \\ &= \mathbf{x}_i\left\langle \mathbf{x}_j, \mathbf{x}_k \right\rangle - (\mathbf{x}_k\left\langle \mathbf{x}_j, \mathbf{x}_i \right\rangle - \left\langle \mathbf{x}_{kj}, \mathbf{x}_i \right\rangle) = \\ &= \mathbf{x}_i\left\langle \mathbf{x}_j, \mathbf{x}_k \right\rangle - \mathbf{x}_k\left\langle \mathbf{x}_j, \mathbf{x}_i \right\rangle + \mathbf{x}_j\langle \mathbf{x}_k, \mathbf{x}_i \rangle - \left\langle \mathbf{x}_k, \mathbf{x}_{ji} \right\rangle. \end{aligned}$$
]

We started out with an extrinsic definition of the covariant derivative and thus the Christoffel symbols. But now we have an intrinsic description of the Christoffel symbols, and so we can give:

Intrinsic Definition of the covariant derivative*. Let* **x** *be a* C^2 *local coordinate chart for the open set U in the n-manifold M. If*

$$\mathbf{X}_p = \sum_i X^i \mathbf{x}_i(p)$$

is a tangent vector at p, and

$$\mathbf{f}(p) = \sum_j F^j(p)\mathbf{x}_j(p)$$

is a differentiable function defined on U, then

$$\nabla_{\mathbf{X}_p}\mathbf{f} = \sum_i X^i \nabla_{\mathbf{x}_i(p)}(\sum_j F^j \mathbf{x}_j) = \sum_i X^i (\sum_j \nabla_{\mathbf{x}_i(p)}(F^j \mathbf{x}_j)) =$$
$$= \sum_i \Big[X^i \sum_k \Big[\mathbf{x}_i(F^k) + \Sigma_j \Big(F^j \tfrac{1}{2} \sum_l g^{kl} \big[\mathbf{x}_j g_{il} - \mathbf{x}_l g_{ij} + \mathbf{x}_i g_{lj} \big] \Big) \Big] \mathbf{x}_k \Big].$$

Note that this definition will work on any n-manifold with a Riemannian metric. The resulting connection

$$\nabla\colon (\mathbf{X},\mathbf{Y}) \to \nabla_{\mathbf{X}}\mathbf{Y}, \text{ where } \mathbf{X}, \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Y} \text{ are vector fields on } M,$$

is often called the ***Riemannian connection***. It is a theorem (see, for example, Theorem 8.6 on page 236 of [**DG:** Millman and Parker]) that this connection is the only connection on M which satisfies:

- ***Metric connection***—Parallel transport with respect to ∇ is an isometry, and
- ***Symmetric*** (or ***Torsion-free***) ***connection***—For all vector fields **X** and **Y** on M and for all real-valued functions f on M, we have

$$(\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X})f = \mathbf{X}_p(\mathbf{Y}f) - \mathbf{Y}_p(\mathbf{X}f).$$

(See Problem **8.5.c**.)

d. *Compute for geodesic rectangular (or polar) coordinates on any surface that*

$$\Gamma^1_{11} = \tfrac{1}{2} g^{11}[\mathbf{x}_1 g_{11} - \mathbf{x}_1 g_{11} + \mathbf{x}_1 g_{11}] = \tfrac{1}{2} h^{-2}\mathbf{x}_1(h^2) = h_1/h,$$
$$\Gamma^2_{11} = -hh_2\,,\ \Gamma^1_{12} = \Gamma^1_{21} = h_2/h\,,$$
$$\textit{all others zero}.$$

Evaluate in the special case of a sphere.

PROBLEM 8.4. Intrinsic Curvature and Geodesics

Now we will use our description of covariant derivatives in terms of local coordinates to find intrinsic local coordinate descriptions of the geodesic (intrinsic) curvature of a curve and thus of geodesics.

a. *If* $\gamma(s)$ *is a curve parametrized by arclength, then, according to Problem* **8.1.b***, the intrinsic curvature at* $\gamma(a)$ *is given by*

$$\kappa_g(a) = \nabla_{\gamma'(a)}\gamma'.$$

Show that if you express the curve in terms of a local coordinates **x** *as*

$$\gamma(s) = \mathbf{x}(\gamma^1(s),\gamma^2(s)),$$

then

$$\gamma'(s) = (\gamma^1)'_s\mathbf{x}_1 + (\gamma^2)'_s\mathbf{x}_2,$$

and the intrinsic curvature is given by

$$\kappa_g(a) = \sum_k\left[(\gamma^k)''_a + \sum_{i,j}\Gamma^k_{ij}(\gamma(a))\,(\gamma^i)'_a(\gamma^j)'_a\right]\mathbf{x}_k\ .$$

[Hint: Use the fact that, for any real-valued function $f(s)$,

$$\gamma'(a)f(s)|_{s=a} = \nabla_{\gamma'(a)}f(s)|_{s=a} = f'(a).]$$

b. *Show that if* $\gamma(s)$ *is a curve parametrized by arclength then,* γ *is a geodesic if and only if*

$$(\gamma^k)''_s + \sum_{i,j}\Gamma^k_{ij}(\gamma(s))\,(\gamma^i)'_s(\gamma^j)'_s = 0\,,$$

for each k and each point along γ*.*

These are the differential equations for a geodesic expressed in local coordinates. This has theoretical importance in analytic treatments of geodesics, but in practice these equations can rarely be solved except approximately.

c. *Express in terms of h the results in Parts* **a** *and* **b** *in the case that* **x** *is geodesic rectangular (or polar) coordinates.*

Problem 8.5. Lie Brackets, Coordinate Vector Fields

We now want to find intrinsic expressions in local coordinates for the curvature of a manifold, but first we must examine the ways in which two tangent vector fields interact.

a. *Let* **x** *be a local chart for the open set U in the* C^2 *manifold M. Show that*

$$\nabla_{\mathbf{x}_i}\mathbf{x}_j = \nabla_{\mathbf{x}_j}\mathbf{x}_i.$$

However, this commutativity does not hold in general. In fact:

b. *In* $\mathbf{R}^2$ *find two (simple) vector fields*

$$\mathbf{A}(x,y) = \mathbf{e}_1 + a(x,y)\mathbf{e}_2 \text{ and } \mathbf{B}(x,y) = \mathbf{e}_2$$

such that

$$\nabla_{\mathbf{A}(0,0)}\mathbf{B} = \mathbf{A}(0,0)\mathbf{B} \neq \mathbf{B}(0,0)\mathbf{A} = \nabla_{\mathbf{B}(0,0)}\mathbf{A}.$$

If **V** and **W** are two vector fields defined on a neighborhood of p in M, then we define the ***Lie bracket*** [**V**,**W**] by setting

$$[\mathbf{V},\mathbf{W}]_p \equiv \nabla_{\mathbf{V}(p)}\mathbf{W} - \nabla_{\mathbf{W}(p)}\mathbf{V}.$$

From **8.5.a** we know that if **V** and **W** are the coordinate vector fields of some coordinate chart, then $[\mathbf{V},\mathbf{W}]\equiv 0$. In part **d** we show that these are the only examples.

c. *Show that, even though* $\mathbf{X}_p\mathbf{Y}$ *is not a tangent vector in general,*

$$\mathbf{X}_p\mathbf{Y} - \mathbf{Y}_p\mathbf{X}$$

is a tangent vector and is equal to $[\mathbf{X},\mathbf{Y}]_p$. This is often the definition of the Lie bracket.

[Hint: Express in terms of local coordinates and use **8.1** and linearity.]

***d.** *On an n-manifold M, show that the n vector fields* $\{\mathbf{V}_i\}$ *are equal to* $\{\mathbf{x}_i\}$ *for some coordinate chart* $\mathbf{x}$ *if and only if*

$$[\mathbf{V}_j,\mathbf{V}_k] \equiv \nabla_{\mathbf{V}_i}\mathbf{V}_k - \nabla_{\mathbf{V}_k}\mathbf{V}_j = 0, \textit{ for all } 1 \le i,j \le n.$$

Outline of a proof:

This outline assumes that the reader has a familiarity with flows defined by vector fields and with the theorem from analysis that a C^1 vector field always has a unique flow. For a discussion of these results, consult [**An**: Strichartz], Chapter 11, or [**DG**: Dodson/Poston], VII.6 and VII.7. In the latter, the details of this outline are filled in.

1. Given a C^1 vector field $\mathbf{V}$ defined and nonzero in a neighborhood of p in M, then there is a coordinate chart $\mathbf{x}$ such that $\mathbf{V} = \mathbf{x}_1$.
2. If $\mathbf{V}$ and $\mathbf{W}$ are two C^1 vector fields on M with flows ϕ_s and ψ_s then the flows commute

   $$\phi_a \circ \psi_b = \psi_b \circ \phi_a \text{ , wherever defined}$$

   if and only if

   $$[\mathbf{V},\mathbf{W}]_p = \mathbf{0}, \text{ for all } p.$$

3. Use the flows to define the coordinate chart $\mathbf{x}$.

Problem 8.6. Riemann Curvature Tensors

We now want to extend the notion of Gaussian (intrinsic) curvature to n-manifolds. First we express the Gaussian curvature of a surface in terms of the covariant derivative in local coordinates. For this problem we assume that $\mathbf{x}$ is a local *orthogonal* C^2 coordinate system on M.

In spite of **8.5.a**, in general,

$$\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{V} \ne \nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_1}\mathbf{V}.$$

In fact, you can prove the following result.

a. *On a surface M with orthogonal coordinates* $\mathbf{x}(u^1,u^2)$, *let* $\mathbf{V}$ *be a tangent vector field such that* $\nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_1}\mathbf{V}$ *and* $\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{V}$ *exist and are continuous. Then, at every p in M, show that*

$$|\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{V} - \nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_1}\mathbf{V}| = |\mathbf{V}||\mathbf{x}_1||\mathbf{x}_2||K(p)|,$$

where K(p) is the Gaussian curvature!

Outline of a proof:

1. Let $p = \mathbf{x}(0,0)$. Since the covariant derivative and the intrinsic curvature can both be defined in terms of parallel transport, look at parallel transport along the coordinate curves and use the following abbreviations:

   $$P_1(\delta,a) = P(t \to \mathbf{x}(t,a), \mathbf{x}(0,a), \mathbf{x}(\delta,a)),$$
   $$P_2(a,\delta) = P(t \to \mathbf{x}(a,t), \mathbf{x}(a,0), \mathbf{x}(a,\delta)).$$

   Look at the situation in Figure 8.4 and define

$$\boldsymbol{P}(\varepsilon,\delta) = P_1(\varepsilon,\delta)[P_2(0,\delta)\boldsymbol{V}(p)] - P_2(\varepsilon,\delta)[P_1(\varepsilon,0)\boldsymbol{V}(p)]\ .$$

Then starting with

$$K(p) = \lim_{R\to 0}(\mathcal{H}(R)/A(R)) = \lim_{R\to 0}(\theta/A(R)),$$

and using the definition of area as an integral (Problem **4.5**), show that

$$|\mathbf{V}||\mathbf{x}_1||\mathbf{x}_2||K(p)| = \lim_{\varepsilon,\delta\to 0}\frac{|\boldsymbol{P}(\varepsilon,\delta)|}{\varepsilon\delta}.$$

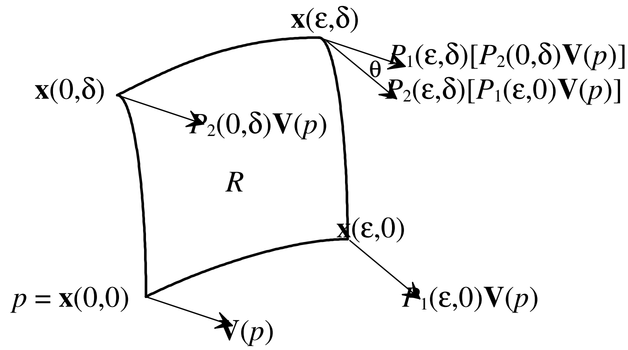

Figure 8.4. Parallel Transport Along Coordinate Curves.

2. Now, denoting $\mathbf{V}(\mathbf{x}(a,b)) = \mathbf{V}(a,b)$, use the limit definition of covariant derivative, the fact that $\nabla_{\mathbf{x}_1}$ is continuous, and the fact that parallel transport is a linear isometry to compute

$$\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{V} - \lim_{\varepsilon,\delta\to 0}\frac{\boldsymbol{P}(\varepsilon,\delta)}{\varepsilon\delta} =$$
$$= \lim_{\delta\to 0}\nabla_{\mathbf{x}_1}\left(\frac{1}{\delta}[\mathbf{V}(a,\delta) - P_2(a,\delta)\mathbf{V}(a,0)]\right) - \lim_{\varepsilon,\delta\to 0}\frac{\boldsymbol{P}(\varepsilon,\delta)}{\varepsilon\delta}.$$

Expand this expression and rearrange until you get it equal to

$$\lim_{\delta\to 0}\frac{1}{\delta}\{(\nabla_{\mathbf{x}_1}\mathbf{V})(0,\delta) - P_2(\varepsilon,\delta)[(\nabla_{\mathbf{x}_1}\mathbf{V})(0,0)]\} = \nabla_{\mathbf{x}_2}((\nabla_{\mathbf{x}_1}\mathbf{V})(0,0)).$$

Then conclude the result.

b. *In part* **a** *we can set* $\mathbf{V}$ *equal to* $\mathbf{x}_1(0,0)$ *and, then after parallel transport around the* (*infinitesimal*) *region, the vector* $\mathbf{x}_1(0,0)$ *will change in the* $\mathbf{x}_2(0,0)$ *direction* (*Why?*) *and thus* (*Why?*)

$$\langle(\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}\mathbf{x}_1 - \nabla_{\mathbf{x}_2}\nabla_{\mathbf{x}_1}\mathbf{x}_1), \mathbf{x}_2\rangle_p = \langle\mathbf{x}_1,\mathbf{x}_1\rangle_p\langle\mathbf{x}_2,\mathbf{x}_2\rangle_p K(p).$$

[Hint: Look in Figure 8.4 at the effect of the parallel transport.]

Now we can find an expression for the Gaussian curvature of the surfaces in an n-manifold that are determined by two coordinates:

Intrinsic Definition of Sectional Curvature. *Let* $\mathbf{x}$ *be a* C^2 *local coordinate chart for the open set U in the n-manifold M. Let*

$$p = \mathbf{x}(a^1, a^2, \ldots, a^n)$$

be any point in U. For each $1 \le i,j \le n$, *the* ***sectional curvature*** *of M at p in the section determined by* $\mathbf{x}_i$ *and* $\mathbf{x}_j$ *is the Gaussian curvature of the surface,* S^{ij}, *in U containing p with local coordinates defined by*

$$S^{ij} = \mathbf{x}(a^1,\ldots,a^{i-1},x^i,a^{i+1},\ldots,a^{j-1},x^j,a^{j+1}).$$

By **8.6.b**, *this sectional curvature is*

$$K_p(\mathbf{x}_i \wedge \mathbf{x}_j) = \frac{1}{|\mathbf{x}_i|^2}\frac{1}{\left|\mathbf{x}_j\right|^2}\left\langle\left(\nabla_{\mathbf{x}_i}\nabla_{\mathbf{x}_j}\mathbf{x}_i - \nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}_i}\mathbf{x}_i\right), \mathbf{x}_j\right\rangle_p.$$

Thus, if **X** and **Y** are two orthogonal unit tangent vectors in an *n*-manifold, it makes sense to try to use the expression

$$\langle \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{X} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{X}, \mathbf{Y}\rangle$$

as the measure of curvature of the two-dimensional section determined by two tangent vectors at *p*, **X** and **Y**. This almost works except for the problem that the expression only makes sense if **X** and **Y** are vector *fields* (otherwise you could not differentiate). You could extend **X** and **Y** to vector fields in a neighborhood of *p*, but unfortunately the result would depend on which vector field you choose. There is a way out this dilemma: Someone discovered (I do not know how or who) the following, which you can prove.

c. *For vector fields,* **X**, **Y**, *and* **Z**, *the expression*

$$\boldsymbol{R}_p(\mathbf{Y},\mathbf{X})\mathbf{Z} \equiv \nabla_{\mathbf{X}_p}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}_p}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]_p}\mathbf{Z}$$

only depends on the vectors $\mathbf{X}_p$, $\mathbf{Y}_p$, $\mathbf{Z}_p$ *and not on the rest of the fields.* (*Note that if* **X** *and* **Y** *are coordinate vector fields then, by* **8.5.a**, $[\mathbf{X},\mathbf{Y}] = 0$.) WARNING: Some books define $\mathbf{R}(\mathbf{Y},\mathbf{X})\mathbf{Z}$ as the negative of our definition.

[Hint: If $\mathbf{F}(\mathbf{X})$ is a vector field that depends linearly on another vector field **X**, then there is a trick that works to check whether $\mathbf{F}_p(\mathbf{X})$ depends only on $\mathbf{X}_p$. Let *k* be any real-valued function defined in a neighborhood of *p* such that if $k(p) = 1$, then $\mathbf{F}_p(\mathbf{X})$ depends only on $\mathbf{X}_p$ if and only if

$$\mathbf{F}_p(k\mathbf{X}) = k(p)\mathbf{F}_p(\mathbf{X}) = \mathbf{F}_p(\mathbf{X}).$$

Note that, in this case, and because **F** is linear,

$$\text{if } \mathbf{X} = \Sigma X^i\mathbf{x}_i \text{ then } \mathbf{F}_p(\mathbf{X}) = \Sigma X^i(p)\mathbf{F}_p(\mathbf{x}_i).]$$

The function $\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z}$ *is called the* ***Riemann Curvature Tensor field of type*** **(1,3)**.

It is called a *tensor of type* (1,3) because it depends linearly on its three variables and because $\mathbf{R}_p(\mathbf{X},\mathbf{Y})\mathbf{Z}$ is a vector (thus type (1,–)) that depends only on the three (thus type (–,3)) vectors $\mathbf{X}_p,\mathbf{Y}_p,\mathbf{Z}_p$. (See *Appendix* **A.8**.)

The function

$$\mathbf{R}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{W}) = \langle \mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{Z},\mathbf{W}\rangle$$

is called the ***Riemann Curvature Tensor field of type*** **(0,4)**.

If $\mathbf{X}$ *and* $\mathbf{Y}$ *are two orthogonal unit tangent vectors at p in an n-manifold, then*

$$K_p(\mathbf{X} \wedge \mathbf{Y}) = \mathbf{R}_p(\mathbf{X},\mathbf{Y},\mathbf{X},\mathbf{Y})$$

is called the ***sectional curvature****.*

The sectional curvature is NOT linear in $\mathbf{X}$ and $\mathbf{Y}$. In fact,

$$K_p((\mathbf{A}+\mathbf{B}) \wedge \mathbf{Y}) = \mathbf{R}_p((\mathbf{A}+\mathbf{B}),\mathbf{Y},(\mathbf{A}+\mathbf{B}),\mathbf{Y}) =$$
$$= \mathbf{R}_p(\mathbf{A},\mathbf{Y},\mathbf{A},\mathbf{Y}) + \mathbf{R}_p(\mathbf{A},\mathbf{Y},\mathbf{B},\mathbf{Y}) + \mathbf{R}_p(\mathbf{B},\mathbf{Y},\mathbf{A},\mathbf{Y}) + \mathbf{R}_p(\mathbf{B},\mathbf{Y},\mathbf{B},\mathbf{Y}),$$

which is a phenomenon that we encountered with normal curvature and the second fundamental form. This is the reason that we have to look at Riemann tensors if we want to express sectional curvature with respect to different local coordinates.

If we have local coordinates $\mathbf{x}$*, then the Riemann curvature tensor of type* (1,3) *is determined by the* n^4 *numbers*

$$R^l_{ijk}(p) = l^{\text{th}} \text{ coordinate of } \boldsymbol{R}_p(\mathbf{x}_i,\mathbf{x}_j)\mathbf{x}_k =$$
$$= l^{\text{th}} \text{ coordinate of } \nabla_{\mathbf{x}_j(p)}\nabla_{\mathbf{x}_i(p)}\mathbf{x}_k - \nabla_{\mathbf{x}_i(p)}\nabla_{\mathbf{x}_j(p)}\mathbf{x}_k.$$

or

$$\mathbf{R}(\mathbf{x}_i,\mathbf{x}_j)\mathbf{x}_k = \sum_l R^l_{ijk}\mathbf{x}_l.$$

(Remember that $[\mathbf{x}_i,\mathbf{x}_j] = 0$.)

d. *Show that the* ***Riemann curvature tensor of type*** **(0,4)** *is determined by the* n^4 *numbers*

$$R_{ijkh} \equiv \mathbf{R}(\mathbf{x}_i,\mathbf{x}_j,\mathbf{x}_k,\mathbf{x}_h) = \sum_l R^l_{ijk}g_{lh}.$$

The ***sectional curvature*** *of the two-dimensional subspace of* T_pM *spanned by the* ***orthonormal*** *vectors*

$$\mathbf{X} = \Sigma X^i\mathbf{x}_i \text{ and } \mathbf{Y} = \Sigma Y^j\mathbf{x}_j$$

is given by

$$K(\mathbf{X}\wedge\mathbf{Y}) = \sum_i\sum_j\sum_k\sum_h R_{ijkh}X^iY^jX^kY^h.$$

Calculation of Curvature Tensors in Local Coordinates

We will now find the coefficients of the Riemann curvature tensor with respect to local coordinates $\mathbf{x}$:

$$\mathbf{R}(\mathbf{x}_i\mathbf{x}_j)\mathbf{x}_k = \sum_l R^l_{ijk}\mathbf{x}_l = \nabla_{\mathbf{x}_j}\nabla_{\mathbf{x}_i}\mathbf{x}_k - \nabla_{\mathbf{x}_i}\nabla_{\mathbf{x}_j}\mathbf{x}_k =$$
$$= \nabla_{\mathbf{x}_j}\Big(\sum_l \Gamma^l_{ik}\mathbf{x}_l\Big) - \nabla_{\mathbf{x}_i}\Big(\sum_l \Gamma^l_{jk}\mathbf{x}_l\Big) = \sum_l\Big[(\mathbf{x}_j\Gamma^l_{ik})\mathbf{x}_l + \Gamma^l_{ik}\big(\nabla_{\mathbf{x}_j}\mathbf{x}_l\big)\Big] - \sum_l\Big[\big(\mathbf{x}_i\Gamma^l_{jk}\big)\mathbf{x}_l + \Gamma^l_{jk}(\nabla_{\mathbf{x}_i}\mathbf{x}_l)\Big] =$$
$$= \sum_l\Big[(\mathbf{x}_j\Gamma^l_{ik})\mathbf{x}_l + \Gamma^l_{ik}\big(\Sigma_h\Gamma^h_{jl}\mathbf{x}_h\big)\Big] - \sum_l\Big[\big(\mathbf{x}_i\Gamma^l_{jk}\big)\mathbf{x}_l + \Gamma^l_{jk}(\Sigma_h\Gamma^h_{il}\mathbf{x}_h)\Big] =$$
$$= \sum_l(\mathbf{x}_j\Gamma^l_{ik})\mathbf{x}_l + \Sigma_l\Sigma_h\Gamma^l_{ik}\Gamma^h_{jl}\mathbf{x}_h - \sum_l\big(\mathbf{x}_i\Gamma^l_{jk}\big)\mathbf{x}_l - \Sigma_l\Sigma_h\Gamma^l_{jk}\Gamma^h_{il}\mathbf{x}_h = \Sigma_h\Big[\mathbf{x}_j\Gamma^h_{ik} + \Sigma_l\Gamma^l_{ik}\Gamma^h_{jl} - \mathbf{x}_i\Gamma^h_{jk} - \Sigma_l\Gamma^l_{jk}\Gamma^h_{il}\Big]\mathbf{x}_h.$$

Therefore,

$$R^h_{ijk} = \mathbf{x}_j\Gamma^h_{ik} + \Sigma_l\,\Gamma^l_{ik}\Gamma^h_{jl} - \mathbf{x}_i\Gamma^h_{jk} - \Sigma_l\,\Gamma^l_{jk}\Gamma^h_{il}.$$

For geodesic rectangular coordinates you can calculate (using **8.3.e**):

$$R^2_{212} = 0 \text{ and } R^1_{212} = K,$$

and in agreement with **8.6.a**,

$$|\mathbf{R}(\mathbf{x}_2\mathbf{x}_1)\mathbf{x}_2| = |\mathbf{x}_1|K.$$

A similar calculation will show that

$$R^1_{121} = 0 \text{ and } R^2_{121} = h^2K.$$

For any orthogonal local coordinates, the ***Riemann Curvature Tensor of Type (4,0)*** is defined by the equation

$$R_{ijkl} = g_{ii}R^i_{jkl}\ .$$

Thus, for geodesic rectangular coordinates

$$R_{1212} = R_{2121} = h^2K\ .$$

If $\mathbf{X} = \mathbf{x}_i/|\mathbf{x}_i|$ and $\mathbf{Y} = \mathbf{x}_j/|\mathbf{x}_j|$ are unit vectors in the coordinate directions, then the ***sectional curvature*** of M is calculated as

$$K\left(\frac{\mathbf{x}_i}{|\mathbf{x}_i|} \wedge \frac{\mathbf{x}_j}{|\mathbf{x}_j|}\right) = R_{ijij}\frac{1}{|\mathbf{x}_i|^2}\frac{1}{|\mathbf{x}_j|^2} = K$$

In particular, for geodesic rectangular coordinates, the sectional curvature (with respect to orthogonal unit vectors) is the same as the Gaussian curvature.

PROBLEM 8.7. Intrinsic Calculations in Examples

Find the Riemannian metric, the Christoffel symbols, the Riemann curvature tensors, and the sectional (Gaussian) curvature for:

a. *the cylinder.*

b. *the sphere.*

c. *the torus* $(\mathbf{S}^1\times\mathbf{S}^1)$ *in* $\mathbf{R}^4$ *with coordinates*

$$\mathbf{x}(u^1,u^2) = (\cos u^1, \sin u^1, \cos u^2, \sin u^2).$$

This is usually called the ***flat torus****. Why is this name appropriate?*

d. *the annular hyperbolic plane with respect to its natural geodesic rectangular coordinate system.* (See Problem **1.8**.)

e. *the 3-manifold* $\mathbf{S}^2\times\mathbf{R}\subset\mathbf{R}^4$*, that is the set of those points*

$$\{\ (x,y,z,w) \in \mathbf{R}^4 \mid (x,y,z) \in \mathbf{S}^2 \subset \mathbf{R}^3\ \}.$$

[Hint: Some of these calculations can be done with or without local coordinates. You will gain more understanding by performing the calculations more than one way.]