COMBINATORICS OF CRYSTAL GRAPHS FOR THE ROOT SYSTEMS OF TYPES A_n, B_n, C_n, D_n AND G_2

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ABSTRACT. This note is devoted to the combinatorics of tableaux for the root systems B_n, C_n, D_n and G_2 defined from Kashiwara's crystal graph theory. We review the definition of tableaux for types B_n, C_n, D_n and G_2 and describe the corresponding bumping and sliding algorithms. We also derive in each case a Robinson-Schensted type correspondence.

1. INTRODUCTION

The Schensted bumping algorithm yields a bijection between words w of length l on the ordered alphabet $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$ and pairs $(P(w), Q(w))$ of tableaux of the same shape containing l boxes, where $P(w)$ is a semi-standard Young tableau on $\mathcal{A}_n$ and $Q(w)$ is a standard tableau. This bijection is called the Robinson-Schensted correspondence (see e.g.[5]). Note that the tableau $P(w)$ may also be constructed from w by using the Schützenberger sliding algorithm (Jeu de Taquin).
We can define a relation $\sim$ on the free monoid $\mathcal{A}_n^*$ by:

$$w_1 \sim w_2 \Longleftrightarrow P(w_1) = P(w_2).$$

Then the quotient $Pl(A_n) := \mathcal{A}_n^*/\sim$ can be described as the quotient of $\mathcal{A}_n^*$ by Knuth relations:

$$zxy = xzy \quad \text{and} \quad yzx = yxz \text{ if } x < y < z,$$
$$xyx = xxy \quad \text{and} \quad xyy = yxy \text{ if } x < y.$$

Hence $Pl(A_n)$, which may be identified with the set of semi-standard Young tableaux, becomes a monoid in a natural way. This monoid is called the "plactic monoid" and has been introduced by Lascoux and Schützenberger in order to give an illuminating proof of the Littlewood-Richardson rule for decomposing tensor products of irreducible gl_n-modules [13].
There have been attempts to find a Robinson-Schensted type correspondence and plactic relations for the other Lie algebras. In [2], Berele has explained a bumping algorithm for sp_{2n} and in [20] Sundaram gives an insertion scheme for so_{2n+1} but it seems difficult to obtain plactic relations from these schemes. More recently Littelmann has used his path model to introduce a plactic algebra for any simple Lie algebra [17]. Sheats [19] has also described a symplectic Jeu de Taquin analogous to Schützenberger's sliding algorithm and has conjectured its compatibility with some plactic relations.

The Robinson-Schensted correspondence has a natural interpretation in terms of Kashiwara's theory of crystal bases [4], [9], [12]. Let V_n denote the vector representation of gl_n. By considering each vertex of the crystal graph of $\bigoplus_{l \geq 0} V_n^{\otimes l}$ as a word on $\mathcal{A}_n$, we have for any words w_1 and w_2:

- $P(w_1) = P(w_2)$ if and only if w_1 and w_2 occur at the same place in two isomorphic connected components of this graph.
- $Q(w_1) = Q(w_2)$ if and only if w_1 and w_2 occur in the same connected component of this graph.

Thanks to crystal basis theory, Kashiwara and Nakashima [8] have obtained a generalization of semi-standard tableaux to types B_n, C_n and D_n. A notion of tableaux for type G_2 has been introduced by Kang and Misra [7]. In [18] Littelmann has also described a labelling of crystal graphs by "defined

chains" for any simple Lie algebra. In particular this description implies that the combinatorics of crystal graphs of finite dimensional irreducible $U_q(\mathfrak{g})$-modules is strongly connected to that of the Bruhat order on the Weyl group of $\mathfrak{g}$. This explains why the combinatorial description of the crystal graphs associated to the root systems of type E_6, E_7, E_8 and F_4 by planar objects like Young tableaux is still an open problem.
This note is concerned with a detailed investigation of the insertion schemes and Robinson-Schensted correspondences for the root systems A_n, B_n, C_n, D_n and G_2. In Section 2 we first introduce the results on semi-standard tableaux we want to generalize. In section 3 we recall some basics on crystal graphs. Section 4 is devoted to the combinatorial descriptions of Kashiwara-Nakashima and Kang-Misra tableaux. These descriptions are related to Littelmann's presentation of tableaux for types B_n, C_n and D_n by "defined chains". In section 5 we introduce the plactic monoids for types B_n, C_n, D_n and G_2 obtained in [14], [15] and [17]. The plactic relations are then interpreted in terms of crystal graph isomorphisms. In section 6 we use theses plactic relations to obtain insertion schemes for tableaux of types B_n, C_n, D_n and G_2. Note that our insertion scheme for the tableaux of type C_n coincides with that described by Baker in [1]. By generalizing the notion of oscillating tableaux due to Sundaram [20], we give in Section 7 a Robinson-Schensted type correspondence for types B_n, C_n, D_n and G_2 compatible with the plactic relations. Section 8 is devoted to the description of the reverse bumping algorithms for tableaux of classical types. Finally in section 9 we introduce Jeux de Taquin for skew tableaux of types B_n and C_n from Sheats' sliding algorithm and sketch their compatibility with the corresponding plactic relations.

Notation: In the sequel we frequently define similar objects for the root systems B_n C_n, D_n and G_2. When they are related to type B_n (resp. C_n, D_n, G_2), we attach to them the label B (resp. the labels C, D, G). To avoid cumbersome repetitions, we sometimes omit the labels B, C, D and G when our definitions or statements are identical for the four root systems.

2. Combinatorics of crystals for the root system A_n

2.1. **Column bumping algorithm on semi-standard tableau.** To each partition $\lambda = (\lambda_1, ..., \lambda_n)$ we associate the Young diagram $Y(\lambda)$ whose i-th row is of length λ_i. A semi-standard tableau of shape λ is a filling T of $Y(\lambda)$ by positive integers between 1 and n considered as the letters of the totally ordered alphabet

$$\mathcal{A}_n = \{1 < \cdots < n\}$$

such that the columns of T strictly increase from top to bottom and the row of T weakly increase from left to right.

Example 2.1.1. $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$ *is a semi-standard tableaux of shape* $\lambda = (3, 2, 1, 0)$.

Given any letter $x \in \mathcal{A}_n$ and any semi-standard tableau T, we denote by $x \to T$, the semi-standard tableau obtained by applying the followings rules.

- If $T = \emptyset$ then $x \to T = \boxed{x}$.
- If $T = C$ has only one column then

$$x \to T = \begin{cases} \begin{array}{|c|}\hline C \\ \hline x \\ \hline\end{array} \text{ if } \begin{array}{|c|}\hline C \\ \hline x \\ \hline\end{array} \text{ is a column,} \\ \begin{array}{|c|c|}\hline C' & y \\ \hline\end{array} \text{ otherwise,} \end{cases}$$

where y is the minimal letter in C such that $x \leq y$ and $C' = C - \{y\} + \{x\}$.

If $T = C_1C_2 \cdots C_r$ has $r \geq 2$ columns then

$$x \rightarrow T = \begin{cases} \begin{array}{|c|}\hline C_1 \\ \hline x \\ \hline \end{array}\, C_2 \cdots C_r \text{ if } x \rightarrow C_1 = \begin{array}{|c|}\hline C_1 \\ \hline x \\ \hline \end{array}\,, \\ C_1'(y \rightarrow C_2 \cdots C_r) \text{ if } x \rightarrow C_1 = \begin{array}{|c|c|}\hline C_1' & y \\ \hline \end{array}\,. \end{cases}$$

The above algorithm is called "column insertion algorithm for semi-standard tableaux".

Example 2.1.2.

$$3 \rightarrow \begin{array}{cccc} 1 & 2 & 2 & 3 \\ 2 & 3 & 5 & 6 \\ 4 & 4 & & \\ 5 & 6 & & \end{array} = \begin{array}{ccccc} 1 & 2 & 2 & 3 & 6 \\ 2 & 3 & 4 & 5 & \\ 3 & 4 & & & \\ 5 & 6 & & & \end{array}$$

2.2. **Robinson-Schensted correspondence.** To any word $w = x_1 \cdots x_l$ of $\mathcal{A}_n^*$ the set of words on $\mathcal{A}_n$, one associates the tableau $P(w)$ defined recursively by

$$\begin{cases} P(x_1) = \begin{array}{|c|}\hline x_1 \\ \hline \end{array} \\ P(x_1 \cdots x_{k+1}) = x_{k+1} \rightarrow P(x_1 \cdots x_k) \end{cases}.$$

Simultaneously, one computes the standard tableau $Q(w)$ which is a semi-standard tableau containing exactly all the letters $1, ..., l$ by putting for any $k = 1, ..., l$, the integer k in the box which is added in the k-th step of the computation of $P(w)$. With $w = 232143$, we obtain the following sequences of tableaux:

$$2,\ \begin{array}{c} 2 \\ 3 \end{array},\ \begin{array}{cc} 2 & 2 \\ 3 & \end{array},\ \begin{array}{ccc} 1 & 2 & 2 \\ 3 & & \end{array},\ \begin{array}{ccc} 1 & 2 & 2 \\ 3 & & \\ 4 & & \end{array},\ \begin{array}{ccc} 1 & 2 & 2 \\ 3 & 3 & \\ 4 & & \end{array} = P(w)$$

and

$$1,\ \begin{array}{c} 1 \\ 2 \end{array},\ \begin{array}{cc} 1 & 3 \\ 2 & \end{array},\ \begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array},\ \begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \\ 5 & & \end{array},\ \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 6 & \\ 5 & & \end{array} = Q(w).$$

Theorem 2.2.1. *(see [5]) The map $w \longmapsto (P(w), Q(w))$ is a one to one correspondence between the set $\mathcal{A}_{n,l}^*$ of words of length l on $\mathcal{A}_n$ to the set of pairs (P, Q) where P and Q are respectively semi-standard and standard tableaux containing l boxes and having the same shape.*

Example 2.2.2. *Suppose $l = 3$ and $n = 3$. We give below the 27 words of $\mathcal{A}_{3,3}^*$ together with their corresponding pair (P, Q).*

111 $\boxed{1\,|\,1\,|\,1}$, $\boxed{1\,|\,2\,|\,3}$
211 $\boxed{1\,|\,1\,|\,2}$, $\boxed{1\,|\,2\,|\,3}$
311 $\boxed{1\,|\,1\,|\,3}$, $\boxed{1\,|\,2\,|\,3}$
221 $\boxed{1\,|\,2\,|\,2}$, $\boxed{1\,|\,2\,|\,3}$
321 $\boxed{1\,|\,2\,|\,3}$, $\boxed{1\,|\,2\,|\,3}$
331 $\boxed{1\,|\,3\,|\,3}$, $\boxed{1\,|\,2\,|\,3}$
222 $\boxed{2\,|\,2\,|\,2}$, $\boxed{1\,|\,2\,|\,3}$
322 $\boxed{2\,|\,2\,|\,3}$, $\boxed{1\,|\,2\,|\,3}$
332 $\boxed{2\,|\,3\,|\,3}$, $\boxed{1\,|\,2\,|\,3}$
333 $\boxed{3\,|\,3\,|\,3}$, $\boxed{1\,|\,2\,|\,3}$,

123 $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$, $\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$

112 $\begin{array}{ll}1&1\\2&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$
113 $\begin{array}{ll}1&1\\3&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$
212 $\begin{array}{ll}1&2\\2&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$
213 $\begin{array}{ll}1&2\\3&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$
312 $\begin{array}{ll}1&3\\2&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$,

121 $\begin{array}{ll}1&1\\2&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$
131 $\begin{array}{ll}1&1\\3&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$
122 $\begin{array}{ll}1&2\\2&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$
231 $\begin{array}{ll}1&2\\3&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$
132 $\begin{array}{ll}1&3\\2&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$
232 $\begin{array}{ll}2&2\\3&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$

223 $\begin{array}{ll}2&2\\3&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$
313 $\begin{array}{ll}1&3\\3&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$
323 $\begin{array}{ll}2&3\\3&\end{array}$, $\begin{array}{ll}1&2\\3&\end{array}$
133 $\begin{array}{ll}1&3\\3&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$
233 $\begin{array}{ll}2&3\\3&\end{array}$, $\begin{array}{ll}1&3\\2&\end{array}$

Remark: There also exists a Robinson-Schensted correspondence based on the bumping procedure and defined on biwords. It associates to each biwords a pair (P, Q) where P and Q are semistandard tableaux with the same shape (see [5]). This general correspondence cannot be extended to types B_n, C_n, D_n or G_2. So we restrict ourselves in the sequel to correspondences defined on words.

2.3. **The plactic monoid.** The plactic monoid $Pl(A_n)$ is the quotient set of the free monoid $\mathcal{A}_n^*$ by Knuth's relations

$$zxy \equiv xzy \quad \text{and} \quad yzx \equiv yxz \text{ if } x < y < z,$$
$$xyx \equiv xxy \quad \text{and} \quad xyy \equiv yxy \text{ if } x < y.$$

Theorem 2.3.1. *(Lascoux-Schützenberger[13]). Given two words w_1 and w_2 of $\mathcal{A}_n^*$, we have*

$$w_1 \equiv w_2 \Longleftrightarrow P(w_1) = P(w_2).$$

We write w(T) for the Japanese reading of the tableau T that is for the word obtained by reading the columns of T from right to left and top to bottom.

2.4. **Schensted correspondence and Kashiwara's crystal graphs.** The crystal graph of the vector representation $V(\Lambda_1)$ of $U_q(sl_n)$ is

$$1 \xrightarrow{1} 2 \cdots \cdots \rightarrow n-1 \xrightarrow{n-1} n.$$

In the sequel we will identify each vertex $u_1 \otimes u_2 \otimes \cdots \otimes u_l$ of the crystal graph $B(\Lambda_1)^{\otimes l}$ with the word $w = u_1 u_2 \cdots u_l$ of $\mathcal{A}_n^*$.
Consider w_1 and w_2 two words on $\mathcal{A}_n^*$. We write

- $w_1 \sim w_2$ when w_1 and w_2 occur at the same place in two isomorphic connected components.
- $w_1 \longleftrightarrow w_2$ when w_1 and w_2 belong to the same connected component.

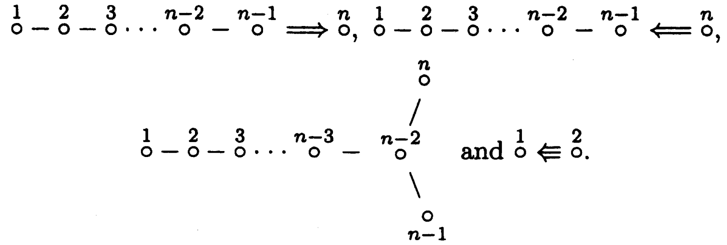
Theorem 2.4.1. *(Kashiwara [9])*

- $w_1 \sim w_2 \Longleftrightarrow P(w_1) = P(w_2) \Longleftrightarrow w_1 \equiv w_2$.
- $w_1 \longleftrightarrow w_2 \Longleftrightarrow Q(w_1) = Q(w_2)$.

The aim of this note is to introduce analogue of Theorems 2.2.1, 2.3.1 and 2.4.1 for the root systems B_n, C_n, D_n and G_2.

3. Basics on crystals for types B_n, C_n, D_n and G_2

3.1. **Crystals and tensor product of crystals.** The Dynkin diagrams for the root systems of types B_n, C_n, D_n and G_2 are respectively



Let $\mathfrak{g}$ be a simple Lie algebra of type B_n, C_n, D_n or G_2. Set $I = \{1, ..., n\}$ where n is the rank of $\mathfrak{g}$. Let α_i, $i \in I$ be the simple roots associated to $\mathfrak{g}$. For any dominant weight λ we denote by $V(\lambda)$ the finite dimensional irreducible $U_q(\mathfrak{g})$-module of highest weight λ. Recall that the crystal graphs of the $U_q(\mathfrak{g})$-modules are oriented colored graphs with colors $i \in I$. An arrow $a \xrightarrow{i} b$ means that $\widetilde{f}_i(a) = b$ and $\widetilde{e}_i(b) = a$ where $\widetilde{e}_i$ and $\widetilde{f}_i$ are the crystal graph operators (for a review of crystal bases and crystal graphs see [9]). Let V, V' be two $U_q(\mathfrak{g})$-modules and B, B' their crystal graphs. A vertex $v^0 \in B$ satisfying $\widetilde{e}_i(v^0) = 0$ for any $i \in I$ is called a highest weight vertex. The decomposition of V into its irreducible components is reflected by the decomposition of B into its connected components. Each connected component of B contains a unique highest weight vertex. We write $B(v)$ for the connected component containing the highest weight vertex v. The crystal graphs of two isomorphic irreducible components are isomorphic as oriented colored graphs. We will say that two vertices b_1 and b_2 of B occur at the same place in two isomorphic connected components Γ_1 and Γ_2 of B if there exist $i_1, ..., i_r \in I$ such that $w_1 = \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_r}(w_1^0)$ and $w_2 = \widetilde{f}_{i_1} \cdots \widetilde{f}_{i_r}(w_2^0)$, where w_1^0 and w_2^0 are respectively the highest weight vertices of Γ_1 and Γ_2.
The action of $\widetilde{e}_i$ and $\widetilde{f}_i$ on $B \otimes B' = \{b \otimes b';\, b \in B, b' \in B'\}$ is given by:

(1)
$$\widetilde{f}_i(u \otimes v) = \begin{cases} \widetilde{f}_i(u) \otimes v \text{ if } \varphi_i(u) > \varepsilon_i(v) \\ u \otimes \widetilde{f}_i(v) \text{ if } \varphi_i(u) \leq \varepsilon_i(v) \end{cases}$$

and

(2)
$$\widetilde{e}_i(u \otimes v) = \begin{cases} u \otimes \widetilde{e}_i(v) \text{ if } \varphi_i(u) < \varepsilon_i(v) \\ \widetilde{e}_i(u) \otimes v \text{ if } \varphi_i(u) \geq \varepsilon_i(v) \end{cases}$$

where $\varepsilon_i(u) = \max\{k; \tilde{e}_i^k(u) \neq 0\}$ and $\varphi_i(u) = \max\{k; \tilde{f}_i^k(u) \neq 0\}$.

3.2. **Crystal of the vector representations.** We denote by $\Lambda_1^B, \ldots, \Lambda_n^B$, $\Lambda_1^C, \ldots, \Lambda_n^C$, $\Lambda_1^D \ldots, \Lambda_n^D$ and Λ_1^G, Λ_2^G the fundamental weights associated to the root systems B_n, C_n, D_n and G_2. The crystal graphs of the fundamental representations of highest weight $\Lambda_1^B, \Lambda_1^C, \Lambda_1^D$ and Λ_1^G are respectively

$$B(\Lambda_1^B) : 1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} 0 \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \rightarrow \bar{2} \xrightarrow{1} \bar{1}$$

$$B(\Lambda_1^C) : 1 \xrightarrow{1} 2 \cdots \rightarrow n-1 \xrightarrow{n-1} n \xrightarrow{n} \bar{n} \xrightarrow{n-1} \overline{n-1} \xrightarrow{n-2} \cdots \rightarrow \bar{2} \xrightarrow{1} \bar{1}$$

$$B(\Lambda_1^D) : 1 \xrightarrow{1} 2 \xrightarrow{2} \cdots \xrightarrow{n-2} n-1 \begin{array}{c} \bar{n} \\ {\scriptstyle n}\nearrow \quad \searrow{\scriptstyle n-1} \\ \\ {\scriptstyle n-1}\searrow \quad \nearrow{\scriptstyle n} \\ n \end{array} \overline{n-1} \xrightarrow{n-2} \cdots \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}$$

and

$$B(\Lambda_1^G) : 1 \xrightarrow{1} 2 \xrightarrow{2} 3 \xrightarrow{1} 0 \xrightarrow{1} \bar{3} \xrightarrow{2} \bar{2} \xrightarrow{1} \bar{1}. \tag{3}$$

In the sequel we will call the corresponding $U_q(\mathfrak{g})$-modules "vector representations". Every vertex $u_1 \otimes u_2 \otimes \cdots \otimes u_l$ of the crystal graphs $B(\Lambda_1^B)^{\otimes l}, B(\Lambda_1^C)^{\otimes l}, B(\Lambda_1^D)^{\otimes l}$ and $B(\Lambda_1^G)^{\otimes l}$ will be identified with the word $u_1 u_2 \cdots u_l$ respectively on the ordered alphabets

$$\mathcal{B}_n = \{1 < \cdots < n-1 < n < 0 < \bar{n} < \overline{n-1} < \cdots < \bar{1}\},$$
$$\mathcal{C}_n = \{1 < \cdots < n-1 < n < \bar{n} < \overline{n-1} < \cdots < \bar{1}\},$$
$$\mathcal{D}_n = \{1 < \cdots < n-1 < \begin{array}{c} n \\ \bar{n} \end{array} < \overline{n-1} < \cdots < \bar{1}\}$$
$$\text{and } \mathcal{G} = \{1 < 2 < 3 < 0 < \bar{3} < \bar{2} < \bar{1}\}.$$

Note that $\mathcal{D}_n$ is only partially ordered. For any $x \in \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n$ or $\mathcal{G}$, we set $|x| = x$ if x is unbarred and $|x| = \bar{x}$ otherwise (with the convention $\bar{\bar{x}} = x$).

Example 3.2.1.
The crystal $B(\Lambda_1^B)^{\otimes 2}$ for type B_3

$$\begin{array}{ccccccccccccc}
11 & \xrightarrow{1} & 21 & \xrightarrow{2} & 31 & \xrightarrow{3} & 01 & \xrightarrow{3} & \bar{3}1 & \xrightarrow{2} & \bar{2}1 & \xrightarrow{1} & \bar{1}1 \\
 & & 1\downarrow & & 1\downarrow & & 1\downarrow & & 1\downarrow & & & & 1\downarrow \\
12 & & 22 & \xrightarrow{2} & 32 & \xrightarrow{3} & 02 & \xrightarrow{3} & \bar{3}2 & \xrightarrow{2} & \bar{2}2 & & \bar{1}2 \\
2\downarrow & & & & 2\downarrow & & 2\downarrow & & & & 2\downarrow & & 2\downarrow \\
13 & \xrightarrow{1} & 23 & & 33 & \xrightarrow{3} & 03 & \xrightarrow{3} & \bar{3}3 & & \bar{2}3 & \xrightarrow{1} & \bar{1}3 \\
3\downarrow & & 3\downarrow & & & & & & 3\downarrow & & 3\downarrow & & 3\downarrow \\
10 & \xrightarrow{1} & 20 & \xrightarrow{2} & 30 & \xrightarrow{3} & 00 & & \bar{3}0 & \xrightarrow{2} & \bar{2}0 & \xrightarrow{1} & \bar{1}0 \\
3\downarrow & & 3\downarrow & & & & 3\downarrow & & 3\downarrow & & 3\downarrow & & 3\downarrow \\
1\bar{3} & \xrightarrow{1} & 2\bar{3} & \xrightarrow{2} & 3\bar{3} & & 0\bar{3} & & \bar{3}\bar{3} & \xrightarrow{2} & \bar{2}\bar{3} & \xrightarrow{1} & \bar{1}\bar{3} \\
2\downarrow & & & & 2\downarrow & & 2\downarrow & & & & 2\downarrow & & 2\downarrow \\
1\bar{2} & \xrightarrow{1} & 2\bar{2} & & 3\bar{2} & \xrightarrow{3} & 0\bar{2} & \xrightarrow{3} & \bar{3}\bar{2} & & \bar{2}\bar{2} & \xrightarrow{1} & \bar{1}\bar{2} \\
 & & 1\downarrow & & 1\downarrow & & 1\downarrow & & 1\downarrow & & & & 1\downarrow \\
1\bar{1} & & 2\bar{1} & \xrightarrow{2} & 3\bar{1} & \xrightarrow{3} & 0\bar{1} & \xrightarrow{3} & \bar{3}\bar{1} & \xrightarrow{2} & \bar{2}\bar{1} & & \bar{1}\bar{1}
\end{array}$$

yields the decomposition $V(\Lambda_1^B)^{\otimes 2} \simeq V(2\Lambda_1^B) \bigoplus V(\Lambda_2^B) \bigoplus V(0)$.

The crystal $B(\Lambda_1^G)^{\otimes 2}$ *for type* G_2

$$
\begin{array}{ccccccccccccc}
11 & \xrightarrow{1} & 21 & \xrightarrow{2} & 31 & \xrightarrow{1} & 01 & \xrightarrow{1} & \bar{3}1 & \xrightarrow{2} & \bar{2}1 & \xrightarrow{1} & \bar{1}1 \\
 & & {\scriptstyle 1}\downarrow & & & & & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow \\
12 & & 22 & \xrightarrow{2} & 32 & \xrightarrow{1} & 02 & & \bar{3}2 & \xrightarrow{2} & \bar{2}2 & & \bar{1}2 \\
{\scriptstyle 2}\downarrow & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
13 & \xrightarrow{1} & 23 & & 33 & \xrightarrow{1} & 03 & \xrightarrow{1} & \bar{3}3 & & \bar{2}3 & \xrightarrow{1} & \bar{1}3 \\
 & & {\scriptstyle 1}\downarrow & & & & & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow \\
10 & & 20 & \xrightarrow{2} & 30 & \xrightarrow{1} & 00 & & \bar{3}0 & \xrightarrow{2} & \bar{2}0 & & \bar{1}0 \\
{\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\
1\bar{3} & & 2\bar{3} & \xrightarrow{2} & 3\bar{3} & & 0\bar{3} & & \bar{3}\bar{3} & \xrightarrow{2} & \bar{2}\bar{3} & & \bar{1}\bar{3} \\
{\scriptstyle 2}\downarrow & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
1\bar{2} & \xrightarrow{1} & 2\bar{2} & & 3\bar{2} & \xrightarrow{1} & 0\bar{2} & \xrightarrow{1} & \bar{3}\bar{2} & & \bar{2}\bar{2} & \xrightarrow{1} & \bar{1}\bar{2} \\
 & & {\scriptstyle 1}\downarrow & & & & & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow \\
1\bar{1} & & 2\bar{1} & \xrightarrow{2} & 3\bar{1} & \xrightarrow{1} & 0\bar{1} & & \bar{3}\bar{1} & \xrightarrow{2} & \bar{2}\bar{1} & & \bar{1}\bar{1}
\end{array} \tag{4}
$$

yields the decomposition $V(\Lambda_1^G)^{\otimes 2} \simeq V(2\Lambda_1^G) \bigoplus V(\Lambda_2^G) \bigoplus V(\Lambda_1^G) \bigoplus V(0)$.

3.3. **The weights** ω_i. Set $G_B = \bigoplus_{l \geq 0} B(\Lambda_1^B)^{\otimes l}$, $G_C = \bigoplus_{l \geq 0} B(\Lambda_1^C)^{\otimes l}$, $G_D = \bigoplus_{l \geq 0} B(\Lambda_1^D)^{\otimes l}$ and $G_G = \bigoplus_{l \geq 0} B(\Lambda_1^G)^{\otimes l}$. Let R be one of the classical root system B_n, C_n or D_n. Then the weight of any vertex $w \in G_R$ is given by:

$$
\mathrm{wt}(w) = d_n \omega_n^R + \sum_{i=1}^{n-1} (d_i - d_{i+1}) \omega_i^R. \tag{5}
$$

where $d_i = \#\{i \in w\} - \#\{\bar{i} \in w\}$ and

$$
\begin{cases} \omega_n^B = 2\Lambda_n^B, \\ \omega_i^B = \Lambda_i^B \text{ for } i = 1, ..., n-1, \end{cases}
$$

$$
\omega_i^C = \Lambda_i^C \text{ for } i = 1, ..., n,
$$

$$
\begin{cases} \omega_n^D = 2\Lambda_n^D, \\ \overline{\omega}_n^D = 2\Lambda_{n-1}^D, \\ \omega_{n-1}^D = \Lambda_n^D + \Lambda_{n-1}^D, \\ \omega_i^D = \Lambda_i^D \text{ for } i = 1, ..., n-2. \end{cases}
$$

Similarly the weight of any vertex $w \in G_G$ is given by

$$
\mathrm{wt}(w) = (d_1 - d_2 + 2d_3)\Lambda_1 + (d_2 - d_3)\Lambda_2
$$

and to make our notation homogeneous we set $\omega_i^G = \Lambda_i^G$ for $i = 1, 2$.

It follows from (5) that there is no connected component of G_B isomorphic to $B(\Lambda_n^B)$ and no connected component of G_D isomorphic to $B(\Lambda_n^D)$ or $B(\Lambda_{n-1}^D)$. So we recover the well known fact that the corresponding fundamental representations can not be obtained as an irreducible component of a tensor power of the vector representation for the orthogonal root systems. Denote by $\Omega_+^B, \Omega_+^C, \Omega_+^D$ and Ω_+^G the sets of dominant weights which can be written as non-negative linear combinations respectively of the weights ω_i^B, ω_i^C, $\omega_i^D, \overline{\omega}_n^D$, and ω_i^G $i \in I$. Then for any dominant weight λ, $V(\lambda)$ is an irreducible component of a tensor power of the vector representation if and only if $\lambda \in \Omega_+$.

In this note we have chosen to describe the Robinson-Schensted correspondence and the related combinatorics only for the irreducible representations of highest weight $\lambda \in \Omega_+$. It is also possible to obtain such correspondences by taking into account the spin representations $V(\Lambda_n^B)$, $V(\Lambda_n^D)$ and $V(\Lambda_{n-1}^D)$ ([15], [16]) but their description requires a large amount of combinatorial technical material [16].

3.4. **The coplactic relation.** For w_1 and $w_2 \in B_n^*$ write $w_1 \overset{B}{\longleftrightarrow} w_2$ if and only if w_1 and w_2 belong to the same connected component of G_B. The coplactic relations $w_1 \overset{C}{\longleftrightarrow} w_2$, $w_1 \overset{D}{\longleftrightarrow} w_2$ and $w_1 \overset{G}{\longleftrightarrow} w_2$ are defined similarly. For any word w, we write $l(w)$ for the length of w, that is the number of letters it contains. We have the useful lemma:

Lemma 3.4.1. *If $w_1 = u_1 v_1$ and $w_2 = u_2 v_2$ with $l(u_1) = l(u_2)$ and $l(v_1) = l(v_2)$, then*

$$w_1 \longleftrightarrow w_2 \Longrightarrow \begin{cases} u_1 \longleftrightarrow u_2 \\ v_1 \longleftrightarrow v_2 \end{cases}.$$

Proof. We have $w_1 \longleftrightarrow w_2$ if and only if $w_2 = \widetilde{H}(w_1)$ where $\widetilde{H}$ is a product of Kashiwara's operators. The Lemma then follows immediately from (1) and (2). □

4. Tableaux for types B_n, C_n, D_n and G_2

4.1. **Columns and admissible columns.** The columns of types B_n, C_n and D_n are respectively column diagrams of the form $C = \begin{array}{|c|}\hline C_+ \\ \hline C_0 \\ \hline C_- \\ \hline\end{array}$, $C = \begin{array}{|c|}\hline C_+ \\ \hline C_- \\ \hline\end{array}$ and $C = \begin{array}{|c|}\hline D_+ \\ \hline D \\ \hline D_- \\ \hline\end{array}$ where C_-, C_+, C_0, D_-, D_+ and D are column shaped Young diagrams such that

$$\begin{cases} C_- \text{ is filled by strictly increasing barred letters} \\ C_+ \text{ is filled by strictly increasing unbarred letters} \\ C_0 \text{ is filled by letters } 0 \\ D_- \text{ is filled by strictly increasing letters} \geq \overline{n-1} \\ D_+ \text{ is filled by strictly increasing letters} \leq n-1 \\ D \text{ is filled by letters } \overline{n} \text{ or } n \text{ different in two adjacent boxes} \end{cases}$$

(recall that the reading order is from top to bottom). The height $h(C)$ of the column C is the number of boxes it contains. The reading of C is the word $\mathrm{w}(C)$ obtained by reading successively from top to bottom the letters of C.
A column C of type G_2 is a Young diagram of column shape and height 1 or 2 filled by letters of $\mathcal{G}$ and such that

$$C = \begin{array}{|c|}\hline a \\ \hline\end{array} \text{ with } a \in \mathcal{G} \text{ or } C = \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \text{ with } a < b \in \mathcal{G} \text{ or } C = \begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline\end{array}.$$

Consider a column C. For any letter $z \leq n$, set

$$N(z) = \#\{x \in C, |x| \leq z\}.$$

The column C is said admissible if and only if the following conditions are satisfied:
(i) : C does not contain any letter $z \leq n$ such that $N(z) > z$ (remind that $0 > n$!).
(ii) : if C is of type B_n and $0 \in C$ then $h(C) \leq n$.
(iii) : if $C = \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$ is of type G_2 and height 2 then $\begin{cases} \mathrm{dist}(a,b) \leq 2 \text{ if } a = 1 \text{ or } 0 \\ \mathrm{dist}(a,b) \leq 3 \text{ otherwise} \end{cases}$ where $\mathrm{dist}(a,b)$ is the number of arrows between the vertices a and b in the crystal (3).

Example 4.1.1.

- *For* $n = 4$, $\begin{array}{|c|}\hline 4\\ \hline 0\\ \hline \bar{4}\\ \hline \bar{2}\\ \hline\end{array}$ *and* $\begin{array}{|c|}\hline 3\\ \hline \bar{4}\\ \hline 4\\ \hline \bar{3}\\ \hline\end{array}$ *are admissible columns respectively of type* B_n *and* D_n *but* $\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline \bar{4}\\ \hline \bar{2}\\ \hline\end{array}$ *is not admissible since* $N(2) = 3$.
- *The admissible columns of type* G_2 *are exactly those whose readings are in* $B(12)$ *in the crystal graph (4).*

4.2. **Combinatorial description of $B(\omega_k)$.** For $k = 1, ..., n$ we set $v^B_{\omega_k} = v^C_{\omega_k} = v^D_{\omega_k} = 1 \cdots k$ and $v^D_{\overline{\omega}_n} = 1 \cdots (n-1)\bar{n}$. Similarly write $v^G_{\omega_1} = 1$ and $v^G_{\omega_2} = 12$. All these vertices are highest weight vertices. Moreover $B(v^B_{\omega_k}), B(v^C_{\omega_k}), B(v^D_{\omega_k}), B(v^G_{\omega_k})$ are respectively isomorphic to $B(\omega^B_k), B(\omega^C_k), B(\omega^D_k), B(\omega^G_k)$ and $B(v^D_{\overline{\omega}_n})$ is isomorphic to $B(\overline{\omega}^D_n)$.

Proposition 4.2.1. *(Kashiwara-Nakashima [8] and Kang-Misra [7])*

- *The vertices of* $B(v^B_{\omega_k})$, $B(v^C_{\omega_k})$ *and* $B(v^G_{\omega_k})$ *with* $k \in I$ *are the readings of the admissible columns of types* B_n, C_n *and* G_2 *with length* k.
- *The vertices of* $B(v^D_{\omega_k})$ *with* $k < n$ *are the readings of the admissible columns of type* D_n *with length* k.
- *The vertices of* $B(v^D_{\omega_n})$ *are the readings of the admissible columns* C *of type* D_n *such that* $\mathrm{w}(C) = x_1 \cdots x_n$ *and* $x_k = n$ *(resp.* $x_k = \bar{n}$*) implies* $n - k$ *is even (resp. odd).*
- *The vertices of* $B(v^D_{\overline{\omega}_n})$ *are the readings of the admissible columns* C *of type* D_n *such that* $\mathrm{w}(C) = x_1 \cdots x_n$ *and* $x_k = \bar{n}$ *(resp.* $x_k = n$*) implies* $n - k$ *is even (resp. odd).*

4.3. **Duplication of an admissible column of classical type.** We say that a column C contains the pair $(z, \bar{z})$ when $z = 0$ and $0 \in C$ or when $z \neq 0$ is unbarred and C contains the two letters $z, \bar{z}$. Note that a letter 0 counts for a pair $(0, \bar{0})$. For each admissible column C of classical type, we compute a pair of columns (lC, rC) without pair $(z, \bar{z})$.
Denote by $I_C = \{z_1 = 0, ..., z_r = 0 > z_{r+1} > \cdots > z_s\}$ the set of letters $z \leq 0$ such that the pair $(z, \bar{z})$ occurs in C. The column C of type B_n or C_n can be split when there exists a set of s unbarred letters $J_C = \{t_1 > \cdots > t_s\} \subset \mathcal{B}_n$ such that:

: t_1 is the greatest letter of $\mathcal{B}_n$ satisfying $t_1 < z_1, t_1 \notin C$ and $\overline{t_1} \notin C$,
: for $i = 2, ..., s$, t_i is the greatest letter of $\mathcal{B}_n$ satisfying $t_i < \min(t_{i-1}, z_i)$, $t_i \notin C$ and $\overline{t_i} \notin C$.

In this case we write:

- rC for the column obtained first by changing in C $\bar{z}_i$ into $\bar{t}_i$ for each letter $z_i \in I_C$, next by reordering if necessary,
- lC for the column obtained first by changing in C z_i into t_i for each letter $z_i \in I_C$, next by reordering if necessary.

Let C be a column of type D_n. Denote by $\widehat{C}$ the column of type B_n obtained by turning in C each factor $\bar{n}n$ into 00. We will say that C can be split when $\widehat{C}$ can be split. In this case we write $lC = l\widehat{C}$ and $rC = r\widehat{C}$.

Example 4.3.1. *Suppose* $n = 9$ *and consider the column* C *of type* B_n *such that* $\mathrm{w}(C) = 458900\bar{8}\bar{5}\bar{4}$. *We have*

$$\mathrm{w}(lC) = 123679\bar{8}\bar{5}\bar{4} \text{ and } \mathrm{w}(rC) = 4589\bar{7}\bar{6}\bar{3}\bar{2}\bar{1}.$$

The duplication of columns is an application of the notion of "dilatation" of crystals introduced by Kashiwara [10]. The sub-crystal of $B(m\lambda)$ generated from $v_{m\lambda}$ and the operators $\widetilde{f}_i^m$'s is isomorphic to $B(\lambda)$. When $m = 2$ and $\lambda = \omega_k$ this isomorphism is simply described by the splitting operation.

Example 4.3.2. *The dilatation of the crystal $B(12)$ for type B_2 and $m=2$.*

$$\begin{array}{ccccccccc} 12 & \overset{2}{\to} & 10 & \overset{2}{\to} & 1\bar{2} & \overset{1}{\to} & 2\bar{2} & \overset{1}{\to} & 2\bar{1} \\ & & \downarrow 1 & & & & & & \downarrow 2 \\ & & 20 & \overset{2}{\to} & 00 & \overset{2}{\to} & 0\bar{2} \overset{1}{\to} 0\bar{1} & \overset{2}{\to} & \bar{2}\bar{1} \end{array}$$

$$\begin{array}{l} (12)\otimes(12) \overset{2^2}{\to} (1\bar{2})\otimes(12) \overset{2^2}{\to} (1\bar{2})\otimes(1\bar{2}) \overset{1^2}{\to} (2\bar{1})\otimes(1\bar{2}) \overset{1^2}{\to} (2\bar{1})\otimes(2\bar{1}) \\ \qquad\qquad\qquad \downarrow 1^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow 2^2 \\ \qquad (2\bar{1})\otimes(12) \overset{2^2}{\to} (\bar{2}\bar{1})\otimes(12) \overset{2^2}{\to} (\bar{2}\bar{1})\otimes(1\bar{2}) \overset{1^2}{\to} (\bar{2}\bar{1})\otimes(2\bar{1}) \overset{2^2}{\to} (\bar{2}\bar{1})\otimes(\bar{2}\bar{1}) \end{array}$$

Proposition 4.3.3. *A column C is admissible if and only if it can be split.*

4.4. **"Young diagrams" associated to a weight λ.** In the sequel we will need to attach to each dominant weight $\lambda \in \Omega_+$ a combinatorial object $Y(\lambda)$. For types B_n, C_n and G_2 it suffices to set $Y(\lambda) = Y_\lambda$ where Y_λ is the Young diagram containing exactly $\widetilde{\lambda}_k$ columns of height k where for any k, $\widetilde{\lambda}_k$ is the k-th coordinate of λ on the basis ω_i, $i \in I$.
For type D_n, λ has a unique decomposition of the form

$$(*) : \lambda = \sum_{i=1}^{n} \widetilde{\lambda}_i \omega_i^D \text{ or } (**) : \lambda = \widetilde{\lambda}_n \bar{\omega}_n^D + \sum_{i=1}^{n-1} \widetilde{\lambda}_i \omega_i^D \text{ with } \widetilde{\lambda}_n \neq 0,$$

where $(\widetilde{\lambda}_1, ..., \widetilde{\lambda}_n) \in \mathbb{N}^n$. This leads us to set:

$$\text{(i)} : Y(\lambda) = (Y_\lambda, +) \text{ in case } (*) \text{ with } \widetilde{\lambda}_n \neq 0,$$
$$\text{(ii)} : Y(\lambda) = (Y_\lambda, 0) \text{ in case } (*) \text{ with } \widetilde{\lambda}_n = 0,$$
$$\text{(iii)} : Y(\lambda) = (Y_\lambda, -) \text{ in case } (**).$$

Note that $Y(\lambda)$ suffices to characterize $\lambda \in \Omega_+$ but not Y_λ for type D_n since for this root system Y_λ is a column diagram of height n for $\lambda = \omega_n$ or $\lambda = \overline{\omega}_n$.

4.5. **Tableaux of types B_n, C_n, D_n and G_2.** Set

$$\begin{cases} v_\lambda^B = (v_{\omega_1^B})^{\otimes \widetilde{\lambda}_1} \otimes \cdots \otimes (v_{\omega_n^B})^{\otimes \widetilde{\lambda}_n}, \\ v_\lambda^C = (v_{\omega_1^C})^{\otimes \widetilde{\lambda}_1} \otimes \cdots \otimes (v_{\omega_n^C})^{\otimes \widetilde{\lambda}_n}, \\ v_\lambda^G = (v_{\omega_1^G})^{\otimes \widetilde{\lambda}_1} \otimes (v_{\omega_2^G})^{\otimes \widetilde{\lambda}_2}, \\ v_\lambda^D = (v_{\omega_1^D})^{\otimes \widetilde{\lambda}_1} \otimes \cdots \otimes (v_{\omega_n^D})^{\otimes \widetilde{\lambda}_n} \text{ in case(i)}, \\ v_\lambda^D = (v_{\omega_1^D})^{\otimes \widetilde{\lambda}_1} \otimes \cdots \otimes (v_{\omega_{n-1}^D})^{\otimes \widetilde{\lambda}_{n-1}} \text{ in case (ii)}, \\ v_\lambda^D = (v_{\omega_1^D})^{\otimes \widetilde{\lambda}_1} \otimes \cdots \otimes (v_{\overline{\omega}_n^D})^{\otimes \widetilde{\lambda}_n} \text{ in case (iii)}. \end{cases}$$

Then v_λ is in each case a highest weight vertex of G of weight λ. Thus $B(v_\lambda)$ is isomorphic to $B(\lambda)$ and we can identify these two crystals.
Consider $C_1, ..., C_r$ columns with the same type. The reading of the juxtaposition of columns $T = C_1 \cdots C_r$ is the word $\mathrm{w}(T) = \mathrm{w}(C_r) \cdots \mathrm{w}(C_1)$ obtained by reading sucessively the columns of T from right to left and top to bottom.

Definition 4.5.1. *A tableau T of shape $Y(\lambda)$ is a juxtaposition of columns such that* $\mathrm{w}(T) \in B(v_\lambda)$.

Consider $\tau = C_1 C_2 \cdots C_r$, a juxtaposition of admissible columns of type B_n, C_n or D_n. The split form of τ is obtained by splitting each column of τ. We write $\mathrm{spl}(\tau) = (lC_1 rC_1)(lC_2 rC_2) \cdots (lC_r rC_r)$.
Since the operation of duplication of a tableau describes the dilatation of $B(v_\lambda)$ in $B(v_{2\lambda})$ we have

Lemma 4.5.2. *τ is a tableau if and only if* $\mathrm{spl}(\tau)$ *is a tableau.*

In order to obtain a complete combinatorial description of the tableaux of type D_n we need the following definition.

Definition 4.5.3. *Consider C_1 and C_2, two columns of type D_n such that $h(C_1) \geq h(C_2)$. Then we say that C_1C_2 contains an a-odd-configuration (with $a \notin \{\overline{n}, n\}$)*

- *$a = x_p, \overline{n} = x_r$ are letters of C_1 and $\overline{a} = y_s, n = y_q$ letters of C_2 such that $r - q + 1$ is odd*

or

- *$a = x_p, n = x_r$ are letters of C_1 and $\overline{a} = y_s, \overline{n} = y_q$ letters of C_2 such that $r - q + 1$ is odd, where the integers p, q, r, s are such that $p \leq q < r \leq s$.*

We say that C_1C_2 contains an a-even-configuration (with $a \notin \{\overline{n}, n\}$) when:

- *$a = x_p, n = x_r$ are letters of C_1 and $\overline{a} = y_s, n = y_q$ letters of C_2 such that $r - q + 1$ is even*

or

- *$a = x_p, \overline{n} = x_r$ are letters of C_1 and $\overline{a} = y_s, \overline{n} = y_q$ letters of C_2 such that $r - q + 1$ is even, where the integers p, q, r, s are such that $p \leq q < r \leq s$.*

Then we denote by $\mu(a)$ the positive integer defined by

$$\mu(a) = s - p.$$

Kashiwara-Nakashima's combinatorial description of a tableau T of type B_n, C_n or D_n is based on the enumeration of configurations that should not occur in two adjacent columns of T. Considering its split form $\mathrm{spl}(T)$, this description becomes simpler because the columns of $\mathrm{spl}(T)$ does not contain any pair $(z, \overline{z})$.

Theorem 4.5.4. *(Kashiwara-Nakashima* [8] *and Kang-Misra* [7]*)*
(i) *The tableaux of types B_n and C_n are the juxtaposition of admissible columns of types B_n and C_n whose duplicated forms are semi-standard for the orders on $\mathcal{B}_n$ and $\mathcal{C}_n$.*
(ii) *The tableaux of type D_n are the juxtaposition of admissible columns of type D_n whose duplicated forms are semi-standard for the order on $\mathcal{D}_n$ and such that rC_ilC_{i+1} does not contain an a-configuration (even or odd) with $\mu(a) = n - a$.*
(iii) *The tableaux of type G_2 are the juxtapositions $C_1 \cdots C_r$ of admissible columns of type G_2 such that for any $i = 1, ..., r-1$, C_iC_{i+1} satisfies one of the following assertions:*

$$\begin{cases} \text{(i) } C_iC_{i+1} = \begin{array}{|c|c|}\hline a & b \\ \hline\end{array} \text{ with } a \leq b \text{ and } (a,b) \neq (0,0), \\[2ex] \text{(ii) } C_iC_{i+1} = \begin{array}{|c|c|}\hline a & c \\ \hline b & \\ \cline{1-1}\end{array} \text{ with } a \leq c \text{ and } (a,c) \neq (0,0), \\[2ex] \text{(iii) } C_iC_{i+1} = \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline\end{array} \text{ with } \begin{cases} a \leq c \text{ and } (a,c) \neq (0,0) \\ b \leq d \text{ and } (b,d) \neq (0,0) \end{cases} \text{ and } \begin{cases} \mathrm{dist}(a,d) \geq 3 \text{ if } a = 2,3,0 \\ \mathrm{dist}(a,d) \geq 2 \text{ if } a = \overline{3}. \end{cases} \end{cases}$$

Example 4.5.5. *Suppose $n = 4$. Then*
$T = \begin{array}{|c|c|c|}\hline 3 & 3 & 4 \\ \hline 4 & 0 & \overline{4} \\ \hline 0 & \overline{2} & \\ \cline{1-2} 0 & & \\ \cline{1-1}\end{array}$ *is a tableau of type B_4 for* $\mathrm{spl}(T) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 3 & 3 & 3 & 4 \\ \hline 2 & 4 & 4 & \overline{4} & \overline{4} & \overline{3} \\ \hline 3 & \overline{2} & \overline{2} & \overline{2} & & \\ \cline{1-4} 4 & \overline{1} & & & & \\ \cline{1-2}\end{array}$ *is semi-standard. But* $\begin{array}{|c|c|}\hline 3 & \overline{4} \\ \hline \overline{4} & \overline{3} \\ \hline\end{array}$ *is not a tableau of type D_3 because it contains a 3-even configuration with $\mu(3) = 1$.*

4.6. **Littelmann's version of tableaux for the classical types.** The Weyl group W acts on the vertices of $B(v_\omega)$. The orbit of v_ω is the readings of the corresponding columns which do not contain a pair $(z,\overline{z})$. Thus there is a one to one correspondence between these columns and the cosets W/W_ω where W_ω is the stabilizer of the weight ω under the action of W. Moreover if C_1, C_2 are two columns without pair $(z,\overline{z})$

$$C_1C_2 \text{ is a tableau} \Longleftrightarrow \tau_{C_1} \lessdot_p \tau_{C_2},$$

where " $\lessdot_p$ " denotes the projection of the Bruhat order on W/W_ω. So $B(v_\omega)$ is labelled by pairs $(\tau_{lC},\tau_{rC}) \in W/W_\omega \times W/W_\omega$ satisfying $\tau_{lC} \lessdot_p \tau_{rC}$.
Let $(C_1,...,C_r)$ be r admissible columns of decreasing heights such that for any $i = 1\cdots r$, $\tau_{lC_i} \in W/W_{\omega_i}$ and $\tau_{rC_i} \in W/W_{\omega_i}$.
We say there exists a defined chain for $(C_1,...,C_r)$, if one can find a sequence $(\sigma_l^1,\sigma_r^1,\sigma_l^2,\sigma_r^2,...,\sigma_l^r,\sigma_r^r)$ of elements in W such that:

$$\sigma_l^1 \lessdot \sigma_r^1 \lessdot \sigma_l^2 \lessdot \sigma_r^2 \lessdot \cdots \lessdot \sigma_l^r \lessdot \sigma_r^r,$$

$$\sigma_l^i \equiv \tau_{lC_i} \mod W_{\omega_i} \quad \text{and} \quad \sigma_r^i \equiv \tau_{rC_i} \mod W_{\omega_i} \text{ for } i = 1,...,r,$$

where $\lessdot$ is the Bruhat order in W. From [8] and Littelmann [18] we deduce the following theorem.

Theorem 4.6.1. *Let $T = C_1\cdots C_r$ be a juxtaposition of admissible columns. Then T is a tableau if and only if there exits a defined chain for $(C_1,...,C_r)$.*

Since the Bruhat order on the Weyl group of type D_n can not be characterized from that of the symmetric group S_{2n}, this explains why the combinatorial description of the tableaux is more complicated in type D_n than in types B_n and C_n.
Note that Littelmann's definition of tableaux by defined chains may be generalized to the exceptional root systems. Except for the root system G_2 no presentation of these tableaux in terms of combinatorial planar objects like Young tableaux is known.

5. Plactic monoids for types B_n, C_n, D_n and G_2

The generalization of the notion of plactic monoid to any simple Lie algebra was first obtained by Littelmann [17] from his path model. In the sequel we have chosen to interpret the plactic relations in terms of isomorphisms of Kashiwara's crystal graphs.

5.1. **Plactic relations.** Let w_1 and w_2 be two words on $\mathcal{B}_n$ (resp. $\mathcal{C}_n,\mathcal{D}_n,\mathcal{G}$). We write $w_1 \overset{B}{\sim} w_2$ (resp. $w_1 \overset{C}{\sim} w_2, w_1 \overset{D}{\sim} w_2, w_1 \overset{G}{\sim} w_2$) when these two words occur at the same place in two isomorphic connected components of the crystal G_B (resp. G_C,G_D,G_G).
The definition of the tableaux implies that for any word $w \in \mathcal{B}_n^*$ (resp. $w \in \mathcal{C}_n^*,\mathcal{D}_n^*,\mathcal{G}^*$) there exists a unique tableau $P^B(w)$ (resp. $P^C(w),P^D(w),P^G(w)$) such that $w \sim \mathrm{w}(P(w))$. So the sets $\mathcal{B}_n^*/\overset{B}{\sim}$ $,\mathcal{C}_n^*/\overset{C}{\sim},\mathcal{D}_n^*/\overset{D}{\sim}$ and $\mathcal{G}^*/\overset{G}{\sim}$ can be identified respectively with the sets of tableaux of type B_n,C_n,D_n and G_2.
For the definitions below, recall that the word $w = x_1x_2\cdots x_{l-1}x_l$ has two strict factors which are the words $x_2\cdots x_{l-1}x_l$ and $x_1x_2\cdots x_{l-1}$.

Definition 5.1.1. *The monoid $Pl(B_n)$ is the quotient of the free monoid $\mathcal{B}_n^*$ by the relations:*
R_1^B : *If $x \neq \overline{z}$ and $x < y < z$:*

$$yzx \overset{B}{\equiv} yxz \quad and \quad xzy \overset{B}{\equiv} zxy.$$

R_2^B : *If $x \neq \overline{y}$ and $x < y$:*

$$xyx \overset{B}{\equiv} xxy \text{ for } x \neq 0 \quad and \quad xyy \overset{B}{\equiv} yxy \text{ for } y \neq 0.$$

R_3^B :*If* $1 < x \leq n$ *and* $x \leq y \leq \overline{x}$:

$$y(\overline{x-1})(x-1) \overset{B}{\equiv} yx\overline{x}, \text{ and } x\overline{x}y \overset{B}{\equiv} (\overline{x-1})(x-1)y,$$

$$0\overline{n}n \equiv \overline{n}n0.$$

R_4^B :*If* $x \leq n$:

$$00x \overset{B}{\equiv} 0x0 \text{ and } 0\overline{x}0 \overset{B}{\equiv} \overline{x}00.$$

CR^B : *Let* $w = \mathrm{w}(C)$ *be a non admissible column word each strict factor of which is admissible. Let* z *be the lowest unbarred letter of* w *such that the pair* $(z, \overline{z})$ *occurs in* w *and* $N(z) > z$*, otherwise set* $z = 0$. *Then* $w \overset{B}{\equiv} \widetilde{w}$*, where* $\widetilde{w}$ *is the column word obtained by erasing the pair* $(z, \overline{z})$ *in* w *if* $z \leq n$*, by erasing* 0 *otherwise.*

Definition 5.1.2. *The monoid* $Pl(C_n)$ *is the quotient of the free monoid* $\mathcal{C}_n^*$ *by the relations:*
R_1^C : $yzx \equiv yxz$ *for* $x \leq y < z$ *with* $z \neq \overline{x}$*, and* $xzy \equiv zxy$ *for* $x < y \leq z$ *with* $z \neq \overline{x}$;
R_2^C : $y(\overline{x-1})(x-1) \equiv yx\overline{x}$ *and* $x\overline{x}y \equiv (\overline{x-1})(x-1)y$ *for* $1 < x \leq n$ *and* $x \leq y \leq \overline{x}$;
CR^C : *let* w *be a non admissible column word such that each strict factor of* w *is an admissible column word. Write* z *for the lowest unbarred letter such that the pair* $(z, \overline{z})$ *occurs in* w *and* $N(z) > z$. *Then* $w \equiv \widetilde{w}$*, where* $\widetilde{w}$ *is the column word obtained by erasing the pair* $(z, \overline{z})$ *in* w.

Definition 5.1.3. *The monoid* $Pl(D_n)$ *is the quotient of the free monoid* $\mathcal{D}_n^*$ *by the relations:*
R_1^D : *If* $x \neq \overline{z}$,

$$yzx \overset{D}{\equiv} yxz \text{ for } x \leq y < z \text{ and } xzy \overset{D}{\equiv} zxy \text{ for } x < y \leq z.$$

R_2^D : *If* $1 < x < n$ *and* $x \leq y \leq \overline{x}$,

$$y(\overline{x-1})(x-1) \overset{D}{\equiv} yx\overline{x} \text{ and } x\overline{x}y \overset{D}{\equiv} (\overline{x-1})(x-1)y.$$

R_3^D : *If* $x \leq n-1$,

$$\begin{cases} \overline{n}\,\overline{x}n \overset{D}{\equiv} \overline{x}\,\overline{n}n \\ n\,\overline{x}\,\overline{n} \overset{D}{\equiv} \overline{x}\,n\overline{n} \end{cases} \text{ and } \begin{cases} \overline{n}nx \overset{D}{\equiv} \overline{n}xn \\ n\overline{n}x \overset{D}{\equiv} nx\overline{n} \end{cases}.$$

R_4^D :

$$\begin{cases} n\overline{n}\,\overline{n} \overset{D}{\equiv} \overline{(n-1)}\,(n-1)\overline{n} \\ \overline{n}\,nn \overset{D}{\equiv} \overline{(n-1)}\,(n-1)n \end{cases} \text{ and } \begin{cases} \overline{n}(\overline{n-1})(n-1) \overset{D}{\equiv} \overline{n}\,\overline{n}n \\ n(\overline{n-1})(n-1) \overset{D}{\equiv} nn\overline{n} \end{cases}.$$

CR^D : *Consider* w *a non admissible column word each strict factor of which is admissible. Let* z *be the lowest unbarred letter such that the pair* $(z, \overline{z})$ *occurs in* w *and* $N(z) > z$. *Then* $w \overset{D}{\equiv} \widetilde{w}$*, where* $\widetilde{w}$ *is the column word obtained by erasing the pair* $(z, \overline{z})$ *in* w *if* $z < n$*, by erasing a pair* $(n, \overline{n})$ *of consecutive letters otherwise.*

Set

$$S = \{21, 31, 01, \overline{3}1, \overline{3}2, \overline{2}1, \overline{2}2, \overline{1}1, \overline{1}2, \overline{2}3, \overline{1}3, \overline{1}0, \overline{1}\,\overline{3}, \overline{1}\,\overline{2}\}.$$

To describe the plactic relations for type G_2 we need the bijection Θ from S to $B^G(12)$ defined by

(6)

w	21	31	01	$\overline{3}1$	$\overline{3}2$	$\overline{2}1$	$\overline{2}2$	$\overline{1}1$	$\overline{1}2$	$\overline{2}3$	$\overline{1}3$	$\overline{1}0$	$\overline{1}\overline{3}$	$\overline{1}\overline{2}$
$\Theta(w)$	12	13	23	20	$2\overline{3}$	30	$3\overline{3}$	00	$0\overline{3}$	$3\overline{2}$	$0\overline{2}$	$\overline{3}\overline{2}$	$\overline{3}\overline{1}$	$\overline{2}\overline{1}$

Definition 5.1.4. *The monoid $Pl(G_2)$ is the quotient of the free monoid $\mathcal{G}^*$ by the relations:*

$$(R_1^G) \qquad 10 \equiv 1,\ 1\bar{3} \equiv 2,\ 1\bar{2} \equiv 3,\ 2\bar{2} \equiv 0,\ 0\bar{1} \equiv \bar{1},\ 3\bar{1} \equiv \bar{2},\ 2\bar{1} \equiv \bar{3}.$$

$$(R_2^G) \qquad 1\bar{1} \equiv \emptyset.$$

$$(R_3^G) \qquad abc \equiv \begin{cases} a\Theta(bc) \text{ if } bc \in S \\ \Theta^{-1}(ab)c \text{ otherwise} \end{cases} \text{ with } ab \in B(12) \text{ and } bc \in B(11).$$

$$(R_4^G) \qquad xyz \equiv \Theta^{-1}(xy)z \text{ with } xy \in B(\Lambda_2) \text{ and } yz \in B(\Lambda_2).$$

5.2. **Interpretation in terms of crystal isomorphisms.**

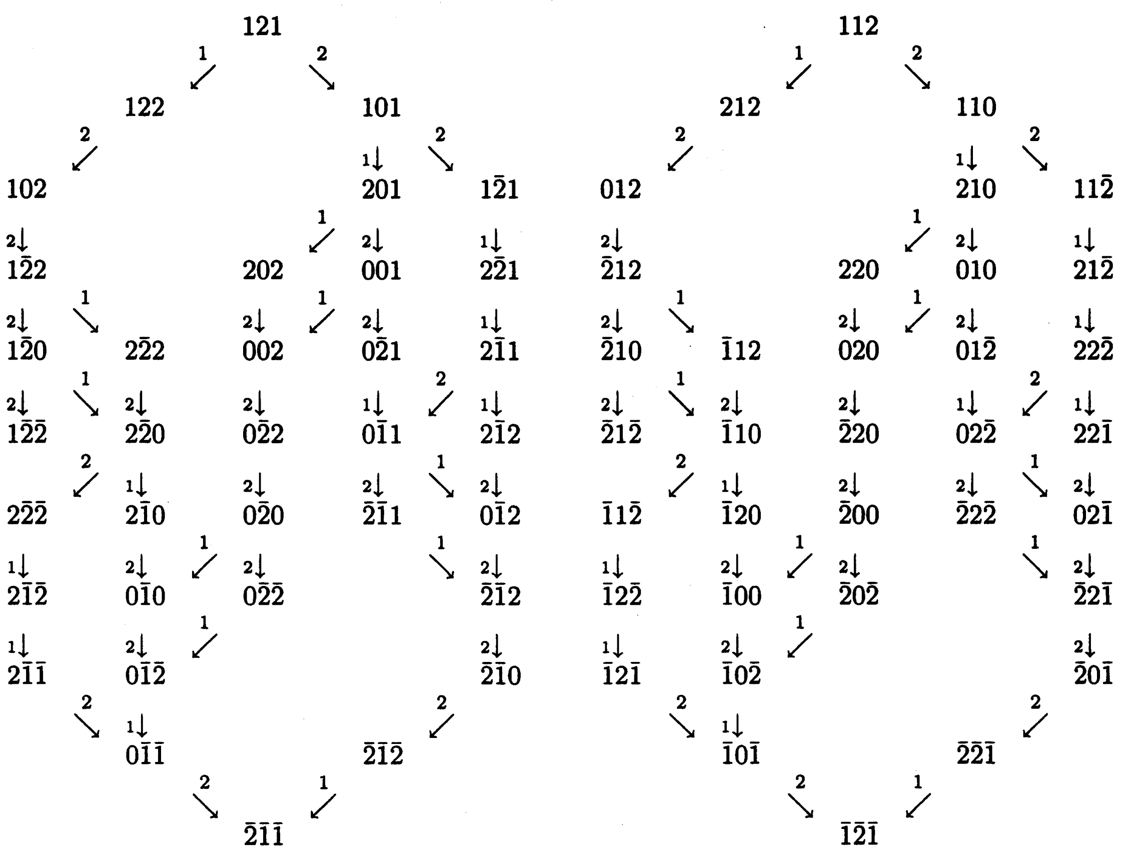
5.2.1. *for classical types B_n, C_n and D_n.* For any word w of length 3 appearing in the left hand side of a relation R above, write $\xi(w)$ for the word appearing in the right hand side of this relation. Similarly for any w of length $p+1$ appearing in the left hand side of a contraction relation CR above, write $\xi_p(w)$ for the word appearing in the right hand side of this relation.

Proposition 5.2.1.

(1) *The map $w \longmapsto \xi(w)$ is the crystal isomorphism from $B(121)$ to $B(112)$.*
(2) *The map $w \longmapsto \xi_p(w)$ is the crystal isomorphism from $B(12\cdots p\bar{p}) \simeq B(12\cdots p-1)$ when $p < n$ and the crystal graph isomorphisms*

$$\begin{cases} \begin{cases} B^B(12\cdots n\bar{n}) \simeq B^B(12\cdots n-1) \\ B^B(12\cdots n0) \simeq B^B(12\cdots n) \end{cases} \\ B^C(12\cdots n\bar{n}) \simeq B^C(12\cdots n-1) \\ \begin{cases} B^D(12\cdots n\bar{n}) \simeq B^D(12\cdots n-1) \\ B^D(12\cdots \bar{n}n) \simeq B^D(12\cdots n) \end{cases} \end{cases}$$

when $p = n$.



The crystals $B^B(121)$ and $B^B(112)$ in G_B

5.2.2. *for type* G_2. For any word w occurring in the left hand side of a relation R_i^G, $i = 1, \dots, 4$ we write $\xi_i(w)$, the word occurring in the right hand side of this relation.

Proposition 5.2.2. *The maps* $\xi_1^G, \xi_2^G, \xi_3^G$ *and* ξ_4^G *are respectively the crystal graph isomorphisms*

$$\text{(i)}: B(10) \xrightarrow{\sim} B(1), \quad \text{(ii)}: B(1\bar{1}) \xrightarrow{\sim} B(\emptyset), \quad \text{(iii)}: B(121) \xrightarrow{\sim} B(112) \textit{ and } \text{(iv)}: B(123) \xrightarrow{\sim} B(110).$$

Remark: Write $(\xi_1^G)'$ for the crystal graph isomorphism $B(110) \xrightarrow{\sim} B(11)$, then $(\xi_4^G)' = (\xi_1^G)'\xi_4^G$ is the crystal isomorphism $B(123) \xrightarrow{\sim} B(11)$.

$$
\begin{array}{ccccccccccccc}
123 & \xrightarrow{1} & 120 & \xrightarrow{2} & 130 & \xrightarrow{1} & 230 & \xrightarrow{1} & 200 & \xrightarrow{2} & 300 & \xrightarrow{1} & 000 \\
 & & {\scriptstyle 1}\downarrow & & & & & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow \\
 & & 12\bar{3} & \xrightarrow{2} & 13\bar{3} & \xrightarrow{1} & 23\bar{3} & & 20\bar{3} & \xrightarrow{2} & 30\bar{3} & & 00\bar{3} \\
 & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
 & & & & 13\bar{2} & \xrightarrow{1} & 23\bar{2} & \xrightarrow{1} & 20\bar{2} & & 30\bar{2} & \xrightarrow{1} & 00\bar{2} \\
 & & & & & & & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow \\
 & & & & & & & & 2\bar{3}\bar{2} & \xrightarrow{2} & 3\bar{3}\bar{2} & & 0\bar{3}\bar{2} \\
 & & & & & & & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\
 & & & & & & & & 2\bar{3}\bar{1} & \xrightarrow{2} & 3\bar{3}\bar{1} & & 0\bar{3}\bar{1} \\
 & & & & & & & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
 & & & & & & & & & & 3\bar{2}\bar{1} & \xrightarrow{1} & 0\bar{2}\bar{1} \\
 & & & & & & & & & & & & {\scriptstyle 1}\downarrow \\
 & & & & & & & & & & & & \bar{3}\bar{2}1
\end{array}
\tag{7}
$$

$$
\begin{array}{ccccccccccccc}
110 & \xrightarrow{1} & 210 & \xrightarrow{2} & 310 & \xrightarrow{1} & 010 & \xrightarrow{1} & \bar{3}10 & \xrightarrow{2} & \bar{2}10 & \xrightarrow{1} & \bar{1}10 \\
 & & {\scriptstyle 1}\downarrow & & & & & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow \\
 & & 21\bar{3} & \xrightarrow{2} & 31\bar{3} & \xrightarrow{1} & 01\bar{3} & & \bar{3}1\bar{3} & \xrightarrow{2} & \bar{2}1\bar{3} & & \bar{1}1\bar{3} \\
 & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
 & & & & 31\bar{2} & \xrightarrow{1} & 01\bar{2} & \xrightarrow{1} & \bar{3}1\bar{2} & & \bar{2}1\bar{2} & \xrightarrow{1} & \bar{1}1\bar{2} \\
 & & & & & & & & {\scriptstyle 1}\downarrow & & & & {\scriptstyle 1}\downarrow \\
 & & & & & & & & \bar{3}2\bar{2} & \xrightarrow{2} & \bar{2}2\bar{2} & & 12\bar{2} \\
 & & & & & & & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle 1}\downarrow \\
 & & & & & & & & \bar{3}2\bar{1} & \xrightarrow{2} & \bar{2}2\bar{1} & & 12\bar{1} \\
 & & & & & & & & & & {\scriptstyle 2}\downarrow & & {\scriptstyle 2}\downarrow \\
 & & & & & & & & & & \bar{2}3\bar{1} & \xrightarrow{1} & 13\bar{1} \\
 & & & & & & & & & & & & {\scriptstyle 1}\downarrow \\
 & & & & & & & & & & & & 0\bar{1}1
\end{array}
\tag{8}
$$

The two isomorphic crystals $B^G(123)$ and $B^G(110)$

5.3. **Congruent words and crystal graphs.** By Propositions 5.2.1 and 5.2.2 the plactic relations above are compatible with Kashiwara's operators, that is, for any words w_1 and w_2 such that $w_1 \equiv w_2$ one has:

$$
\begin{cases} \tilde{e}_i(w_1) \equiv \tilde{e}_i(w_2) \text{ and } \varepsilon_i(w_1) = \varepsilon_i(w_2), \\ \tilde{f}_i(w_1) \equiv \tilde{f}_i(w_2) \text{ and } \varphi_i(w_1) = \varphi_i(w_2). \end{cases} \tag{9}
$$

So we obtain:

$$
w_1 \equiv w_2 \Longrightarrow w_1 \sim w_2. \tag{10}
$$

To prove the converse we need the two following lemmas:

Lemma 5.3.1. *For any words v_1 and v_2, the word $v_1 v_2$ is a highest weight vertex if and only if:*

- *v_1 is a highest weight vertex*
- *for any $i = 1, \ldots, n$, $\varepsilon_i(v_2) \leq \varphi_i(v_1)$.*

Proof. The proof follows immediately from (1) and (2). □

Lemma 5.3.2. *Let w be a highest weight vertex. Then* $\mathrm{w}(P(w)) \equiv w$.

Proof. For classical types, the proof follows by induction on $l(w)$. When $l(w) = 1$, $\mathrm{w}(P(w)) = w$. By writing $w = vx$, one shows that $\mathrm{w}(P(w))$ may be obtained from the word $\mathrm{w}(P(v))x$ by applying only Knuth relations and contraction relations of type $12 \cdots r\overline{p} \equiv 12 \cdots \widehat{p} \cdots r$ with $p \leq r \leq n$ (the hat means removal of the letter p). The proof is similar up to minor modifications for type G_2. □

Theorem 5.3.3. *For any words w_1 and w_2*

$$w_1 \equiv w_2 \Longleftrightarrow w_1 \sim w_2.$$

Proof. From Lemma 5.3.2, we obtain that two highest weight vertices w_1^0 and w_2^0 with the same weight λ satisfy $w_1^0 \equiv w_2^0$. Indeed, since there is only one tableau whose reading is a highest weight vertex of weight λ, we must have $P(w_1^0) = P(w_2^0)$. Now suppose that $w_1 \sim w_2$ and denote by w_1^0 and w_2^0 the highest weight vertices of $B(w_1)$ and $B(w_2)$. We have $w_1^0 \equiv w_2^0$. Set $w_1 = \widetilde{F} w_1^0$ where $\widetilde{F}$ is a product of Kashiwara's operators $\widetilde{f}_i$, $i \in I$. Then $w_2 = \widetilde{F} w_2^0$ because $w_1 \sim w_2$. So by (9) we obtain

$$w_1^0 \equiv w_2^0 \Longrightarrow \widetilde{F} w_1^0 \equiv \widetilde{F} w_2^0 \Longrightarrow w_1 \equiv w_2.$$

Then the theorem follows from (10). □

6. Bumping algorithms for types B_n, C_n, D_n and G_2

For any dominant weight $\lambda \in \Omega_+$, write

$$B(\lambda) \otimes B(\Lambda_1) \simeq \bigoplus_{\nu \in \Omega_+} B(\nu).$$

This decomposition is multiplicity free. Given any letter $x \in B(\Lambda_1)$ and any tableau S such that $\mathrm{w}(S) \in B(\lambda)$ we want to compute the unique tableau T such that $\mathrm{w}(T) \equiv \mathrm{w}(S)x$. We will set $T = x \rightarrow S$ and call this combinatorial operation "insertion of the letter x in the tableau S".

6.1. **Bumping algorithm on an admissible column of type B_n, C_n or D_n.** Consider a word $w = \mathrm{w}(C)x$, where x and C are respectively a letter and an admissible column of height p. Denote by $w^0 = u^0 x^0$ the highest weight vertex of $B(w)$. Only three situations can happen:

(1) $w^0 = v_{\omega_{p+1}}$ with $h(C) = p + 1$: then there is nothing to do and $P(w) = \begin{array}{|c|}\hline C\\\hline x\\\hline\end{array}$ is an admissible column,

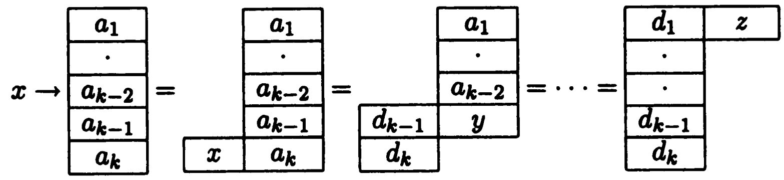
(2) $w^0 = v_{\omega_p}\overline{p}$: then $\begin{array}{|c|}\hline C\\\hline x\\\hline\end{array}$ is a non admissible column of height $p + 1$ and $P(w) = \widetilde{C}$ obtained by applying a contraction relation to $\mathrm{w}(C)x$.

(3) $w^0 = v_{\omega_p}1$ (or $w^0 = v_{\overline{\omega}_n}1$ for type D_n): then $P(w)$ must be a tableau of two columns of heights p and 1 since $B(v_{\omega_p}1) \simeq B(1v_{\omega_p})$.

The relations of length 3 of the plactic monoids are precisely those which are needed to describe the insertion $x \rightarrow C$ of a letter x in an admissible column $C = \begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}$ such that $\begin{array}{|c|}\hline a\\\hline b\\\hline x\\\hline\end{array}$ is not a column. This can be written

$$x \rightarrow \begin{array}{|c|}\hline a\\\hline b\\\hline\end{array} = \begin{array}{cc}\cline{2-2} & \multicolumn{1}{|c|}{a}\\\hline \multicolumn{1}{|c|}{x} & \multicolumn{1}{c|}{b}\\\hline\end{array} = \begin{array}{|c|c|}\hline a' & x'\\\hline b' & \multicolumn{1}{c}{}\\\cline{1-1}\end{array}.$$

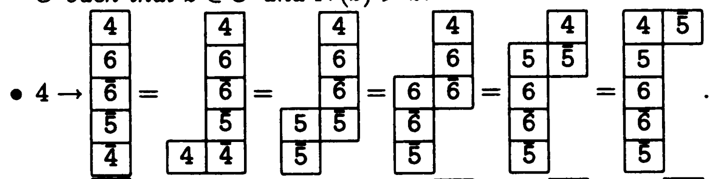
In case 3 above the insertion in an admissible column C of arbitrary height can then be depicted by

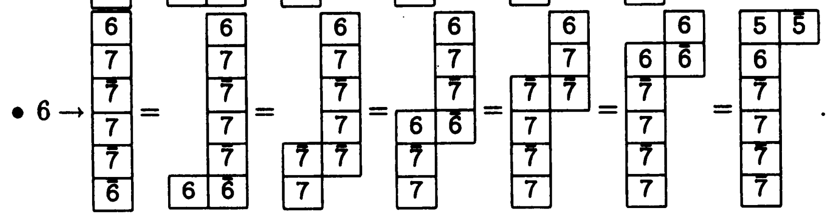
(11) 

that is, one elementary transformation is applied at each step.

Example 6.1.1. *Suppose* $n = 7$.

- $\bar{3} \rightarrow \begin{array}{|c|}\hline 4\\\hline 6\\\hline \bar{6}\\\hline \bar{5}\\\hline \bar{4}\\\hline\end{array} = \begin{array}{|c|}\hline 4\\\hline 6\\\hline \bar{6}\\\hline \bar{5}\\\hline \bar{4}\\\hline \bar{3}\\\hline\end{array}$ *and* $\bar{3} \rightarrow \begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline \bar{5}\\\hline \bar{4}\\\hline\end{array} = \begin{array}{|c|}\hline 3\\\hline 4\\\hline \bar{4}\\\hline \bar{3}\\\hline\end{array}$. *Indeed* $\begin{array}{|c|}\hline 3\\\hline 4\\\hline 5\\\hline \bar{5}\\\hline \bar{4}\\\hline \bar{3}\\\hline\end{array}$ *is not admissible and* 5 *is the lowest letter* z *of* C *such that* $z \in C$ *and* $N(z) > z$.





6.2. **Bumping algorithm on an admissible column of type G_2.** When $h(C) = 1$ and $C = \boxed{a}$ we have

$$x \rightarrow C = \begin{cases} \text{(i)} : \boxed{a}\boxed{x} \text{ if } ax \in B(11), \\ \text{(ii)} : \begin{array}{|c|}\hline a\\\hline x\\\hline\end{array} \text{ if } ax \in B(12), \\ \text{(iii)} : \boxed{a'} \text{ with } a' = \xi_1(ax) \text{ if } ax \in B(10), \\ \text{(iv)} : \emptyset \text{ if } ax = 1\bar{1}. \end{cases} \qquad (12)$$

Indeed in each case (i) to (iv), $x \rightarrow C$ is the unique tableau of type G_2 such that $\mathrm{w}(x \rightarrow C) \equiv \mathrm{w}(C)x$.

When $h(C) = 2$ and $C = \begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$ we have

$$x \to C = \begin{cases} \text{(v)} : \begin{array}{|c|c|}\hline a' & x' \\ \hline b' \\ \cline{1-1}\end{array} \text{ with } x'a'b' = \xi_3^G(abx) \text{ if } bx \text{ is not a column word,} \\ \text{(vi)} : \begin{array}{|c|c|}\hline x' & y' \\ \hline\end{array} \text{ with } a'x' = \xi_1^G\xi_4^G(abx) \text{ if } bx \text{ is an admissible column word,} \\ \text{(vii)} : \begin{array}{|c|}\hline x' \\ \hline\end{array} \text{ with } x' = \xi_1^G(a\xi_1^G(bx)) \text{ if } bx \text{ is a non admissible column word.} \end{cases}$$

Indeed in cases (v) and (vi), $x \to C$ is a tableau of type G_2 such that $\mathrm{w}(x \to C) \equiv \mathrm{w}(C)x$ by Proposition 5.2.2. In case (vii), we obtain by (2) that the highest weight vertex of $B(abx)$ may be written $12u$ with u a letter such that $\varepsilon_1(u) = 0$ and $\varepsilon_2(u) \leq 1$. So $u \in \{1, 3, \bar{2}\}$. We have $u = \bar{2}$, otherwise $B(abx) = B(121)$ and $x \leq b$, or $B(abx) = B(123)$ and bx is an admissible column word. Hence $B(abx) = B(12\bar{2})$. We have

$$B(12\bar{2}) : 12\bar{2} \overset{1}{\to} 12\bar{1} \overset{2}{\to} 13\bar{1} \overset{1}{\to} 23\bar{1} \overset{1}{\to} 20\bar{1} \overset{2}{\to} 30\bar{1} \overset{1}{\to} 00\bar{1}$$

and it is easy to verify that $\xi_1^G(a\xi_1^G(bx))$ is the image of abx by the crystal isomorphism $B(12\bar{2}) \overset{\sim}{\to} B(1)$. In cases (iii), (iv), (vi) and (vii) we have $l(x \to C) < l(\mathrm{w}(C)x)$. Then the insertion procedure causes a contraction. Note that if the words $\mathrm{w}(C_1)x_1$ and $\mathrm{w}(C_2)x_2$ (where C_1, C_2 are admissible columns and x_1, x_2 are letters) belong to the same connected component, the insertions $x_1 \to C_1$ and $x_2 \to C_2$ are of the same type (i) to (vii).

6.3. **The P-symbol $P(w)$.** Set $T = C_1 \cdots C_r$ where C_i, $i = 1, ..., r$ are the admissible columns of T.

(1) When $\begin{array}{|c|}\hline C_1 \\ \hline x \\ \hline\end{array}$ is not a column, write $x \to C = \begin{array}{|c|c|}\hline C_1' & y \\ \hline\end{array}$, where C_1' is an admissible column of height $h(C_1)$ and y a letter. Then $x \to T = C_1'(y \to C_2 \cdots C_r)$ that is, $x \to T$ is the juxtaposition of C_1' with the tableau $\widehat{T}$ obtained by inserting y in the tableau $C_2 \cdots C_r$.

(2) When $\begin{array}{|c|}\hline C_1 \\ \hline x \\ \hline\end{array}$ is an admissible column, $x \to T$ is the tableau obtained by adding a box containing x on bottom of C_1.

(3) When $\begin{array}{|c|}\hline C_1 \\ \hline x \\ \hline\end{array}$ is a column which is not admissible, write $x \to C_1 = \widetilde{C_1}$ and set $\mathrm{w}(\widetilde{C_1}) = y_1 \cdots y_s$. Then $x \to T = y_s \to (y_{s-1} \to (\cdots y_1 \to \widehat{T}))$ that is $x \to T$ is obtained by inserting successively the letters of $\widetilde{C_1}$ into the tableau $\widehat{T} = C_2 \cdots C_r$.

Finally for any vertex w

$$\begin{cases} P(w) = \begin{array}{|c|}\hline w \\ \hline\end{array} \text{ if } w \text{ is a letter,} \\ P(w) = x \to P(u) \text{ if } w = ux \text{ with } u \text{ a word and } x \text{ a letter.} \end{cases}$$

Example 6.3.1.

- *Let* $T = \begin{array}{|c|c|c|c|}\hline 1&3&\bar{3}&\bar{2}\\\hline \bar{3}&\bar{3}&\bar{2}\\\cline{1-3} \bar{2}&\bar{1}\\\cline{1-2}\end{array}$ *of type* C_3 *and* $x = \bar{1}$. *The column* $\begin{array}{|c|}\hline C_1\\\hline x\\\hline\end{array} = \begin{array}{|c|}\hline 1\\\hline \bar{3}\\\hline \bar{2}\\\hline \bar{1}\\\hline\end{array}$ *is not admissible and* $\widetilde{C}_1 = \begin{array}{|c|}\hline \bar{3}\\\hline \bar{2}\\\hline\end{array}$. *So we have to insert* $\bar{3}$ *next* $\bar{2}$ *in the tableau* $\widehat{T} = \begin{array}{|c|c|c|}\hline 3&\bar{3}&\bar{2}\\\hline \bar{3}&\bar{2}\\\cline{1-2} \bar{1}\\\cline{1-1}\end{array}$. *The insertion of* $\bar{3}$ *gives the tableau* $\begin{array}{|c|c|c|c|}\hline 2&\bar{3}&\bar{2}&\bar{2}\\\hline \bar{3}&\bar{2}\\\cline{1-2} \bar{1}\\\cline{1-1}\end{array}$ *then by inserting* $\bar{2}$ *we obtain* $\bar{1} \to T = \begin{array}{|c|c|c|c|}\hline 2&\bar{3}&\bar{2}&\bar{2}\\\hline \bar{3}&\bar{2}\\\cline{1-2} \bar{2}&\bar{1}\\\cline{1-2}\end{array}$.
- $T = \begin{array}{|c|c|c|c|}\hline 2&3&\bar{3}&\bar{2}\\\hline 0&\bar{3}&\bar{2}\\\cline{1-3} 0&\bar{1}\\\cline{1-2}\end{array}$ *of type* B_3. *Then* $\bar{2} \to T = \begin{array}{|c|c|c|c|}\hline 2&3&\bar{3}&\bar{2}\\\hline 0&\bar{3}&\bar{2}\\\cline{1-3} \bar{2}&\bar{1}\\\cline{1-2}\end{array}$ *since* $\bar{2} \to \begin{array}{|c|}\hline 2\\\hline 0\\\hline 0\\\hline\end{array} = \begin{array}{|c|}\hline 2\\\hline 0\\\hline \bar{2}\\\hline\end{array}$.
- $T = \begin{array}{|c|c|c|}\hline 2&0&\bar{3}\\\hline 0&\bar{2}&\bar{1}\\\hline\end{array}$ *of type* G_2. *Then the insertion* $\bar{2} \to T = \bar{2} \to \begin{array}{|c|c|c|}\hline 2&0&\bar{3}\\\hline 0&\bar{2}&\bar{1}\\\hline\end{array} = 3 \to \left(\bar{3} \to \begin{array}{|c|c|}\hline 0&\bar{3}\\\hline \bar{2}&\bar{1}\\\hline\end{array}\right) = 3 \to \left(\begin{array}{|c|}\hline 3\\\hline \bar{3}\\\hline\end{array}\left(\bar{1} \to \begin{array}{|c|}\hline \bar{3}\\\hline \bar{1}\\\hline\end{array}\right)\right) = 3 \to \begin{array}{|c|c|c|}\hline 3&\bar{3}&\bar{1}\\\hline \bar{3}&\bar{1}\\\cline{1-2}\end{array} = \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\left(\bar{2} \to \begin{array}{|c|c|}\hline \bar{3}&\bar{1}\\\hline \bar{1}\\\cline{1-1}\end{array}\right) = \begin{array}{|c|c|c|c|}\hline 2&\bar{3}&\bar{1}&\bar{1}\\\hline 3&\bar{2}\\\cline{1-2}\end{array}$.

Remark: All the insertion schemes described in this note are column insertion algorithms. For semi-standard tableaux of type A_n, there also exists a row insertion algorithm compatible with Knuth relations. Moreover the row and column readings of a semi-standard tableau belong to the same plactic class. This is not true for the tableaux of types B_n, C_n, D_n and G_2. For example in type C_3, the row reading of $T = \begin{array}{|c|c|}\hline 2&\bar{3}\\\hline 3&\bar{2}\\\hline\end{array}$ is the word $w = \bar{3}2\bar{2}3$, but $P^C(w) = \begin{array}{|c|c|}\hline 1&\bar{3}\\\hline 3&\bar{1}\\\hline\end{array} \neq T$. This explains why the relevant insertion schemes for Kashiwara-Nakashima tableaux are column insertion algorithms.

7. Robinson-Schensted type correspondences

7.1. Oscillating tableaux.

An oscillating tableau Q of type B_n and length l is a sequence of Young diagrams $(Q_1, ..., Q_l)$ whose columns have height $\leq n$ and such that any two consecutive diagrams are equal or differ by exactly one box (i.e. $Q_{k+1} = Q_k$, $Q_{k+1}/Q_k = (\square)$ or $Q_k/Q_{k+1} = (\square)$).
An oscillating tableau Q of type C_n and length l is a sequence of Young diagrams $(Q_1, ..., Q_l)$ whose columns have height $\leq n$ and such that any two consecutive diagrams differ by exactly one box (i.e. $Q_{k+1}/Q_k = (\square)$ or $Q_k/Q_{k+1} = (\square)$).
An oscillating tableau Q of type D_n and length l is a sequence $(Q_1, ..., Q_l)$ of pairs $Q_k = (O_k, \varepsilon_k)$ where O_k is a Young diagram whose columns have height $\leq n$ and $\varepsilon_k \in \{-, 0, +\}$, satisfying for $k = 1, ..., l$,

- $O_{k+1}/O_k = (\square)$ or $O_k/O_{k+1} = (\square)$,
- $\varepsilon_{k+1} \neq 0$ and $\varepsilon_k \neq 0$ imply $\varepsilon_{k+1} = \varepsilon_k$.
- $\varepsilon_k = 0$ if and only if O_k has no columns of height n.

An oscillating tableau Q of type G_2 and length l is a sequence $(Q_1, ..., Q_l)$ of Young diagrams whose columns have height 1 or 2 satisfying for $k = 1, ..., l$ one of the following assertions:

- Q_{k+1} is obtained by adding one box to Q_k.
- Q_{k+1} is obtained by deleting one box in Q_k.
- $Q_{k+1} = Q_k$.
- Q_{k+1} is obtained from Q_k by moving one box from height 2 to height 1.
- Q_{k+1} is obtained from Q_k by moving one box from height 1 to height 2.

7.2. **The Q-symbol $Q(w)$.** Let $w = x_1 \cdots x_l$ be a word. The construction of $P(w)$ involves the construction of the l tableaux defined by $P_i = P(x_1 \cdots x_i)$. For $w \in \mathcal{B}_n^*$ (resp. $w \in \mathcal{C}_n^*, w \in \mathcal{D}_n^*, w \in \mathcal{G}^*$) we denote by $Q_B(w)$ (resp. $Q_C(w), Q_D(w), Q_G(w)$) the sequence of shapes of the tableaux $P_1, ..., P_l$.

Proposition 7.2.1. *$Q_B(w), Q_C(w), Q_D(w)$ and $Q_G(w)$ are respectively oscillating tableaux of type B_n, C_n, D_n and G_2.*

Proof. We give the proof for the orthogonal types. The arguments are essentially the same for types C_n and G_2.

Each Q_i is the shape of an orthogonal tableau so it suffices to prove that for any letter x and any orthogonal tableau T, the shape of $x \to T$ differs from the shape of T by at most one box according to the above definition of oscillating tableaux of types B_n and D_n.

The highest weight vertex of the connected component containing $\mathrm{w}(T)x$ may be written $\mathrm{w}(T^0)x^0$ where T^0 is an orthogonal tableau. It follows from Lemma 3.4.1 that $\mathrm{w}(T) \longleftrightarrow \mathrm{w}(T^0)$. So $\mathrm{wt}(\mathrm{w}(T^0))$ is given by the shape of T. Then the shape of $x \to T$ is given by the coordinates of $\mathrm{wt}(\mathrm{w}(T^0)x^0)$ on the basis $(\omega_1^B, ..., \omega_n^B)$ for type B_n, on the base $(\omega_1^D, ..., \omega_n^D)$ or $(\omega_1^D, ..., \omega_{n-1}^D, \overline{\omega}_n^D)$ for type D_n.

Suppose that $x \in \mathcal{B}_n^*$ and T is orthogonal of type B_n. Let $(\lambda_1, ..., \lambda_n)$ be the coordinates of $\mathrm{wt}(T^0)$ on the basis of the ω_i^B's. If $x^0 = \bar{i} > 0$ then $\mathrm{wt}(x^0) = \omega_{i-1}^B - \omega_i^B$. So $\lambda_i > 0$ and $\mathrm{wt}(\mathrm{w}(T^0)x^0) = (\lambda_1, ..., \lambda_{i-1} + 1, \lambda_i - 1, ..., \lambda_{n-1})$. Hence during the insertion of the letter x in T, a column of height i (corresponding to the weight ω_i) is turned into a column of height $i - 1$ (corresponding to the weight ω_{i-1}). So the shape of $x \to T$ is obtained by erasing one box in the shape of T. If $x^0 = i < 0$, then we can prove by similar arguments that the shape of $x \to T$ is obtained by adding one box to the shape of T. When $x^0 = 0$, $\mathrm{wt}(x^0) = 0$, so $\mathrm{wt}(\mathrm{w}(T^0)x^0) = \mathrm{wt}(\mathrm{w}(T^0))$. Hence the shapes of T and $x \to T$ are the same.

Suppose $x \in \mathcal{D}_n^*$ and T orthogonal of type D_n. When $|x^0| \neq n$, the proof is the same as above. If $x^0 = n$, $\mathrm{wt}(x^0) = \Lambda_n - \Lambda_{n-1} = \omega_n - \omega_{n-1} = \omega_{n-1} - \overline{\omega}_n$. We have to consider three cases, (i): $\varepsilon_T = -$, (ii): $\varepsilon_T = 0$ and (iii): $\varepsilon_T = +$. Denote by $(\lambda_1, ..., \lambda_n)$ the positive decomposition of $\mathrm{wt}(\mathrm{w}(T^0))$ on the basis $(\omega_1^D, ..., \omega_n^D)$ or on the basis $(\omega_1^D, ..., \overline{\omega}_n^D)$.

In the first case, $\lambda_n > 0$ and the positive decomposition of $\mathrm{wt}(x^0\mathrm{w}(T^0))$ on the basis $(\omega_1^D, ..., \overline{\omega}_n^D)$ is $(\lambda_1, ..., \lambda_{n-2}, \lambda_{n-1} + 1, \lambda_n - 1)$. It means that during the insertion of x in T a column of height n (corresponding to $\overline{\omega}_n$) is turned into a column of height $n - 1$ (corresponding to ω_{n-1}). Moreover $\varepsilon_{x \to T} = \varepsilon_T$ if $\lambda_n > 1$ and $\varepsilon_{x \to T} = 0$ otherwise.

In the second case, $\lambda_{n-1} > 0$, $\lambda_n = 0$ and the positive decomposition of $\mathrm{wt}(x^0\mathrm{w}(T^0))$ on the base $(\omega_1^D, ..., \omega_n^D)$ is $(\lambda_1, \lambda_2, ..., \lambda_{n-1} - 1, 1)$. It means that during the insertion of x in T a column of height $n - 1$ (corresponding to ω_{n-1}) is turned into a column of height n (corresponding to ω_n). Moreover $\varepsilon_{x \to T} = +$.

In the last case, $\lambda_{n-1} > 0$, $\lambda_n > 0$ and the positive decomposition of $\mathrm{wt}(x^0\mathrm{w}(T^0))$ on $(\omega_1^D, ..., \omega_n^D)$ is $(\lambda_1, \lambda_2, ..., \lambda_{n-1} - 1, \lambda_n + 1)$. It means that during the insertion of x in T a column of height $n - 1$ (corresponding to ω_{n-1}) is turned into a column of height n (corresponding to ω_n). Moreover $\varepsilon_{x \to T} = \varepsilon_T$.

When $x^0 = \overline{n}$, the proof is similar. □

Remark: The above proposition implies in particular that there should be at most one contraction during the insertion procedure $x \to T$.

Theorem 7.2.2. *For any vertices w_1 and w_2*

$$w_1 \longleftrightarrow w_2 \Leftrightarrow Q(w_1) = Q(w_2).$$

Proof. We proceed by induction on the length l of the words w_1 and w_2. If $l = 1$ the result is immediate. If w_1 and w_2 have length $l > 1$, we can write $w_1 = u_1x_1$ and $w_2 = u_2x_2$ with x_1, x_2 letters and u_1, u_2 words of length $l-1$. Let $w_1^0 = u_1^0 x_1^0$ and $w_2^0 = u_2^0 x_2^0$ be the highest weight vertices of $B(w_1)$ and $B(w_2)$. Write Q_1 and Q_2 for the shapes of $P(w_1)$ and $P(w_2)$ (that is those of $P(w_1^0)$ and $P(w_2^0)$). We suppose the Proposition is true for the words of length $l-1$. First we have:

$$w_1 \longleftrightarrow w_2 \Longleftrightarrow \begin{cases} u_1 \longleftrightarrow u_2, \\ Q_1 = Q_2. \end{cases}$$

Indeed if $w_1 \longleftrightarrow w_2$ then $u_1 \longleftrightarrow u_2$ follows from Lemma 3.4.1 and we obtain $Q_1 = Q_2$ because the readings of $P(w_1)$ and $P(w_2)$ are in the same connected component. Conversely, $u_1 \longleftrightarrow u_2$ implies that $u_1^0 = u_2^0$ and it follows from the equality $Q_1 = Q_2$ that $\mathrm{wt}(w_1^0) = \mathrm{wt}(w_2^0)$ (because the shape of $P(w_i^0)$ $i = 1, 2$ coincides with the weight $\mathrm{wt}(w_i^0)$). So $x_1^0 = x_2^0$. This means that $w_1^0 = w_2^0$ i.e. $w_1 \longleftrightarrow w_2$. Finally we obtain by induction:

$$w_1 \longleftrightarrow w_2 \Longleftrightarrow \begin{cases} Q(u_1) = Q(u_2) \\ Q_1 = Q_2 \end{cases} \Longleftrightarrow Q(w_1) = Q(w_2).$$

□

Write respectively $\mathcal{O}_l^B, \mathcal{O}_l^C, \mathcal{O}_l^D$ and $\mathcal{O}_l^G$ for the sets of pairs (P, Q) where P is a tableau and Q an oscillating tableau respectively of type B_n, C_n, D_n and G_2 and length l such that P has shape Q_l (Q_l is the last shape of Q). Let $\mathcal{B}_{n,l}^*, \mathcal{C}_{n,l}^*, \mathcal{D}_{n,l}^*$ and $\mathcal{G}_l^*$ be the subsets of words of length l respectively in $\mathcal{B}_n^*, \mathcal{C}_n^*, \mathcal{D}_n^*$ and $\mathcal{G}^*$.

Corollary 7.2.3. *The maps:*

$$\begin{matrix} \Psi^B : B_{n,l}^* \to \mathcal{O}_l^B \\ w \mapsto (P^B(w), Q^B(w)) \end{matrix}, \quad \begin{matrix} \Psi^C : C_{n,l}^* \to \mathcal{O}_l^C \\ w \mapsto (P^C(w), Q^C(w)) \end{matrix} \quad \begin{matrix} \Psi^D : \mathcal{D}_{n,l}^* \to \mathcal{O}_l^D \\ w \mapsto (P^D(w), Q^D(w)) \end{matrix} \quad \textit{and} \quad \begin{matrix} \Psi^G : \mathcal{G}_l^* \to \mathcal{O}_l^G \\ w \mapsto (P^G(w), Q^G(w)) \end{matrix}$$

are bijections.

Proof. Type C_n : by Theorems 5.3.3 and 7.2.2, we obtain that Ψ is injective. Consider an oscillating tableau Q of length l. Set $x_1 = 1$ and for $i = 2, ..., l$, $x_i = k$ if Q_i differs from Q_{i-1} by adding a box in row k and $x_i = \bar{k}$ if Q_i differs from Q_{i-1} by removing a box in row k. Consider $w_Q = x_l \cdots x_2 1$. Then $Q(w_Q) = Q$. By Theorem 4.5.4, the image of $B(w_Q)$ by Ψ is the pair (P, Q) where P is a symplectic tableau of shape Q_l. We deduce immediately that Ψ is surjective.
The proof is similar for types B_n, D_n and G_2. □

8. Reverse bumping algorithm for the classical types

For any dominant weight $\lambda \in \Omega_+$, recall the decomposition

$$B(\lambda) \otimes B(\Lambda_1) \simeq \bigoplus_{\nu \in \Omega_+} B(\nu).$$

Since this decomposition is multiplicity free, it must be possible, starting from T with $\mathrm{w}(T) \in B(\nu)$ and λ, to determine the unique pair (x, S) where x is a letter and $\mathrm{w}(S) \in B(\lambda)$ such that $x \to S = T$. This procedure is called "reverse bumping algorithm". To describe the reverse bumping steps, it suffices to interpret each plactic relation on words, read from right hand side to left hand side, as a combinatorial operation. We give below a complete description of these operations on the tableaux of classical types. The reverse bumping algorithm for type G_2 can be obtained similarly from the plactic relations of Definition 5.1.4.

8.1. **Reverse bumping on column tableaux.** Given an admissible column D with $h(D) < n-1$, there exist two pairs (x_1, C_1) and (x_2, C_2) such that $x_1 \to C_1 = x_2 \to C_2 = D$ and $h(C_1) = h(D) - 1$, $h(C_2) = h(D) + 1$ depending on whether a contraction happens or not in the insertion scheme. The pair (x_1, C_1) is immediately computed since x_1 is the bottom letter of D and $C_1 = D - \{x_1\}$. Since there is a contraction during the insertion $x_2 \to C_2 = D$, the column $\begin{array}{|c|}\hline C_2 \\ \hline x_2 \\ \hline\end{array}$ is obtained by adding to D the pair $(z, \bar{z})$ with z minimal such that $N(z) > z$ in $\begin{array}{|c|}\hline C_2 \\ \hline x_2 \\ \hline\end{array}$ and C_2 is admissible.
When $h(D) = n-1$,, we obtain two pairs (x_1, C_1) and (x_2, C_2) as above for types B_n and C_n. For type D_n, the pair (x_1, C_1) is determined as above but there exist two pairs (x_2', C_2') and (x_2, C_2) with $h(C_2) = h(C_2') = h(D) + 1$ and $x_2 \to C_2 = x_2' \to C_2' = D$. Indeed we have two isomorphisms $B(1 \cdots (n-1)) \simeq B(1 \cdots n\bar{n}) \simeq B(1 \cdots \bar{n}n)$. In this case we have $z = n$ and the columns $\begin{array}{|c|}\hline C_2 \\ \hline x_2 \\ \hline\end{array}$, $\begin{array}{|c|}\hline C_2' \\ \hline x_2' \\ \hline\end{array}$ are obtained by adding two boxes $\begin{array}{|c|}\hline n \\ \hline \bar{n} \\ \hline\end{array}$, $\begin{array}{|c|}\hline \bar{n} \\ \hline n \\ \hline\end{array}$ in D.
When $h(D) = n$, only the pair (x_1, C_1) exists for types C_n and D_n. For type B_n, we also have to consider the pair (x_3, C_3) where x_3 is the bottom letter of D and $C_3 = D - \{x_3\} + \{0\}$.

Example 8.1.1. *Suppose* $n = 6$.

- *Consider* $D = \begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline \bar{5} \\ \hline \bar{3} \\ \hline \bar{1} \\ \hline\end{array}$. *Then we have* $x_2 = \bar{1}$ *and* $C_2 = \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{3} \\ \hline\end{array}$.
- *Consider* $D = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline 0 \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline \bar{1} \\ \hline\end{array}$. *Then for type* B_n *we have* $x_3 = \bar{1}$ *and* $C_3 = \begin{array}{|c|}\hline 4 \\ \hline 5 \\ \hline 0 \\ \hline 0 \\ \hline \bar{5} \\ \hline \bar{4} \\ \hline\end{array}$.
- *Consider* $D = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 5 \\ \hline \bar{5} \\ \hline \bar{2} \\ \hline\end{array}$. *Then we have for type* D_n, $x_2 = x_2' = \bar{2}$ *and* $C_2 = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 5 \\ \hline 6 \\ \hline \bar{6} \\ \hline \bar{5} \\ \hline\end{array}$, $C_2' = \begin{array}{|c|}\hline 1 \\ \hline 4 \\ \hline 5 \\ \hline \bar{6} \\ \hline 6 \\ \hline \bar{5} \\ \hline\end{array}$.

8.2. **Fundamental reverse bumping step.** Consider $T = D\,\boxed{x}$ a tableau of type B_n, C_n or D_n with two columns such that $h(D) = p$ and $\boxed{x}$ contains only the letter x. Then by applying procedure (11) from right to left, that is, applying to $x\mathrm{w}(D)$ plactic relations of length 3 read from right to left, we can obtain the unique pair (x, C) where x is a letter and C an admissible column of height p such that $x \to C = D\,\boxed{x}$.

Example 8.2.1. *From* $T = \begin{array}{cc} 4 & \bar{5} \\ 5 & \\ 6 & \\ \bar{6} & \\ \bar{5} & \end{array}$ *we obtain* $\begin{array}{cc} 4 & \bar{5} \\ 5 & \\ 6 & \\ \bar{6} & \\ \bar{5} & \end{array} = \begin{array}{cc} & 4 \\ 5 & \bar{5} \\ 6 & \\ \bar{6} & \\ \bar{5} & \end{array} = \begin{array}{cc} & 4 \\ & 6 \\ 6 & \bar{6} \\ \bar{6} & \\ \bar{5} & \end{array} = \begin{array}{cc} & 4 \\ & 6 \\ & \bar{6} \\ 5 & \bar{5} \\ \bar{5} & \end{array} = \begin{array}{cc} & 4 \\ & 6 \\ & \bar{6} \\ & \bar{5} \\ 4 & \bar{4} \end{array} = 4 \rightarrow \begin{array}{c} 4 \\ 6 \\ \bar{6} \\ \bar{5} \\ \bar{4} \end{array}$

.

8.3. Reverse bumping algorithm on a tableau.

8.3.1. *type* C_n. Consider $T = C_1 \cdots C_p$ a tableau and Y a "Young diagram" such that the shape $Y(T)$ of T and Y differ by exactly one box. We look for the letter x and the tableau S such that $x \rightarrow S = T$. Suppose first that Y has one box less than $Y(T)$. Let C_k be the column of T corresponding to this box and y the letter it contains. If $k = 1$ we find (x, S) immediately, otherwise we apply the reverse bumping algorithm to the tableau $\boxed{C_{k-1} \mid y}$ (see section 8.2) .This gives a new column C'_{k-1} and a letter y_1. When $C_{k-1} = C_1$ we have $x = y_1$ and $S = C'_1 C_2 \cdots C_p$. When $k > 2$, we apply the reverse bumping algorithm to $\boxed{C_{k-2} \mid y_1}$ and so on until we obtain the letter y_{k-1}. Then $x = y_{k-1}$ and $S = C'_1 \cdots C'_{k-1} C_k \cdots C_p$.

Example 8.3.1. *Consider* $T = \begin{array}{cccc} 2 & 2 & 3 & \bar{1} \\ 4 & 4 & \bar{3} & \\ \bar{4} & \bar{2} & \bar{1} & \\ \bar{3} & & & \end{array}$ *and* $Y = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & & \\ \square & & & \end{array}$. *We obtain* $x = y_2 = \bar{3}$, $y_1 = \bar{1}$ *and*

$S = \begin{array}{cccc} 3 & 2 & 3 & \bar{1} \\ 4 & 4 & \bar{3} & \\ \bar{4} & \bar{1} & & \\ \bar{3} & & & \end{array}$.

Now suppose that Y has one box more than $Y(T)$. Denote by Y' the Young diagram obtained by deleting the first column of Y. Since Y has one box more than $Y(T)$, a contraction happens during the insertion $x \rightarrow S = T$. This is case 3 of the insertion procedure described in section 6.3. The tableau S can certainly be written in the form $S = D_1 S'$ where D_1 is the leftmost column of S and $S' = S - D_1$. Then $D = \begin{array}{|c|} \hline D_1 \\ \hline x \\ \hline \end{array}$ is a non admissible column . Denote by $\widetilde{D}$ the column obtained by contracting D and set $s = h(\widetilde{D})$. During the insertion $x \rightarrow S = T$, D is contracted and next the letters of $\widetilde{D}$ are inserted in the tableau S'. This forces Y', the shape of S', to be contained in $Y(T)$. More precisely $Y(T) - Y'$ contains s boxes corresponding to the insertion of the letters of $\widetilde{D}$. Since the insertion of the letters of the column $\widetilde{D}$ does not induce new contractions, the s boxes of $Y(T) - Y'$ belong to different rows of $Y(T)$. This is verified easily by a detailed analysis of the highest weight vertices implied in the previous combinatorial operations on tableaux.

This gives a procedure to compute S. Indeed, it suffices to consider the letters of T which do not belong to Y'. These letters appear on different rows of T. One then applies the previous reverse bumping algorithm to these s letters starting from the letter appearing in the lowest row until terminating with the letter appearing in the highest row. One can prove that this computation is always possible. The letters then obtained form the reading of an admissible column (namely the reading of $\widetilde{D}$ with the previous notation). Moreover the resulting tableau is the tableau S'. Finally we obtain D from $\widetilde{D}$ as in section 8.1. Then $S = DS'$ is the juxtaposition of D and S'.

Example 8.3.2. *Consider* $T = \begin{array}{|c|c|c|c|}\hline 2 & \bar{3} & \bar{2} & \bar{2} \\ \hline \bar{3} & \bar{2} \\ \cline{1-2} \bar{2} & \bar{1} \\ \cline{1-2}\end{array}$ *and* $Y = \begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ \\ \cline{1-3} \ & \ \\ \cline{1-2}\end{array}$. *We obtain* $Y' = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}$. *So we have to remove the letters* $\bar{1}$ *next* $\bar{2}$ *in* T. *This gives* $T = \bar{2} \rightarrow \bar{3} \rightarrow \begin{array}{|c|c|c|}\hline 3 & \bar{3} & \bar{2} \\ \hline \bar{3} & \bar{2} \\ \cline{1-2} \bar{1} \\ \cline{1-1}\end{array}$. *Thus* $\widetilde{D} = \begin{array}{|c|}\hline \bar{3} \\ \hline \bar{2} \\ \hline\end{array}$ *and* $S' = \begin{array}{|c|c|c|}\hline 3 & \bar{3} & \bar{2} \\ \hline \bar{3} & \bar{2} \\ \cline{1-2} \bar{1} \\ \cline{1-1}\end{array}$. *We obtain* $D = \begin{array}{|c|}\hline 1 \\ \hline \bar{3} \\ \hline \bar{2} \\ \hline \bar{1} \\ \hline\end{array}$. *Finally* $x = \bar{1}$ *and* $S = \begin{array}{|c|c|c|c|}\hline 1 & 3 & \bar{3} & \bar{2} \\ \hline \bar{3} & \bar{3} & \bar{2} \\ \cline{1-3} \bar{2} & \bar{1} \\ \cline{1-2}\end{array}$.

8.3.2. *Type* B_n. We proceed similarly. The only difference is when $Y(T) = Y(S)$. Then $\begin{array}{|c|}\hline C \\ \hline x \\ \hline\end{array}$ is the column obtained by adding a letter 0 in C_1, the first column of T.

8.3.3. *Type* D_n. Write $Y(T) = (Y_T, \varepsilon_T)$ and $Y(S) = (Y_S, \varepsilon_S)$ where Y_T, Y_S are Young diagrams and $\varepsilon_S, \varepsilon_T$ belong to $\{-, 0, +\}$ (see section 4.4). When Y_T has one box more than Y_S, we proceed as for type C_n. When Y_T has one box less than Y_S, the reverse algorithm is the same except for the computation of $\begin{array}{|c|}\hline C \\ \hline x \\ \hline\end{array}$ from D when $h(D) = n - 1$, that is, when $\varepsilon_S \neq 0$. Indeed we have seen that there are two possibilities for $\begin{array}{|c|}\hline C \\ \hline x \\ \hline\end{array}$ in this case. If $\varepsilon = +$, we choose $\begin{array}{|c|}\hline C \\ \hline x \\ \hline\end{array}$ such that $\mathrm{w}(C) \in B(\omega_n^B)$, otherwise we choose it to have $\mathrm{w}(C) \in B(\overline{\omega}_n^B)$.

9. Sliding algorithm for types B_n and C_n

This section is concerned with a symplectic Jeu de Taquin (or sliding algorithm) introduced by J. T. Sheats [19] in order to obtain an explicit bijection between King's and De Concini's symplectic tableaux.

Recall that the Schützenberger sliding algorithm is a procedure which yields a semistandard tableau starting from a given skew semistandard tableau T by a sequence of successive horizontal and vertical slides. The reading $\mathrm{w}(T)$ of T is obtained by reading the columns of T from right to left and top to bottom. One can prove that the readings of the skew semistandard tableaux successively obtained from T by a sequence of slides all belongs to the plactic congruence class of $\mathrm{w}(T)$. This implies that the sliding algorithm is confluent, that is the resulting semistandard tableau does not depend on the order in which the inner corners of T are evacuated (see [5]).

An analogous property has been conjectured in [19] and proved in [15]. This yields an alternative way to compute $P_C(w)$ for any word w. By considering the splitting form of the admissible columns of type B_n, it is also possible to obtain a sliding algorithm for type B_n. In the sequel we only summarize the main definitions and results concerning these Jeux de Taquin and refer the reader to [15] and [16] for the proofs.

9.1. Sheats sliding algorithm.

9.1.1. *Skew admissible tableaux.* Let $\lambda = \sum_{i=1}^{n} \widetilde{\lambda}_i \Lambda_i^C$ and $\mu = \sum_{i=1}^{n} \widetilde{\mu}_i \Lambda_i^C$ be two dominant weights such that $\widetilde{\mu}_i \leq \widetilde{\lambda}_i$ for $i = 1, ..., n$. A skew tableau of shape λ/μ over $\mathcal{C}_n$ is a filling of letters of $\mathcal{C}_n$ in the skew Young diagram Y_λ / Y_μ making columns strictly increasing from top to bottom.

Definition 9.1.1. *A skew tableau over C_n is admissible if its columns are admissible and the rows of its split form (obtained by turning each column C into its split form (lC, rC)) are weakly increasing from left to right.*

Example 9.1.2. $T = \begin{array}{cccc} & & 4 & 4 \\ & 3 & \bar{4} & \bar{3} \\ \bar{3} & \bar{3} & \bar{2} & \\ \bar{2} & \bar{2} & & \end{array}$ *is an admissible skew tableau. Its split form is*

$$\begin{array}{cccccccc} & & & & 4 & 4 & 4 & 4 \\ & & 1 & 3 & \bar{4} & \bar{3} & 3 & \bar{3} \\ \bar{3} & \bar{3} & \bar{3} & \bar{2} & \bar{2} & \bar{2} & & \\ \bar{2} & \bar{2} & \bar{2} & \bar{1} & & & & \end{array}.$$

Remark: One can prove that the set of readings of the admissible skew tableaux of shape λ/μ is a sub-crystal of G_n^C, that is, a disjoint union of connected components of G_n^C.

We denote by $\mathcal{T}_{(\lambda/\mu)}$ the set of admissible skew tableaux of shape (λ/μ) and by $\mathcal{U}_{(\lambda/\mu)}$ the set of readings of these skew tableaux.
Consider an admissible skew tableau of shape λ/μ. An inner corner is a box of Y_μ such that the boxes down and to the right are not in Y_μ. An outside corner is a box of Y_λ such that the boxes down and to the right are not in Y_λ.
A skew tableau is said to be punctured if one of its box contains the symbol $*$ called the puncture.
A punctured column C is admissible if the column C' obtained by ignoring the puncture is admissible. Then the punctured columns rC and lC are respectively obtained by replacing the letters of C (except the puncture) by the letters of rC' and lC'. The split form of C is $lCrC$.
A punctured skew tableau is admissible if its columns are admissible and the rows of its split form (obtained by splitting its columns) are weakly increasing (ignoring the puncture).

Example 9.1.3. $T = \begin{array}{cccc} & & 4 & 4 \\ & 3 & * & \bar{3} \\ \bar{3} & \bar{3} & \bar{2} & \\ \bar{2} & \bar{2} & & \end{array}$ *is an admissible skew punctured tableau of split form* $spl(T) =$

$$\begin{array}{cccccccc} & & & & 3 & 4 & 4 & 4 \\ & & 1 & 3 & * & * & \bar{3} & \bar{3} \\ \bar{3} & \bar{3} & \bar{3} & \bar{2} & \bar{2} & \bar{2} & & \\ \bar{2} & \bar{2} & \bar{2} & \bar{1} & & & & \end{array}.$$

9.1.2. *Coadmissible columns.* A column C of type C_n is called coadmissible if for each pair $(z, \bar{z})$ in C, the number $N^*(z)$ of letters x in C such that $x \geq z$ and $x \leq \bar{z}$ satisfies

$$N^*(z) \leq n - z + 1. \tag{13}$$

Let C be an admissible column of type C_n. Denote by C^* the column obtained by filling the shape of C (from top to bottom) with the unbarred letters of lC in increasing order followed by the barred letters of rC in increasing order. Then it is easy to prove that C^* is coadmissible. More precisely the map:

$$\Phi : C \to C^* \tag{14}$$

is a bijection between the sets of admissible and coadmissible columns of the same height. Starting from a coadmissible column C^* we can compute the pair (lC, rC) associated to the unique admissible column C such that $C^* = \Phi(C)$ by reversing the duplication algorithm in section 4.3. Then C is the column containing the unbarred letters of rC and the barred letters of lC.

Example 9.1.4.

$$\text{If } C = \begin{array}{|c|}\hline 1\\\hline 4\\\hline \overline{5}\\\hline \overline{4}\\\hline \overline{3}\\\hline\end{array}, \text{ then } (lC, rC) = \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline \overline{5}\\\hline \overline{4}\\\hline \overline{3}\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 4\\\hline \overline{5}\\\hline \overline{3}\\\hline \overline{2}\\\hline\end{array}\right) \text{ and } C^* = \begin{array}{|c|}\hline 1\\\hline 2\\\hline \overline{5}\\\hline \overline{3}\\\hline \overline{2}\\\hline\end{array}.$$

$$\text{If } D^* = \begin{array}{|c|}\hline 1\\\hline 2\\\hline \overline{2}\\\hline \overline{1}\\\hline\end{array}, \text{ then } (lD, rD) = \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline \overline{4}\\\hline \overline{3}\\\hline\end{array}, \begin{array}{|c|}\hline 3\\\hline 4\\\hline \overline{2}\\\hline \overline{1}\\\hline\end{array}\right) \text{ and } D = \begin{array}{|c|}\hline 3\\\hline 4\\\hline \overline{4}\\\hline \overline{3}\\\hline\end{array}.$$

9.1.3. *Elementary step of the sliding algorithm* . Let us consider an admissible punctured skew tableau T containing two columns C_1 and C_2 with the puncture in C_1. To apply an elementary step of the sliding algorithm to T we first have to consider the split form of T. In this split form we have a configuration of the type:

$$\begin{array}{|c|c|c|c|}\hline \cdots & \cdots & \cdots & \cdots\\\hline * & * & b & b'\\\hline a & a' & \cdots & \cdots\\\hline \cdots & \cdots & \multicolumn{2}{c}{}\\\cline{1-2}\end{array}$$ where the boxes containing a, a' and b, b' may be empty.

An elementary step of the Symplectic Jeu de Taquin (SJDT) consists of the following transformations:

(1) If $a' \leq b$ or the double box $b\ b'$ is empty, then the double boxes $a\ a'$ and $*\ *$ are permuted.
(2) If $a' > b$ or the double box $a\ a'$ is empty then:
   (i): when b is a barred letter, b slides into rC_1 to the box containing $*$ and $D_1 = \Phi(C_1) - \{*\} + \{b\}$ is a coadmissible column (see (14)). Simultaneously the symbol $*$ slides into lC_2 to the box containing b and $C_2' = C_2 - \{b\} + \{*\}$ is a punctured admissible column. Then we obtain a new punctured skew tableau $C_1'C_2'$ by setting $C_1' = \Phi^{-1}(D_1)$.
   (ii): when b is an unbarred letter, b slides into rC_1 to the box containing $*$ and give a new column $C_1' = C_1 - \{*\} + \{b\}$. Simultaneously the symbol $*$ slides into lC_2 to the box containing b and $D_2 = \Phi(C_2) - \{b\} + \{*\}$ is a punctured coadmissible column. Then we obtain a new punctured skew tableau $C_1'C_2'$ by setting $C_2' = \Phi^{-1}(D_2)$.

Remark: In case 2 (i) the coadmissibility of D_1 is not immediate and in case 2 (ii) the column C_1' may be not admissible.

Lemma 9.1.5. *We can always apply an elementary step of the* SJDT *to an admissible punctured skew tableau (i.e. D_1 is a coadmissible column in case* 2 (i)).

Example 9.1.6.

$$\text{For } T_1 = \begin{array}{|c|c|}\hline 2 & 4\\\hline 4 & 5\\\hline * & \overline{4}\\\hline \overline{3} & \overline{1}\\\hline \overline{1} & \multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ spl(T_1) = \begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 4\\\hline 4 & 4 & 5 & 5\\\hline * & * & \overline{4} & \overline{3}\\\hline \overline{3} & \overline{3} & \overline{1} & \overline{1}\\\hline \overline{1} & \overline{1} & \multicolumn{2}{c}{}\\\cline{1-2}\end{array}. \text{ We are in case 2 (i) and } C_1'C_2' = \begin{array}{|c|c|}\hline 2 & 4\\\hline 5 & 5\\\hline \overline{5} & *\\\hline \overline{3} & \overline{1}\\\hline \overline{1} & \multicolumn{1}{c}{}\\\cline{1-1}\end{array}.$$

$$\text{For } T_2 = \begin{array}{|c|c|c|}\hline 1 & * & 1\\\hline \overline{5} & \overline{1} & \multicolumn{1}{c}{}\\\cline{1-2}\end{array} \text{ we obtain } \begin{array}{|c|c|}\hline 1 & 1\\\hline \overline{5} & \overline{1}\\\hline\end{array} \text{ as in case 2 (ii).}$$

For $T_3 = \begin{array}{|c|c|c|}\hline 4 & * & 4 \\ \hline \bar{5} & \bar{4} & \bar{3} \\ \hline \end{array}$ *we obtain* $\begin{array}{|c|c|c|}\hline 4 & 4 & * \\ \hline \bar{5} & \bar{4} & \bar{3} \\ \hline \end{array}$.

The punctured skew tableau obtained by computing a step of the SJDT on an admissible punctured skew tableau is not always admissible. In the second example above the second column of the resulting tableau is not admissible. In the third the first row of the split form after sliding is not increasing (we will see that this last problem does not occur in the complete SJDT algorithm).

9.1.4. *Complete symplectic Jeu de Taquin (*SJDT*).* Let T be an admissible skew tableau and c an inner corner in T. In order to apply the complete sliding algorithm let us puncture the corner c. We obtain an admissible punctured skew tableau. To see what happens when we apply successively elementary steps of SJDT to this skew tableau, we need to compute the split form for each intermediate punctured tableau. We have seen that a horizontal move of an unbarred letter may give a new non admissible column C_1' such that all the strict factors of $\mathrm{w}(C_1')$ are admissible. So it is impossible to compute its split form using letters of $\mathcal{C}_n$. To overcome this problem, we embed the alphabet $\mathcal{C}_n$ into

$$\mathcal{C}_{n+1}' = \{a_1 < 1 < \cdots < n < \bar{n} < \cdots < \bar{1} < \overline{a_1}\}.$$

To compute the split form of a non admissible column C such that all the strict subwords of $\mathrm{w}(C)$ are admissible, we extend the duplication algorithm of section 4.3 by using the new letter a_1. For example

$$\text{if } C = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \bar{4} \\ \hline \bar{1} \\ \hline \end{array} \ (lC, rC) = \begin{array}{|c|c|}\hline a_1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline \bar{4} & \bar{1} \\ \hline \bar{1} & \overline{a_1} \\ \hline \end{array} \text{ in } \mathcal{C}_{n+1}'.$$

So all the columns that may be obtained when we apply an elementary step of SJDT to an admissible skew tableau (defined on $\mathcal{C}_n$) can be split in $\mathcal{C}_{n+1}'$. We say that a skew punctured tableau is a_1-admissible if all its columns can be split in $\mathcal{C}_{n+1}'$ and the rows of the obtained split form are weakly increasing.

Theorem 9.1.7. *(Sheats* [19]*)*

- *Elementary steps of* SJDT *can be applied to* T *until the puncture* $*$ *becomes an outside corner.*
- *All the skew punctured tableaux obtained as steps in the algorithm are* a_1*-admissible. Moreover* $\overline{a_1}$ *and* a_1 *only appear simultaneously in the split form of the column containing the inner corner* c *of* T *at which the slide started.*

Example 9.1.8. *Suppose* $T = \begin{array}{ccc} & * & 2 \\ & 2 & 3 \\ 3 & 3 & 4 \\ \bar{5} & \bar{4} & \bar{1} \\ \bar{3} & \bar{1} & \end{array}$, $spl(T) = \begin{array}{cccccc} & & * & * & 2 & 2 \\ & & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 & 4 & 4 \\ \bar{5} & \bar{5} & \bar{4} & \bar{4} & \bar{1} & \bar{1} \\ \bar{3} & \bar{2} & \bar{1} & \bar{1} & & \end{array}$. *We compute successively*

the split form of the a_1*-admissible punctured skew tableaux:*

$$\begin{array}{cccccc} & & 2 & 2 & 2 & 2 \\ & & * & * & 3 & 3 \\ 2 & 3 & 3 & 3 & 4 & 4 \\ \bar{5} & \bar{5} & \bar{4} & \bar{4} & \bar{1} & \bar{1} \\ \bar{3} & \bar{2} & \bar{1} & \bar{1} & & \end{array} \qquad \begin{array}{cccccc} & & 2 & 2 & 2 & 2 \\ & & 3 & 3 & 3 & 3 \\ 2 & 3 & * & * & 4 & 4 \\ \bar{5} & \bar{5} & \bar{4} & \bar{4} & \bar{1} & \bar{1} \\ \bar{3} & \bar{2} & \bar{1} & \bar{1} & & \end{array} \qquad \begin{array}{cccccc} & & a_1 & 2 & 2 & 2 \\ & & 2 & 3 & 3 & 3 \\ 2 & 3 & 3 & 4 & * & * \\ \bar{5} & \bar{5} & \bar{4} & \bar{1} & \bar{1} & \bar{1} \\ \bar{3} & \bar{2} & \bar{1} & \overline{a_1} & & \end{array}$$

$$\begin{array}{cccccc} & & a_1 & 2 & 2 & 2 \\ & & 2 & 3 & 3 & 3 \\ 2 & 3 & 3 & 4 & \bar{1} & \bar{1} \\ \bar{5} & \bar{5} & \bar{4} & \bar{1} & * & * \\ \bar{3} & \bar{2} & \bar{1} & \overline{a_1} & & \end{array}$$ *. Then we obtain the a_1-admissible skew tableau:* $$\begin{array}{ccc} & 2 & 2 \\ & 3 & 3 \\ 3 & 4 & \bar{1} \\ \bar{5} & \bar{4} & \\ \bar{3} & \bar{1} & \end{array}.$$

9.2. **Sliding algorithm on $\mathcal{C}_n$.** Let T be an admissible skew tableau and c be an inner corner. If we denote by T' the skew tableau obtained by applying the complete SJDT to T, then T' may be only a_1-admissible (see Theorem 9.1.7). Suppose that, in the split form, $\overline{a_1}$ and a_1 occur in the k-th split column $lC'_k\, rC'_k$ of T'. Then the column C'_k is not admissible. Write $\widetilde{\mathrm{w}(C'_k)}$ for the column word obtained by applying a contraction relation to $\mathrm{w}(C'_k)$. In order to obtain an admissible skew tableau, we are led to consider the skew tableau $\widetilde{T}'$ obtained by erasing the top and bottom boxes of C'_k and filling this new column with the letters of the word $\widetilde{\mathrm{w}(C'_k)}$. We denote this new column by $\widetilde{C}_k$.

Example 9.2.1. *Continuing the previous example we obtain:*

$$\widetilde{T} = \begin{array}{ccc} & & 2 \\ & 2 & 3 \\ 3 & 3 & \bar{1} \\ \bar{5} & \bar{1} & \\ \bar{3} & & \end{array}.$$

By using the notations introduced above, we have:

Proposition 9.2.2. *$\widetilde{T}'$ is an admissible skew tableau and* $\mathrm{w}(T') \equiv \mathrm{w}(\widetilde{T}')$.

Given an admissible skew tableau T and an inside corner c in T, we can apply elementary steps of SJDT to obtain a skew tableau T'. We set:

$$\mathrm{SJDT}(T,c) = \begin{cases} T' \text{ if } T' \text{ is admissible,} \\ \widetilde{T} \text{ if } T' \text{ is only } a_1\text{-admissible.} \end{cases}$$

During the algorithm an inner corner is filled or $\mathrm{SJDT}(T,c)$ has two boxes less than T. By choosing a new inner corner at each step, we can iterate the procedure $T \to \mathrm{SJDT}(T,c)$ to construct a symplectic tableau from any admissible skew tableau. In [14] we have proved that each elementary sliding operation can,be interpreted in terms of crystal isomorphisms, thus it is compatible with the plactic relations of Definition 5.1.2. So we obtain the following theorem:

Theorem 9.2.3. *Let T be an admissible skew tableau. Then by applying the* SJDT *successively to the inner corners of T, we obtain a symplectic tableau independent of the order in which these inner corners are filled. Moreover this tableau coincides with* $P^C(\mathrm{w}(T))$.

9.3. **Jeu de Taquin for type B_n.** Consider $\mathcal{C}_n = \{1 < \cdots < n < \bar{n} < \cdots < \bar{1}\} \subset \mathcal{B}_n$. The tableaux of type C_n can be regarded as tableaux of type B_n on the alphabet $\mathcal{C}_n$ instead of $\mathcal{B}_n$. Moreover for two words w_1 and w_2 of $\mathcal{C}_n^*$, we have:

$$w_1 \overset{C}{\equiv} w_2 \Longrightarrow w_1 \overset{B}{\equiv} w_2.$$

A skew tableau of type B_n is a skew Young diagram filled by letters of $\mathcal{B}_n$ whose columns are admissible of type B_n and the rows of its split form (obtained by splitting its columns) are weakly increasing from left to right.

Example 9.3.1. *For $n = 3$,*

$$T = \begin{array}{|c|c|c|}\hline & & 2 \\ \hline & 3 & 0 \\ \hline 0 & \bar{3} & \bar{1} \\ \hline 0 & & \\ \hline\end{array}$$ *is a skew orthogonal tableau of type* B_3 *because* $$\mathrm{spl}(T) = \begin{array}{|c|c|c|c|c|c|}\hline & & & & 2 & 2 \\ \hline & & 2 & 3 & 3 & \bar{3} \\ \hline 2 & \bar{3} & \bar{3} & \bar{2} & \bar{1} & \bar{1} \\ \hline 3 & \bar{2} & & & & \\ \hline\end{array}.$$

The relation $0\bar{n}n \equiv \bar{n}n0$ has no natural interpretation in terms of horizontal or vertical slidings in skew orthogonal tableaux. To overcome this problem we are going to work on the split form of the skew tableaux instead of the skew tableaux themselves, that is, we are going to obtain a Jeu de Taquin for type B_n by applying the symplectic Jeu de Taquin on the split form of the skew orthogonal tableaux of type B_n.

Lemma 9.3.2. *Let T and T' be two skew orthogonal tableaux of type B_n. Then:*

$$\mathrm{w}(T) \overset{B}{\equiv} \mathrm{w}(T') \Longleftrightarrow \mathrm{w}[\mathrm{spl}(T)] \overset{B}{\equiv} \mathrm{w}[\mathrm{spl}(T')].$$

If T is a skew orthogonal tableau of type B_n with r columns, then $\mathrm{spl}(T)$ is a symplectic skew tableau with $2r$ columns. We can apply the symplectic Jeu de Taquin to $\mathrm{spl}(T)$ to obtain a symplectic tableau $\mathrm{spl}(T)'$. We will have $\mathrm{w}[\mathrm{spl}(T)'] \overset{C}{\equiv} \mathrm{w}[\mathrm{spl}(T)]$ so $\mathrm{w}[\mathrm{spl}(T)'] \overset{B}{\equiv} \mathrm{w}[\mathrm{spl}(T)]$.

Proposition 9.3.3. *$\mathrm{spl}(T)'$ is the split form of the orthogonal tableau $P^B(T)$.*

The columns of the split form of a skew orthogonal tableau T of type B_n contain no letters 0 and no pairs of letters $(x, \bar{x})$ with $x \leq n$. In this particular case most of the elementary steps of the symplectic Jeu de Taquin applied on T are simple slidings identical to those of the original Jeu de Taquin of Lascoux and Schützenberger (complexity of the symplectic Jeu de Taquin are not needed in these slidings).

Example 9.3.4. *From* $$\mathrm{spl}\left(\begin{array}{|c|c|c|}\hline & 1 & 2 \\ \hline 1 & 0 & \bar{3} \\ \hline 3 & \bar{3} & \bar{2} \\ \hline\end{array}\right) = \begin{array}{|c|c|c|c|c|c|}\hline \ast & \ast & 1 & 1 & 1 & 2 \\ \hline 1 & 1 & 2 & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{3} & \bar{2} & \bar{2} & \bar{1} \\ \hline\end{array},$$ *we compute successively:*

$$\begin{array}{|c|c|c|c|c|c|}\hline \ast & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & \ast & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{3} & \bar{2} & \bar{2} & \bar{1} \\ \hline\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline \ast & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & \ast & \bar{2} & \bar{1} \\ \hline\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline \ast & 1 & 1 & 1 & 1 & 2 \\ \hline 1 & 2 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & \bar{2} & \ast & \bar{1} \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \ast & 1 & 1 & 1 & 2 & 2 \\ \hline 1 & 2 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & \bar{2} & \bar{2} & \ast \\ \hline\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & \ast & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & 3 & \bar{2} & \bar{2} & \bar{2} & \ast \\ \hline\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & \bar{2} & \ast & \bar{2} & \bar{2} & \ast \\ \hline\end{array},$$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & \bar{2} & \bar{2} & \ast & \bar{2} & \ast \\ \hline\end{array}, \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & \bar{3} & \bar{3} & \bar{3} & \bar{3} \\ \hline 3 & \bar{2} & \bar{2} & \bar{2} & \ast & \ast \\ \hline\end{array} = \mathrm{spl}\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & \bar{3} & \bar{3} \\ \hline 0 & \bar{2} & \\ \hline\end{array}\right).$$

Note that the sliding applied in the fourth duplicated tableau above is the unique sliding which is not identical to the original Jeu de Taquin step.

The split form of a skew orthogonal tableau of type D_n (defined in the same way as for type B_n) is still a symplectic skew tableau. But

$$w_1 \overset{C}{\equiv} w_2 \not\Longleftrightarrow w_1 \overset{D}{\equiv} w_2,$$

so we can not use the same idea to obtain a Jeu de Taquin for type D_n. Moreover one verifies for $n = 3$ that

$$P^D(3\bar{2}\bar{1}1\bar{3}) = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \bar{3} & \bar{2} \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \text{ and } P^D(\bar{3}\bar{2}\bar{1}1\bar{3}) = \begin{array}{|c|c|}\hline \bar{3} & \bar{3} \\ \hline 3 & \bar{2} \\ \hline \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

By interpreting the words $3\bar{2}\bar{1}1\bar{3}$ and $\bar{3}\bar{2}\bar{1}1\bar{3}$ as readings of skew tableaux, we obtain:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline \bar{3} & \bar{2} \\ \hline * & \bar{1} \\ \hline\end{array} \equiv \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \bar{3} & \bar{2} \\ \hline \bar{2} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \text{ and } \begin{array}{|c|c|}\hline 1 & \bar{3} \\ \hline \bar{3} & \bar{2} \\ \hline * & \bar{1} \\ \hline\end{array} \equiv \begin{array}{|c|c|}\hline \bar{3} & \bar{3} \\ \hline 3 & \bar{2} \\ \hline \bar{3} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

This shows that it is not enough to know what letter x slides from the second column C_2 to the first C_1 to be able to compute a horizontal sliding. Indeed the result depends on the whole column C_2. Thus, to give a combinatorial description of a sliding algorithm for type D_n would probably be very complicated.

References

[1] Baker, T. H.: *An insertion scheme for C_n crystals*, in M. Kashiwara and T. Miwa, eds., Physical Combinatorics, Birkhäuser, Boston, 2000, **191**: 1-48.

[2] Berele, A.: *A Schensted-type correspondence for the symplectic group*, J. of Comb Th (A), **43**, 320-328 (1986).

[3] De Concini, C.: *Symplectic standard tableaux*, Adv. in Math. **34**,1-27 (1979).

[4] Date, E., Jimbo, M. and Miwa, T.: *Representations of $U_q(gl(n,C))$ at $q = 0$ and the Robinson-Schensted correspondence*, in L. Brink, D. Friedman and A.M. Polyakov (eds), Physics and Mathematics of Strings, Word Scientific, Teaneck, NJ, 1990, pp. 185-211.

[5] Fulton, W.: *Young tableaux*, London Mathematical Society, Student Text **35**.

[6] Fulton, W, Harris, J.: *Representation theory*, Graduate Texts in Mathematics, Springer-Verlag.

[7] Kang, S.J, Misra, K.C.: *Crystal bases and tensor product decompositions of $U_q(G_2)$-modules*, Journal of Algebra, **163** (1994), 675-691.

[8] Kashiwara, M. and Nakashima T.: *Crystal graphs for representations of the q-analogue of classical Lie algebras*, J. Algebra, **165** (1994), 295-345.

[9] Kashiwara, M.: *On crystal bases*: Canadian Mathematical Society, Conference Proceedings, Volume **16** (1995).

[10] Kashiwara, M.: *Similarity of crystal bases: AMS Contemporary Math. 194, 177-186 (1996).*

[11] King, R. C.: *Weight Multiplicities for the Classical Groups*, Lectures Notes in Physics **50** (New York; Springer 1975) 490-499.

[12] Lascoux, A., Leclerc, B., Thibon, J.Y.: *Crystal graph and q-analogues of weight multiplicities for the roots system A_n^**, Lett. Math. Phys. **35**: 359-374, 1994.

[13] Lascoux, A. and Schützenberger, M. P.: *Le monoïde plaxique*, in non commutative structures in algebra and geometric combinatorics A. de Luca Ed., Quaderni della Ricerca Scientifica del C.N.R., Roma, 1981.

[14] Lecouvey, C.: *Schensted-type correspondence, plactic monoid and Jeu de Taquin for type C_n*: J. Algebra **247**, 295-331 (2002).

[15] Lecouvey, C.: *Schensted-type correspondences and plactic monoids for types B_n* and D_n, Journal of Algebraic Combinatorics, vol **18** n° **2**, 99-133 (2003).

[16] Lecouvey, C.: *Algorithmique et combinatoire des algèbres enveloppantes quantiques de type classique,* Thèse, Université de Caen, 2001.

[17] Littelmann P.: *A plactic algebra for semisimple Lie algebras*, Adv. in Math. **124**,312-331 (1996).

[18] Littelmann P.: *Crystal graph and Young tableaux, J.* Algebra, **175**, 65-87 (1995).

[19] Sheats J.T.: *A symplectic Jeu de Taquin bijection between the tableaux of King and of De Concini*, Trans. A.M.S. ,**351**, n°7, 3569-3607 (1999).

[20] Sundaram S.: *Orthogonal tableaux and an insertion scheme for SO_{2n+1}*, J. Combin. Theory, ser. A 53, *239-256 (1990).*

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