CHAPTER IV

PARTIAL DIFFERENTIATION; EXPLICIT FUNCTIONS

43. Functions of two or more variables. The definitions and theorems about functions of more than one independent variable are to a large extent similar to those given in Chap. II for functions of a single variable, and the changes and difficulties which occur are for the most part amply illustrated by the case of two variables. The work in the text will therefore be confined largely to this case and the generalizations to functions involving more than two variables may be left as exercises.

If the value of a variable z is uniquely determined when the values (x, y) of two variables are known, z is said to be a function $z = f(x, y)$ of the two variables. The set of values $[(x, y)]$ or of points $P(x, y)$ of the xy-plane for which z is defined may be any set, but usually consists of all the points in a certain area or region of the plane bounded by a curve which may or may not belong to the region, just as the end points of an interval may or may not belong to it. Thus the function $1/\sqrt{1 - x^2 - y^2}$ is defined for all points within the circle $x^2 + y^2 = 1$, but not for points on the perimeter of the circle. For most purposes it is sufficient to think of the boundary of the region of definition as a polygon whose sides are straight lines or such curves as the geometric intuition naturally suggests.

The first way of representing the function $z = f(x, y)$ geometrically is by *the surface* $z = f(x, y)$, just as $y = f(x)$ was represented by a curve. This method is not available for $u = f(x, y, z)$, a function of three variables, or for functions of a greater number of variables; for space has only three dimensions. A second method of representing the function $z = f(x, y)$ is by its *contour lines* in the xy-plane, that is, the curves $f(x, y) = \text{const.}$ are plotted and to each curve is attached the value of the constant. This is the method employed on maps in marking heights above sea level or depths of the ocean below sea level. It is evident that these contour lines are nothing but the projections on the xy-plane of the curves in which the surface $z = f(x, y)$ is cut by the planes $z = \text{const.}$ This method is applicable to functions $u = f(x, y, z)$ of three variables. The *contour surfaces* $u = \text{const.}$ which are thus obtained are frequently called *equipotential surfaces.* If the function is single valued, the contour lines or surfaces cannot intersect one another.

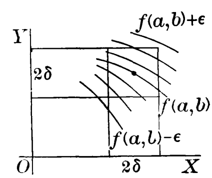
The *function* $z = f(x, y)$ *is continuous for* (a, b) when either of the following equivalent conditions is satisfied:

1°. $\lim f(x, y) = f(a, b)$ *or* $\lim f(x, y) = f(\lim x, \lim y)$, *no matter how the variable point* $P(x, y)$ *approaches* (a, b).

2°. *If for any assigned* ϵ, *a number* δ *may be found so that*

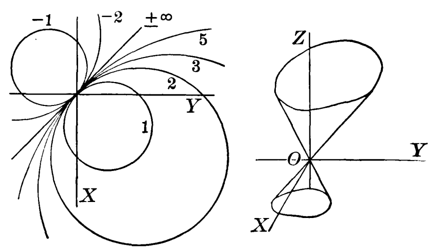
$$|f(x, y) - f(a, b)| < \epsilon \quad \textit{when} \quad |x - a| < \delta,\ |y - b| < \delta.$$

Geometrically this means that if a square with (a, b) as center and with sides of length 2δ parallel to the axes be drawn, the portion of the surface $z = f(x, y)$ above the square will lie between the two planes $z = f(a, b) \pm \epsilon$. Or if contour lines are used, no line $f(x, y) =$ const. where the constant differs from $f(a, b)$ by so much as ϵ will cut into the square. It is clear that in place of a square surrounding (a, b) a circle of radius δ or any other figure which lay within the square might be used.



44. *Continuity examined.* From the definition of continuity just given and from the corresponding definition in § 24, it follows that if $f(x, y)$ is a continuous function of x and y for (a, b), then $f(x, b)$ is a continuous function of x for $x = a$ and $f(a, y)$ is a continuous function of y for $y = b$. That is, if f is continuous in x and y jointly, it is continuous in x and y severally. It might be thought that conversely if $f(x, b)$ is continuous for $x = a$ and $f(a, y)$ for $y = b$, $f(x, y)$ would be continuous in (x, y) for (a, b). That is, if f is continuous in x and y severally, it would be continuous in x and y jointly. A simple example will show that this is *not necessarily true.* Consider the case

$$z = f(x, y) = \frac{x^2 + y^2}{x + y}$$
$$f(0, 0) = 0$$



and examine z for continuity at $(0, 0)$. The functions $f(x, 0) = x$, and $f(0, y) = y$ are surely continuous in their respective variables. But the surface $z = f(x, y)$ is a conical surface (except for the points of the z-axis other than the origin) and it is clear that $P(x, y)$ may approach the origin in such a manner that z shall approach any desired value. Moreover, a glance at the contour lines shows that they all enter any circle or square, no matter how small, concentric with the origin. If P approaches the origin along one of these lines, z remains constant and its limiting value is that constant. In fact by approaching the origin along a set of points which jump from one contour line to another, a method of approach may be found such that z approaches no limit whatsoever but oscillates between wide limits or becomes infinite. Clearly the conditions of continuity are not at all fulfilled by z at $(0, 0)$.

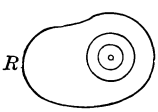
Double limits. There often arise for consideration expressions like

$$\lim_{y \doteq b}\left[\lim_{x \doteq a} f(x, y)\right], \qquad \lim_{x \doteq a}\left[\lim_{y \doteq b} f(x, y)\right], \tag{1}$$

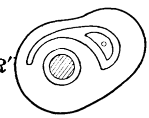
where the limits exist whether x first approaches its limit, and then y its limit, or vice versa, and where the question arises as to whether the two limits thus obtained are equal, that is, whether the order of taking the limits in the double limit may be interchanged. It is clear that if the function $f(x, y)$ is continuous at (a, b), the limits approached by the two expressions will be equal; for the limit of $f(x, y)$ is $f(a, b)$ no matter how (x, y) approaches (a, b). If f is discontinuous at (a, b), it may still happen that the order of the limits in the double limit may be interchanged, as was true in the case above where the value in either order was zero; but this cannot be affirmed in general, and special considerations must be applied to each case when f is discontinuous.

*Varieties of regions.** For both pure mathematics and physics the classification of regions according to their *connectivity* is important. Consider a finite region R bounded by a curve which nowhere cuts itself. (For the present purposes it is not necessary to enter upon the subtleties of the meaning of "curve" (see §§ 127–128); ordinary intuition will suffice.) It is clear that if any closed curve drawn in this region had an unlimited tendency to contract, it could draw together to a point and disappear. On the other hand, if R' be a region like R except that a portion has been removed so that R' is bounded by two curves one within the other, it is clear that some closed curves, namely those which did not encircle the portion removed, could shrink away to a point, whereas other closed curves, namely those which encircled that portion, could at most shrink down into coincidence with the boundary of that portion. Again, if two portions are removed so as to give rise to the region R'', there are circuits around each of the portions which at most can only shrink down to the boundaries of those portions and circuits around both portions which can shrink down to the boundaries and a line joining them. A region like R, where *any* closed curve or circuit may be shrunk away to nothing is called a *simply connected region;* whereas regions in which there are circuits which cannot be shrunk away to nothing are called *multiply connected* regions.

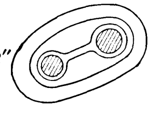
R



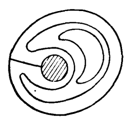
R'



R''



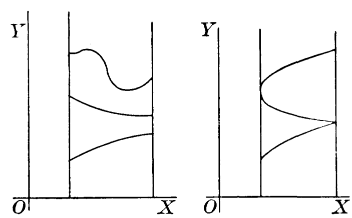
A multiply connected region may be made simply connected by a simple device and convention. For suppose that in R' a line were drawn connecting the two bounding curves and it were agreed that no curve or circuit drawn within R' should cross this line. Then the entire region would be surrounded by a single boundary, part of which would be counted twice. The figure indicates the situation. In like manner if two lines were drawn in R'' connecting both interior boundaries to the exterior or connecting the two interior boundaries together and either of them to the outer boundary, the region would be rendered simply connected. The entire region would have a single boundary of which parts would be counted twice, and any circuit which did not cross the lines could be shrunk away to nothing. The lines



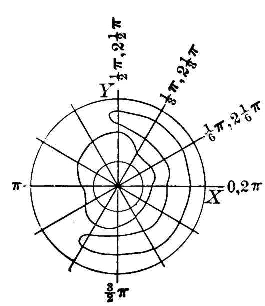
* The discussion from this point to the end of § 45 may be connected with that of §§ 123–126.

thus drawn in the region to make it simply connected are called *cuts*. There is no need that the region be finite; it might extend off indefinitely in some directions like the region between two parallel lines or between the sides of an angle, or like the entire half of the xy-plane for which y is positive. In such cases the cuts may be drawn either to the boundary or off indefinitely in such a way as not to meet the boundary.

45. *Multiple valued functions.* If more than one value of z corresponds to the pair of values (x, y), the function z is multiple valued, and there are some noteworthy differences between multiple valued functions of one variable and of several variables. It was stated (§ 23) that multiple valued functions were divided into branches each of which was single valued. There are two cases to consider when there is one variable, and they are illustrated in the figure. Either there is no value of x in the interval for which the different values of the function are equal and there is consequently a number D which gives the least value of the difference between any two branches, or there is a value of x for which different branches have the same value. Now in the first case, if x changes its value continuously and if $f(x)$ be constrained also to change continuously, there is no possibility of passing from one branch of the function to another; but in the second case such change is possible for, when x passes through the value for which the branches have the same value, the function while constrained to change its value continuously may turn off onto the other branch, although it need not do so.

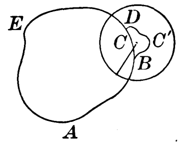


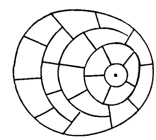
In the case of a function $z = f(x, y)$ of two variables, it is not true that if the values of the function nowhere become equal in or on the boundary of the region over which the function is defined, then it is impossible to pass continuously from one branch to another, and if $P(x, y)$ describes any continuous closed curve or circuit in the region, the value of $f(x, y)$ changing continuously must return to its original value when P has completed the description of the circuit. For suppose the function z be a helicoidal surface $z = a \tan^{-1}(y/x)$, or rather the portion of that surface between two cylindrical surfaces concentric with the axis of the helicoid, as is the case of the surface of the screw of a jack, and the circuit be taken around the inner cylinder. The multiple numbering of the contour lines indicates the fact that the function is multiple valued. Clearly, each time that the circuit is described, the value of z is increased by the amount between the successive branches or leaves of the surface (or decreased by that amount if the circuit is described in the opposite direction). The region here dealt with is not simply connected and the circuit cannot be shrunk to nothing — which is the key to the situation.



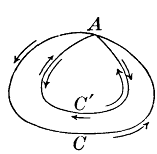
Theorem. If the difference between the different values of a continuous multiple valued function is never less than a finite number D for any set (x, y) of values of the variables whether in or upon the boundary of the region of definition, then the value $f(x, y)$ of the function, constrained to change continuously, will return to its initial value when the point $P(x, y)$, describing a closed curve which can be shrunk to nothing, completes the circuit and returns to its starting point.

Now owing to the continuity of f throughout the region, it is possible to find a number δ so that $|f(x, y) - f(x', y')| < \epsilon$ when $|x - x'| < \delta$ and $|y - y'| < \delta$ no matter what points of the region (x, y) and (x', y') may be. Hence the values of f at any two points of a small region which lies within any circle of radius $\frac{1}{2}\delta$ cannot differ by so much as the amount D. If, then, the circuit is so small that it may be inclosed within such a circle, there is no possibility of passing from one value of f to another when the circuit is described and f must return to its initial value. Next let there be given any circuit such that the value of f starting from a given value $f(x, y)$ returns to that value when the circuit has been completely described. Suppose that a modification were introduced in the circuit by enlarging or diminishing the inclosed area by a small area lying wholly within a circle of radius $\frac{1}{2}\delta$. Consider the circuit $ABCDEA$ and the modified circuit $ABC'DEA$. As these circuits coincide except for the arcs BCD and $BC'D$, it is only necessary to show that f takes on the same value at D whether D is reached from B by the way of C or by the way of C'. But this is necessarily so for the reason that both arcs are within a circle of radius $\frac{1}{2}\delta$. Then the value of f must still return to its initial value $f(x, y)$ when the modified circuit is described. Now to complete the proof of the theorem, it suffices to note that any circuit which can be shrunk to nothing can be made up by piecing together a number of small circuits as shown in the figure. Then as the change in f around any one of the small circuits is zero, the change must be zero around 2, 3, 4, $\cdots$ adjacent circuits, and thus finally around the complete large circuit.

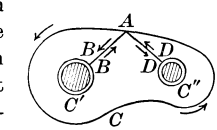




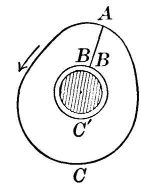
Reducibility of circuits. If a circuit can be shrunk away to nothing, it is said to be *reducible;* if it cannot, it is said to be *irreducible.* In a simply connected region all circuits are reducible; in a multiply connected region there are an infinity of irreducible circuits. Two circuits are said to be *equivalent* or reducible to each other when either can be expanded or shrunk into the other. The change in the value of f on passing around two equivalent circuits from A to A is the same, provided the circuits are described in the same direction. For consider the figure and the equivalent circuits ACA and $AC'A$ described as indicated by the large arrows. It is clear that either may be modified little by little, as indicated in the proof above, until it has been changed into the other. Hence the change in the value of f around the two circuits is the same. Or, as another proof, it may be observed that the combined circuit $ACAC'A$, where the second is described as indicated by the small arrows, may be regarded as a reducible circuit which touches itself at A. Then the change of f around the circuit is zero and f must lose as much on passing from A to A by C' as it gains in passing from A to A by C. Hence on passing from A to A by C' in the direction of the large arrows the gain in f must be the same as on passing by C.



It is now possible to see that *any circuit ABC may be reduced to circuits around the portions cut out of the region combined with lines going to and from A and the boundaries.* The figure shows this; for the circuit $ABC'BADC''DA$ is clearly reducible to the circuit ACA. It must not be forgotten that although the lines AB and BA coincide, the values of the function are not necessarily the same on AB as on BA but differ by the amount of change introduced in f on passing around the irreducible circuit $BC'B$. One of the cases which arises most frequently in practice is that in which the successive branches of $f(x, y)$ differ by a constant amount as in the case $z = \tan^{-1}(y/x)$ where 2π is the difference between successive values of z for the same values of the variables.



If now a circuit such as $ABC'BA$ be considered, where it is imagined that the origin lies within $BC'B$, it is clear that the values of z along AB and along BA differ by 2π, and whatever z gains on passing from A to B will be lost on passing from B to A, although the values through which z changes will be different in the two cases by the amount 2π. Hence the circuit $ABC'BA$ gives the same changes for z as the simpler circuit $BC'B$. In other words the result is obtained that *if the different values of a multiple valued function for the same values of the variables differ by a constant independent of the values of the variables, any circuit may be reduced to circuits about the boundaries of the portions removed;* in this case the lines going from the point A to the boundaries and back may be discarded.



EXERCISES

1. Draw the contour lines and sketch the surfaces corresponding to

$$(\alpha)\ z = \frac{x+y}{x-y}, \quad z(0, 0) = 0, \qquad (\beta)\ z = \frac{xy}{x+y}, \quad z(0, 0) = 0.$$

Note that here and in the text only one of the contour lines passes through the origin although an infinite number have it as a frontier point between two parts of the same contour line. Discuss the double limits $\lim_{x \doteq 0} \lim_{y \doteq 0} z$, $\lim_{y \doteq 0} \lim_{x \doteq 0} z$.

2. Draw the contour lines and sketch the surfaces corresponding to

$$(\alpha)\ z = \frac{x^2 + y^2 - 1}{2y}, \qquad (\beta)\ z = \frac{y^2}{x}, \qquad (\gamma)\ z = \frac{x^2 + 2y^2 - 1}{2x^2 + y^2 - 1}.$$

Examine particularly the behavior of the function in the neighborhood of the apparent points of intersection of different contour lines. Why apparent?

3. State and prove for functions of two independent variables the generalizations of Theorems 6–11 of Chap. II. Note that the theorem on uniformity is proved for two variables by the application of Ex. 9, p. 40, in almost the identical manner as for the case of one variable.

4. Outline definitions and theorems for functions of three variables. In particular indicate the contour surfaces of the functions

$$(\alpha)\ u = \frac{x + y + 2z}{x - y - z}, \qquad (\beta)\ u = \frac{x^2 + y^2 + z^2}{x + y + z}, \qquad (\gamma)\ u = \frac{xy}{z},$$

and discuss the triple limits as x, y, z in different orders approach the origin.

5. Let $z = P(x, y)/Q(x, y)$, where P and Q are polynomials, be a rational function of x and y. Show that if the curves $P = 0$ and $Q = 0$ intersect in any points, all the contour lines of z will converge toward these points; and conversely show that if two different contour lines of z apparently cut in some point, all the contour lines will converge toward that point, P and Q will there vanish, and z will be undefined.

6. If D is the minimum difference between different values of a multiple valued function, as in the text, and if the function returns to its initial value plus $D' \geqq D$ when P describes a circuit, show that it will return to its initial value plus $D' \geqq D$ when P describes the new circuit formed by piecing on to the given circuit a small region which lies within a circle of radius $\frac{1}{2}\delta$.

7. Study the function $z = \tan^{-1}(y/x)$, noting especially the relation between contour lines and the surface. To eliminate the origin at which the function is not defined draw a small circle about the point (0, 0) and observe that the region of the whole xy-plane outside this circle is not simply connected but may be made so by drawing a cut from the circumference off to an infinite distance. Study the variation of the function as P describes various circuits.

8. Study the contour lines and the surfaces due to the functions

$$(\alpha)\ z = \tan^{-1} xy, \qquad (\beta)\ z = \tan^{-1}\frac{1-x^2}{1-y^2}, \qquad (\gamma)\ z = \sin^{-1}(x-y).$$

Cut out the points where the functions are not defined and follow the changes in the functions about such circuits as indicated in the figures of the text. How may the region of definition be made simply connected?

9. Consider the function $z = \tan^{-1}(P/Q)$ where P and Q are polynomials and where the curves $P = 0$ and $Q = 0$ intersect in n points $(a_1, b_1), (a_2, b_2), \cdots, (a_n, b_n)$ but are not tangent (the polynomials have common solutions which are not multiple roots). Show that the value of the function will change by $2k\pi$ if (x, y) describes a circuit which includes k of the points. Illustrate by taking for P/Q the fractions in Ex. 2.

10. Consider regions or volumes in space. Show that there are regions in which some circuits cannot be shrunk away to nothing; also regions in which all circuits may be shrunk away but not all closed surfaces.

46. First partial derivatives. Let $z = f(x, y)$ be a single valued function, or one branch of a multiple valued function, defined for (a, b) and for all points in the neighborhood. If y be given the value b, then z becomes a function $f(x, b)$ of x alone, and if that function has a derivative for $x = a$, that derivative is called the *partial derivative* of $z = f(x, y)$ with respect to x at (a, b). Similarly, if x is held fast and equal to a and if $f(a, y)$ has a derivative when $y = b$, that derivative is called the partial derivative of z with respect to y at (a, b). To obtain these derivatives formally in the case of a given function $f(x, y)$ it is merely necessary to differentiate the function by the ordinary rules, treating y as a constant when finding the derivative with respect to x and x as a constant for the derivative with respect to y. Notations are

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x'' = f_x' = z_x' = D_x f = D_x z = \left(\frac{dz}{dx}\right)_y$$

for the x-derivative with similar ones for the y-derivative. The partial derivatives are the limits of the quotients

$$\lim_{h \doteq 0} \frac{f(a+h, b) - f(a, b)}{h}, \quad \lim_{k \doteq 0} \frac{f(a, b+k) - f(a, b)}{k}, \tag{2}$$

provided those limits exist. The application of the Theorem of the Mean to the functions $f(x, b)$ and $f(a, y)$ gives

$$\begin{aligned} f(a+h, b) - f(a, b) &= hf'_x(a + \theta_1 h, b), \quad 0 < \theta_1 < 1, \\ f(a, b+k) - f(a, b) &= kf'_y(a, b + \theta_2 k), \quad 0 < \theta_2 < 1, \end{aligned} \tag{3}$$

under the proper but evident restrictions (see § 26).

Two comments may be made. First, some writers denote the partial derivatives by the same symbols dz/dx and dz/dy as if z were a function of only one variable and were differentiated with respect to that variable; and if they desire especially to call attention to the other variables which are held constant, they affix them as subscripts as shown in the last symbol given (p. 93). This notation is particularly prevalent in thermodynamics. As a matter of fact, it would probably be impossible to devise a simple notation for partial derivatives which should be satisfactory for all purposes. The only safe rule to adopt is to use a notation which is sufficiently explicit for the purposes in hand, and at all times to pay careful attention to what the derivative actually means in each case. Second, it should be noted that for points on the boundary of the region of definition of $f(x, y)$ there may be merely right-hand or left-hand partial derivatives or perhaps none at all. For it is necessary that the lines $y = b$ and $x = a$ cut into the region on one side or the other in the neighborhood of (a, b) if there is to be a derivative even one-sided; and at a corner of the boundary it may happen that neither of these lines cuts into the region.

THEOREM. If $f(x, y)$ and its derivatives f'_x and f'_y are continuous functions of (x, y) in the neighborhood of (a, b), the increment Δf may be written in any of the three forms

$$\begin{aligned} \Delta f &= f(a+h, b+k) - f(a, b) \\ &= hf'_x(a + \theta_1 h, b) + kf'_y(a+h, b + \theta_2 k) \\ &= hf'_x(a + \theta h, b + \theta k) + kf'_y(a + \theta h, b + \theta k) \\ &= hf'_x(a, b) + kf'_y(a, b) + \zeta_1 h + \zeta_2 k, \end{aligned} \tag{4}$$

where the θ's are proper fractions, the ζ's infinitesimals.

To prove the first form, add and subtract $f(a+h, b)$; then

$$\begin{aligned} \Delta f &= [f(a+h, b) - f(a, b)] + [f(a+h, b+k) - f(a+h, b)] \\ &= hf'_x(a + \theta_1 h, b) + kf'_y(a+h, b + \theta_2 k) \end{aligned}$$

by the application of the Theorem of the Mean for functions of a single variable (§§ 7, 26). The application may be made because the function is continuous and the indicated derivatives exist. Now if the derivatives are also continuous, they may be expressed as

$$f'_x(a + \theta_1 h, b) = f'_x(a, b) + \zeta_1, \qquad f'_y(a+h, b + \theta_2 k) = f'_y(a, b) + \zeta_2$$

where ζ_1, ζ_2 may be made as small as desired by taking h and k sufficiently small. Hence the third form follows from the first. The second form, which is symmetric in the increments h, k, may be obtained by writing $x = a + th$ and $y = b + tk$. Then $f(x, y) = \Phi(t)$. As f is continuous in (x, y), the function Φ is continuous in t and its increment is

$$\Delta\Phi = f(a + \overline{t + \Delta t}h, b + \overline{t + \Delta t}k) - f(a + th, b + tk).$$

This may be regarded as the increment of f taken from the point (x, y) with $\Delta t \cdot h$ and $\Delta t \cdot k$ as increments in x and y. Hence $\Delta\Phi$ may be written as

$$\Delta\Phi = \Delta t \cdot hf'_x(a + th, b + tk) + \Delta t \cdot kf'_y(a + th, b + tk) + \zeta_1 \Delta t \cdot h + \zeta_2 \Delta t \cdot k.$$

Now if $\Delta\Phi$ be divided by Δt and Δt be allowed to approach zero, it is seen that

$$\lim \frac{\Delta\Phi}{\Delta t} = hf'_x(a + th, b + tk) + kf'_y(a + th, b + tk) = \frac{d\Phi}{dt}.$$

The Theorem of the Mean may now be applied to Φ to give $\Phi(1) - \Phi(0) = 1 \cdot \Phi'(\theta)$, and hence

$$\begin{aligned}\Phi(1) - \Phi(0) &= f(a + h, b + k) - f(a, b) \\ &= \Delta f = hf'_x(a + \theta h, b + \theta k) + kf'_y(a + \theta h, b + \theta k).\end{aligned}$$

47. The *partial differentials* of f may be defined as

$$\begin{aligned} d_x f &= f'_x \Delta x, \quad \text{so that} \quad dx = \Delta x, \quad \frac{d_x f}{dx} = \frac{\partial f}{\partial x}, \\ d_y f &= f'_y \Delta y, \quad \text{so that} \quad dy = \Delta y, \quad \frac{d_y f}{dy} = \frac{\partial f}{\partial y}, \end{aligned} \tag{5}$$

where the indices x and y introduced in $d_x f$ and $d_y f$ indicate that x and y respectively are alone allowed to vary in forming the corresponding partial differentials. The *total differential*

$$df = d_x f + d_y f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy, \tag{6}$$

which is the sum of the partial differentials, may be defined as that sum; but it is better defined as that part of the increment

$$\Delta f = \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \zeta_1 \Delta x + \zeta_2 \Delta y \tag{7}$$

which is obtained by neglecting the terms $\zeta_1 \Delta x + \zeta_2 \Delta y$, which are of higher order than Δx and Δy. The total differential may therefore be computed by finding the partial derivatives, multiplying them respectively by dx and dy, and adding.

The total differential of $z = f(x, y)$ may be formed for (x_0, y_0) as

$$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y - y_0), \tag{8}$$

where the values $x - x_0$ and $y - y_0$ are given to the independent differentials dx and dy, and $df = dz$ is written as $z - z_0$. This, however, is the equation of a plane since x and y are independent. The difference $\Delta f - df$ which measures the distance from the plane to the surface along a parallel to the z-axis is of higher order than $\sqrt{\Delta x^2 + \Delta y^2}$; for

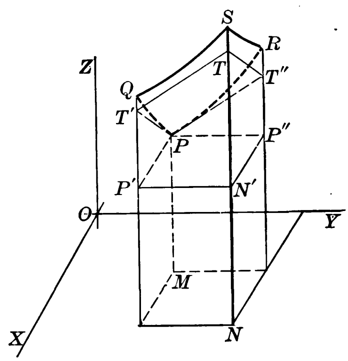
$$\left|\frac{\Delta f - df}{\sqrt{\Delta x^2 + \Delta y^2}}\right| = \left|\frac{\zeta_1 \Delta x + \zeta_2 \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}\right| < |\zeta_1| + |\zeta_2| \doteq 0.$$

Hence the plane (8) will be defined as the *tangent plane* at (x_0, y_0, z_0) to the surface $z = f(x, y)$. The normal to the plane is

$$\frac{x - x_0}{\left(\dfrac{\partial f}{\partial x}\right)_0} = \frac{y - y_0}{\left(\dfrac{\partial f}{\partial y}\right)_0} = \frac{z - z_0}{-1}, \tag{9}$$

which will be defined as the *normal to the surface* at (x_0, y_0, z_0). The tangent plane will cut the planes $y = y_0$ and $x = x_0$ in lines of which the slope is f'_{x_0} and f'_{y_0}. The surface will cut these planes in curves which are tangent to the lines.

In the figure, $PQSR$ is a portion of the surface $z = f(x, y)$ and $PT'TT''$ is a corresponding portion of its tangent plane at $P(x_0, y_0, z_0)$. Now the various values may be read off.



$$PP' = \Delta x, \qquad P'Q = \Delta_x f,$$
$$P'T'/PP' = f'_x, \qquad P'T' = d_x f,$$
$$PP'' = \Delta y, \qquad P''R = \Delta_y f,$$
$$P''T''/PP'' = f'_y, \qquad P''T'' = d_y f,$$
$$P'T' + P''T'' = N'T, \qquad N'S = \Delta f,$$
$$N'T = df = d_x f + d_y f.$$

48. If the variables x and y are expressed as $x = \phi(t)$ and $y = \psi(t)$ so that $f(x, y)$ becomes a function of t, the derivative of f with respect to t is found from the expression for the increment of f.

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t} + \zeta_1 \frac{\Delta x}{\Delta t} + \zeta_2 \frac{\Delta y}{\Delta t}$$

or

$$\lim_{\Delta t \doteq 0} \frac{\Delta f}{\Delta t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{10}$$

The conclusion requires that x and y should have finite derivatives with respect to t. The differential of f as a function of t is

$$df = \frac{df}{dt}dt = \frac{\partial f}{\partial x}\frac{dx}{dt}dt + \frac{\partial f}{\partial y}\frac{dy}{dt}dt = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{11}$$

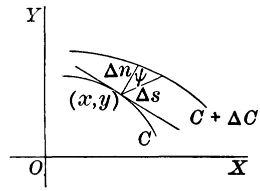
and hence it appears that *the differential has the same form as the total differential.* This result will be generalized later.

As a particular case of (10) suppose that x and y are so related that the point (x, y) moves along a line inclined at an angle τ to the x-axis. If s denote distance along the line, then

$$x = x_0 + s \cos\tau, \quad y = y_0 + s \sin\tau, \quad dx = \cos\tau ds, \quad dy = \sin\tau ds \tag{12}$$

and
$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} = f'_x \cos\tau + f'_y \sin\tau. \tag{13}$$

The derivative (13) is called the *directional derivative* of f in the direction of the line. The partial derivatives f'_x, f'_y are the particular directional derivatives along the directions of the x-axis and y-axis. The directional derivative of f in any direction is the rate of increase of f along that direction; if $z = f(x, y)$ be interpreted as a surface, the directional derivative is the slope of the curve in which a plane through the line (12) and perpendicular to the xy-plane cuts the surface. If $f(x, y)$ be represented by its contour lines, the derivative at a point (x, y) in any direction is the limit of the ratio $\Delta f/\Delta s = \Delta C/\Delta s$ of the increase of f, from one contour line to a neighboring one, to the distance between the lines in that direction. It is therefore evident that the derivative along any contour line is zero and that the derivative along the normal to the contour line is greater than in any other direction because the element dn of the normal is less than ds in any other direction. In fact, apart from infinitesimals of higher order,



$$\frac{\Delta n}{\Delta s} = \cos\psi, \quad \frac{\Delta f}{\Delta s} = \frac{\Delta f}{\Delta n}\cos\psi, \quad \frac{df}{ds} = \frac{df}{dn}\cos\psi. \tag{14}$$

Hence it is seen that *the derivative along any direction may be found by multiplying the derivative along the normal by the cosine of the angle between that direction and the normal.* The derivative along the normal to a contour line is called the *normal derivative* of f and is, of course, a function of (x, y).

49. Next suppose that $u = f(x, y, z, \cdots)$ is a function of any number of variables. The reasoning of the foregoing paragraphs may be repeated without change except for the additional number of variables. The increment of f will take any of the forms

$$\begin{aligned}\Delta f &= f(a + h, b + k, c + l, \cdots) - f(a, b, c, \cdots) \\ &= hf'_x(a + \theta_1 h, b, c, \cdots) + kf'_y(a + h, b + \theta_2 k, c, \cdots) \\ &\quad + lf'_z(a + h, b + k, c + \theta_3 l, \cdots) + \cdots \\ &= [hf'_x + kf'_y + lf'_z + \cdots]_{a+\theta h,\, b+\theta k,\, c+\theta l,\, \cdots} \\ &= hf'_x + kf'_y + lf'_z + \cdots + \zeta_1 h + \zeta_2 k + \zeta_3 l + \cdots,\end{aligned}$$

and the total differential will naturally be defined as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \cdots, \tag{16}$$

and finally if $x, y, z, \cdots$ be functions of t, it follows that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \cdots \tag{17}$$

and the differential of f as a function of t is still (16).

If the variables $x, y, z, \cdots$ were expressed in terms of several new variables $r, s, \cdots$, the function f would become a function of those variables. To find the partial derivative of f with respect to one of those variables, say r, the remaining ones, $s, \cdots$, would be held constant and f would for the moment become a function of r alone, and so would x, $y, z, \cdots$. Hence (17) may be applied to obtain the partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r} + \cdots, \\ \text{and} \quad \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} + \cdots, \text{ etc.} \end{aligned} \tag{18}$$

These are the formulas for *change of variable* analogous to (4) of § 2. If these equations be multiplied by Δr, Δs, $\cdots$ and added,

$$\frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial s}\Delta s + \cdots = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}\Delta r + \frac{\partial x}{\partial s}\Delta s + \cdots\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial r}\Delta r + \cdots\right) + \cdots,$$

or

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \cdots;$$

for when $r, s, \cdots$ are the independent variables, the parentheses above are $dx, dy, dz, \cdots$ and the expression on the left is df.

Theorem. The expression of the total differential of a function of $x, y, z, \cdots$ as $df = f'_x dx + f'_y dy + f'_z dz + \cdots$ is the same whether x, y, $z, \cdots$ are the independent variables or functions of other independent variables $r, s, \cdots$; it being assumed that all the derivatives which occur, whether of f by $x, y, z, \cdots$ or of $x, y, z, \cdots$ by $r, s, \cdots$, are continuous functions.

By the same reasoning or by virtue of this theorem the rules

$$\begin{aligned} &d(cu) = cdu, \quad d(u + v - w) = du + dv - dw, \\ &d(uv) = udv + vdu, \quad d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}, \end{aligned} \tag{19}$$

of the differential calculus will apply to calculate the total differential of combinations or functions of several variables. If by this means, or any other, there is obtained an expression

$$df = R(r, s, t, \cdots)dr + S(r, s, t, \cdots)ds + T(r, s, t, \cdots)dt + \cdots \quad (20)$$

for the total differential in which $r, s, t, \cdots$ are *independent* variables, the coefficients $R, S, T, \cdots$ are the derivatives

$$R = \frac{\partial f}{\partial r}, \quad S = \frac{\partial f}{\partial s}, \quad T = \frac{\partial f}{\partial t}, \cdots. \quad (21)$$

For in the equation $df = Rdr + Sds + Tdt + \cdots = f'_r dr + f'_s ds + f'_t dt + \cdots$, the variables $r, s, t, \cdots$, being independent, may be assigned increments absolutely at pleasure and if the particular choice $dr = 1$, $ds = dt = \cdots = 0$, be made, it follows that $R = f'_r$; and so on. The single equation (20) is thus equivalent to the equations (21) in number equal to the number of the independent variables.

As an example, consider the case of the function $\tan^{-1}(y/x)$. By the rules (19),

$$d\tan^{-1}\frac{y}{x} = \frac{d(y/x)}{1+(y/x)^2} = \frac{dy/x - ydx/x^2}{1+(y/x)^2} = \frac{xdy - ydx}{x^2+y^2}.$$

Then $$\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x} = -\frac{y}{x^2+y^2}, \quad \frac{\partial}{\partial y}\tan^{-1}\frac{y}{x} = \frac{x}{x^2+y^2}, \quad \text{by (20)–(21).}$$

If y and x were expressed as $y = \sinh rst$ and $x = \cosh rst$, then

$$d\tan^{-1}\frac{y}{x} = \frac{xdy - ydx}{x^2+y^2} = \frac{[stdr + rtds + rsdt][\cosh^2 rst - \sinh^2 rst]}{\cosh^2 rst + \sinh^2 rst}$$

and $$\frac{\partial f}{\partial r} = \frac{st}{\cosh 2\,rst}, \quad \frac{\partial f}{\partial s} = \frac{rt}{\cosh 2\,rst}, \quad \frac{\partial f}{\partial t} = \frac{rs}{\cosh 2\,rst}.$$

EXERCISES

1. Find the partial derivatives f'_x, f'_y or f'_x, f'_y, f'_z of these functions:

(α) $\log(x^2+y^2)$, (β) $e^x \cos y \sin z$, (γ) $x^2 + 3xy + y^3$,

(δ) $\dfrac{xy}{x+y}$, (ϵ) $\dfrac{e^{xy}}{e^x + e^y}$, ($\zeta$) $\log(\sin x + \sin^2 y + \sin^3 z)$,

(η) $\sin^{-1}\dfrac{y}{x}$, (θ) $\dfrac{z}{x}e^{\frac{y}{x}}$, ($\iota$) $\tanh^{-1}\sqrt{2}\left(\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right)^{\frac{1}{2}}$.

2. Apply the definition (2) directly to the following to find the partial derivatives at the indicated points:

(α) $\dfrac{xy}{x+y}$ at (1, 1), (β) $x^2 + 3xy + y^3$ at (0, 0), and (γ) at (1, 1),

(δ) $\dfrac{x-y}{x+y}$ at (0, 0); also try differentiating and substituting (0, 0).

3. Find the partial derivatives and hence the total differential of:

(α) $\dfrac{e^{xy}}{x^2+y^2}$, (β) $x\log yz$, (γ) $\sqrt{a^2 - x^2 - y^2}$,

(δ) $e^{-x}\sin y$, (ϵ) $e^{z^2}\sinh xy$, (ζ) $\log\tan\left(x + \dfrac{\pi}{4}y\right)$,

(η) $\left(\dfrac{y}{z}\right)^x$, ($\theta$) $\dfrac{x-y}{x+z}$, (ι) $\log\left(\dfrac{3x}{y^2} + \sqrt{1 + \dfrac{z^2x^2}{y^4}}\right)$.

4. Find the general equations of the tangent plane and normal line to these surfaces and find the equations of the plane and line for the indicated (x_0, y_0):

(α) the helicoid $z = k\tan^{-1}(y/x)$, $(1, 0), (1, -1), (0, 1)$,
(β) the paraboloid $4pz = (x^2 + y^2)$, $(0, p), (2p, 0), (p, -p)$,
(γ) the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$, $(0, -\frac{1}{2}a), (\frac{1}{2}a, \frac{1}{2}a), (\frac{1}{2}\sqrt{3}\,a, 0)$,
(δ) the cubic $xyz = 1$, $(1, 1, 1), (-\frac{1}{2}, -\frac{1}{2}, 4), (4, \frac{1}{2}, \frac{1}{2})$.

5. Find the derivative with respect to t in these cases by (10):

(α) $f = x^2 + y^2$, $x = a\cos t$, $y = b\sin t$, (β) $\tan^{-1}\sqrt{\dfrac{y}{x}}$, $y = \cosh t$, $x = \sinh t$,

(γ) $\sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, (δ) $\cos 2xy$, $x = \tan^{-1} t$, $y = \cot^{-1} t$.

6. Find the directional derivative in the direction indicated and obtain its numerical value at the points indicated:

(α) x^2y, $\tau = 45°$, $(1, 2)$, (β) $\sin^2 xy$, $\tau = 60°$, $(\sqrt{3}, -2)$.

7. (α) Determine the maximum value of df/ds from (13) by regarding τ as variable and applying the ordinary rules. Show that the direction that gives the maximum is

$$\tau = \tan^{-1}\frac{f_y'}{f_x'}, \quad \text{and then} \quad \frac{df}{dn} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

(β) Show that the sum of the squares of the derivatives along any two perpendicular directions is the same and is the square of the normal derivative.

8. Show that $(f_x' + y'f_y')/\sqrt{1 + y'^2}$ and $(f_x'y' - f_y')/\sqrt{1 + y'^2}$ are the derivatives of f along the curve $y = \phi(x)$ and normal to the curve.

9. If df/dn is defined by the work of Ex. 7 (α), prove (14) as a consequence.

10. Apply the formulas for the change of variable to the following cases:

(α) $r = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}\dfrac{y}{x}$. Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2}$.

(β) $x = r\cos\phi$, $y = r\sin\phi$. Find $\dfrac{\partial f}{\partial r}, \dfrac{\partial f}{\partial \phi}, \left(\dfrac{\partial f}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial f}{\partial \phi}\right)^2$.

(γ) $x = 2r - 3s + 7$, $y = -r + 8s - 9$. Find $\dfrac{\partial u}{\partial r} = 4x + 2y$ if $u = x^2 - y^2$.

(δ) $\begin{cases} x = x'\cos\alpha - y'\sin\alpha, \\ y = x'\sin\alpha + y'\cos\alpha. \end{cases}$ Show $\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 = \left(\dfrac{\partial f}{\partial x'}\right)^2 + \left(\dfrac{\partial f}{\partial y'}\right)^2$.

(ϵ) Prove $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = 0$ if $f(u, v) = f(x - y, y - x)$.

(ζ) Let $x = ax' + by' + cz'$, $y = a'x' + b'y' + c'z'$, $z = a''x' + b''y' + c''z'$, where $a, b, c, a', b', c', a'', b'', c''$ are the direction cosines of new rectangular axes with respect to the old. This transformation is called an *orthogonal transformation*. Show

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{\partial f}{\partial x'}\right)^2 + \left(\frac{\partial f}{\partial y'}\right)^2 + \left(\frac{\partial f}{\partial z'}\right)^2 = \left(\frac{df}{dn}\right)^2.$$

11. Define directional derivative in space; also normal derivative and establish (14) for this case. Find the normal derivative of $f = xyz$ at $(1, 2, 3)$.

12. Find the total differential and hence the partial derivatives in Exs. 1, 3, and

(α) $\log(x^2 + y^2 + z^2)$, (β) y/x, (γ) $x^2ye^{xy^2}$, (δ) $xyz\log xyz$,

(ϵ) $u = x^2 - y^2$, $x = r\cos st$, $y = s\sin rt$. Find $\partial u/\partial r$, $\partial u/\partial s$, $\partial u/\partial t$.

(ζ) $u = y/x$, $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$. Find u_r', u_ϕ', u_θ'.

(η) $u = e^{xy}$, $x = \log\sqrt{r^2+s^2}$, $y = \tan^{-1}(s/r)$. Find u_r', u_s'.

13. If $\dfrac{\partial f}{\partial x} = \dfrac{\partial g}{\partial y}$ and $\dfrac{\partial f}{\partial y} = -\dfrac{\partial g}{\partial x}$, show $\dfrac{\partial f}{\partial r} = \dfrac{1}{r}\dfrac{\partial g}{\partial \phi}$ and $\dfrac{1}{r}\dfrac{\partial f}{\partial \phi} = -\dfrac{\partial g}{\partial r}$ if r, ϕ are polar coördinates and f, g are any two functions.

14. If $p(x, y, z, t)$ is the pressure in a fluid, or $\rho(x, y, z, t)$ is the density, depending on the position in the fluid and on the time, and if u, v, w are the velocities of the particles of the fluid along the axes,

$$\frac{dp}{dt} = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} + \frac{\partial p}{\partial t} \quad \text{and} \quad \frac{d\rho}{dt} = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \frac{\partial \rho}{dt}.$$

Explain the meaning of each derivative and prove the formula.

15. If $z = xy$, interpret z as the area of a rectangle and mark d_xz, Δ_yz, Δz on the figure. Consider likewise $u = xyz$ as the volume of a rectangular parallelepiped.

16. *Small errors.* If $f(x, y)$ be a quantity determined by measurements on x and y, the error in f due to small errors dx, dy in x and y may be estimated as $df = f_x'dx + f_y'dy$ and the relative error may be taken as $df \div f = d\log f$. Why is this?

(α) Suppose $S = \frac{1}{2}ab\sin C$ be the area of a triangle with $a = 10$, $b = 20$, $C = 30°$. Find the error and the relative error if a is subject to an error of 0.1. *Ans.* 0.5, 1%.

(β) In (α) suppose C were liable to an error of 10′ of arc. *Ans.* 0.27, $\frac{1}{2}$%.

(γ) If a, b, C are liable to errors of 1%, the combined error in S may be 3.1%.

(δ) The radius r of a capillary tube is determined from $13.6\pi r^2 l = w$ by finding the weight w of a column of mercury of length l. If $w = 1$ gram with an error of 10^{-3} gr. and $l = 10$ cm. with an error of 0.2 cm., determine the possible error and relative error in r. *Ans.* 1.2%, 6×10^{-4}, mostly due to error in l.

(ϵ) The formula $c^2 = a^2 + b^2 - 2ab\cos C$ is used to determine c where $a = 20$, $b = 20$, $C = 60°$ with possible errors of 0.1 in a and b and 30′ in C. Find the possible absolute and relative errors in c. *Ans.* $\frac{1}{4}$, $1\frac{1}{4}$%.

(ζ) The possible percentage error of a product is the sum of the percentage errors of the factors.

(η) The constant g of gravity is determined from $g = 2st^{-2}$ by observing a body fall. If s is set at 4 ft. and t determined at about $\frac{1}{2}$ sec., show that the error in g is almost wholly due to the error in t, that is, that s can be set very much more accurately than t can be determined. For example, find the error in t which would make the same error in g as an error of $\frac{1}{8}$ inch in s.

(θ) The constant g is determined by $gt^2 = \pi^2 l$ with a pendulum of length l and period t. Suppose t is determined by taking the time 100 sec. of 100 beats of the pendulum with a stop watch that measures to $\frac{1}{5}$ sec. and that l may be measured as 100 cm. accurate to $\frac{1}{2}$ millimeter. Discuss the errors in g.

17. Let the coördinate x of a particle be $x = f(q_1, q_2)$ and depend on two independent variables q_1, q_2. Show that the velocity and kinetic energy are

$$v = f_{q_1}'\frac{dq_1}{dt} + f_{q_2}'\frac{dq_2}{dt}, \qquad T = \tfrac{1}{2}mv^2 = a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2,$$

where dots denote differentiation by t, and a_{11}, a_{12}, a_{22} are functions of (q_1, q_2). Show $\dfrac{\partial v}{\partial \dot{q}_i} = \dfrac{\partial x}{\partial q_i}$, $i = 1, 2$, and similarly for any number of variables q.

18. The helix $x = a\cos t$, $y = a\sin t$, $z = t\tan\alpha$ cuts the sphere $x^2 + y^2 + z^2 = a^2\sec^2\beta$ at $\sin^{-1}(\sin\alpha\sin\beta)$.

19. Apply the Theorem of the Mean to prove that $f(x, y, z)$ is a constant if $f'_x = f'_y = f'_z = 0$ is true for all values of x, y, z. Compare Theorem 16 (§ 27) and make the statement accurate.

20. Transform $\dfrac{df}{dn} = \sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2}$ to (α) cylindrical and (β) polar coördinates (§ 40).

21. Find the angle of intersection of the helix $x = 2\cos t$, $y = 2\sin t$, $z = t$ and the surface $xyz = 1$ at their first intersection, that is, with $0 < t < \frac{1}{4}\pi$.

22. Let f, g, h be three functions of (x, y, z). In cylindrical coördinates (§ 40) form the combinations $F = f\cos\phi + g\sin\phi$, $G = -f\sin\phi + g\cos\phi$, $H = h$. Transform

$$(\alpha)\ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}, \qquad (\beta)\ \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}, \qquad (\gamma)\ \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$$

to cylindrical coördinates and express in terms of F, G, H in simplest form.

23. Given the functions y^x and $(z^y)^x$ and $z^{(y^x)}$. Find the total differentials and hence obtain the derivatives of x^x and $(x^x)^x$ and $x^{(x^x)}$.

50. Derivatives of higher order. If the first derivatives be again differentiated, there arise four derivatives $f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy}$ of the second order, where the first subscript denotes the first differentiation. These may also be written

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f''_{xy} = \frac{\partial^2 f}{\partial y \partial x}, \quad f''_{yx} = \frac{\partial^2 f}{\partial x \partial y}, \quad f''_{yy} = \frac{\partial^2 f}{\partial y^2},$$

where the derivative of $\partial f/\partial y$ with respect to x is written $\partial^2 f/\partial x \partial y$ with the variables in the same order as required in $D_x D_y f$ and opposite to the order of the subscripts in f''_{yx}. This matter of order is usually of no importance owing to the theorem: *If the derivatives f'_x, f'_y have derivatives f''_{xy}, f''_{yx} which are continuous in (x, y) in the neighborhood of any point (x_0, y_0), the derivatives f''_{xy} and f''_{yx} are equal*, that is, $f''_{xy}(x_0, y_0) = f''_{yx}(x_0, y_0)$:

The theorem may be proved by repeated application of the Theorem of the Mean. For

$$[f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)] - [f(x_0 + h, y_0) - f(x_0, y_0)] = [\phi(y_0 + k) - \phi(y_0)]$$
$$= [f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)] - [f(x_0, y_0 + k) - f(x_0, y_0)] = [\psi(x_0 + h) - \psi(x_0)]$$

where $\phi(y)$ stands for $f(x_0 + h, y) - f(x_0, y)$ and $\psi(x)$ for $f(x, y_0 + k) - f(x, y_0)$. Now

$$\phi(y_0 + k) - \phi(y_0) = k\phi'(y_0 + \theta k) = k[f'_y(x_0 + h, y_0 + \theta k) - f'_y(x_0, y_0 + \theta k)],$$
$$\psi(x_0 + h) - \psi(x_0) = h\psi'(x_0 + \theta' h) = h[f'_x(x_0 + \theta' h, y_0 + k) - f'_x(x_0 + \theta' h, y_0)]$$

by applying the Theorem of the Mean to $\phi(y)$ and $\psi(x)$ regarded as functions of a single variable and then substituting. The results obtained are necessarily equal to each other; but each of these is in form for another application of the theorem.

$$k[f'_y(x_0+h,\, y_0+\theta k) - f'_y(x_0,\, y_0+\theta k)] = khf''_{yx}(x_0+\eta h,\, y_0+\theta k),$$
$$h[f'_x(x_0+\theta' h,\, y_0+k) - f'_x(x_0+\theta' h,\, y_0)] = hkf''_{xy}(x_0+\theta' h,\, y_0+\eta' k).$$

Hence $$f''_{yx}(x_0+\eta h,\, y_0+\theta k) = f''_{xy}(x_0+\theta' h,\, y_0+\eta' k).$$

As the derivatives f''_{yx}, f''_{xy} are supposed to exist and be continuous in the variables (x, y) at and in the neighborhood of (x_0, y_0), the limit of each side of the equation exists as $h \doteq 0$, $k \doteq 0$ and the equation is true in the limit. Hence

$$f''_{yx}(x_0, y_0) = f''_{xy}(x_0, y_0).$$

The differentiation of the three derivatives $f''_{xx}, f''_{xy} = f''_{yx}, f''_{yy}$ will give six derivatives of the third order. Consider f'''_{xxy} and f'''_{xyx}. These may be written as $(f'_x)''_{xy}$ and $(f'_x)''_{yx}$ and are equal by the theorem just proved (provided the restrictions as to continuity and existence are satisfied). A similar conclusion holds for f'''_{yxy} and f'''_{yyx}; the number of distinct derivatives of the third order reduces from six to four, just as the number of the second order reduces from four to three. In like manner for derivatives of any order, *the value of the derivative depends not on the order in which the individual differentiations with respect to x and y are performed, but only on the total number of differentiations with respect to each,* and the result may be written with the differentiations collected as

$$D_x^m D_y^n f = \frac{\partial^{m+n} f}{\partial x^m \partial y^n} = f^{(m+n)}_{x^m y^n}, \text{ etc.} \tag{22}$$

Analogous results hold for functions of any number of variables. If several derivatives are to be found and added together, a symbolic form of writing is frequently advantageous. For example,

$$(D_x^2 D_y D_z^3 + D_y^6) f = \frac{\partial^6 f}{\partial x^2 \partial y \partial z^3} + \frac{\partial^6 f}{\partial y^6}$$

or $$(D_x + D_y)^2 f = (D_x^2 + 2\,D_x D_y + D_y^2) f = f''_{xx} + 2 f''_{xy} + f''_{yy}.$$

51. It is sometimes necessary to *change the variable* in higher derivatives, particularly in those of the second order. This is done by a repeated application of (18). Thus f''_{rr} would be found by differentiating the first equation with respect to r, and f''_{rs} by differentiating the first by s or the second by r, and so on. Compare p. 12. The exercise below illustrates the method. It may be remarked that the use of *higher differentials* is often of advantage, although these differentials, like the higher differentials of functions of a single variable (Exs. 10, 16–19, p. 67), have the disadvantage that their form depends on what the independent variables are. This is also illustrated below. It should be particularly borne in mind that the great value of the first differential lies in the facts that it may be treated like a finite quantity and that its form is independent of the variables.

To change the variable in $v''_{xx} + v''_{yy}$ to polar coördinates and show

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2}, \qquad \begin{cases} x = r\cos\phi, & y = r\sin\phi, \\ r = \sqrt{x^2+y^2}, & \phi = \tan^{-1}(y/x). \end{cases}$$

Then
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial x}, \qquad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial y}$$

by applying (18) directly with x, y taking the place of $r, s, \cdots$ and r, ϕ the place of $x, y, z, \cdots$. These expressions may be reduced so that

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial v}{\partial \phi}\frac{-y}{x^2+y^2} = \frac{\partial v}{\partial r}\frac{x}{r} + \frac{\partial v}{\partial \phi}\frac{-y}{r^2}.$$

Next
$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial x} = \frac{\partial}{\partial r}\frac{\partial v}{\partial x}\cdot\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial v}{\partial x}\cdot\frac{\partial \phi}{\partial x}$$
$$= \left[\frac{\partial^2 v}{\partial r^2}\frac{x}{r} + \frac{\partial v}{\partial r}\frac{\partial}{\partial r}\frac{x}{r} + \frac{\partial^2 v}{\partial r\partial \phi}\frac{-y}{r^2} + \frac{\partial v}{\partial \phi}\frac{\partial}{\partial r}\frac{-y}{r^2}\right]\frac{x}{r}$$
$$+ \left[\frac{\partial^2 v}{\partial \phi\partial r}\frac{x}{r} + \frac{\partial v}{\partial r}\frac{\partial}{\partial \phi}\frac{x}{r} + \frac{\partial^2 v}{\partial \phi^2}\frac{-y}{r^2} + \frac{\partial v}{\partial \phi}\frac{\partial}{\partial \phi}\frac{-y}{r^2}\right]\frac{-y}{r^2}.$$

The differentiations of x/r and $-y/r^2$ may be performed as indicated with respect to r, ϕ, remembering that, as r, ϕ are independent, the derivative of r by ϕ is 0. Then

$$\frac{\partial^2 v}{\partial x^2} = \frac{x^2}{r^2}\frac{\partial^2 v}{\partial r^2} + \frac{y^2}{r^3}\frac{\partial v}{\partial r} - 2\frac{xy}{r^3}\frac{\partial^2 v}{\partial r\partial \phi} + 2\frac{xy}{r^4}\frac{\partial v}{\partial \phi} + \frac{y^2}{r^4}\frac{\partial^2 v}{\partial \phi^2}.$$

In like manner $\partial^2 v/\partial y^2$ may be found, and the sum of the two derivatives reduces to the desired expression. This method is long and tedious though straightforward.

It is considerably shorter to start with the expression in polar coördinates and transform by the same method to the one in rectangular coördinates. Thus

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\cos\phi + \frac{\partial v}{\partial y}\sin\phi = \frac{1}{r}\left(\frac{\partial v}{\partial x}x + \frac{\partial v}{\partial y}y\right),$$
$$\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \left(\frac{\partial^2 v}{\partial x^2}\cos\phi + \frac{\partial^2 v}{\partial y\partial x}\sin\phi\right)x + \left(\frac{\partial^2 v}{\partial x\partial y}\cos\phi + \frac{\partial^2 v}{\partial y^2}\sin\phi\right)y + \frac{\partial v}{\partial x}\cos\phi + \frac{\partial v}{\partial y}\sin\phi,$$
$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \phi} = -\frac{\partial v}{\partial x}r\sin\phi + \frac{\partial v}{\partial y}r\cos\phi = -\frac{\partial v}{\partial x}y + \frac{\partial v}{\partial y}x,$$
$$\frac{1}{r}\frac{\partial^2 v}{\partial \phi^2} = \left(\frac{\partial^2 v}{\partial x^2}\sin\phi - \frac{\partial^2 v}{\partial y\partial x}\cos\phi\right)y + \left(-\frac{\partial^2 v}{\partial x\partial y}\sin\phi + \frac{\partial^2 v}{\partial y^2}\cos\phi\right)x$$
$$-\frac{\partial v}{\partial x}\cos\phi - \frac{\partial v}{\partial y}\sin\phi.$$

Then
$$\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 v}{\partial \phi^2} = \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)r$$

or
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2}. \tag{23}$$

The definitions $d_x^2 f = f''_{xx}dx^2$, $d_x d_y f = f''_{xy}dxdy$, $d_y^2 f = f''_{yy}dy^2$ would naturally be given for *partial differentials of the second order*, each of which would vanish if f reduced to either of the independent variables x, y or to any linear function of them. Thus the second differentials of the independent variables are zero. The second total differential would be obtained by differentiating the first total differential.

$$d^2f = ddf = d\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = d\frac{\partial f}{\partial x}dx + d\frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial x}d^2x + \frac{\partial f}{\partial y}d^2y\,;$$

but $$d\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial y \partial x}dy, \qquad d\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}dx + \frac{\partial^2 f}{\partial y^2}dy,$$

and $$d^2f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial f}{\partial x}d^2x + \frac{\partial f}{\partial y}d^2y. \tag{24}$$

The last two terms vanish and the total differential reduces to the first three terms if x and y are the independent variables; and in this case the second derivatives, $f''_{xx}, f''_{xy}, f''_{yy}$, are the coefficients of dx^2, $2\,dxdy$, dy^2, which enables those derivatives to be found by an extension of the method of finding the first derivatives (§ 49). The method is particularly useful when all the second derivatives are needed.

The problem of the change of variable may now be treated. Let

$$\begin{aligned} d^2v &= \frac{\partial^2 v}{\partial x^2}dx^2 + 2\frac{\partial^2 v}{\partial x^2}dxdy + \frac{\partial^2 v}{\partial y^2}dy^2 \\ &= \frac{\partial^2 v}{\partial r^2}dr^2 + 2\frac{\partial^2 v}{\partial r \partial \phi}drd\phi + \frac{\partial^2 v}{\partial \phi^2}d\phi^2 + \frac{\partial v}{\partial r}d^2r + \frac{\partial v}{\partial \phi}d^2\phi, \end{aligned}$$

where x, y are the independent variables and r, ϕ other variables dependent on them — in this case, defined by the relations for polar coördinates. Then

$$dx = \cos\phi dr - r\sin\phi d\phi, \qquad dy = \sin\phi dr + r\cos\phi d\phi$$

or $$dr = \cos\phi dx + \sin\phi dy, \qquad rd\phi = -\sin\phi dx + \cos\phi dy. \tag{25}$$

Then $$d^2r = (-\sin\phi dx + \cos\phi dy)\,d\phi = rd\phi d\phi = rd\phi^2,$$

$$drd\phi + rd^2\phi = -(\cos\phi dx + \sin\phi dy)\,d\phi = -drd\phi,$$

where the differentials of dr and $rd\phi$ have been found subject to $d^2x = d^2y = 0$. Hence $d^2r = rd\phi^2$ and $rd^2\phi = -2\,drd\phi$. These may be substituted in d^2v which becomes

$$d^2v = \frac{\partial^2 v}{\partial r^2}dr^2 + 2\left(\frac{\partial^2 v}{\partial r \partial \phi} - \frac{1}{r}\frac{\partial v}{\partial \phi}\right)drd\phi + \left(\frac{\partial^2 v}{\partial \phi^2} + r\frac{\partial v}{\partial r}\right)d\phi^2.$$

Next the values of dr^2, $drd\phi$, $d\phi^2$ may be substituted from (25) and

$$\begin{aligned} d^2v = &\left[\frac{\partial^2 v}{\partial r^2}\cos^2\phi - \frac{2}{r}\left(\frac{\partial^2 v}{\partial r \partial \phi} - \frac{1}{r}\frac{\partial v}{\partial \phi}\right)\cos\phi\sin\phi + \left(\frac{\partial^2 v}{\partial \phi^2} + r\frac{\partial v}{\partial r}\right)\frac{\sin^2\phi}{r^2}\right]dx^2 \\ &+ 2\left[\frac{\partial^2 v}{\partial r^2}\cos\phi\sin\phi + \left(\frac{\partial^2 v}{\partial r \partial \phi} - \frac{1}{r}\frac{\partial v}{\partial \phi}\right)\frac{\cos^2\phi - \sin^2\phi}{r} - \frac{\partial^2 v}{\partial \phi^2}\frac{\cos\phi\sin\phi}{r^2}\right]dxdy \\ &+ \left[\frac{\partial^2 v}{\partial r^2}\sin^2\phi + \frac{2}{r}\left(\frac{\partial^2 v}{\partial r \partial \phi} - \frac{1}{r}\frac{\partial v}{\partial \phi}\right)\cos\phi\sin\phi + \left(\frac{\partial^2 v}{\partial \phi^2} + r\frac{\partial v}{\partial r}\right)\frac{\cos^2\phi}{r^2}\right]dy^2. \end{aligned}$$

Thus finally the derivatives v''_{xx}, v''_{xy}, v''_{yy} are the three brackets which are the coefficients of dx^2, $2\,dxdy$, dy^2. The value of $v''_{xx} + v''_{yy}$ is as found before.

52. The condition $f''_{xy} = f''_{yx}$ which subsists in accordance with the fundamental theorem of § 50 gives *the condition that*

$$M(x, y)\,dx + N(x, y)\,dy = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = df$$

be the total differential of some function $f(x, y)$. In fact

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

and
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad or \quad \left(\frac{dM}{dy}\right)_x = \left(\frac{dN}{dx}\right)_y. \tag{26}$$

The second form, where the variables which are constant during the differentiation are explicitly indicated as subscripts, is more common in works on thermodynamics. It will be proved later that conversely if this relation (26) holds, the expression $Mdx + Ndy$ is the total differential of some function, and the method of finding the function will also be given (§§ 92, 124). In case $Mdx + Ndy$ is the differential of some function $f(x, y)$ it is usually called an *exact differential.*

The application of the condition for an exact differential may be made in connection with a problem in thermodynamics. Let S and U be the entropy and energy of a gas or vapor inclosed in a receptacle of volume v and subjected to the pressure p at the temperature T. The fundamental equation of thermodynamics, connecting the differentials of energy, entropy, and volume, is

$$dU = TdS - pdv; \quad \text{and} \quad \left(\frac{dT}{dv}\right)_S = -\left(\frac{dp}{dS}\right)_v \tag{27}$$

is the condition that dU be a total differential. Now, any two of the five quantities U, S, v, T, p may be taken as independent variables. In (27) the choice is S, v; if the equation were solved for dS, the choice would be U, v; and U, S if solved for dv. In each case the cross differentiation to express the condition (26) would give rise to a relation between the derivatives.

If p, T were desired as independent variables, the change of variable

$$dS = \left(\frac{dS}{dp}\right)_T dp + \left(\frac{dS}{dT}\right)_p dT, \qquad dv = \left(\frac{dv}{dp}\right)_T dp + \left(\frac{dv}{dT}\right)_p dT$$

with
$$dU = \left[T\left(\frac{dS}{dp}\right)_T - p\left(\frac{dv}{dp}\right)_T\right]dp + \left[T\left(\frac{dS}{dT}\right)_p - p\left(\frac{dv}{dT}\right)_p\right]dT$$

should be made. The expression of the condition is then

$$\left\{\frac{d}{dT}\left[T\left(\frac{dS}{dp}\right)_T - p\left(\frac{dv}{dp}\right)_T\right]\right\}_p = \left\{\frac{d}{dp}\left[T\left(\frac{dS}{dT}\right)_p - p\left(\frac{dv}{dT}\right)_p\right]\right\}_T$$

or
$$\left(\frac{dS}{dp}\right)_T + T\frac{\partial S}{\partial T\partial p} - p\frac{\partial^2 v}{\partial T\partial p} = T\frac{\partial^2 S}{\partial p\partial T} - \left(\frac{dv}{dT}\right)_p - p\frac{\partial^2 v}{\partial p\partial T},$$

where the differentiation on the left is made with p constant and that on the right with T constant and where the subscripts have been dropped from the second derivatives and the usual notation adopted. Everything cancels except two terms which give

$$\left(\frac{dS}{dp}\right)_T = -\left(\frac{dv}{dT}\right)_p \quad \text{or} \quad \frac{1}{T}\left(\frac{TdS}{dp}\right)_T = -\left(\frac{dv}{dT}\right)_p. \tag{28}$$

The importance of the test for an exact differential lies not only in the relations obtained between the derivatives as above, but also in the fact that in applied mathematics a great many expressions are written as differentials which are not the total differentials of any functions and which must be distinguished from exact differentials. For instance if dH denote the infinitesimal portion of heat added to the gas or vapor above considered, the fundamental equation is expressed as $dH = dU + pdv$. That is to say, the amount of heat added is equal to the increase in the energy plus the work done by the gas in expanding. Now dH is not the differential of any function $H(U, v)$; it is $dS = dH/T$ which is the differential, and this is one reason for introducing the entropy S. Again if the forces X, Y act on a particle, the *work* done during the displacement through the arc $ds = \sqrt{dx^2 + dy^2}$ is written $dW = Xdx + Ydy$. It may happen that this is the total differential of some function; indeed, if

$$dW = -dV(x, y), \quad Xdx + Ydy = -dV, \quad X = -\frac{\partial V}{\partial x}, \quad Y = -\frac{\partial V}{\partial y},$$

where the negative sign is introduced in accordance with custom, the function V is called the *potential energy* of the particle. In general, however, there is no potential energy function V, and dW is not an exact differential; this is always true when part of the work is due to forces of friction. A notation which should distinguish between exact differentials and those which are not exact is much more needed than a notation to distinguish between partial and ordinary derivatives; but there appears to be none.

Many of the physical magnitudes of thermodynamics are expressed as derivatives and such relations as (26) establish relations between the magnitudes. Some definitions:

specific heat at constant volume is $C_v = \left(\frac{dH}{dT}\right)_v = T\left(\frac{dS}{dT}\right)_v$,

specific heat at constant pressure is $C_p = \left(\frac{dH}{dT}\right)_p = T\left(\frac{dS}{dT}\right)_p$,

latent heat of expansion is $L_v = \left(\frac{dH}{dv}\right)_T = T\left(\frac{dS}{dv}\right)_T$,

coefficient of cubic expansion is $\alpha_p = \frac{1}{v}\left(\frac{dv}{dT}\right)_p$,

modulus of elasticity (isothermal) is $E_T = -v\left(\frac{dp}{dv}\right)_T$,

modulus of elasticity (adiabatic) is $E_S = -v\left(\frac{dp}{dv}\right)_S$.

53. A polynomial is said to be homogeneous when each of its terms is of the same order when all the variables are considered. A definition of homogeneity which includes this case and is applicable to more general cases is: *A function $f(x, y, z, \cdots)$ of any number of variables is called homogeneous if the function is multiplied by some power of λ when all the variables are multiplied by λ*; and the power of λ which factors out is called the order of homogeneity of the function. In symbols the condition for homogeneity of order n is

$$f(\lambda x, \lambda y, \lambda z, \cdots) = \lambda^n f(x, y, z, \cdots). \tag{29}$$

Thus
$$xe^{\frac{y}{x}} + \frac{y^2}{x}, \qquad \frac{xy}{z^2} + \tan^{-1}\frac{x}{z}, \qquad \frac{1}{\sqrt{x^2+y^2}} \tag{29'}$$

are homogeneous functions of order 1, 0, -1 respectively. To test a function for homogeneity it is merely necessary to replace all the variables by λ times the variables and see if λ factors out completely. The homogeneity may usually be seen without the test.

If the identity (29) be differentiated with respect to λ,

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \cdots\right) f(\lambda x, \lambda y, \lambda z, \cdots) = n\lambda^{n-1} f(x, y, z, \cdots).$$

A second differentiation with respect to λ would give

$$\left(x^2\frac{\partial^2}{\partial x^2} + xy\frac{\partial^2}{\partial x\partial y} + xz\frac{\partial^2}{\partial x\partial z} + \cdots\right)f + \left(yx\frac{\partial^2}{\partial y\partial x} + y^2\frac{\partial^2}{\partial y^2} + yz\frac{\partial^2}{\partial z^2} + \cdots\right)f$$
$$+\left(zx\frac{\partial^2}{\partial z\partial x} + zy\frac{\partial^2}{\partial z\partial y} + z^2\frac{\partial^2}{\partial z^2} + \cdots\right)f + \cdots = n(n-1)\lambda^{n-2} f(x, y, z, \cdots)$$

or
$$\left(x^2\frac{\partial^2}{\partial x^2} + 2xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2} + \cdots\right)f = n(n-1)\lambda^{n-2} f(x, y, z, \cdots).$$

Now if λ be set equal to 1 in these equations, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} + \cdots = nf(x, y, z, \cdots), \tag{30}$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} + 2xz\frac{\partial^2 f}{\partial x\partial z} + \cdots = n(n-1)f(x, y, z, \cdots).$$

In words, these equations state that the sum of the partial derivatives each multiplied by the variable with respect to which the differentiation is performed is n times the function if the function is homogeneous of order n; and that the sum of the second derivatives each multiplied by the variables involved and by 1 or 2, according as the variable is repeated or not, is $n(n-1)$ times the function. The general formula obtained by differentiating any number of times with respect to λ may be expressed symbolically in the convenient form

$$(xD_x + yD_y + zD_z + \cdots)^k f = n(n-1)\cdots(n-k+1)f. \tag{31}$$

This is known as *Euler's Formula* on homogeneous functions.

It is worth while noting that in a certain sense every equation which represents a geometric or physical relation is homogeneous. For instance, in geometry the magnitudes that arise may be lengths, areas, volumes, or angles. These magnitudes are expressed as a number times a unit; thus, $\sqrt{2}$ ft., 3 sq. yd., π cu. ft.

In adding and subtracting, the terms must be like quantities; lengths added to lengths, areas to areas, etc. The *fundamental unit* is taken as length. The units of area, volume, and angle are *derived* therefrom. Thus the area of a rectangle or the volume of a rectangular parallelepiped is

$$A = a \text{ ft.} \times b \text{ ft.} = ab \text{ ft.}^2 = ab \text{ sq ft.}, \quad V = a \text{ ft.} \times b \text{ ft.} \times c \text{ ft.} = abc \text{ ft.}^3 = abc \text{ cu. ft.},$$

and the units sq. ft., cu. ft. are denoted as ft.^{2}, ft.^{3} just as if the simple unit ft. had been treated as a literal quantity and included in the multiplication. An area or volume is therefore considered as a compound quantity consisting of a number which gives its magnitude and a unit which gives its quality or dimensions. If L denote length and $[L]$ denote "of the dimensions of length," and if similar notations be introduced for area and volume, the equations $[A] = [L]^2$ and $[V] = [L]^3$ state that the dimensions of area are squares of length, and of volumes, cubes of lengths. If it be recalled that for purposes of analysis an angle is measured by the ratio of the arc subtended to the radius of the circle, the dimensions of angle are seen to be nil, as the definition involves the ratio of like magnitudes and must therefore be a *pure number*.

When geometric facts are represented analytically, either of two alternatives is open: 1°, the equations may be regarded as existing between mere numbers; or 2°, as between actual magnitudes. Sometimes one method is preferable, sometimes the other. Thus the equation $x^2 + y^2 = r^2$ of a circle may be interpreted as 1°, the sum of the squares of the coördinates (numbers) is constant; or 2°, the sum of the squares on the legs of a right triangle is equal to the square on the hypotenuse (Pythagorean Theorem). The second interpretation better sets forth the true inwardness of the equation. Consider in like manner the parabola $y^2 = 4px$. Generally y and x are regarded as mere numbers, but they may equally be looked upon as lengths and then the statement is that the square upon the ordinate equals the rectangle upon the abscissa and the constant length $4p$; this may be interpreted into an actual construction for the parabola, because a square equivalent to a rectangle may be constructed.

In the last interpretation the constant p was assigned the dimensions of length so as to render the equation homogeneous in dimensions, with each term of the dimensions of area or $[L]^2$. It will be recalled, however, that in the definition of the parabola, the quantity p actually has the dimensions of length, being half the distance from the fixed point to the fixed line (focus and directrix). This is merely another corroboration of the initial statement that the equations which actually arise in considering geometric problems are homogeneous in their dimensions, and must be so for the reason that in stating the first equation like magnitudes must be compared with like magnitudes.

The question of dimensions may be carried along through such processes as differentiation and integration. For let y have the dimensions $[y]$ and x the dimensions $[x]$. Then Δy, the difference of two y's, must still have the dimensions $[y]$ and Δx the dimensions $[x]$. The quotient $\Delta y/\Delta x$ then has the dimensions $[y]/[x]$. For example the relations for area and for volume of revolution,

$$\frac{dA}{dx} = y, \quad \frac{dV}{dx} = \pi y^2, \quad \text{give} \quad \left[\frac{dA}{dx}\right] = \frac{[A]}{[L]} = [L], \quad \left[\frac{dV}{dx}\right] = \frac{[V]}{[L]} = [L]^2,$$

and the dimensions of the left-hand side check with those of the right-hand side. As integration is the limit of a sum, the dimensions of an integral are the product of the dimensions of the function to be integrated and of the differential dx. Thus if

$$y=\int_0^x \frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

were an integral arising in actual practice, the very fact that a^2 and x^2 are added would show that they must have the same dimensions. If the dimensions of x be $[L]$, then

$$\left[\int_0^x \frac{dx}{a^2+x^2}\right]=\left[\frac{1}{a^2+x^2}\right][dx]=\frac{1}{[L]^2}[L]=\frac{1}{[L]}=[y],$$

and this checks with the dimensions on the right which are $[L]^{-1}$, since angle has no dimensions. As a rule, the theory of dimensions is neglected in pure mathematics; but it can nevertheless be made exceedingly useful and instructive.

In mechanics the *fundamental units* are length, mass, and time; and are denoted by $[L]$, $[M]$, $[T]$. The following table contains some derived units:

velocity	$\frac{[L]}{[T]}$,	acceleration	$\frac{[L]}{[T]^2}$,	force	$\frac{[M][L]}{[T]^2}$,
areal velocity	$\frac{[L]^2}{[T]}$,	density	$\frac{[M]}{[L]^3}$,	momentum	$\frac{[M][L]}{[T]}$,
angular velocity	$\frac{1}{[T]}$,	moment	$\frac{[M][L]^2}{[T]^2}$,	energy	$\frac{[M][L]^2}{[T]^2}$.

With the aid of a table like this it is easy to convert magnitudes in one set of units as ft., lb., sec., to another system, say cm., gm., sec. All that is necessary is to substitute for each individual unit its value in the new system. Thus

$$g=32\tfrac{1}{6}\,\frac{\text{ft.}}{\text{sec.}^2},\qquad 1\text{ ft.}=30.48\text{ cm.},\qquad g=32\tfrac{1}{6}\times 30.48\,\frac{\text{cm.}}{\text{sec.}^2}=980\tfrac{1}{2}\,\frac{\text{cm.}}{\text{sec.}^2}.$$

EXERCISES

1. Obtain the derivatives f''_{xx}, f''_{xy}, f''_{yx}, f''_{yy} and verify $f''_{xy}=f''_{yx}$.

$$(\alpha)\ \sin^{-1}\frac{y}{x},\qquad (\beta)\ \log\frac{x^2+y^2}{xy},\qquad (\gamma)\ \phi\left(\frac{y}{x}\right)+\psi(xy).$$

2. Compute $\partial^2 v/\partial y^2$ in polar coördinates by the straightforward method.

3. Show that $a^2\dfrac{\partial^2 v}{\partial x^2}=\dfrac{\partial^2 v}{\partial t^2}$ if $v=f(x+at)+\phi(x-at)$.

4. Show that this equation is unchanged in form by the transformation:

$$\frac{\partial^2 f}{\partial x^2}+2xy^2\frac{\partial f}{\partial x}+2(y-y^3)\frac{\partial f}{\partial y}+x^2y^2f=0;\qquad u=xy,\quad v=1/y.$$

5. In polar coördinates $z=r\cos\theta$, $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ in space

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial y^2}=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)\right].$$

The work of transformation may be shortened by substituting successively

$$x=r_1\cos\phi,\quad y=r_1\sin\phi,\quad\text{and}\quad z=r\cos\phi,\quad r_1=r\sin\phi.$$

6. Let x, y, z, t be four independent variables and $x=r\cos\phi$, $y=r\sin\phi$, $z=z$ the equations for transforming x, y, z to cylindrical coördinates. Let

$$X = -\frac{\partial^2 f}{\partial x \partial z}, \quad Y = -\frac{\partial^2 f}{\partial y \partial z}, \quad Z = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}, \quad F = \frac{\partial^2 f}{\partial y \partial t}, \quad G = -\frac{\partial^2 f}{\partial x \partial t};$$

show $Z = \frac{1}{r}\frac{\partial Q}{\partial r}$, $\quad X\cos\phi + Y\sin\phi = -\frac{1}{r}\frac{\partial Q}{\partial z}$, $\quad F\sin\phi - G\cos\phi = \frac{1}{r}\frac{\partial Q}{\partial t}$,

where $r^{-1}Q = \partial f/\partial r$. (Of importance for the Hertz oscillator.)

7. Apply the test for an exact differential to each of the following, and write by inspection the functions corresponding to the exact differentials:

(α) $3xdx + y^2dy$, $\quad(\beta)$ $3xydx + x^3dy$, $\quad(\gamma)$ $x^2ydx + y^2dy$,

(δ) $\dfrac{xdx + ydy}{x^2 + y^2}$, $\quad(\epsilon)$ $\dfrac{xdx - ydy}{x^2 + y^2}$, $\quad(\zeta)$ $\dfrac{ydx - xdy}{x^2 + y^2}$,

(η) $(4x^3 + 3x^2y + y^2)dx + (x^3 + 2xy + 3y^3)dy$, $\quad(\theta)$ $x^2y^2(dx + dy)$.

8. Express the conditions that $P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$ be an exact differential $dF(x, y, z)$. Apply these conditions to the differentials:

(α) $3x^2y^2zdx + 2x^3yzdy + x^3y^2dz$, $\quad(\beta)$ $(y + z)dx + (x + z)dy + (x + y)dz$.

9. Obtain $\left(\dfrac{dp}{dT}\right)_v = \left(\dfrac{dS}{dv}\right)_T$ and $\left(\dfrac{dv}{dS}\right)_p = \left(\dfrac{dT}{dp}\right)_S$ from (27) with proper variables.

10. If three functions (called thermodynamic potentials) be defined as

$$\psi = U - TS, \qquad \chi = U + pv, \qquad \zeta = U - TS + pv,$$

show $\quad d\psi = -SdT - pdv, \quad d\chi = TdS + vdp, \quad d\zeta = -SdT + vdp,$

and express the conditions that $d\psi$, $d\chi$, $d\zeta$ be exact. Compare with Ex. 9.

11. State in words the definitions corresponding to the defining formulas, p. 107.

12. If the sum $(Mdx + Ndy) + (Pdx + Qdy)$ of two differentials is exact and one of the differentials is exact, the other is. Prove this.

13. Apply Euler's Formula (31), for the simple case $k = 1$, to the three functions (29′) and verify the formula. Apply it for $k = 2$ to the first function.

14. Verify the homogeneity of these functions and determine their order:

(α) $y^2/x + x(\log x - \log y)$, $\quad(\beta)$ $\dfrac{x^my^n}{\sqrt{x^2 + y^2}}$, $\quad(\gamma)$ $\dfrac{xyz}{ax + by + cz}$,

(δ) $xye^{\frac{xz}{y^2}} + z^2$, $\quad(\epsilon)$ $\sqrt{x}\cot^{-1}\dfrac{y}{z}$, $\quad(\zeta)$ $\dfrac{\sqrt[5]{x} - \sqrt[5]{y}}{\sqrt[4]{x} + \sqrt[4]{y}}$.

15. State the dimensions of moment of inertia and convert a unit of moment of inertia in ft.-lb. into its equivalent in cm.-gm.

16. Discuss for dimensions Peirce's formulas Nos. 93, 124–125, 220, 300.

17. Continue Ex. 17, p. 101, to show $\dfrac{d}{dt}\dfrac{\partial x}{\partial q_i} = \dfrac{\partial v}{\partial q_i}$ and $\dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{q}_i} = m\dot{v}\dfrac{\partial x}{\partial q_i} + \dfrac{\partial T}{\partial q_i}$.

18. If $p_i = \dfrac{\partial T}{\partial \dot{q}_i}$ in Ex. 17, p. 101, show without analysis that $2T = \dot{q}_1p_1 + \dot{q}_2p_2$. If T' denote $T' = T$, where T' is considered as a function of p_1, p_2 while T is considered as a function of $\dot{q}_1$, $\dot{q}_2$, prove from $T' = \dot{q}_1p_1 + \dot{q}_2p_2 - T$ that

$$\frac{\partial T'}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial T'}{\partial q_i} = -\frac{\partial T}{\partial q_i}.$$

19. If (x_1, y_1) and (x_2, y_2) are the coördinates of two moving particles and

$$m_1\frac{d^2x_1}{dt^2} = X_1, \quad m_1\frac{d^2y_1}{dt^2} = Y_1, \quad m_2\frac{d^2x_2}{dt^2} = X_2, \quad m_2\frac{d^2y_2}{dt^2} = Y_2$$

are the equations of motion, and if x_1, y_1, x_2, y_2 are expressible as

$$x_1 = f_1(q_1, q_2, q_3), \quad y_1 = g_1(q_1, q_2, q_3), \quad x_2 = f_2(q_1, q_2, q_3), \quad y_2 = g_2(q_1, q_2, q_3)$$

in terms of three independent variables q_1, q_2, q_3, show that

$$Q_1 = X_1\frac{\partial x_1}{\partial q_1} + Y_1\frac{\partial y_1}{\partial q_1} + X_2\frac{\partial x_2}{\partial q_1} + Y_2\frac{\partial y_2}{\partial q_1} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1},$$

where $T = \frac{1}{2}(m_1v_1^2 + m_2v_2^2) = T(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$ and is homogeneous of the second degree in $\dot{q}_1, \dot{q}_2, \dot{q}_3$. The work may be carried on as a generalization of Ex. 17, p. 101, and Ex. 17 above. It may be further extended to any number of particles whose positions in space depend on a number of variables q.

20. In Ex. 19 if $p_i = \dfrac{\partial T}{\partial \dot{q}_i}$, generalize Ex. 18 to obtain

$$\dot{q}_i = \frac{\partial T'}{\partial p_i}, \quad \frac{\partial T'}{\partial q_i} = -\frac{\partial T}{\partial q_i}, \quad Q_1 = \frac{dp_1}{dt} + \frac{\partial T'}{\partial q_1}.$$

The equations $Q_i = \dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{q}_i} - \dfrac{\partial T}{\partial q_i}$ and $Q_i = \dfrac{dp_i}{dt} + \dfrac{\partial T'}{\partial q_i}$ are respectively the Lagrangian and Hamiltonian equations of motion.

21. If $rr' = k^2$ and $\phi' = \phi$ and $v'(r', \phi') = v(r, \phi)$, show

$$\frac{\partial^2 v'}{\partial r'^2} + \frac{1}{r'}\frac{\partial v'}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2 v'}{\partial \phi'^2} = \frac{r^2}{r'^2}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \phi^2}\right).$$

22. If $rr' = k^2$, $\phi' = \phi$, $\theta' = \theta$, and $v'(r', \phi', \theta') = \dfrac{k}{r'}v(r, \phi, \theta)$, show that the expression of Ex. 5 in the primed letters is kr^2/r'^3 of its value for the unprimed letters. (Useful in § 198.)

23. If $z = x\phi\left(\dfrac{y}{x}\right) + \psi\left(\dfrac{y}{x}\right)$, show $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 0$.

24. Make the indicated changes of variable:

(α) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = e^{-2u}\left(\dfrac{\partial^2 V}{\partial u^2} + \dfrac{\partial^2 V}{\partial v^2}\right)$ if $x = e^u \cos v$, $y = e^u \sin v$,

(β) $\dfrac{\partial^2 V}{\partial u^2} + \dfrac{\partial^2 V}{\partial v^2} = \left(\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2}\right)\left[\left(\dfrac{\partial f}{\partial u}\right)^2 + \left(\dfrac{\partial f}{\partial v}\right)^2\right]$, where

$$x = f(u, v), \quad y = \phi(u, v), \quad \frac{\partial f}{\partial u} = \frac{\partial \phi}{\partial v}, \quad \frac{\partial f}{\partial v} = -\frac{\partial \phi}{\partial u}.$$

25. For an orthogonal transformation (Ex. 10 (ζ), p. 100)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial x'^2} + \frac{\partial^2 v}{\partial y'^2} + \frac{\partial^2 v}{\partial z'^2}.$$

54. Taylor's Formula and applications. The development of $f(x, y)$ is found, as was the Theorem of the Mean, from the relation (p. 95)

$$\Delta f = \Phi(1) - \Phi(0) \quad \text{if} \quad \Phi(t) = f(a + th,\ b + tk).$$

If $\Phi(t)$ be expanded by Maclaurin's Formula to n terms;

$$\Phi(t) - \Phi(0) = t\Phi'(0) + \frac{t^2}{2!}\Phi''(0) + \cdots + \frac{t^{n-1}}{(n-1)!}\Phi^{(n-1)}(0) + \frac{t^n}{n!}\Phi^{(n)}(\theta t).$$

The expressions for $\Phi'(t)$ and $\Phi'(0)$ may be found as follows by (10):

$$\Phi'(t) = hf'_x + kf'_y, \quad \Phi'(0) = [hf'_x + kf'_y]_{\substack{x=a\\y=b}},$$

then
$$\begin{aligned}\Phi''(t) &= h(hf''_{xx} + kf''_{yx}) + k(hf''_{xy} + kf''_{yy})\\ &= h^2 f''_{xx} + 2hkf''_{xy} + k^2 f''_{yy} = (hD_x + kD_y)^2 f,\end{aligned}$$
$$\Phi^{(i)}(t) = (hD_x + hD_y)^i f, \quad \Phi^{(i)}(0) = [(hD_x + kD_y)^i f]_{\substack{x=a\\y=b}}.$$

And
$$\begin{aligned}f(a+h, b+k) - f(a, b) &= \Delta f = \Phi(1) - \Phi(0) = (hD_x + kD_y)f(a, b)\\ &+ \frac{1}{2!}(hD_x + kD_y)^2 f(a, b) + \cdots + \frac{1}{(n-1)!}(hD_x + kD_y)^{n-1} f(a, b)\\ &+ \frac{1}{n!}(hD_x + kD_y)^n f(a + \theta h,\ b + \theta k). \qquad (32)\end{aligned}$$

In this expansion, the increments h and k may be replaced, if desired, by $x - a$ and $y - b$ and then $f(x, y)$ will be expressed in terms of its value and the values of its derivatives at (a, b) in a manner entirely analogous to the case of a single variable. In particular if the point (a, b) about which the development takes place be $(0, 0)$ the development becomes Maclaurin's Formula for $f(x, y)$.

$$\begin{aligned}f(x, y) &= f(0, 0) + (xD_x + yD_y)f(0, 0) + \frac{1}{2!}(xD_x + yD_y)^2 f(0, 0) + \cdots\\ &+ \frac{1}{(n-1)!}(xD_x + yD_y)^{n-1} f(0, 0) + \frac{1}{n!}(xD_x + yD_y)^n f(\theta x, \theta y). \qquad (32')\end{aligned}$$

Whether in Maclaurin's or Taylor's Formula, the successive terms are homogeneous polynomials of the 1st, 2d, $\cdots$, $(n - 1)$st order in x, y or in $x - a$, $y - b$. The formulas are unique as in § 32.

Suppose $\sqrt{1 - x^2 - y^2}$ is to be developed about $(0, 0)$. The successive derivatives are

$$f'_x = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad f'_y = \frac{-y}{\sqrt{1 - x^2 - y^2}}, \quad f'_x(0, 0) = 0, \quad f'_y(0, 0) = 0,$$

$$f''_{xx} = \frac{-1 + y^2}{(1 - x^2 - y^2)^{\frac{3}{2}}}, \quad f''_{xy} = \frac{xy}{(1 - x^2 - y^2)^{\frac{3}{2}}}, \quad f''_{yy} = \frac{-1 + x^2}{(1 - x^2 - y^2)^{\frac{3}{2}}},$$

$$f'''_{x^3} = \frac{\frac{3}{2}(1 - y^2)x}{(1 - x^2 - y^2)^{\frac{5}{2}}}, \quad f'''_{x^2y} = \frac{y^3 - 2xy^2 - y}{(1 - x^2 - y^2)^{\frac{5}{2}}}, \quad \cdots,$$

and $\sqrt{1 - x^2 - y^2} = 1 + (0x + 0y) + \frac{1}{2}(-x^2 + 0xy - y^2) + \frac{1}{6}(0x^3 + \cdots) + \cdots,$

or $\sqrt{1 - x^2 - y^2} = 1 - \frac{1}{2}(x^2 + y^2) + \text{terms of fourth order} + \cdots.$

In this case the expansion may be found by treating $x^2 + y^2$ as a single term and expanding by the binomial theorem. The result would be

$$[1-(x^2+y^2)]^{\frac{1}{2}} = 1 - \tfrac{1}{2}(x^2+y^2) - \tfrac{1}{8}(x^4+2x^2y^2+y^4) - \tfrac{1}{16}(x^2+y^2)^3 - \cdots.$$

That the development thus obtained is identical with the Maclaurin development that might be had by the method above, follows from the uniqueness of the development. Some such short cut is usually available.

55. The condition that a function $z = f(x, y)$ have a minimum or maximum at (a, b) is that $\Delta f > 0$ or $\Delta f < 0$ for all values of $h = \Delta x$ and $k = \Delta y$ which are sufficiently small. From either geometrical or analytic considerations it is seen that if the surface $z = f(x, y)$ has a minimum or maximum at (a, b), the curves in which the planes $y = b$ and $x = a$ cut the surface have minima or maxima at $x = a$ and $y = b$ respectively. Hence the partial derivatives f'_x and f'_y must both vanish at (a, b), provided, of course, that exceptions like those mentioned on page 7 be made. The two simultaneous equations

$$f'_x = 0, \quad f'_y = 0, \tag{33}$$

corresponding to $f'(x) = 0$ in the case of a function of a single variable, may then be solved to find the positions (x, y) of the minima and maxima. Frequently the geometric or physical interpretation of $z = f(x, y)$ or some special device will then determine whether there is a maximum or a minimum or neither at each of these points.

For example let it be required to find the maximum rectangular parallelepiped which has three faces in the coördinate planes and one vertex in the plane $x/a + y/b + z/c = 1$. The volume is

$$V = xyz = cxy\left(1 - \frac{x}{a} - \frac{y}{b}\right).$$

$$\frac{\partial V}{\partial x} = -2\frac{c}{a}xy - \frac{c}{b}y^2 + cy = 0 \qquad \frac{\partial V}{\partial y} = -2\frac{c}{b}xy - \frac{c}{a}x^2 + cx = 0.$$

The solution of these equations is $x = \frac{1}{3}a$, $y = \frac{1}{3}b$. The corresponding z is $\frac{1}{3}c$ and the volume V is therefore $abc/9$ or $\frac{2}{3}$ of the volume cut off from the first octant by the plane. It is evident that this solution is a maximum. There are other solutions of $V'_x = V'_y = 0$ which have been discarded because they give $V = 0$.

The conditions $f'_x = f'_y = 0$ may be established analytically. For

$$\Delta f = (f'_x + \zeta_1)\Delta x + (f'_y + \zeta_2)\Delta y.$$

Now as ζ_1, ζ_2 are infinitesimals, the signs of the parentheses are determined by the signs of f'_x, f'_y unless these derivatives vanish; and hence unless $f'_x = 0$, the sign of Δf for Δx sufficiently small and positive and $\Delta y = 0$ would be opposite to the sign of Δf for Δx sufficiently small and negative and $\Delta y = 0$. Therefore *for a minimum or maximum* $f'_x = 0$; *and in like manner* $f'_y = 0$. Considerations like these will serve to establish a criterion for distinguishing between maxima and minima analogous to the criterion furnished by $f''(x)$ in the case of one variable. For if $f'_x = f'_y = 0$, then

$$\Delta f = \tfrac{1}{2}(h^2 f''_{xx} + 2\,hk f''_{xy} + k^2 f''_{yy})_{x=a+\theta h,\, y=b+\theta k},$$

by Taylor's Formula to two terms. Now if the second derivatives are continuous functions of (x, y) in the neighborhood of (a, b), each derivative at $(a + \theta h, b + \theta k)$ may be written as its value at (a, b) plus an infinitesimal. Hence

$$\Delta f = \tfrac{1}{2}(h^2 f''_{xx} + 2\,hk f''_{xy} + k^2 f''_{yy})_{(a,b)} + \tfrac{1}{2}(h^2\zeta_1 + 2\,hk\zeta_2 + k^2\zeta_3).$$

Now the sign of Δf for sufficiently small values of h, k must be the same as the sign of the first parenthesis provided that parenthesis does not vanish. Hence if the quantity

$$(h^2 f''_{xx} + 2\,hk f''_{xy} + k^2 f''_{yy})_{(a,b)} \begin{matrix} > 0 \text{ for every } (h, k), \text{ a minimum} \\ < 0 \text{ for every } (h, k), \text{ a maximum.} \end{matrix}$$

As the derivatives are taken at the point (a, b), they have certain constant values, say A, B, C. The question of distinguishing between minima and maxima therefore reduces to the discussion of the possible signs of a *quadratic form* $Ah^2 + 2\,Bhk + Ck^2$ for different values of h and k. The examples

$$h^2 + k^2, \quad -h^2 - k^2, \quad h^2 - k^2, \quad \pm(h-k)^2$$

show that a quadratic form may be: either 1°, positive for every (h, k) except $(0, 0)$; or 2°, negative for every (h, k) except $(0, 0)$; or 3°, positive for some values (h, k) and negative for others and zero for others; or finally 4°, zero for values other than $(0, 0)$, but either never negative or never positive. Moreover, the four possibilities here mentioned are the only cases conceivable except 5°, that $A = B = C = 0$ and the form always is 0. In the first case the form is called a *definite positive* form, in the second a *definite negative* form, in the third an *indefinite* form, and in the fourth and fifth a *singular* form. The first case assures a minimum, the second a maximum, the third neither a minimum nor a maximum (sometimes called a minimax); but the case of a singular form leaves the question entirely undecided just as the condition $f''(x) = 0$ did.

The conditions which distinguish between the different possibilities may be expressed in terms of the coefficients A, B, C.

$$\begin{matrix} 1^\circ \text{ pos. def.,} & B^2 < AC, & A,\ C > 0; & 3^\circ \text{ indef.,} & B^2 > AC; \\ 2^\circ \text{ neg. def.,} & B^2 < AC, & A,\ C < 0; & 4^\circ \text{ sing.,} & B^2 = AC. \end{matrix}$$

The conditions for distinguishing between maxima and minima are:

$$\left.\begin{matrix} f'_x = 0 \\ f'_y = 0 \end{matrix}\right\} \; f''^2_{xy} < f''_{xx} f''_{yy}, \; \left\{\begin{matrix} f''_{xx}, f''_{yy} > 0 \text{ minimum}; \\ f''_{xx}, f''_{yy} < 0 \text{ maximum}; \end{matrix}\right. \tag{34}$$

$$f''^2_{xy} > f''_{xx} f''_{yy}, \text{ minimax}; \quad f''^2_{xy} = f''_{xx} f''_{yy} \ (?).$$

It may be noted that in applying these conditions to the case of a definite form it is sufficient to show that either f''_{xx} or f''_{yy} is positive or negative because they necessarily have the same sign.

EXERCISES

1. Write at length, without symbolic shortening, the expansion of $f(x, y)$ by Taylor's Formula to and including the terms of the third order in $x - a$, $y - b$. Write the formula also with the terms of the third order as the remainder.

2. Write by analogy the proper form of Taylor's Formula for $f(x, y, z)$ and prove it. Indicate the result for any number of variables.

3. Obtain the quadratic and lower terms in the development

$$(\alpha)\ \text{of } xy^2 + \sin xy \text{ at } (1, \tfrac{1}{2}\pi) \quad \text{and} \quad (\beta)\ \text{of } \tan^{-1}(y/x) \text{ at } (1, 1).$$

4. A rectangular parallelepiped with one vertex at the origin and three faces in the coördinate planes has the opposite vertex upon the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Find the maximum volume.

5. Find the point within a triangle such that the sum of the squares of its distances to the vertices shall be a minimum. Note that the point is the intersection of the medians. Is it obvious that a minimum and not a maximum is present?

6. A floating anchorage is to be made with a cylindrical body and equal conical ends. Find the dimensions that make the surface least for a given volume.

7. A cylindrical tent has a conical roof. Find the best dimensions.

8. Apply the test by second derivatives to the problem in the text and to any of Exs. 4–7. Discuss for maxima or minima the following functions:

$$\begin{array}{ll} (\alpha)\ x^2y + xy^2 - x, & (\beta)\ x^3 + y^3 - x^2y^2 - \tfrac{1}{2}(x^2 + y^2), \\ (\gamma)\ x^2 + y^2 + x + y, & (\delta)\ \tfrac{1}{3}y^3 - xy^2 + x^2y - x, \\ (\epsilon)\ x^3 + y^3 - 9xy + 27, & (\zeta)\ x^4 + y^4 - 2x^2 + 4xy - 2y^2. \end{array}$$

9. State the conditions on the first derivatives for a maximum or minimum of function of three or any number of variables. Prove in the case of three variables.

10. A wall tent with rectangular body and gable roof is to be so constructed as to use the least amount of tenting for a given volume. Find the dimensions.

11. Given any number of masses $m_1, m_2, \cdots, m_n$ situated at (x_1, y_1), $(x_2, y_2), \cdots$, (x_n, y_n). Show that the point about which their moment of inertia is least is their center of gravity. If the points were $(x_1, y_1, z_1), \cdots$ in space, what point would make Σmr^2 a minimum?

12. A test for maximum or minimum analogous to that of Ex. 27, p. 10, may be given for a function $f(x, y)$ of two variables, namely: If a function is positive all over a region and vanishes upon the contour of the region, it must have a maximum within the region at the point for which $f'_x = f'_y = 0$. If a function is finite all over a region and becomes infinite over the contour of the region, it must have a minimum within the region at the point for which $f'_x = f'_y = 0$. These tests are subject to the proviso that $f'_x = f'_y = 0$ has only a single solution. Comment on the test and apply it to exercises above.

13. If a, b, c, r are the sides of a given triangle and the radius of the inscribed circle, the pyramid of altitude h constructed on the triangle as base will have its maximum surface when the surface is $\frac{1}{2}(a + b + c)\sqrt{r^2 + h^2}$.