Chapter 3. The analogue of the measure W for a class of linear diffusions.

In Chapters 1 and 2, we have, starting from penalisation results, associated to Wiener measure in dimensions 1 and 2 a positive and σ-finite measure $\mathbf{W}$ (resp. : $\mathbf{W}^{(2)}$ in dimension 2) on the canonical space $(\Omega, \mathcal{F}_\infty)$. In this 3rd Chapter, we shall prove the existence of a measure which is analogous to $\mathbf{W}$, in the more general situation of a large class of linear diffusions. This class is described in Section 3.2. Our approach in this Chapter does not use any penalisation result. Then, in Section 3.3, we shall particularize these results about linear diffusions to the situation of Bessel processes with dimension $d = 2(1-\alpha)$ ($0 < d < 2$, or $0 < \alpha < 1$). Thus, we shall obtain the existence of the measure $\mathbf{W}^{(-\alpha)}(0 < \alpha < 1)$ on $\big(\mathcal{C}(\mathbb{R} \to \mathbb{R}_+),\ \mathcal{F}_\infty\big)$ and we shall then indicate its relationship with penalisation problems. Section 3.1 is devoted to a presentation of our hypotheses and notations.

3.1 Main hypotheses and notations.

3.1.1 Our framework is that of Salminen-Vallois-Yor. [SVY], that is :
$(X_t,\ t \geq 0)$ is a $\mathbb{R}_+ = [0, \infty[$ valued diffusion, with 0 an instantaneously reflecting barrier. The infinitesimal generator $\mathcal{G}$ of $(X_t,\ t \geq 0)$ is given by :

$$\mathcal{G}\, f(x) = \frac{d}{dm}\, \frac{d}{dS}\, f(x) \qquad (x \geq 0) \tag{3.1.1}$$

where the scale function S is a continuous, strictly increasing function s.t. :

$$S(0) = 0, \quad S(+\infty) = +\infty \tag{3.1.2}$$

and $m(dx)$ is the speed measure of X ; we assume $m(\{0\}) = 0$.

3.1.2 The semi-group of $(X_t,\ t \geq 0)$ admits $p(t,x,y)$ as density with respect to m :

$$P_x(X_t \in dy) = p(t,x,y)m(dy) \tag{3.1.3}$$

with p continuous in the 3 variables, and $p(t,x,y) = p(t,y,x)$. $\widehat{X}$ denotes the process X, killed at $T_0 = \inf\{t\ ;\ X_t = 0\}$. We denote by $\widehat{p}$ its density with respect to m :

$$\widehat{P}_x(\widehat{X}_t \in dy) = P_x(X_t \in dy\ ;\ 1_{t<T_0}) := \widehat{p}(t,x,y)m(dy) \tag{3.1.4}$$

with $\widehat{p}(t,x,y) = p(t,x,y)P_x(T_0 > t | X_t = y)$.

3.1.3 The local time process
We denote by $\{L_t^y\ ;\ t \geq 0,\ y \geq 0\}$ the jointly continuous family of local times of X, which satisfy the density of occupation formula :

$$\int_0^t h(X_s)ds = \int_0^\infty h(y) L_t^y\, m(dy) \tag{3.1.5}$$

for any $h : \mathbb{R}_+ \to \mathbb{R}_+$, Borel. It is easily deduced from (3.1.5) and (3.1.3) that :

$$E_x(d_t\, L_t^y) = p(t,x,y)dt \tag{3.1.6}$$

We denote by $P_0^{\tau_l}$ the law, under P_0, of $(X_t,\ t \leq \tau_l)$ with $\tau_l := \inf\{t \geq 0\ ;\ L_t^0 > l\}$. We have also :

$$\big(S(X_t) - L_t,\ t \geq 0\big) \quad \text{is a martingale} \tag{3.1.7}$$

a property which results from (3.1.1) and (3.1.5) $\big($see [DM, RVY] for such a property in the context of Bessel processes$\big)$.

3.1.4 The process X, conditioned not to vanish, is a Doob h-transform of $\widehat{X}$, with $h(x) = S(x)$. In other terms : if $P_x^{\uparrow}$ is the law of X conditioned not to vanish :

$$P_x^{\uparrow}(F_t) = \frac{1}{S(x)}\, E_x[F_t\, S(X_t) 1_{t<T_0}] \tag{3.1.8}$$

for any $F_t \in b(\mathcal{F}_t)$. In particular, the semi-group of the conditioned process is given by :

$$P_x^{\uparrow}(X_t \in dy) = \frac{\widehat{p}(t,x,y)}{S(x)}\, S(y)\, m(dy) \qquad (x \geq 0) \tag{3.1.9}$$

Later, it will be interesting to use the following :

$$P_0^{\uparrow}(X_t \in dy) = f_{y,0}(t)\; S(y)\, m(dy) \tag{3.1.10}$$

where $f_{y,0}(t)$ admits the following description :

$$f_{y,0}(t) = \lim_{x\downarrow 0} \frac{\widehat{p}(t,x,y)}{S(x)} \tag{3.1.11}$$

$$f_{y,0}(t)dt \underset{(a)}{=} P_y(T_0 \in dt) \underset{(b)}{=} P_0^{\uparrow}(g_y \in dt) \tag{3.1.12}$$

with

$$g_y := \sup\{t\ ;\ X_t = y\}$$

We indicate here that (3.1.12) is a partial expression of the time reversal result :

$$P_y\big(\{X_{T_0-t},\ t \leq T_0\}\big) = P_0^{\uparrow}\big(\{X_u,\ u \leq g_y\}\big) \tag{3.1.13}$$

Furthermore :

$$P_0^{\uparrow}\big(\{X_u,\ u \leq g_y\}\big| g_y = t\big) = P_0^{\uparrow}\big(\{X_u,\ u \leq t\}\big| X_t = y\big) \tag{3.1.14}$$

where in (3.1.13) and in (3.1.14) we have used the notation $P\big(\{X_u,\ u \leq a\}\big)$ to denote the law of the process $(X_u,\ u \leq a)$ under P. All these facts, as well as those presented in the following Proposition may be found in [SVY], [BS], [PY2], ... which all deal with properties of linear diffusions.

3.1.5 A useful Proposition :
We shall use the following :
Proposition 3.1.1. *Let $g^{(t)} := \sup\{s \leq t,\, X_s = 0\}$.*
1) *Under P_0, conditionally on $g^{(t)}$, the processes $(X_u,\ u \leq g^{(t)})$ and $(X_{g^{(t)}+u},\ u \leq t - g^{(t)})$ are independent.*
2) *Conditionally on $g^{(t)} = s$, $(s \leq t)$, the process $(X_u,\ u \leq s)$ is distributed as $\Pi_0^{(s)}$, the law of the bridge of X under P_0, with length s, ending at $x = 0$ at time s.*
3) *The law of the couple $\big(g^{(t)}, (X_{g^{(t)}+u},\ u \leq t - g^{(t)})\big)$ under P_0 may be described as follows :*

i) $$P_0(g^{(t)} \in ds,\ X_t \in dy)\, S(y) = p(s,0,0)\, 1_{s<t} P_0^{\uparrow}(X_{t-s} \in dy) ds \tag{3.1.15}$$

or equivalently, with the help of (3.1.10) :

i') $$P_0(g^{(t)} \in ds,\ X_t \in dy) = p(s,0,0)\, f_{y,0}(t-s)\, 1_{s<t} ds\, m(dy) \tag{3.1.16}$$

and, on the other hand :

ii) $P_0\big(\{X_{g_t+u},\ u \le t-g_t\}|X_t = y,\ g_t = s\big)$
$= P_0^{\uparrow}\big(\{X_u,\ u \le t-s\}|X_{t-s} = y\big)$ (3.1.17)

These different properties are established in [SVY], [Sa1] and [Sa2].

3.2 The σ-finite measure $\mathbf{W}^*$.

3.2.1 Definition of $\mathbf{W}^*$:

Here is the main result of this Section.

Theorem 3.2.1.

1) *There exists a unique σ-finite measure, which we denote by $\mathbf{W}^*$, on $\big(\mathcal{C}(\mathbb{R}_+ \to \mathbb{R}_+),\ \mathcal{F}_\infty\big)$ such that :*

$$\forall t \ge 0,\ \forall F_t \in b(\mathcal{F}_t) :\quad E_0\big(F_t\, S(X_t)\big) = \mathbf{W}^*(F_t\, 1_{g\le t}) \tag{3.2.1}$$

with $g := \sup\{t \ge 0\ ;\ X_t = 0\}$

2) $$\mathbf{W}^* = \int_0^\infty dl (P_0^{\tau_l} \circ P_0^{\uparrow}) \tag{3.2.2}$$

3) $$\mathbf{W}^* = \int_0^\infty dt\, p(t,0,0)\, (\Pi_0^{(t)} \circ P_0^{\uparrow}) \tag{3.2.3}$$

In particular, if we denote $\mathbf{W}^*_g$ the restriction of $\mathbf{W}^*$ to $\mathcal{F}_g$, we have :

$$\mathbf{W}^*_g = \int_0^\infty dl\, P_0^{\tau_l} = \int_0^\infty dt\, p(t,0,0)\, \Pi_0^{(t)} \tag{3.2.4}$$

Of course, this Theorem 3.2.1. has been guessed from the comparison with the Brownian situation described in Chapters 1 and 2.

Proof of Theorem 3.2.1.

i) First of all, it is not difficult to show that, starting from equation (3.2.1), where $\mathbf{W}^*$ is the unknown, this problem admits at most one solution such that $g < \infty$, $\mathbf{W}^*$ a.s.

ii) Define

$$\mathbf{W}_* = \int_0^\infty dl\, (P_0^{\tau_l} \circ P_0^{\uparrow}) \tag{3.2.5}$$

We shall now prove that $\mathbf{W}_*$ satisfies (3.2.3) and (3.2.4). Since, under $P_0^{\uparrow}$, the process $(X_t,\ t \ge 0)$ remains in $\mathbb{R}_+ \setminus \{0\}$, it follows immediately, from the definition (3.2.5) of $\mathbf{W}_*$ that

$$\mathbf{W}_{*,g} = \int_0^\infty dl\, P_0^{\tau_l} \tag{3.2.6}$$

where $\mathbf{W}_{*,g}$ denotes the restriction of $\mathbf{W}_*$ to $\mathcal{F}_g$.

On the other hand, a classical argument, which hinges on the fact that the random measure (dL_s) is carried by the zeros of X, allows to show easily that :

$$\int_0^\infty dl\, P_0^{\tau_l} = \int_0^\infty dt\, p(t,0,0)\, \Pi_0^{(t)} \tag{3.2.7}$$

Indeed, by integrating $F := \big(F_t := F(X_u,\ u \le t),\ t \ge 0\big)$ a positive measurable functional, we obtain on the LHS of (3.2.6)

$$\int_0^\infty dl\, P_0^{\tau_l}(F) = \int_0^\infty dl\, P_0(F_{\tau_l}) = P_0\left(\int_0^\infty dL_s \cdot F_s\right)$$

(by time change $l = L_s$)

$$\begin{aligned}
&= P_0\left(\int_0^\infty dL_s\, P_0(F_s | X_s = 0)\right)\\
&= \int_0^\infty P_0(dL_s) P_0(F_s|X_s = 0)\\
&= \int_0^\infty dt\, p(t,0,0)\, \Pi_0^{(t)}(F)
\end{aligned}$$

by (3.1.6), with $x = y = 0$.

ii) We now prove that $\mathbf{W}_*$ satisfies (3.2.1), by showing this equality for the test functionals :

$$F_t = \Phi(X_u,\ u \le g^{(t)})\, \varphi(g^{(t)})\, \psi(X_{g^{(t)}+u},\ u \le t - g^{(t)}) \tag{3.2.8}$$

From (3.2.3), the RHS of (3.2.1) is equal to (with $\mathbf{W}_*$ instead of $\mathbf{W}^*$) :

$$\begin{aligned}
R^F &:= \mathbf{W}_*(F_t\, 1_{g \le t})\\
&= \int_0^t ds\, p(s,0,0)\, \Pi_0^{(s)}\big(\Phi(X_u,\ u \le s)\big)\varphi(s)\, P_0^{\uparrow}\big(\psi(X_u,\ u \le t-s)\big)
\end{aligned} \tag{3.2.9}$$

On the other hand, the LHS of (3.2.1) is equal to :

$$\begin{aligned}
L^F &:= E_0\big[F_t\, S(X_t)\big]\\
&= E_0\big[\Phi(X_u,\ u \le g^{(t)})\, \varphi(g^{(t)})\, \psi(X_{g^{(t)}+u},\ u \le t - g^{(t)})\, S(X_t)\big]\\
&= \int_0^t P_0(g^{(t)} \in ds)\varphi(s)\, E_0\big[\Phi(X_u,\ u \le s)|X_s = 0\big]\, E_0\big[\psi(X_{s+u},\ u \le t-s)\\
&\qquad\qquad S(X_t)|g^{(t)} = s\big]
\end{aligned} \tag{3.2.10}$$

where we have used a part of the results presented in the Proposition 3.1.1. Comparing (3.2.9) and (3.2.10), we now see that showing equality $R^F = L^F$ (i.e. the proof of (3.2.1)) has now been reduced to showing :

$$\begin{aligned}
&P_0^{\uparrow}\big(\psi(X_u,\ u \le t-s)\big)\, p(s,0,0)\, 1_{s<t} ds\\
= \;& P_0(g^{(t)} \in ds)\, E_0\big(\psi(X_{s+u}\ ;\ u \le t-s) \cdot S(X_t)|g^{(t)} = s\big)
\end{aligned} \tag{3.2.11}$$

But (3.2.11) is an easy consequence of point 3 of Proposition 3.1.1.

3.2.2 Some properties of $\mathbf{W}^*$.

The end of this subsection 3.2.2 is devoted to the statement of some results related to the measure $\mathbf{W}^*$. These results are presented without proofs since those are close to the ones found in Chapter 1. These theorems (below) are due to Christophe Profeta ([Pr], thesis in preparation).

3.2.2.1 The probabilities $P_{x,\infty}^{(\lambda)}$.

Theorem 3.2.2.
1) *Let, for $\lambda \geq 0$ and $x \geq 0$:*

$$M_t^{(\lambda,x)} := \frac{1+\frac{\lambda}{2}\,S(X_t)}{1+\frac{\lambda}{2}\,S(x)}\; e^{-\frac{\lambda}{2}\,L_t} = 1 + \frac{\lambda}{2+\lambda S(x)} \int_0^t e^{-\frac{\lambda}{2}\,L_s} dN_s \tag{3.2.12}$$

where $\big(N_s := S(X_s) - L_s,\ s \geq 0\big)$ is the martingale defined by (3.1.7). Then, $(M_t^{(\lambda,x)},\ t \geq 0)$ is a $\big((\mathcal{F}_t,\ t \geq 0),\ P_x\big)$ positive martingale such that : $M_t^{(\lambda,x)} \underset{t\to\infty}{\longrightarrow} 0$, a.s.

2) *Let us define the probability $P_{x,\infty}^{(\lambda)}$ by :*

$$P_{x,\infty}^{(\lambda)}\big|_{\mathcal{F}_t} = M_t^{(\lambda,x)} \cdot P_x\big|_{\mathcal{F}_t} \tag{3.2.13}$$

Then, under $P_{x,\infty}^{(\lambda)}$:

- *The canonical process $(X_t,\ t \geq 0)$ is a transient diffusion with infinitesimal generator $\mathcal{G}_\infty^{(\lambda)}$:*

$$\begin{aligned} \mathcal{G}_\infty^{(\lambda)} f(x) &= \frac{2}{2+\lambda S(x)} \left(\mathcal{G}f(x) + \frac{\lambda}{2}\,\mathcal{G}(Sf)(x) \right) \\ &= \mathcal{G}f(x) + \frac{2\lambda}{2+\lambda S(x)}\, \frac{df}{dm}\,(x) \end{aligned} \tag{3.2.14}$$

  and scale function $S_\infty^{(\lambda)}$:

$$S_\infty^{(\lambda)} := -\frac{2}{2+\lambda S} \tag{3.2.15}$$

- *If $\alpha < \lambda$:*

$$E_{x,\infty}^{(\lambda)} \left(e^{\frac{\alpha}{2}\,L_\infty} \right) < \infty \tag{3.2.16}$$

  and if $\alpha \geq \lambda$:

$$E_{x,\infty}^{(\lambda)} \left(e^{\frac{\alpha}{2}\,L_\infty} \right) = \infty \tag{3.2.17}$$

- *The law of L_∞ is given by :*

$$P_{x,\infty}^{(\lambda)}(L_\infty \in dl) = \frac{\lambda}{2+\lambda S(x)}\; e^{-\frac{\lambda}{2}\,l} dl + \frac{\lambda S(x)}{2+\lambda S(x)} \delta_0(dl) \tag{3.2.18}$$

- *$P_{x,\infty}^{(\lambda)}$ admits the following decomposition :*

$$P_{x,\infty}^{(\lambda)} = \frac{\lambda}{2+\lambda S(x)} \int_0^\infty du\, p(u,x,0) e^{-\frac{\lambda}{2}\,L_u} \,.\, (\Pi_{x,0}^{(u)} \circ P_0^{\uparrow}) + \frac{\lambda S(x)}{2+\lambda S(x)}\, P_x^{\uparrow} \tag{3.2.19}$$

$$= \frac{\lambda}{2+\lambda S(x)} \int_0^\infty e^{-\frac{\lambda l}{2}} dl\, (P_x^{\tau_l} \circ P_0^{\uparrow}) + \frac{\lambda S(x)}{2+\lambda S(x)}\, P_x^{\uparrow} \tag{3.2.20}$$

3.2.2.2 The measures $(\mathbf{W}_x^*,\ x \in \mathbb{R}_+)$.

Theorem 3.2.3.

1) *For any $\lambda > 0$, the σ-finite measure $\left(\dfrac{2}{\lambda} + S(x)\right) \cdot e^{\frac{\lambda}{2} L_\infty} \cdot P_{x,\infty}^{(\lambda)}$ does not depend on λ. We define :*

$$\mathbf{W}_x^* := \left(\frac{2}{\lambda} + S(x)\right) e^{\frac{\lambda}{2} L_\infty} \cdot P_{x,\infty}^{(\lambda)} \tag{3.2.21}$$

We have the decompositions :

$$\mathbf{W}_x^* = \int_0^\infty du\, p(u,x,0)\, (\Pi_{x,0}^{(u)} \circ P_0^{\uparrow}) + S(x) P_x^{\uparrow} \tag{3.2.22}$$

$$\text{and} \quad \mathbf{W}_x^* = \int_0^\infty dl\, (P_x^{\tau_l} \circ P_0^{\uparrow}) + S(x) P_x^{\uparrow} \tag{3.2.23}$$

In particular, $\mathbf{W}_0^* = \mathbf{W}^*$, where $\mathbf{W}^*$ is defined by (3.2.2) or (3.2.3).

2) *i) For every $(\mathcal{F}_t,\ t \geq 0)$ stopping time T and $\Gamma_T \in b(\mathcal{F}_T)$:*

$$E_x\big(\Gamma_T S(X_T) 1_{T<\infty}\big) = \mathbf{W}_x^*(\Gamma_T 1_{g \leq T < \infty}) \tag{3.2.24}$$

where $g := \sup\{s \geq 0\ ;\ X_s = 0\}$

ii) The law of g under $\mathbf{W}_x^*$ is given by :

$$\mathbf{W}_x^*(g \in dt) = p(t,x,0)dt + S(x)\delta_0(dt) \quad (t \geq 0) \tag{3.2.25}$$

and for every $(\mathcal{F}_t,\ t \geq 0)$ stopping time T, we have :

$$\mathbf{W}_x^*(1_{T<\infty}, L_\infty - L_T \in dl) = P_x(T < \infty)\, 1_{[0,\infty[}(l) dl + E_x\big[S(X_T) 1_{T<\infty}\big] \delta_0(dl) \tag{3.2.26}$$

3) *For every previsible and positive process $(\Phi_s,\ s \geq 0)$, we have :*

$$\mathbf{W}_x^*(\Phi_g) = S(x)\Phi_0 + E_x\left(\int_0^\infty \Phi_s dL_s\right) \tag{3.2.27}$$

We note that, from (3.2.19), (3.2.20), (3.2.22) and (3.2.23), we have :

$$\lim_{\lambda \to 0} \frac{2}{\lambda} P_{x,\infty}^{(\lambda)} = \mathbf{W}_x^* \tag{3.2.27'}$$

3.2.2.3 Martingales associated with $(\mathbf{W}_x^*,\ x \in \mathbb{R}_+)$.

Theorem 3.2.4.

Let $F \in L^1_+(\Omega, \mathcal{F}_\infty, \mathbf{W}_x^*)$. There exists a positive $\big((\mathcal{F}_t,\ t \geq 0),\ P_x\big)$ martingale $\big(M_t^*(F),\ t \geq 0\big)$ such that :

1) *For every $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:*

$$\mathbf{W}_x^*(F \cdot \Gamma_t) = E_x\big(M_t^*(F)\Gamma_t\big) \tag{3.2.28}$$

In particular, $\mathbf{W}_x^*(F) = E_x\big(M_t^*(F)\big)$

2) *For every $\lambda > 0$:*

$$\begin{aligned} M_t^*(F) &= \left(\frac{2}{\lambda} + S(X_t)\right) e^{-\frac{\lambda}{2} L_t} E_{x,\infty}^{(\lambda)}\big(F e^{\frac{\lambda}{2} L_\infty} | \mathcal{F}_t\big) \\ &= \widehat{\mathbf{W}}_{X_t}^*\big(F(\omega_t, \widehat{\omega}^t)\big) \end{aligned} \tag{3.2.29}$$

3) $M_t^*(F) \underset{t\to\infty}{\longrightarrow} 0 \qquad P_x \qquad \text{a.s.}$ (3.2.30)

Examples :

• Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\displaystyle\int_0^\infty h(u)du < \infty$. Then :

$$M_t^*\big(h(L_\infty)\big) = h(L_t)S(X_t) + \int_{L_t}^\infty h(l)dl \tag{3.2.31}$$

In particular, if $h(y) = e^{-\frac{\lambda}{2}y}$ $(y \geq 0)$:

$$M_t^*(e^{-\frac{\lambda}{2}L_\infty}) = \left(\frac{2}{\lambda} + S(X_t)\right)e^{-\frac{\lambda}{2}L_t} = \frac{2}{\lambda}\,M_t^{(\lambda,0)} \qquad (x=0)$$

• Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ Borel such that $\displaystyle\int_0^\infty \Phi(u)p(u,x,0)du < \infty$. Then

$$M_t^*\big(\Phi(g)\big) = \Phi(g^{(t)})S(X_t) + \int_0^\infty \Phi(t+u)p(u,X_t,0)du \tag{3.2.32}$$

3.2.2.4 A decomposition Theorem of $\big((\mathcal{F}_t,\ t\geq 0),\ P_x\big)$ positive supermartingales.

Theorem 3.2.5.
Let $(Z_t,\ t\geq 0)$ a positive $\big((\mathcal{F}_t,\ t\geq 0),\ P_x\big)$ supermartingale. We denote

$$Z_\infty := \lim_{t\to\infty} Z_t \qquad P_x \qquad \text{a.s.}$$

Then :
1) $z_\infty := \displaystyle\lim_{t\to\infty}\frac{Z_t}{1+S(X_t)}$ *exists* $\mathbf{W}_x^*$ *a.s. and* $\mathbf{W}_x^*(z_\infty) < \infty$
2) $(Z_t,\ t\geq 0)$ *admits the following decomposition :*

$$Z_t = M_t^*(z_\infty) + E_x\big(Z_\infty|\mathcal{F}_t\big) + \xi_t \tag{3.2.33}$$

where $\big(M_t^*(z_\infty),\ t\geq 0\big)$ *and* $\big(E_x\big(Z_\infty|\mathcal{F}_t\big),\ t\geq 0\big)$ *denote two positive* $\big((\mathcal{F}_t,\ t\geq 0),\ P_x\big)$ *martingales and* $(\xi_t,\ t\geq 0)$ *is a positive supermartingale such that :*
• $Z_\infty \in L^1_+(\mathcal{F}_\infty, P_x)$*, hence* $\big(E_x(Z_\infty|\mathcal{F}_t),\ t\geq 0\big)$ *is a uniformly integrable martingale converging towards* Z_∞.
• $\dfrac{E_x\big(Z_\infty|\mathcal{F}_t\big)+\xi_t}{1+S(X_t)} \underset{t\to\infty}{\longrightarrow} 0 \quad \mathbf{W}_x^*$ *a.s.*
• $M_t^*(z_\infty) + \xi_t \underset{t\to\infty}{\longrightarrow} 0 \quad P_x$ *a.s.*
This decomposition (3.2.33) is unique.

Corollary 3.2.6.
A positive martingale $(Z_t,\ t\geq 0)$ is equal to $\big(M_t^*(F),\ t\geq 0\big)$ for some $F \in L^1_+(\mathcal{F}_\infty, \mathbf{W}_x^*)$ if and only if :

$$Z_0 = \mathbf{W}_x^*\left(\lim_{t\to\infty}\frac{Z_t}{1+S(X_t)}\right) \tag{3.2.34}$$

In the present framework of linear diffusions, it is possible to state a decomposition theorem for the martingales $\big(M_t^*(F),\ t\geq 0\big)$ $\big(F\in L^1(\mathcal{F}_\infty, \mathbf{W}_x^*)\big)$ which is similar to the result stated in Theorem 1.2.11. We leave this task to the interested reader.

3.2.3 Relation between the measure $\mathbf{W}^*$ and penalisations.
In a submitted article, P. Salminen and P. Vallois (see [SV]) obtain the following result involving, as weight functional, the local time of a diffusion, under a certain subexponentiality hypothesis. We now summarize their results.
Let $(\tau_l,\ l \geq 0)$ denote the right continuous inverse of the local time process $(L_t,\ t \geq 0)$ at level 0 associated to $(X_t,\ t \geq 0)$:

$$\tau_l := \inf\{t \geq 0\ ;\ L_t > l\}$$

This subordinator $(\tau_l,\ l \geq 0)$ admits as its Levy measure a mesure ν with density, which we denote here by $\overset{\bullet}{\nu}$ (see [KS]) :

$$E(e^{-\lambda \tau_l}) = \exp\left\{-l \int_0^\infty (1 - e^{-\lambda x})\, \overset{\bullet}{\nu}(x) dx\right\} \qquad (\lambda, l \geq 0)$$

P. Salminen and P. Vallois then make the following hypothesis : the function $F : [1, \infty[\to [0, 1]$ defined by :

$$F(x) := \frac{\nu\big(]1, x[\big)}{\nu\big(]1, \infty[\big)} = \frac{\displaystyle\int_1^x \overset{\bullet}{\nu}(y) dy}{\displaystyle\int_1^\infty \overset{\bullet}{\nu}(y) dy} \tag{3.2.35}$$

is sub-exponential [1], i.e. :

$$\lim_{x \to \infty} \frac{\overline{F * F}(x)}{\overline{F}(x)} = 2 \tag{3.2.36}$$

where $\overline{F}(x) := 1 - F(x),\ x \geq 1$ and where $*$ indicates the convolution operation.
One of the main consequences of the subexponentiality of F is :

$$\frac{\overline{F}(x + y)}{\overline{F}(x)} \underset{x \to \infty}{\longrightarrow} 1 \qquad \text{uniformly on compacts (in } y\text{)}$$

Thus, here

$$\frac{\nu\big(]x + y, \infty[\big)}{\nu\big(]x, \infty[\big)} \underset{x \to \infty}{\longrightarrow} 1 \qquad \text{uniformly on compacts (in } y\text{)} \tag{3.2.37}$$

Under this subexponentiality hypothesis, P. Salminen and P. Vallois [SV] then prove the following Theorem.

Theorem 3.2.7. *(Penalisation by $(1_{(L_t < l)},\ t \geq 0)$*
Let $l > 0$ be fixed. Then, for every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$:

$$\lim_{t \to \infty} \frac{E_x\big(\Gamma_s\, 1_{(L_t < l)}\big)}{P_x(L_t < l)} = E_x(\Gamma_s \cdot M_s^{(l)}) := P_{x,\infty}^{(l)}(\Gamma_s) \tag{3.2.38}$$

where $(M_s^{(l)},\ s \geq 0)$ is the positive martingale defined by :

$$M_s^{(l)} := \frac{S(X_s) - L_s + l}{S(x) + l} \cdot 1_{L_s < l}$$

[1] This notion has little to do with the sub-exponential functions, i.e. functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ which satisfy : $f(x) \leq c_1 e^{-c_2 x}$ for some constants $c_1, c_2 > 0$, and which are considered in [RY, IX].

Let us remark that for $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_0^\infty \left(1 + \frac{1}{l}\right) f(l)\, dl < \infty.$$

Then, we have :

$$\begin{aligned} M_s^{(f)} &:= \int_0^\infty M_s^{(l)} f(l)\, dl \\ &= (S(X_s) - L_s) \int_{L_s}^\infty \frac{f(l)dl}{S(x)+l} + \int_{L_s}^\infty \frac{f(l)l\, dl}{S(x)+l} \end{aligned}$$

and, for $x = 0$,

$$\begin{aligned} M_s^{(f)} &= (S(X_s) - L_s) \int_{L_s}^\infty \frac{f(l)}{l}\, dl + \int_{L_s}^\infty f(l)\, dl \\ &= S(X_s)h(L_s) + \int_{L_s}^\infty h(y)dy \end{aligned}$$

with

$$h(y) := \int_y^\infty \frac{f(l)}{l}\, dl.$$

Thus, $(M_s^{(f)}, s \geq 0)$ is the Azéma-Yor martingale associated to h (see [AY1]).
The key point of the proof of Theorem 3.2.7 is the following

Lemma 3.2.8. ([SV])

$$P_x(L_t < l) \underset{t\to\infty}{\sim} \big(S(x) + l\big)\, \nu\big(]t, \infty[\big) \tag{3.2.39}$$

Theorem 3.2.7 now follows easily from Lemma 3.2.8 and from relation (3.2.37).
From this Theorem 3.2.7, we deduce the following relation between the probability $P_{0,\infty}^{(l)}$ defined by (3.2.38) and the σ-finite measure $\mathbf{W}^*$ defined by (3.2.2) or (3.2.3) :

$$1_{L_\infty < l} \cdot \mathbf{W}^* = \mathbf{W}^*(L_\infty < l) \cdot P_{0,\infty}^{(l)} \tag{3.2.39}$$

(We note that $P_{0,\infty}^{(l)}(L_\infty < l) = 1$). The reader may compare relation (3.2.39) with relation (1.1.107) of Theorem 1.1.11' and with relation (3.2.21) of Theorem 3.2.3. From (3.2.39), we also deduce, with the notation of Theorem 3.2.4, that :

$$M_t^*(1_{(L_\infty < l)}) = \mathbf{W}^*(L_\infty < l) \cdot \left(\frac{S(X_t) + l - L_t}{l}\right) 1_{L_t < l} \qquad (x = 0) \tag{3.2.40}$$

Finally, we indicate that, in further works in progress, C. Profeta (see [Pr]) studies the penalisation of a linear diffusion reflected in 0 and 1 (the subexponentiality hypothesis is not satisfied) with the functional $(e^{\alpha L_t}, t \geq 0)$ $(\alpha \in \mathbb{R})$. He proves that the penalised process is still a linear diffusion reflected in 0 and 1 and computes the scale function and the speed measure of this new process.

3.3 The example of Bessel processes with dimension d $(0 < d < 2)$

3.3.1 Transcription of our notation in the context of Bessel processes.
Let $d = 2(1-\alpha)$ with $0 < d < 2$ (or $0 < \alpha < 1$). We now study the particular case of the process $(X_t,\ t \geq 0)$ described in Section 3.1 with :

$$m(dx) = \frac{x^{1-2\alpha}}{\alpha}\, 1_{[0,\infty[}(x)dx \tag{3.3.1}$$

$$S(x) = x^{2\alpha} \quad (x \geq 0) \tag{3.3.2}$$

Then, the process $(X_t,\ t \geq 0)$ described in Section 3.1 is a Bessel process with dimension d, and index $\frac{d}{2} - 1 = -\alpha$. We denote by $(P_x^{(-\alpha)},\ x \in \mathbb{R}_+)$ the family of its laws. We note $\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{R}_+),\ (R_t, \mathcal{F}_t),\ t \geq 0,\ \mathcal{F}_\infty,\ P_x^{(-\alpha)}\ (x \in \mathbb{R}_+)\big)$ the canonical realisation of the Bessel process with index $(-\alpha)$. Here, the probability $P_x^{\uparrow}$ defined in 3.1.4 is the law of Bessel process with dimension $4 - d = 2(1+\alpha)$, i.e. : index α. We shall denote this law by $P_x^{(\alpha)}$. The formulae of subsection 3.1 now become :

$$(R_t^{2\alpha} - L_t,\ t \geq 0) \text{ is a martingale} \tag{3.3.3}$$

$$P_x^{\uparrow} = P_x^{(\alpha)} \tag{3.3.4}$$

$$\int_0^t h(R_s)ds = \frac{1}{\alpha}\int_0^\infty h(x)\, L_t^x\, x^{1-2\alpha}dx \tag{3.3.5}$$

$$E_0^{(-\alpha)}(L_t^0) = t^\alpha\, E_0^{(-\alpha)}(L_1) = \frac{2^\alpha\, t^\alpha}{\Gamma(1-\alpha)} \tag{3.3.6}$$

$$L^{(\alpha)} f(r) = \frac{1}{2} f''(r) + \frac{1+2\alpha}{2r}\, f'(r) \tag{3.3.7}$$

The reader may refer to [DMRVY] for these formulae.

3.3.2 The measure $\mathbf{W}^{(-\alpha)}$.
In this framework, Theorem 3.2.1 becomes :
Theorem 3.3.1. *For every* $\alpha \in]0,1[$ *:*
1) *There exists a unique positive and* σ*-finite measure* $\mathbf{W}^{(-\alpha)}$ *on* $\big(\Omega = \mathcal{C}(\mathbb{R}_+ \to \mathbb{R}_+),\ \mathcal{F}_\infty\big)$ *such that, for every* $F_t \in b(\mathcal{F}_t)$ *:*

$$\mathbf{W}^{(-\alpha)}(F_t\, 1_{g \leq t}) = P_0^{(-\alpha)}(F_t \cdot R_t^{2\alpha}) \tag{3.3.8}$$

2) $$\mathbf{W}^{(-\alpha)} = \int_0^\infty (P_0^{(-\alpha,\,\tau_l)} \circ P_0^{(\alpha)})dl \tag{3.3.9}$$

3) *i)* $$\mathbf{W}^{(-\alpha)}(g \in dt) = \frac{\alpha\, 2^\alpha}{\Gamma(1-\alpha)}\, t^{\alpha-1}dt \qquad (t \geq 0)$$

ii) Conditionally on $g = t$*, under* $\mathbf{W}^{(-\alpha)}$*,* $(R_u,\ u \leq g)$ *is a Bessel bridge with index* $(-\alpha)$ *and of length* t

iii) $$\mathbf{W}^{(-\alpha)} = \int_0^\infty \frac{\alpha\, 2^\alpha\, t^{\alpha-1}}{\Gamma(1-\alpha)}\, dt(\Pi_0^{(-\alpha,t)} \circ P_0^{(+\alpha)}) \tag{3.3.10}$$

In this Theorem :
$\Pi_0^{(-\alpha,t)}$ denotes the law of a Bessel bridge with index $(-\alpha)$ and of length t.
$P_0^{(-\alpha,\tau_l)}$ denotes the law of a Bessel process with index $(-\alpha)$ starting at 0 and stopped at τ_l, with :

$$\tau_l = \inf\{t \geq 0\ ;\ L_t^0 > l\} \tag{3.3.11}$$

3.3.3 Relations between $\mathbf{W}^{(-\alpha)}\big(d = 2(1-\alpha)\big)$ and Feynman-Kac penalisations.

Remark 3.3.2. The measure $\mathbf{W}^{(-\alpha)}$ which we just described is also related to a penalisation problem. More precisely, one can prove $\big($see [RVY, I or V]$\big)$:

i) Let q be a positive Radon measure on $\mathbb{R}_+$, with compact support. Then :

$$2^\alpha \Gamma(1+\alpha) t^\alpha \, P_r^{(-\alpha)} \left(\exp\left(-\frac{1}{2} A_t^{(q)} \right) \right) \underset{t\to\infty}{\longrightarrow} \varphi_q^{(-\alpha)}(r) \tag{3.3.12}$$

with

$$A_t^{(q)} := \int_0^\infty q(R_s) ds = \frac{1}{\alpha} \int_0^\infty L_t^x \, x^{1-2\alpha} q(dx) \tag{3.3.13}$$

ii) The function $\varphi_q^{(-\alpha)}$ defined by (3.3.12) is characterised as the unique solution of :

$$\begin{cases} \dfrac{1}{2} f''(r) + \dfrac{1-2\alpha}{2r} f'(r) = \dfrac{1}{2} f(r) q(r) \\ \text{(in the sense of Schwartz distributions)} \\ f(r) \underset{r\to\infty}{\sim} r^{2\alpha} \end{cases} \tag{3.3.14}$$

iii) For every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$:

$$E_r^{(-\alpha)} \left(\Gamma_s \, \frac{\exp -\frac{1}{2} A_t^{(q)}}{E_r^{(-\alpha)}\big(\exp -\frac{1}{2} A_t^{(q)} \big)} \right) \underset{t\to\infty}{\longrightarrow} P_{r,\infty}^{(-\alpha,q)}(\Gamma_s) \tag{3.3.15}$$

where the probability $P_{r,\infty}^{(-\alpha,q)}$ satisfies :

$$P_{r,\infty}^{(-\alpha,q)}|_{\mathcal{F}_s} = M_s^{(-\alpha,q)} P_r^{(-\alpha)}|_{\mathcal{F}_s} \tag{3.3.16}$$

with

$$M_s^{(-\alpha,q)} = \frac{\varphi_q^{(-\alpha)}(R_s)}{\varphi_q^{(-\alpha)}(r)} \exp\left(-\frac{1}{2} A_s^{(q)} \right) \tag{3.3.17}$$

and $(M_s^{(-\alpha,q)}, \ s \geq 0)$ is a $\big((\mathcal{F}_s, \ s \geq 0), \ P^{(-\alpha)}\big)$ martingale.

iv) Under $P_{r,\infty}^{(-\alpha,q)}$ $(r \geq 0)$, the canonical process $(R_t, \ t \geq 0)$ is a transient diffusion with infinitesimal generator $\mathcal{G}^{(-\alpha,q)}$ given by :

$$\mathcal{G}^{(-\alpha,q)} f(r) = \frac{1}{2} f''(r) + \left(\frac{1-2\alpha}{2r} + \frac{(\varphi_q^{(-\alpha)})'}{\varphi_q^{(-\alpha)}}(r) \right) f'(r) \tag{3.3.18}$$

Remark 3.3.3.

With the notation of Remark 3.3.2, in the particular case where q is the measure q_0 such that $\dfrac{1}{\alpha} x^{1-2\alpha} q_0(dx)$ is Dirac mass in 0 (of course, this is somewhat informal : we need to choose a sequence $q_0^{(n)}$ such that $\dfrac{1}{\alpha} x^{1-2\alpha} q_0^{(n)}(dx)$ converges towards δ_0 as $n \to \infty$), we obtain :

$$\varphi_{q_0}^{(-\alpha)}(r) = 2 + r^{2\alpha}, \ \varphi_{q_0}^{(-\alpha)}(0) = 2 \tag{3.3.19}$$

$$\text{and} \quad M_t^{(-\alpha,q_0)} = \left(1 + \frac{R_t^{2\alpha}}{2} \right) e^{-\frac{1}{2} L_t} \tag{3.3.20}$$

Now, the analogue of Theorem 1.1.5 is :

Theorem 3.3.4.

Under $P_\infty^{(-\alpha,q_0)}$, the canonical process $(R_t,\ t\geq 0)$ satisfies :

i) Let $g=\sup\{s\geq 0,\ R_s=0\}$. Then :

$$g<\infty \qquad P_\infty^{(-\alpha,q_0)} \qquad a.s. \qquad and \tag{3.3.21}$$

ii) $L_\infty (=L_g)$ admits as density :

$$f_{L_\infty}^{P_\infty^{(-\alpha,q_0)}}(l)=\frac{1}{2}\,e^{-\frac{l}{2}}\,1_{[0,\infty[}(l)dl \tag{3.3.22}$$

iii) Conditionally on g, $(R_s,\ s\leq g)$ and $(R_{g+s},\ s\geq 0)$ are independent.

iv) $(R_{g+s},\ s\geq 0)$ is a $(4-d)$ dimensional Bessel process starting at 0 $\big($i.e. admits $P_0^{(+\alpha)}$ as its law$\big)$.

v) Conditionally on $L_\infty(=L_g)=l$, $(R_s,\ s\leq g)$ is a d-dimensional Bessel process stopped at τ_l. Its law is $P_0^{(-\alpha,\tau_l)}$.

Remark 3.3.5.

1) Since, for $\alpha=\dfrac{1}{2}$, $(R_t,\ t\geq 0)$ under $P^{(-\alpha)}$ is a reflected Brownian motion, one has :

$$\mathbf{W}\big(F(|X_s|,\ s\geq 0)\big)=\mathbf{W}^{\left(-\frac{1}{2}\right)}\big(F(R_s,\ s\geq 0)\big) \tag{3.3.23}$$

(where $\mathbf{W}$ is defined by Theorem 1.1.2).

2) In the same spirit, since the modulus of a 2-dimensional Brownian motion is a 2-dimensional Bessel process, hence has index 0, we conjecture that, in a sense to be made precise :

$$\mathbf{W}^{(2)}\big(F(|X_s|,\ s\geq 0)\big)=\lim_{\alpha\downarrow 0}\quad \mathbf{W}^{(-\alpha)}\big(F(R_s,\ s\geq 0)\big) \tag{3.3.24}$$

(where $\mathbf{W}^{(2)}$ is defined by Theorem 2.1.2).

Remark 3.3.6.

We have given, in Subsection 1.1.6, a proof of Theorem 1.1.6 (this is precisely Theorem 1.1.10) which hinges upon the disintegration of Wiener measure restricted to $\mathcal{F}_t$, with respect to the law of $g^{(t)}$ $\big($see (1.1.82)$\big)$. Formula (3.3.10) may be proven in a quite similar way by using the following :

i) For fixed time t, the three following random elements are independent :

- $\left(r_u:=\dfrac{1}{\sqrt{g^{(t)}}}\,R_{u\,g^{(t)}},\ u\leq 1\right)$ which is a Bessel bridge with dimension $d=2(1-\alpha)$
- $g^{(t)}:=\sup\{u<t\ ;\ R_u=0\}$ which is distributed as :

$$P_0^{(-\alpha)}(g^{(t)}\in du)=\frac{du}{b_\alpha u^{1-\alpha}(t-u)^\alpha} \qquad (0\leq u\leq t) \tag{3.3.25}$$

with : $b_\alpha=B(\alpha,1-\alpha)=\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin(\pi\alpha)}$.

- $\left(m_u := \dfrac{1}{\sqrt{t - g^{(t)}}} \, R_{g^{(t)} + u(1 - g^{(t)})}, \ u \le 1 \right)$, which is a Bessel meander (with dimension d).

ii) Imhof's absolute continuity relationship between the laws of the Bessel meander $(m_u, \ u \le \ 1)$ and the Bessel process with dimension $2(1+\alpha)$ (i.e. : with index α) is :

$$E^{(-\alpha)}\big(F(m_u, \ u \le 1)\big) = E_0^{(\alpha)} \left(F(R_u, \ u \le 1) \, \frac{2^\alpha \Gamma(1+\alpha)}{R_1^{2\alpha}} \right) \tag{3.3.26}$$

3.4 Another description of $\mathbf{W}^{(-\alpha)}$ (and of $\mathbf{W}_g^*$).

3.4.1 We recall that $\big($see (3.2.4), (3.3.9) and (3.3.10)$\big)$:

$$\mathbf{W}_g^* = \int_0^\infty dl \, P_0^{\tau_l} = \int_0^\infty dt \, p(t,0,0) \, \Pi_0^{(t)} \tag{3.4.1}$$

in the context of general linear diffusions and :

$$\mathbf{W}_g^{(-\alpha)} = \int_0^\infty dl \, P_0^{(-\alpha, \tau_l)} = \int_0^\infty \frac{\alpha 2^\alpha t^{\alpha - 1}}{\Gamma(1-\alpha)} \, \Pi_0^{(-\alpha, t)} \, dt \tag{3.4.2}$$

in the context of the Bessel processes with index $(-\alpha)$ $(0 < \alpha < 1)$
We shall now give a new description of $\mathbf{W}_g^{(-\alpha)}$ (resp. $\mathbf{W}_g^*$) which is the restriction of $\mathbf{W}^{(-\alpha)}$ (resp. $\mathbf{W}^*$) to $\mathcal{F}_g$. This new description is simply the transcript in the Bessel framework of results found in Pitman-Yor $\big($see [PY2]$\big)$.

3.4.2 We begin by recalling in the framework of Bessel processes some of the results from [PY2].
We denote by $\widehat{\Omega}$ the space of continuous functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ with finite lifetime ξ :

$$\begin{aligned} \widehat{\Omega} = \big\{ \omega \ : \mathbb{R}_+ \to \mathbb{R}_+ \ ; \ \exists \xi(\omega) < \infty \text{ s.t. } \omega(0) = 0 = \omega(\xi), \\ \text{and} \qquad \omega(u) = 0 \quad \text{for every } u \ge \xi(\omega) \big\} \end{aligned}$$

We denote by $(R_t, \ t \ge 0)$ the set of coordinates on this space :

$$R_t(\omega) = \omega(t), \ \omega \in \widehat{\Omega}$$

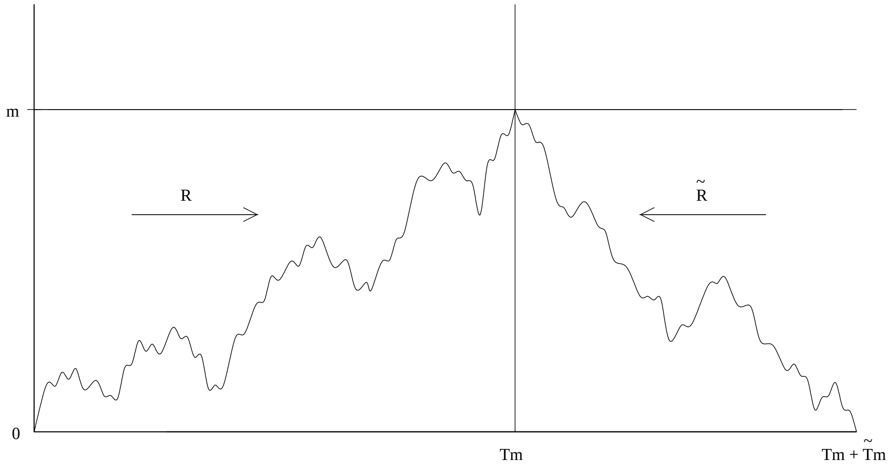
The result of Pitman-Yor which we use $\big($Theorem 1.1 of [PY2]$\big)$ asserts the existence, for every $\delta > 0$, of a positive and σ-finite measure on $(\widehat{\Omega}, \mathcal{F}_\infty)$, denoted as $\mathbf{\Lambda}_{0,0}^{(\delta)}$ and which may be described in either of the following manners :

First description

$$\mathbf{\Lambda}_{0,0}^{(\delta)} = \int_0^\infty \frac{2^{-\frac{\delta}{2}}}{\Gamma(\delta/2)} \, t^{-\frac{\delta}{2}} \quad \Pi_0^{\left(\frac{\delta}{2} - 1, \, t\right)} dt \tag{3.4.3}$$

where $\Pi_0^{\left(\frac{\delta}{2} - 1, \, t\right)}$ denotes the law of the Bessel bridge with index $\dfrac{\delta}{2} - 1$, i.e. with dimension δ, and length t.

Second description



Let $m>0$ fixed and let $P_0^{\left(\frac{\delta}{2}-1,m,\nearrow\nwarrow\right)}$ denote the law of the process obtained by putting two Bessel processes with index $\left(\dfrac{\delta}{2}-1\right)$ (i.e. : with dimension δ), back to back starting from 0, and stopped when they first reach level m. These two processes R and $\widetilde{R}$ are assumed to be independent. In other terms, $P_0^{\left(\frac{\delta}{2}-1,m,\nearrow\nwarrow\right)}$ is the law of the process $(Y_t,\ t\geq 0)$ defined by :

$$Y_t=\begin{cases} R_t & \text{if} \quad t\leq T_m \\ \widetilde{R}_{T_m+\widetilde{T}_m-t} & \text{if} \quad T_m\leq t\leq T_m+\widetilde{T}_m \\ 0 & \text{if} \quad t\geq T_m+\widetilde{T}_m \end{cases} \tag{3.4.4}$$

where T_m (resp $\widetilde{T}_m$) is the first hitting time of m by $(R_t,\ t\geq 0)$ $\big($resp. by $(\widetilde{R}_t,\ t\geq 0)\big)$.
Then :

$$\mathbf{\Lambda}_{0,0}^{(\delta)}=\int_0^\infty m^{1-\delta}dm\, P^{\left(\frac{\delta}{2}-1,m,\nearrow\nwarrow\right)} \tag{3.4.5}$$

The measure $\mathbf{\Lambda}_{0,0}^{(\delta)}$ is called the "generalized excursion measure" in Pitman-Yor. When $\delta=3$, $\mathbf{\Lambda}_{0,0}^{(3)}$ is the Itô measure of (positive) Brownian excursions. Formula (3.4.3) is Itô's description of Itô's measure (see [ReY], Chap. XII), whereas formula (3.4.5) is Williams' description of that measure (see [Wi]).
3.4.3 Here is now, in the framework of the Bessel processes, the announced transcription :
Theorem 3.4.1 *For every $\alpha\in]0,1[$:*

$$\mathbf{W}^{(-\alpha)}\big|_{\mathcal{F}_g}=\mathbf{W}_g^{(-\alpha)}=2\alpha\mathbf{\Lambda}_{0,0}^{(2(1-\alpha))} \tag{3.4.6}$$

In particular :

$$\mathbf{W}^{(-\alpha)}\big|_{\mathcal{F}_g}=\frac{\alpha 2^\alpha}{\Gamma(1-\alpha)}\int_0^\infty t^{\alpha-1}dt\,\Pi_0^{(-\alpha,t)} \tag{3.4.7}$$

$$\mathbf{W}^{(-\alpha)}\big|_{\mathcal{F}_g}=2\alpha\int_0^\infty m^{2\alpha-1}dm\, P^{(-\alpha,m,\ \nearrow\nwarrow)} \tag{3.4.8}$$

Thus, formula (3.4.8) provides us with a new description of the measure $\mathbf{W}_g^{(-\alpha)}$.

Proof of Theorem 3.4.1 Of course, from (3.4.3), and (3.4.5), it suffices to show (3.4.6). Note that, from (3.3.8), for $\Gamma_t \in b(\mathcal{F}_t)$, one has :

$$\mathbf{W}^{(-\alpha)}(\Gamma_t\, 1_{g\le t}) = P_0^{(-\alpha)}(\Gamma_t\, R_t^{2\alpha}) \tag{3.4.9}$$

Thus, for every $s \le t$ and $\Gamma_s \in b(\mathcal{F}_s)$, since $(R_t^{2\alpha} - L_t,\ t \ge 0)$ is a martingale $\big($see (3.3.3)$\big)$, we have :

$$\begin{aligned}\mathbf{W}^{(-\alpha)}(\Gamma_s\, 1_{s\le g\le t}) &= P_0^{(-\alpha)}\big(\Gamma_s(R_t^{2\alpha} - R_s^{2\alpha})\big)\\ &= P_0^{(-\alpha)}\big(\Gamma_s(L_t - L_s)\big)\end{aligned} \tag{3.4.10}$$

We deduce from the monotone class theorem and from (3.4.10) that, for every positive previsible process $(\Phi_u,\ u \ge 0)$, one has :

$$\begin{aligned}\mathbf{W}^{(-\alpha)}(\Phi_g) &= P_0^{(-\alpha)}\left(\int_0^\infty \Phi_u\, dL_u\right)\\ &= \int_0^\infty P_0^{(-\alpha)}(\Phi_u | R_u = 0) P_0^{(-\alpha)}(dL_u)\\ &= \int_0^\infty \Pi^{(-\alpha,u)}(\Phi_u)\, \frac{\alpha 2^\alpha u^{\alpha-1}}{\Gamma(1-\alpha)}\, du\end{aligned}$$

from (3.3.6). Hence :

$$\begin{aligned}\mathbf{W}^{(-\alpha)}(\Phi_g) &= \left(\int_0^\infty \Pi_0^{(-\alpha,u)}\frac{\alpha 2^\alpha u^{\alpha-1}}{\Gamma(1-\alpha)}\, du\right)(\Phi_g)\\ &= \left(2\alpha \int_0^\infty du\, \frac{2^{-\frac{\delta}{2}}}{\Gamma(\frac{\delta}{2})}\, u^{-\frac{\delta}{2}}\, \Pi_0^{(\frac{\delta}{2}-1,u)}\right)(\Phi_g)\\ &\quad \big(\text{since } \delta = 2(1-\alpha)\big)\\ &= 2\alpha \boldsymbol{\Lambda}_{0,0}^{(2(1-\alpha))}(\Phi_g) \quad (\text{from (3.4.3)})\end{aligned}$$

3.4.4 <u>In the general framework of linear diffusions, formulae (3.4.7) and (3.4.8) become</u> :

$$\mathbf{W}_g^* = \int_0^\infty dt\, p(t,0,0)\, \Pi_0^{(t)}$$

$\big($this is formula (3.2.4)$\big)$ and :

$$\mathbf{W}_g^* = \int_0^\infty P_0^{(m,\nearrow\nwarrow)} dS(m) \tag{3.4.11}$$

The reader may refer to $\big($[PY2], 2.2, Corollary 2.1, p. 298$\big)$ where the probability $P_0^{(m,\nearrow\nwarrow)}$ is defined in terms of the law $P_0^{\uparrow}$ $\big($of the process $(X_t,\ t\ge 0)$ conditioned to remain $> 0\big)$ just as $P_0^{(-\alpha,m,\nearrow\nwarrow)}$ is, in terms of the law $P_0^{(\alpha)}$. $\big($see (3.4.4) with $\delta = 2(1+\alpha)\big)$.

3.5 Penalisations of α-stable symmetric Lévy process $(1 < \alpha \le 2)$

In this subsection, we summarize the results by K. Yano, Y. Yano and M. Yor [YYY] which bears upon the penalisation of the α-stable symmetric Lévy process, with $1 < \alpha \leq 2$. This summary is not exhaustive ; rather, it is an invitation to read [YYY].

3.5.1 Notation and classical results. (see, e.g., [Be], [C], [SY])

3.5.1.1 $\big(\Omega, (X_t, \mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, P_x,\ x \in \mathbb{R}\big)$ denotes the canonical realization of the α-stable symmetric Lévy process, with $1 < \alpha \leq 2$. The notations are the same as in 1.0.1, with the difference that Ω now denotes the space of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}$. α being fixed once and for all, the dependency in α will be mostly omitted in our notation. This Lévy process $(X_t,\ t \geq 0)$ is characterised via :

$$E_0(e^{i\lambda X_t}) = \exp\big(-t|\lambda|^\alpha\big) \qquad (t \geq 0,\ \lambda \in \mathbb{R}) \tag{3.5.1}$$

The case $\alpha = 2$ corresponds to $(X_t,\ t \geq 0) \equiv (B_{2t},\ t \geq 0)$ where $(B_t,\ t \geq 0)$ is a standard 1-dimensional Brownian motion.

3.5.1.2 $p_t(x)$ denotes the density (with respect to Lebesgue measure on $\mathbb{R}$) of the law of the r.v. X_t and u_λ $(\lambda > 0)$ the resolvent kernel :

$$P_x(X_t \in dy) = p_t(x-y)dy = p_t(y-x)dy \tag{3.5.2}$$

$$p_t(0) = \frac{1}{\alpha\pi}\,\Gamma\left(\frac{1}{\alpha}\right)\,t^{-\frac{1}{\alpha}} \qquad (t > 0) \tag{3.5.3}$$

$$u_\lambda(x) := \int_0^\infty e^{-\lambda t} p_t(x)dt = \frac{1}{\pi}\int_0^\infty \frac{\cos(xy)}{\lambda + y^\alpha}\,dy \tag{3.5.4}$$

$$u_\lambda(0) = \frac{1}{\pi}\,B\left(1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right)\,\lambda^{\frac{1}{\alpha}-1} \tag{3.5.5}$$

Let, for any $a \in \mathbb{R}$, $T_a := \inf\{t \geq 0\ ;\ X_t = a\}$. Then :

$$E_x[e^{-\lambda T_0}] = \frac{u_\lambda(x)}{u_\lambda(0)} \tag{3.5.6}$$

3.5.1.3 We denote by $(L_t^x,\ t \geq 0,\ x \in \mathbb{R})$ the jointly continuous process of local times of $(X_t,\ t \geq 0)$, $(L_t,\ t \geq 0)$ stands for $(L_t^0,\ t \geq 0)$, the local time process at 0, and $(\tau_l,\ l \geq 0)$ its right continuous inverse. We have :

$$E_0(e^{-\lambda\tau_l}) = \exp\left(-\frac{l}{u_\lambda(0)}\right) \tag{3.5.7}$$

so that, from (3.5.5), $(\tau_l,\ l \geq 0)$ is a stable subordinator with index $1 - \dfrac{1}{\alpha}\cdot$ On the other hand :

$$E_0\left(\int_0^\infty e^{-\lambda t} dL_t\right) = \int_0^\infty E_0(e^{-\lambda\tau_l})dl = u_\lambda(0) \tag{3.5.8}$$

$$\text{and} \qquad E_0(dL_t) = p_t(0)dt = \frac{1}{\alpha\pi}\,\Gamma\left(\frac{1}{\alpha}\right)\,t^{-\frac{1}{\alpha}}dt \tag{3.5.9}$$

$$\text{More generally : } E_x(dL_t) = E_0(d_t\,L_t^x) = p_t(x)dt \tag{3.5.10}$$

3.5.1.4 We denote by h the function defined by :

$$h(x) := \frac{1}{2\,\Gamma(\alpha)\sin\left[\frac{(\alpha-1)\pi}{2}\right]}\,|x|^{\alpha-1} \qquad (x\in\mathbb{R}) \tag{3.5.11}$$

This function is harmonic for the process $(X_t,\ t\geq 0)$ killed when it reaches 0, i.e. : for every $x\in\mathbb{R}$, and $t\geq 0$:

$$E_x\big[h(X_t)1_{T_0\geq t}\big] = h(x) \tag{3.5.12}$$

Moreover, there exists a constant $c>0$ such that, for every $x\in\mathbb{R}$:

$$\big(N_t^x := h(X_t) - h(x) - cL_t^x,\ t\geq 0\big) \tag{3.5.13}$$

is a square integrable P_x-martingale $\big($this formula may be compared with (3.1.7)$\big)$.

3.5.1.5 Since 0 is a regular and recurrent point for $(X_t,\ t\geq 0)$, Itô's excursion theory may be applied. We denote by $\widetilde{\Omega}$ the excursions space, where $(Y_t,\ t\geq 0)$ is the process of coordinates, ξ the lifetime of the generic excursion and $\mathbf{n}$ Itô's excursion measure. The master formula from excursion theory implies :

$$E_0\left[\int_0^\infty e^{-\lambda t}f(X_t)dt\right] = E_0\left(\int_0^\infty e^{-\lambda\,\tau_l}dl\right)\cdot\int_0^\infty e^{-\lambda t}\mathbf{n}\big(f(Y_t)\big)dt \tag{3.5.14}$$

for any $f:\mathbb{R}\to\mathbb{R}_+$ Borel, such that $f(0)=0$. In particular :

$$\mathbf{n}(\xi>t) = \frac{\alpha\pi}{B\left(1-\frac{1}{\alpha},\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}\right)}\,t^{\frac{1}{\alpha}-1} \tag{3.5.15}$$

There exists a function $\rho(t,x)$ which is positive and jointly measurable such that :

$$\mathbf{n}(Y_t\in dx) = \rho(t,x)dx \tag{3.5.16}$$

$$\text{and}\qquad P_x(T_0\in dt) = \rho(t,x)dt \tag{3.5.17}$$

3.5.2 Definition of the σ-finite measure $\mathbf{P}$ [2]

The measure $\mathbf{P}$ is defined on $(\Omega,\mathcal{F}_\infty)$ by :

$$\mathbf{P} := \int_0^\infty P_0(dL_u)\,(Q^{(u)}\circ P_0^{\uparrow}) \tag{3.5.18}$$

$$= \frac{1}{\alpha\pi}\,\Gamma\left(\frac{1}{\alpha}\right)\int_0^\infty du\,u^{-\frac{1}{\alpha}}\,(Q^{(u)}\circ P_0^{\uparrow}) \tag{3.5.19}$$

$\big($from (3.5.3)$\big)$. We now explain the notations in (3.5.18) :

- $Q^{(u)}$ denotes the law of the α-stable symmetric bridge with length u :

$$Q^{(u)}(\Gamma_u) = P_0\big(\Gamma_u|X_u=0\big) \qquad (\Gamma_u\in\mathcal{F}_u) \tag{3.5.20}$$

[2]We take up the notation from [YYY].

- We denote by $(P_x^0,\ x \neq 0)$ the law of the process $(X_t,\ t \geq 0)$ starting from x and killed in T_0 :
$$P_x^0(\Gamma_t) = E_x(\Gamma_t\, 1_{T_0>t}) \qquad \Gamma_t \in b(\mathcal{F}_t)$$
and by $P_x^{\uparrow}$ the law obtained from that of P_x^0 by Doob's h-transform $\big($recall that h is defined by (3.5.11) and that it is harmonic for the process $(X_t,\ t \geq 0)$ killed in $T_0\big)$:
$$P^{\uparrow}_{x\,|\mathcal{F}_t} := \frac{h(X_t)}{h(x)} \cdot P^0_{x\,|\mathcal{F}_t} \qquad x \neq 0 \tag{3.5.21}$$
Letting x tend to 0 in (3.5.21), we obtain :
$$P^{\uparrow}_{0\,|\mathcal{F}_t} := \lim_{x\to 0} \frac{h(X_t)}{h(x)} \cdot P^0_{x\,|\mathcal{F}_t} = h(X_t)\, \mathbf{n}_{\,|\mathcal{F}_t} \tag{3.5.22}$$

- Another manner to define $P_0^{\uparrow}$ consists in first defining the law $M^{(t)}$ of the stable meander (with duration t) :
$$M^{(t)}(\Gamma_t) := \mathbf{n}\big(\Gamma_t | \xi > t\big) = \frac{\mathbf{n}\big(\Gamma_t \cap (\xi > t)\big)}{\mathbf{n}(\xi > t)} \qquad \big(\Gamma_t \in b(\mathcal{F}_t)\big) \tag{3.5.23}$$
then to show that :
$$M^{(t)} \underset{t\to\infty}{\longrightarrow} P_0^{\uparrow} \tag{3.5.24}$$
with the preceding convergence taking place along $(\mathcal{F}_s)$, i.e. : for every $s \geq 0$ and $\Gamma_s \in b(\mathcal{F}_s)$:
$$M^{(t)}(\Gamma_s) \underset{t\to\infty}{\longrightarrow} P_0^{\uparrow}(\Gamma_s) \tag{3.5.25}$$

- The measure $\mathbf{P}$ defined by (3.5.18) plays for the symmetric α-stable Lévy process the same role as the measure $\mathbf{W}$ for standard Brownian motion. Indeed, for $\alpha = 2$, (3.5.18) becomes
$$\mathbf{P} = \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{du}{\sqrt{u}}\, (Q^{(u)} \circ P_0^{\uparrow}) = \frac{1}{\sqrt{2}}\, \mathbf{W}$$
where $\mathbf{W}$ is defined by (1.1.16), or (1.1.43). The multiplication factor $\dfrac{1}{\sqrt{2}}$ arises from the fact that, for $\alpha = 2$, the 2-stable symmetric Lévy process $(X_t,\ t \geq 0)$ is the process $(B_{2t},\ t \geq 0)$ and not $(B_t,\ t \geq 0)$ $\big($see (3.5.1)$\big)$.

3.5.3 The martingales $\big(M_t(F),\ t \geq 0\big)$ associated with $\mathbf{P}$

3.5.3.1 In the same manner that we have associated to the σ-finite measures $\mathbf{W}$, $\mathbf{W}^{(2)}$ and $\mathbf{W}^*$ introduced in Section 1.2, and in (3.2.2) and (3.2.3), a family of martingales, we associate here to every r.v. $F \in L^1_+(\Omega, \mathcal{F}_\infty, \mathbf{P})$ the $\big((\mathcal{F}_t)_{t\geq 0},\ P_0\big)$ martingale $\big(M_t(F),\ t \geq 0\big)$ characterized by : for any $t \geq 0$ and $\Gamma_t \in b(\mathcal{F}_t)$:
$$E_{\mathbf{P}}[F \cdot \Gamma_t] = E_0\big(M_t(F) \cdot \Gamma_t\big) \tag{3.5.26}$$
In particular, for every $t \geq 0$:
$$E_0\big[M_t(F)\big] = E_{\mathbf{P}}(F) \tag{3.5.27}$$

3.5.3.2 Example 1. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ Borel such that $\displaystyle\int_0^\infty f(y)dy < \infty$. Then :

$$M_t\big(f(L_\infty)\big) = f(L_t)h(X_t) + \int_{L_t}^\infty f(x)dx \qquad (t \geq 0) \tag{3.5.28}$$

where, in (3.5.28), the function h is defined by (3.5.11). It is not difficult to see, thanks to (3.5.13), that $\big(M_t\big(f(L_\infty)\big),\ t \geq 0\big)$ defined by (3.5.28) is indeed a martingale. We also note the analogy between (3.5.28) and formula (3.2.31) obtained in the framework of linear diffusions :

$$M_t^*\big(f(L_\infty)\big) = f(L_t)S(X_t) + \int_{L_t}^\infty f(y)dy \tag{3.5.29}$$

Thus, we shift from (3.5.29) to (3.5.28) by replacing simply the scale function S by the function h $\big($these two functions are such that, in both cases, $\big(S(X_t)1_{t<T_0},\ t \geq 0\big)$ and $\big(h(X_t)1_{t<T_0}\big)$ are martingales$\big)$.

3.5.3.3 Example 2. (Feynman-Kac martingales)

Let q denote a Radon measure on $\mathbb{R}$ such that :

$$0 < \int_{\mathbb{R}} \big(1 + h(x)\big)\, q(dx) < \infty \quad \text{(with } h \text{ defined by (3.5.11))} \tag{3.5.30}$$

Let

$$A_t^{(q)} := \int_{\mathbb{R}} L_t^x q(dx) \tag{3.5.31}$$

and $A_\infty^{(q)} := \lim\limits_{t\to\infty} A_t^{(q)}$. Then

$$M_t\big(\exp(-A_\infty^{(q)})\big) = \varphi_q(X_t) \cdot \exp(-A_t^{(q)}) \tag{3.5.32}$$

with

$$\varphi_q(x) := \lim_{t\to\infty} \frac{E_x(\exp -A_t^{(q)})}{\mathbf{n}(\xi > t)} \qquad (x \in \mathbb{R}) \tag{3.5.33}$$

We note that : $E_{\mathbf{P}}\big(\exp(-A_\infty^{(q)})\big) = \varphi_q(0)$.

Other descriptions of the function φ_q are found in [YYY]. The reader will have noticed the complete analogy between the definition of $M_t(\exp -A_\infty^{(q)})$ given by (3.5.32) and that, in the Brownian case, of $M_t(\exp -A_\infty^{(q)})$ which is given by (1.2.19) :

$$M_t\left(\exp -\frac{1}{2}\,A_\infty^{(q)}\right) = \varphi_q(X_t)\exp\left(-\frac{1}{2}\,A_t^{(q)}\right)$$

3.5.4 Relations between $\mathbf{P}$ and penalisations

The following penalisation theorems, which we now present, are found in [YYY] :

Theorem 3.5.1 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ Borel such that $\displaystyle\int_0^\infty f(y)dy < \infty$. Then :*

1) *For every $s \geq 0$, $\Gamma_s \in b(\mathcal{F}_s)$:*

$$\lim_{t\to\infty} \frac{E_0\big[\Gamma_s f(L_t)\big]}{\mathbf{n}(\xi > t)} = E_0\big[\Gamma_s\, M_s\big(f(L_\infty)\big)\big] \tag{3.5.34}$$

where $\big(M_t\big(f(L_\infty),\ t\geq 0\big)$ *is the positive martingale defined by (3.5.28).*

2) *Let* $P_{0,\infty}^{f(L)}$ *the probability induced on* $(\Omega,\mathcal{F}_\infty)$ *by :*

$$P_{0,\infty\ |\mathcal{F}_t}^{f(L)} := \frac{M_t\big(f(L_\infty)\big)}{M_0\big(f(L_\infty)\big)}\cdot P_{0\ |\mathcal{F}_t} \tag{3.5.35}$$

Then, the absolute continuity formula :

$$f(L_\infty)\cdot \mathbf{P} = E_{\mathbf{P}}\big(f(L_\infty)\big)\cdot P_{0,\infty}^{f(L)} \quad \textit{holds} \tag{3.5.36}$$

$\Big($*Note that :* $E_{\mathbf{P}}\big(f(L_\infty)\big) = \int_0^\infty f(y)dy = E_0\big(M_t\big(f(L_\infty)\big).\big)$

Clearly, this formula (3.5.36) is formally identical to formula (1.1.107) obtained in the Brownian set-up (with $h^+ = h^- = f$).

Theorem 3.5.2 *Let* q *denote a Radon measure on* $\mathbb{R}$ *such that* $0 < \int_{\mathbb{R}} \big(1+h(x)\big)q(dx) < \infty$ $\big($*with* h *defined by (3.5.11)*$\big)$ *and let* $A_t^{(q)} := \int_{\mathbb{R}} L_t^x q(dx)$. *Then :*

1) *For every* $s\geq 0$ *and* $\Gamma_s\in b(\mathcal{F}_s)$ *:*

$$\lim_{t\to\infty} \frac{E_0\big(\Gamma_s \exp\big(-A_t^{(q)}\big)\big)}{\mathbf{n}(\xi>t)} = E_0\big[\Gamma_s\, M_s\big(\exp(-A_\infty^{(q)})\big)\big] \tag{3.5.37}$$

where $\big(M_t\big(\exp(-A_\infty^{(q)})\big),\ t\geq 0\big)$ *is the positive martingale defined by (3.5.32).*

2) *Let* $P_{0,\infty}^{(q)}$ *denote the probability induced on* $(\Omega,\mathcal{F}_\infty)$ *by :*

$$P_{0,\infty\ |\mathcal{F}_t}^{(q)} = \frac{M_t\big(\exp(-A_\infty^{(q)})\big)}{M_0\big(\exp -(A_\infty^{(q)})\big)}\cdot P_{0\ |\mathcal{F}_t} \tag{3.5.38}$$

Then, the absolute continuity formula :

$$\exp(-A_\infty^{(q)})\cdot\mathbf{P} = E_{\mathbf{P}}(\exp(-A_\infty^{(q)}))\cdot P_{0,\infty}^{(q)} \quad \textit{holds} \tag{3.5.39}$$

Of course, this formula is formally identical to formula (1.1.16') obtained in the Brownian framework (one should note that $E_{\mathbf{P}}(\exp - A_\infty^{(q)}) = \varphi_q(0) = E_0\big(M_t\big(\exp(-A_\infty^{(q)})\big)$ where φ_q is defined by (3.5.33)).

Throughout the preceding discussion, a particular role was played by the point $x=0$. However, since any Lévy process enjoys a property of invariance by translation, we may define, for every $x\in\mathbb{R}$ the σ-finite measure $\mathbf{P}_x$ by the formula :

$$E_{\mathbf{P}_x}\big[F(X_t,\ t\geq 0)\big] = E_{\mathbf{P}}\big[F(x+X_t,\ t\geq 0)\big]$$

for every positive measurable functional ; thus, the knowledge of $\mathbf{P}$ induces that of $\mathbf{P}_x$, for any $x\neq 0$.

The reader will have noticed the quasi complete analogy between, on one hand, the results of [YYY] which we just described in the set-up of the α-stable symmetric Lévy process with $1<\alpha\leq 2$ and the results of Chapter 1 of this monograph, in the Brownian set-up. We refer the interested reader to [YYY] where the proofs of the results announced above are found, as well as many other informations.