35. The Expression of a Double Inclusion by Means of an Indeterminate.—THEOREM. *The double inclusion*

$$b < x < a$$

is equivalent to the equality $x = au + bu'$ *together with the condition* $(b < a)$, u *being a term absolutely indeterminate.*

Demonstration.—Let us develop the equality in question,

$$x(a'u + b'u') + x'(au + bu') = 0,$$

$$(a'x + ax')u + (b'x + bx')u' = 0.$$

Eliminating u from it,

$$a'b'x + abx' = 0.$$

This equality is equivalent to the double inclusion

$$ab < x < a + b.$$

But, by hypothesis, we have

$$(b < a) = (ab = b) = (a + b = a).$$

The double inclusion is therefore reduced to

$$b < x < a.$$

So, whatever the value of u, the equality under consideration involves the double inclusion. Conversely, the double inclusion involves the equality, whatever the value of x may be, for it is equivalent to

$$a'x + bx' = 0,$$

and then the equality is simplified and reduced to

$$ax'u + b'xu' = 0.$$

---

from which (by a formula to be demonstrated later on) we derive the solutions

$$u = ab + w\,(a + b'), \qquad v = a'b + w\,(a + b),$$

or simply

$$u = ab + wb', \qquad v = a'b + wa,$$

w being absolutely indeterminate. We would arrive at these solutions simply by asking: By what term must we multiply b in order to obtain a? By a term which contains ab plus any part of b'. What term must we add to a in order to obtain b? A term which contains $a'b$ plus any part of a. In short, u can vary between ab and $a + b'$, v between $a'b$ and $a + b$.

We can always derive from this the value of u in terms of x, for the resultant $(ab'xx' = 0)$ is identically verified. The solution is given by the double inclusion

$$b'x < u < a' + x.$$

Remark.—There is no contradiction between this result, which shows that the value of u lies between certain limits, and the previous assertion that u is absolutely indeterminate; for the latter assumes that x is any value that will verify the double inclusion, while when we evaluate u in terms of x the value of x is supposed to be determinate, and it is with respect to this particular value of x that the value of u is subjected to limits.[1]

In order that the value of u should be completely determined, it is necessary and sufficient that we should have

$$b'x = a' + x,$$

that is to say,

$$b'xax' + (b + x')(a' + x) = 0$$

or

$$bx + a'x' = 0.$$

Now, by hypothesis, we already have

$$a'x + bx' = 0.$$

If we combine these two equalities, we find

$$(a' + b = 0) = (a = 1)(b = 0).$$

This is the case when the value of x is absolutely indeterminate, since it lies between the limits 0 and 1.

In this case we have

$$u = b'x = a + x = x.$$

In order that the value of u be absolutely indeterminate, it is necessary and sufficient that we have at the same time

---

[1] Moreover, if we substitute for x its inferior limit b in the inferior limit of u, this limit becomes $bb' = 0$; and, if we substitute for x its superior limit a in the superior limit of u, this limit becomes $a + a' = 1$.

$$b'x = 0, \quad a' + x = 1,$$

or

$$b'x + ax' = 0,$$

that is

$$a < x < b.$$

Now we already have, by hypothesis,

$$b < x < a;$$

so we may infer

$$b = x = a.$$

This is the case in which the value of x is completely determinate.

36. Solution of an Equation Involving One Unknown Quantity.—The solution of the equation

$$ax + bx' = 0$$

may be expressed in the form

$$x = a'u + bu',$$

u being an indeterminate, on condition that the resultant of the equation be verified; for we can prove that this equality implies the equality

$$ab'x + a'bx' = 0,$$

which is equivalent to the double inclusion

$$a'b < x < a' + b.$$

Now, by hypothesis, we have

$$(ab = 0) = (a'b = b) = (a' + b = a').$$

Therefore, in this hypothesis, the proposed solution implies the double inclusion

$$b < x < a';$$

which is equivalent to the given equation.

Remark.—In the same hypothesis in which we have

$$(ab = 0) = (b < a'),$$

we can always put this solution in the simpler but less symmetrical forms

$$x = b + a'u, \quad x = a'(b + u).$$