But now, according to Ax. X, we have

$$(a = 1) + (a = 0) = a + a' = 1.$$

"A proposition is either true or false". From these two formulas combined we deduce directly that the propositions $(a = 1)$ and $(a = 0)$ are contradictory, *i. e.*,

$$(a \neq 1) = (a = 0), \qquad (a \neq 0) = (a = 1).$$

From the point of view of calculation Ax. X makes it possible to reduce to its first member every equality whose second member is 1, and to transform inequalities into equalities. Of course these equalities and inequalities must have propositions as their members. Nevertheless all the formulas of this section are also valid for classes in the particular case where the universe of discourse contains only one element, for then there are no classes but 0 and 1. In short, the special calculus of propositions is equivalent to the calculus of classes when the classes can possess only the two values 0 and 1.

57. Equivalence of an Implication and an Alternative. —The fundamental equivalence

$$(a < b) = (a' + b = 1)$$

gives rise, by Ax. X, to the equivalence

$$(a < b) = (a' + b),$$

which is no less fundamental in the calculus of propositions. To say that a implies b is the same as affirming "not-a or b", *i. e.*, "either a is false or b is true." This equivalence is often employed in every day conversation.

Corollary.—For any equality, we have the equivalence

$$(a = b) = ab + a'b'.$$

Demonstration:

$$(a = b) = (a < b)\ (b < a) = (a' + b)\ (b' + a) = ab + a'b'.$$

"To affirm that two propositions are equal (equivalent) is the same as stating that either both are true or both are false".

The fundamental equivalence established above has important consequences which we shall enumerate.

In the first place, it makes it possible to reduce secondary, tertiary, etc., propositions to primary propositions, or even to sums (alternatives) of elementary propositions. For it makes it possible to suppress the copula of any proposition, and consequently to lower its order of complexity. An implication $(A < B)$, in which A and B represent propositions more or less complex, is reduced to the sum $A' + B$, in which only copulas within A and B appear, that is, propositions of an inferior order. Likewise an equality $(A = B)$ is reduced to the sum $(AB + A'B')$ which is of a lower order.

We know that the principle of composition makes it possible to combine several *simultaneous* inclusions or equalities, but we cannot combine alternative inclusions or equalities, or at least the result is not equivalent to their alternative but is only a consequence of it. In short, we have only the *implications*

$$(a < c) + (b < c) < (ab < c),$$
$$(c < a) + (c < b) < (c < a + b),$$

which, in the special cases where $c = 0$ and $c = 1$, become

$$(a = 0) + (b = 0) < (ab = 0),$$
$$(a = 1) + (b = 1) < (a + b = 1).$$

In the calculus of classes, the converse implications are not valid, for, from the statement that the class ab is null, we cannot conclude that one of the classes a or b is null (they can be not-null and still not have any element in common); and from the statement that the sum $(a + b)$ is equal to 1 we cannot conclude that either a or b is equal to 1 (these classes can *together* comprise all the elements of the universe without any of them *alone* comprising all). But these converse implications are true in the calculus of propositions

$$(ab < c) < (a < c) + (b < c),$$
$$(c < a + b) < (c < a) + (c < b);$$

for they are deduced from the equivalence established above, or rather we may deduce from it the corresponding equalities which imply them,

(1) $$(ab < c) = (a < c) + (b < c),$$

(2) $$(c < a + b) = (c < a) + (c < b).$$

Demonstration:

(1) $$(ab < c) = a' + b' + c,$$

$$(a < c) + (b < c) = (a' + c) + (b' + c) = a' + b' + c;$$

(2) $$(c < a + b) = c' + a + b,$$

$$(c < a) + (c < b) = (c' + a) + (c' + b) = c' + a + b.$$

In the special cases where $c = 0$ and $c = 1$ respectively, we find

(3) $$(ab = 0) = (a = 0) + (b = 0),$$

(4) $$(a + b = 1) = (a = 1) + (b = 1).$$

P. I.: (1) To say that two propositions united imply a third is to say that one of them implies this third proposition.

(2) To say that a proposition implies the alternative of two others is to say that it implies one of them.

(3) To say that two propositions combined are false is to say that one of them is false.

(4) To say that the alternative of two propositions is true is to say that one of them is true.

The paradoxical character of the first three of these statements will be noted in contrast to the self-evident character of the fourth. These paradoxes are explained, on the one hand, by the special axiom which states that a proposition is either true or false; and, on the other hand, by the fact that the false implies the true and that *only* the false is not implied by the true. For instance, if both premises in the first statement are true, each of them implies the consequence, and if one of them is false, it implies the consequence (true or false). In the second, if the alternative is true, one of its terms must be true, and consequently will, like the alternative, be implied by the premise (true or false).

Finally, in the third, the product of two propositions cannot be false unless one of them is false, for, if both were true, their product would be true (equal to 1).

58. Law of Importation and Exportation.—The fundamental equivalence $(a < b) = a' + b$ has many other interesting consequences. One of the most important of these is *the law of importation and exportation,* which is expressed by the following formula:

$$[a < (b < c)] = (ab < c).$$

"To say that if a is true b implies c, is to say that a and b imply c".

This equality involves two converse implications: If we infer the second member from the first, we *import* into the implication $(b < c)$ the hypothesis or condition a; if we infer the first member from the second, we, on the contrary, *export* from the implication $(ab < c)$ the hypothesis a.

Demonstration:

$$[a < (b < c)] = a' + (b < c) = a' + b' + c,$$
$$(ab < c) = (ab)' + c = a' + b' + c.$$

Cor. 1.—Obviously we have the equivalence

$$[a < (b < c)] = [b < (a < c)],$$

since both members are equal to $(ab < c)$, by the commutative law of multiplication.

Cor. 2.—We have also

$$[a < (a < b)] = (a < b),$$

for, by the law of importation and exportation,

$$[a < (a < b)] = (aa < b) = (a < b).$$

If we apply the law of importation to the two following formulas, of which the first results from the principle of identity and the second expresses the principle of contraposition,