Advanced Studies in Pure Mathematics 76, 2018
Representation Theory, Special Functions and Painlevé Equations — RIMS 2015
pp. 125–153

Branching rules for symmetric hypergeometric polynomials

Jan Felipe van Diejen and Erdal Emsiz

Abstract.

Starting from a recently found branching rule for the six-parameter family of symmetric Macdonald-Koornwinder polynomials, we arrive by degeneration at corresponding branching formulas for symmetric hypergeometric orthogonal polynomials of Wilson, continuous Hahn, Jacobi, Laguerre, and Hermite type.

§1. Introduction

Let $M_\lambda(x_1,\ldots,x_n)$ with

$$\lambda \in \Lambda_n := \{(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n \mid \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0\}$$

denote the monomial basis of an algebra of symmetric (trigonometric) polynomials. We are concerned with branching rules for families of hypergeometric polynomials of the form

$$\begin{aligned}(1\text{a})\qquad P_\lambda(x_1,\ldots,x_n) =& M_\lambda(x_1,\ldots,x_n)\\ &+\sum_{\substack{\mu\in\Lambda_n\\ \mu<\lambda}} c_{\lambda,\mu}M_\mu(x_1,\ldots,x_n)\qquad (\lambda\in\Lambda_n),\end{aligned}$$

with $c_{\lambda,\mu}\in\mathbb{C}$ such that

$$\begin{aligned}(1\text{b})\qquad \int_{\mathcal{D}} P_\lambda(x_1,\ldots,x_n)\overline{M_\mu(x_1,\ldots,x_n)}\Delta(x_1,\ldots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n&\\ = 0 \quad \text{if} \quad \mu<\lambda.&\end{aligned}$$

Received October 20, 2015.
Revised April 1, 2016.
2010 *Mathematics Subject Classification.* 33C52, 05E05.
Key words and phrases. hypergeometric polynomials, symmetric functions, branching rules.

Here the partitions are partially ordered in accordance with the (nonhomogeneous) dominance order

$$\mu \leq \lambda \quad \text{iff} \quad \mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k \quad \text{for } k = 1, \ldots, n,$$

and the polynomial family is characterized by a Selberg-type orthogonality weight function $\Delta(x_1, \ldots, x_n)$ supported on $\mathcal{D} = [-\pi, \pi]^n$ (in the case of trigonometric polynomials) or $\mathcal{D} = \mathbb{R}^n$ (otherwise).

The branching rules under consideration are expansion formulas for the polynomials in $n+1$ variables in terms of the n-variable polynomials of the form

$$P_\lambda(x_1, \ldots, x_n, x) = \sum_{\substack{\mu \in \Lambda_n \\ \mu \preceq \lambda}} P_\mu(x_1, \ldots, x_n) P_{\lambda/\mu}(x) \qquad (\lambda \in \Lambda_{n+1}), \tag{2}$$

where $\mu \preceq \lambda$ iff there exists a $\nu \in \Lambda_n$ with $\mu \subset \nu \subset \lambda$ such that the skew diagrams λ/ν and ν/μ are horizontal strips. Here Λ_n is thought of as being embedded in Λ_{n+1} 'by adding a part of size zero', and we recall that for $\lambda, \mu \in \Lambda_n$ one has that $\mu \subset \lambda$ iff $\mu_j \leq \lambda_j$ for $j = 1, \ldots, n$, while the corresponding skew diagram λ/μ is a horizontal strip provided the parts of λ and μ interlace as follows:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_n \geq \mu_n.$$

By iterating the branching rule (2)

$$P_\lambda(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+l}) = \sum_{\substack{\mu^{(n+i)} \in \Lambda_{n+i}, i=0,\ldots,l \\ \mu^{(n)} \preceq \mu^{(n+1)} \preceq \cdots \preceq \mu^{(n+l)} = \lambda}} P_{\mu^{(n)}}(x_1, \ldots, x_n) \prod_{1 \leq i \leq l} P_{\mu^{(n+i)}/\mu^{(n+i-1)}}(x_{n+i}), \tag{3}$$

one can build the polynomials in n variables

$$P_\lambda(x_1, \ldots, x_n) = \sum_{\substack{\mu^{(i)} \in \Lambda_i, i=1,\ldots,n \\ \mu^{(1)} \preceq \mu^{(2)} \preceq \cdots \preceq \mu^{(n)} = \lambda}} P_{\mu^{(1)}}(x_1) \prod_{1 < i \leq n} P_{\mu^{(i)}/\mu^{(i-1)}}(x_i) \tag{4}$$

starting from the corresponding one-variable polynomials $P_m(x)$, $m = 0, 1, 2, \ldots$.

To make the above construction effective, it suffices to determine the branching polynomials $P_{\lambda/\mu}(x)$ in Eq. (2) explicitly. In [DE3] this was realized for the symmetric Macdonald-Koornwinder polynomials [K1, M3], which for $n = 1$ amount to the Askey-Wilson polynomials [KLS]. Our present aim is to degenerate from the Askey-Wilson level and identify the pertinent branching polynomials needed for the explicit construction of the symmetric hypergeometric polynomials of Wilson type [D2, D5, Z, G], continuous Hahn type [D2, D5], Jacobi type [V, HO, H, M1, D, L1, BO, BF, DLM, Z, OO2, DES, SV, HL, K2], Laguerre type [M1, L2, BF, D4, X, ADO, DES, HL, BB, Ol], and Hermite type [M1, L3, BF, D4, X, DES, WNU, HL], respectively. Hence, we concentrate on hypergeometric families that (i) are obtained from the Askey-Wilson level via limit transitions rather than parameter specializations and (ii) are endowed with a *continuous* orthogonality measure.

For all these families it turns out that the relevant branching polynomials can be conveniently written in terms of expansion coefficients $C_{\lambda,r}^{\mu,n}$ arising from Pieri formulas [D3, D5, D4]

$$E_r(x_1,\ldots,x_n)P_\lambda(x_1,\ldots,x_n) = \sum_{\substack{\mu\in\Lambda_n\\ \mu\sim_r\lambda}} C_{\lambda,r}^{\mu,n} P_\mu(x_1,\ldots,x_n) \qquad (r=1,\ldots,n), \tag{5}$$

associated with a suitable choice of generators

$$E_1(x_1,\ldots,x_n),\ldots,E_n(x_1,\ldots,x_n)$$

for our algebra of (trigonometric) symmetric polynomials. The nonvanishing expansion coefficents on the RHS of Eq. (5) are in these cases governed by the following proximity relation within Λ_n: $\mu\sim_r\lambda$ iff there exists a partition $\nu\in\Lambda_n$ with $\nu\subset\lambda$ and $\nu\subset\mu$ such that the skew diagrams λ/ν and μ/ν are vertical strips with $|\lambda/\nu|+|\mu/\nu|\leq r$. Here $|\cdot|$ denotes the number of boxes of the diagram and (recall) the skew diagram λ/ν is a vertical strip iff $\nu_j\leq\lambda_j\leq\nu_j+1$ $(j=1,\ldots,n)$.

After recalling—in Section 2—the explicit branching polynomials from [DE3] that enable the recursive construction of the Macdonald-Koornwinder polynomials starting from the one-variable Askey-Wilson polynomials, we will work our way down Askey's scheme and provide the corresponding branching polynomials for the Wilson level (Section 3), the continuous Hahn level (Section 4), the Jacobi level (Section 5), the Laguerre level (Section 6), and the Hermite level (Section 7). At the bottom level of the symmetric Hermite polynomials, to date only special instances of the corresponding Pieri coefficients are available in closed form in the literature. Our proof of the branching formula mimics in this situation the proof from [DE3] for the Macdonald-Koornwinder case and relies on the Hermite degeneration of a (dual) Cauchy identity due to Mimachi detailed in the appendix at the end of the paper.

Notation. i) For future reference, we associated to any pair of partitions $\lambda, \mu \in \Lambda_n$ the following subsets of $\{1, \ldots, n\}$:

$$\begin{aligned} J = J(\lambda,\mu) &:= \{1 \le j \le n \mid \lambda_j \neq \mu_j\}, \\ J^c = J^c(\lambda,\mu) &:= \{1 \le j \le n \mid \lambda_j = \mu_j\}, \\ J_+ = J_+(\lambda,\mu) &:= \{1 \le j \le n \mid \mu_j > \lambda_j\}, \\ J_- = J_-(\lambda,\mu) &:= \{1 \le j \le n \mid \mu_j < \lambda_j\}, \end{aligned}$$

and we also define

$$\epsilon_j = \epsilon_j(J_+, J_-) := \begin{cases} 1 & \text{if } j \in J_+, \\ -1 & \text{if } j \in J_-, \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate from these definitions that if $\mu \sim_r \lambda$, then the cardinality $|J|$ of $J = J(\lambda, \mu)$ is at most r and $\epsilon_j = \mu_j - \lambda_j$ for $j = 1, \ldots, n$.

ii) Following standard conventions, shifted factorials and their q-versions are denoted by:

$$(a)_k := (a)(a+1)\cdots(a+k-1), \ (a;q)_k := (1-a)(1-aq)\cdots(1-aq^{k-1}),$$

with $(a)_0 = (a;q)_0 := 1$, and

$$(a_1, \ldots, a_l)_k := (a_1)_k \cdots (a_l)_k, \quad (a_1, \ldots, a_l; q)_k := (a_1;q)_k \cdots (a_l;q)_k.$$

iii) Finally, we will employ the (principal specialization) vectors $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ with components given by

$$\tau_j = t^{n-j} t_0, \qquad \hat{\tau}_j = t^{n-j} \hat{t}_0$$

and

$$\rho_j = (n-j)g + g_0, \qquad \hat{\rho}_j = (n-j)g + \hat{g}_0$$

$(j = 1, \ldots, n)$, where t, t_0, $\hat{t}_0$ and g, g_0, $\hat{g}_0$ denote parameters to be specified below.

§2. Askey-Wilson level

2.1. Symmetric Macdonald-Koornwinder polynomials [K1, M3]

The symmetric Macdonald-Koornwinder polynomials are trigonometric polynomials

$$P_\lambda(x_1, \ldots, x_n) = P_\lambda(x_1, \ldots, x_n; q, t, t_l) \qquad (\lambda \in \Lambda_n) \tag{6}$$

determined by the properties in Eqs. (1a), (1b), with

$$M_\lambda(x_1,\ldots,x_n) = m_\lambda(e^{ix_1}+e^{-ix_1},\ldots,e^{ix_n}+e^{-ix_n}),$$

$$m_\lambda(z_1,\ldots,z_n) := \sum_{\nu\in S_n(\lambda)} z_1^{\nu_1}\cdots z_n^{\nu_n}$$

(where the sum is over the orbit of λ with respect to the action of the symmetric group S_n), and

(7)

$$\begin{aligned}&\Delta(x_1,\ldots,x_n) = \Delta(x_1,\ldots,x_n;q,t,t_l)\\ &:= \prod_{1\leq j\leq n}\left|\frac{(e^{2ix_j};q)_\infty}{\prod_{0\leq l\leq 3}(t_le^{ix_j};q)_\infty}\right|^2 \prod_{1\leq j<k\leq n}\left|\frac{(e^{i(x_j+x_k)},e^{i(x_j-x_k)};q)_\infty}{(te^{i(x_j+x_k)},te^{i(x_j-x_k)};q)_\infty}\right|^2\end{aligned}$$

supported on $\mathcal{D} = [-\pi,\pi]^n$. Here it is assumed that the parameters belong to the domain $0<q,|t|,|t_l|<1$, with t being real and possibly non-real parameters t_l ($l=0,1,2,3$) occurring in complex conjugate pairs.

2.2. Pieri coefficients [D3, D5, Sa]

Let $\boldsymbol{e}_m(z_1,\ldots,z_n)$ and $\boldsymbol{h}_m(z_1,\ldots,z_n)$ denote the elementary and the complete symmetric polynomials of degree m:

$$\boldsymbol{e}_m(z_1,\ldots,z_n) := \sum_{1\leq i_1<\cdots<i_m\leq n} z_{i_1}\cdots z_{i_m} = \sum_{\substack{I\subset\{1,\ldots,n\}\\ |I|=m}}\prod_{i\in I} z_i,$$

$$\boldsymbol{h}_m(z_1,\ldots,z_n) := \sum_{1\leq i_1\leq\cdots\leq i_m\leq n} z_{i_1}\cdots z_{i_m},$$

with the convention that $\boldsymbol{e}_0=\boldsymbol{h}_0\equiv 1$, and let

$$\hat{t}_0^2 = q^{-1}t_0t_1t_2t_3, \qquad \hat{t}_0\hat{t}_l = t_0t_l \quad (l=1,2,3).$$

The Pieri coefficients $C_{\lambda,r}^{\mu,n}$ (5) for the Macdonald-Koornwinder polynomials associated with the multiplication of $P_\lambda(x_1,\ldots,x_n;q,t,t_l)$ by

(8) $$E_r(x_1,\ldots,x_n) = E_r(x_1,\ldots,x_n;t,t_0) :=$$

$$\sum_{0\leq m\leq r}(-1)^{r+m}\boldsymbol{e}_m(e^{ix_1}+e^{-ix_1},\ldots,e^{ix_n}+e^{-ix_n})\boldsymbol{h}_{r-m}(\tau_r+\tau_r^{-1},\ldots,\tau_n+\tau_n^{-1})$$

are given by

$$
\begin{aligned}
(9)\qquad C^{\mu,n}_{\lambda,r} &= C^{\mu,n}_{\lambda,r}(q,t,t_l)\\
&:= \frac{p_\lambda(q,t,t_l)}{p_\mu(q,t,t_l)} V^n_{J_+,J_-}(\lambda;q,t,t_l) U^n_{J^c,r-|J|}(\lambda;q,t,t_l),
\end{aligned}
$$

where

$$
p_\lambda(q,t,t_l) := \prod_{1\le j\le n} \frac{\prod_{0\le l\le 3}(\hat{t}_l\hat{\tau}_j;q)_{\lambda_j}}{\tau_j^{\lambda_j}(\hat{\tau}_j^2;q)_{2\lambda_j}} \prod_{1\le j<k\le n} \frac{(t\hat{\tau}_j\hat{\tau}_k;q)_{\lambda_j+\lambda_k}(t\hat{\tau}_j\hat{\tau}_k^{-1};q)_{\lambda_j-\lambda_k}}{(\hat{\tau}_j\hat{\tau}_k;q)_{\lambda_j+\lambda_k}(\hat{\tau}_j\hat{\tau}_k^{-1};q)_{\lambda_j-\lambda_k}}
$$

(which corresponds to the principal specialization value of the polynomial $P_\lambda(x_1,\ldots,x_n;q,t,t_l)$),

$$
\begin{aligned}
V^n_{J_+,J_-}(\lambda;q,t,t_l) := \prod_{j\in J} &\frac{\prod_{0\le l\le 3}(1-\hat{t}_l\hat{\tau}_j^{\epsilon_j}q^{\epsilon_j\lambda_j})}{t_0(1-\hat{\tau}_j^{2\epsilon_j}q^{2\epsilon_j\lambda_j})(1-\hat{\tau}_j^{2\epsilon_j}q^{2\epsilon_j\lambda_j+1})}\\
&\times \prod_{\substack{j,j'\in J\\ j<j'}} \frac{(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}})(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}+1})}{t(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}})(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}+1})}\\
&\times \prod_{j\in J,k\in J^c} \frac{(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k q^{\epsilon_j\lambda_j+\lambda_k})(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k^{-1}q^{\epsilon_j\lambda_j-\lambda_k})}{t(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k q^{\epsilon_j\lambda_j+\lambda_k})(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k^{-1}q^{\epsilon_j\lambda_j-\lambda_k})}
\end{aligned}
$$

with $\epsilon_j \equiv \epsilon_j(J_+,J_-)$, and

$$
\begin{aligned}
U^n_{K,p}(\lambda;q,t,t_l) := (-1)^p \sum_{\substack{I_+,I_-\subset K\\ I_+\cap I_-=\emptyset\\ |I_+|+|I_-|=p}} \Biggl(&\prod_{j\in I} \frac{\prod_{0\le l\le 3}(1-\hat{t}_l\hat{\tau}_j^{\epsilon_j}q^{\epsilon_j\lambda_j})}{t_0(1-\hat{\tau}_j^{2\epsilon_j}q^{2\epsilon_j\lambda_j})(1-\hat{\tau}_j^{2\epsilon_j}q^{2\epsilon_j\lambda_j+1})}\\
&\times \prod_{\substack{j,j'\in I\\ j<j'}} \frac{(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}})(1-t^{-1}\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}+1})}{(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}})(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_{j'}^{\epsilon_{j'}}q^{\epsilon_j\lambda_j+\epsilon_{j'}\lambda_{j'}+1})}\\
&\times \prod_{j\in I,k\in K\setminus I} \frac{(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k q^{\epsilon_j\lambda_j+\lambda_k})(1-t\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k^{-1}q^{\epsilon_j\lambda_j-\lambda_k})}{t(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k q^{\epsilon_j\lambda_j+\lambda_k})(1-\hat{\tau}_j^{\epsilon_j}\hat{\tau}_k^{-1}q^{\epsilon_j\lambda_j-\lambda_k})}\Biggr)
\end{aligned}
$$

with $\epsilon_j \equiv \epsilon_j(I_+,I_-)$, $I = I_+\cup I_-$ and $p = 0,\ldots,|K|$. In these formulas it is assumed—by convention—that $V^n_{J_+,J_-}(\lambda;q,t,t_l) = 1$ if J is empty and that $U^n_{K,p}(\lambda;q,t,t_l) = 1$ if $p = 0$; furthermore, the index sets $J = J(\lambda,\mu)$, $J^c = J^c(\lambda,\mu)$ and $J_\pm = J_\pm(\lambda,\mu)$, the signs $\epsilon_j = \epsilon_j(J_+,J_-)$, and the principal specialization vectors τ_j, $\hat{\tau}_j$ ($j = 1,\ldots,n$), are all defined in accordance with the conventions detailed at the end of the introduction.

Remark 1. *The polynomials* $E_r(x_1, \ldots, x_n; t, t_0)$ (8) *first arose in* [D1] *as eigenvalues of a family of commuting difference operators diagonalized by the Macdonald-Koornwinder polynomials. The explicit formula in Eq.* (8) *was derived in Section 3.3 and Appendix B of* [D1] *by means of certain vanishing properties* (*cf. also Lemmas 3.1 and 4.2 of* [I]). *Via Macdonald's duality symmetry* [D3, Sa], *the polynomials in question then entered as multiplying elements in the Pieri formulas for the Macdonald-Koornwinder polynomials* (*cf.* [D3, Thm 2] *and* [D5, Sec. 6.1]). *In* [KNS, Sec. 5] *a different representation of* $E_r(x_1, \ldots, x_n; t, t_0)$ (8) *was found and it was observed that this polynomial amounts in fact to a special instance of Okounkov's hyperoctahedral interpolation polynomial* [O, R] *corresponding to the partition consisting of a single column of size* r. *While we used the representation of* $E_r(x_1, \ldots, x_n; t, t_0)$ *from* [KNS, Eq. (5.1)] *when recalling the Pieri formula for the Macdonald-Koornwinder polynomials in* [DE3], *here we have found it convenient to return to the original expression of the type in* [D5, Eq. (5.6)]. *The equivalence of both representations for* $E_r(x_1, \ldots, x_n; t, t_0)$ *is seen by comparing the recurrences in* [D1, Lem. B.2] *and* [KNS, Eq. (5.5)].

2.3. Branching polynomials [DE3]

For $\lambda \in \Lambda_{n+1}$ and $\mu \in \Lambda_n$ with $\mu \preceq \lambda$, the branching polynomial $P_{\lambda/\mu}(x) = P_{\lambda/\mu}(x; q, t, t_l)$—which arises as the expansion coefficient for the Macdonald-Koornwinder polynomial in $(n+1)$ variables in terms of the n-variable polynomials (2)—is given explicitly by

$$\boxed{P_{\lambda/\mu}(x; q, t, t_l) = \sum_{0 \leq k \leq d} B^k_{\lambda/\mu}(q, t, t_l) \langle x; t_0 \rangle_{q,k}} \tag{10a}$$

where

$$\langle x; t_0 \rangle_{q,k} := \prod_{1 \leq j \leq k} (e^{ix} + e^{-ix} - q^{j-1} t_0 - q^{-(j-1)} t_0^{-1})$$

(with $\langle x; t_0 \rangle_{q,0} := 1$),

$$d = d(\lambda, \mu) := |\{1 \leq j \leq m \mid \lambda'_j = \mu'_j + 1\}|,$$

and
(10b)

$$\boxed{B^k_{\lambda/\mu}(q, t, t_l) = (-1)^{k+|\lambda|-|\mu|} C^{(n+1)^m - \lambda', m}_{n^m - \mu', m-k}(t, q, t_l) \quad (k = 0, \ldots, d).}$$

Here $|\lambda| = \lambda_1 + \cdots + \lambda_n$, $m = \ell(\lambda')$ $(= \lambda_1)$, and λ' $(\in \Lambda_m)$ denotes the conjugate partition of λ, while $m^n - \mu$ with $\mu \subset m^n$ stands for the partition such that $(m^n - \mu)_j = m - \mu_{n+1-j}$ $(j = 1, \ldots, n)$. The formulas in Eqs. (10a), (10b) reveal that the coefficients of the branching polynomial are given by Pieri coefficients stemming from the multiplication of dual ($q \leftrightarrow t$, $\lambda \leftrightarrow n^m - \lambda'$) Macdonald-Koornwinder polynomials by corresponding one-column interpolation polynomials $E_r(x_1, \ldots, x_n; q, t_0)$.

Apart from exploiting the above Pieri formulas, the proof in [DE3] of this branching rule is based on Mimachi's Cauchy identity for the Macdonald-Koornwinder polynomials [Mi, Thm. 2.1] (cf. Eq. (47) below) as well as on a special 'column-row' case [KNS, Lem. 5.1] of the Cauchy identity for Okounkov's hyperoctahedral interpolation polynomials [O, Thm. 6.2] with shifted variables in accordance with [R, Thm. 3.16] (cf. Remark 4 at the end of Section 7 below).

Remark 2. *It is known that the highest-degree leading homogeneous terms of the Macdonald-Koornwinder polynomial $P_\lambda(x_1, \ldots, x_n; q, t, t_l)$ consist of the (monic) Macdonald polynomial $P_\lambda(x_1, \ldots, x_n; q, t)$* [D1, §5.2]. *This ties in with the recent observation in* [K2, Rem. 6.1] *that—by filtering the terms of leading degree on both sides of the above branching formula for the Macdonald-Koornwinder polynomials—a celebrated branching rule for the Macdonald polynomials* [M2, LW, Su] *is recovered:*

$$P_\lambda(x_1, \ldots, x_n, x; q, t) = \sum_{\substack{\mu \in \Lambda_n,\, \mu \subset \lambda \\ \lambda/\mu \text{ horizontal strip}}} P_\mu(x_1, \ldots, x_n; q, t) P_{\lambda/\mu}(x; q, t)$$

$(\lambda \in \Lambda_{n+1})$, *where*

$$P_{\lambda/\mu}(x; q, t) = e^{ix(|\lambda| - |\mu|)} B_{\lambda/\mu}(q, t)$$

and

$$\begin{aligned} B_{\lambda/\mu}(q,t) =& B^d_{\lambda/\mu}(q, t, t_l) = C^{(n+1)^m - \lambda', m}_{n^m - \mu', m-d}(t, q, t_l) \quad \textit{with } d = |\lambda| - |\mu| \\ =& \frac{p_{n^m - \mu'}(t, q, t_l)}{p_{(n+1)^m - \lambda'}(t, q, t_l)} V^m_{J(n^m - \mu', (n+1)^m - \lambda'), \emptyset}(n^m - \mu'; t, q, t_l) \\ =& \prod_{1 \le j < k \le m} \frac{(q^{1+k-j}; t)_{\mu'_j - \mu'_k}}{(q^{k-j}; t)_{\mu'_j - \mu'_k}} \frac{(q^{k-j}; t)_{\lambda'_j - \lambda'_k}}{(q^{1+k-j}; t)_{\lambda'_j - \lambda'_k}} \\ & \times \prod_{\substack{1 \le j,k \le m \\ \mu'_j \ne \lambda'_j \\ \mu'_k = \lambda'_k}} \left(\frac{1 - q^{1+k-j} t^{\mu'_j - \mu'_k}}{1 - q^{k-j} t^{\mu'_j - \mu'_k}} \right) \prod_{\substack{1 \le j < k \le m \\ \mu'_j = \lambda'_j \\ \mu'_k \ne \lambda'_k}} q^{-1} \end{aligned}$$

$$= \prod_{\substack{1\le j<k\le m\\ \mu'_j=\lambda'_j\\ \mu'_k\neq\lambda'_k}} \left(\frac{1-q^{1+k-j}t^{\lambda'_j-\lambda'_k}}{1-q^{k-j}t^{\lambda'_j-\lambda'_k}}\right)\left(\frac{1-q^{-1+k-j}t^{\mu'_j-\mu'_k}}{1-q^{k-j}t^{\mu'_j-\mu'_k}}\right)$$

$$\stackrel{*}{=} \prod_{1\le j\le k\le \ell(\mu)} \frac{(q^{\mu_j-\mu_k}t^{1+k-j}, q^{1+\mu_j-\lambda_{k+1}}t^{k-j};q)_{\lambda_j-\mu_j}}{(q^{1+\mu_j-\mu_k}t^{k-j}, q^{\mu_j-\lambda_{k+1}}t^{1+k-j};q)_{\lambda_j-\mu_j}}$$

* *cf. Eq. (6.13), Rem. 2., and Example 2.(b) of* [M2, Ch. VI.6].

2.4. Whittaker limit ($t\to 0$)

For $t\to 0$ the Macdonald-Koornwinder polynomial degenerates into a deformed q-Whittaker function $P_\lambda(x_1,\ldots,x_n;q,0,t_l)$ that diagonalizes Ruijsenaars's q-difference Toda chain with one-sided integrable boundary interactions of Askey-Wilson type [DE1]. The corresponding branching polynomial for $P_\lambda(x_1,\ldots,x_n;q,0,t_l)$—obtained from Eq. (10a) in the limit $t\to 0$—is given by

$$\text{(11a)}\qquad \boxed{P_{\lambda/\mu}(x;q,0,t_l)=\sum_{0\le k\le d} B^k_{\lambda/\mu}(q,0,t_l)\langle x;t_0\rangle_{q,k}}$$

with coefficients

$$\text{(11b)}\qquad \boxed{B^k_{\lambda/\mu}(q,0,t_l)=(-1)^{k+|\lambda|-|\mu|}C^{(n+1)^m-\lambda',m}_{n^m-\mu',m-k}(0,q,t_l)}$$

governed by the Pieri coefficients for the Macdonald-Koornwinder polynomials at $q=0$:

$$\text{(12)}\qquad C^{\mu,n}_{\lambda,r}(0,t,t_l):=\frac{p_\lambda(0,t,t_l)}{p_\mu(0,t,t_l)}V^n_{J_+,J_-}(\lambda;0,t,t_l)U^n_{J^c,r-|J|}(\lambda;0,t,t_l).$$

These Pieri coefficients are given explicitly by [DE2]:

$$p_\lambda(0,t,t_l):=\prod_{\substack{1\le j\le n\\ \lambda_j>0}}\Bigl(\tau_j^{-\lambda_j}\prod_{1\le l\le 3}(1-t_0t_lt^{n-j})\Bigr)$$
$$\times\prod_{\substack{1\le j\le n\\ \lambda_j=1}}(1-t_0t_1t_2t_3t^{n-j+m_0(\lambda)})^{-1}\prod_{\substack{1\le j<k\le n\\ \lambda_j>\lambda_k}}\frac{1-t^{1+k-j}}{1-t^{k-j}},$$

$$V^n_{J_+,J_-}(\lambda;0,t,t_l) :=$$
$$\prod_{\substack{j\in J_+\\ \lambda_j=0}} \frac{(1-t_0t_1t_2t_3t^{n-j+m_0(\lambda)+m_1(\lambda)-m_1^+(\lambda)})\prod_{1\leq l\leq 3}(1-t_0t_lt^{n-j})}{(1-t_0t_1t_2t_3t^{2(n-j)})(1-t_0t_1t_2t_3t^{2(n-j)+1})}$$
$$\times \prod_{\substack{j\in J_-\\ \lambda_j=1}} \frac{(1-t_0t_1t_2t_3t^{n-j-1})\prod_{1\leq l<m\leq 3}(1-t_lt_mt^{n-j})}{(1-t_0t_1t_2t_3t^{2(n-j)})(1-t_0t_1t_2t_3t^{2(n-j)-1})}$$
$$\times \prod_{\substack{j\in J_+\\ \lambda_j=1}} (1-t_0t_1t_2t_3t^{n-j+m_0(\lambda)}) \prod_{\substack{1\leq j<k\leq n\\ \lambda_j=\lambda_k,\epsilon_j>\epsilon_k}} \frac{1-t^{1+k-j}}{1-t^{k-j}} \prod_{1\leq j\leq n} \tau_j^{-\epsilon_j},$$

with $\epsilon_j \equiv \epsilon_j(J_+,J_-)$, and

$$U^n_{K,p}(\lambda;0,t,t_l) := (-1)^p \times$$
$$\sum_{\substack{I_+,I_-\subset K\\ I_+\cap I_-=\emptyset\\ |I_+|+|I_-|=p}} \Biggl(\prod_{\substack{j\in I_+\\ \lambda_j=0}} \frac{\prod_{1\leq l\leq 3}(1-t_0t_lt^{n-j})}{1-t_0t_1t_2t_3t^{2(n-j)}} \prod_{\substack{j\in I_+\\ \lambda_j=1}} (1-t_0t_1t_2t_3t^{n-j})$$
$$\times \prod_{\substack{j\in I_-\\ \lambda_j=1}} \frac{\prod_{1\leq l<m\leq 3}(1-t_lt_mt^{n-j})}{1-t_0t_1t_2t_3t^{2(n-j)}} \prod_{\substack{j,k\in K\\ \lambda_j=\lambda_k,\epsilon_j>\epsilon_k}} \frac{1-t^{1+k-j}}{1-t^{k-j}} \prod_{\substack{j\in I_-,\, k\in I_+\\ \lambda_j=\lambda_k+1}} \frac{1-t^{1+k-j}}{1-t^{k-j}}$$
$$\times \prod_{\substack{j,k\in K, j<k\\ \epsilon_j+\epsilon_k\in\{-2,1,2\}\\ \lambda_j=1,\lambda_k=\delta_{1+\epsilon_k}}} \frac{1-t_0t_1t_2t_3t^{2n+1-j-k}}{1-t_0t_1t_2t_3t^{2n-j-k}} \prod_{\substack{j\in I_+\cup I_-,k\in K\setminus I_-\\ j<k,\epsilon_k-\epsilon_j\in\{0,1\}\\ \lambda_j=\delta_{1+\epsilon_j},\lambda_k=0}} \frac{1-t_0t_1t_2t_3t^{2n-1-j-k}}{1-t_0t_1t_2t_3t^{2n-j-k}}$$
$$\times \prod_{j\in K} t_0^{-\epsilon_j} \prod_{\substack{j,k\in K,\, j<k\\ \epsilon_j\neq\epsilon_k=0}} t^{-\epsilon_j} \prod_{\substack{j,k\in K,\, j<k\\ \lambda_j=\lambda_k,\, \epsilon_k-\epsilon_j=1}} t^{-1} \Biggr),$$

with $\epsilon_j \equiv \epsilon_j(I_+,I_-)$, where we have employed the additional notation $m_l(\lambda) := |\{1\leq j\leq n \mid \lambda_j = l\}|$, $m_l^+(\lambda) := |\{j\in J_+ \mid \lambda_j = l\}|$, and $\delta_m := 1$ if $m=0$ and $\delta_m := 0$ otherwise.

Remark 3. *It is evident from these explicit Pieri coefficients that the branching polynomial* $P_{\lambda/\mu}(x;q,0,t_l)$ (11a) *corresponding to the deformed q-Whittaker function* $P_\lambda(x_1,\ldots,x_n;q,0,t_l)$ *simplifies considerably when one or more of the parameters* t_1,t_2 *or* t_3 *vanish. From the perspective of Ruijsenaars' q-difference Toda chain, such parameter reductions amount to a degenerations of the interaction at the boundary* [DE1, Sec. 7].

§3. Wilson level

3.1. Symmetric Wilson polynomials [D2, D5, G]

The symmetric Wilson polynomials are even polynomials

$$P_\lambda(x_1,\ldots,x_n) = P_\lambda^W(x_1,\ldots,x_n;g,g_l) \qquad (\lambda\in\Lambda_n) \tag{13}$$

of the form in Eqs. (1a), (1b), with

$$M_\lambda(x_1,\ldots,x_n) = m_\lambda(x_1^2,\ldots,x_n^2)$$

and

$$\begin{aligned}\Delta(x_1,\ldots,x_n) =&\Delta^W(x_1,\ldots,x_n;g,g_l) := \\ \prod_{1\le j\le n}\left|\frac{\prod_{0\le l\le 3}\Gamma(g_l+ix_j)}{\Gamma(2ix_j)}\right|^2 &\prod_{1\le j<k\le n}\left|\frac{\Gamma(g+i(x_j+x_k))\Gamma(g+i(x_j-x_k))}{\Gamma(i(x_j+x_k))\Gamma(i(x_j-x_k))}\right|^2\end{aligned} \tag{14}$$

supported on $\mathcal{D}=\mathbb{R}^n$. Here $\Gamma(\cdot)$ denotes the gamma function and it is assumed that $g,\mathrm{Re}(g_l)>0$, with possibly non-real parameters g_l occurring in complex conjugate pairs ($l=0,1,2,3$).

3.2. Pieri coefficients [D5]

Let

$$\hat g_0 = \frac{1}{2}(g_0+g_1+g_2+g_3-1) \quad\text{and}\quad \hat g_0+\hat g_l = g_0+g_l \quad (l=1,2,3).$$

The Pieri coefficients $C_{\lambda,r}^{\mu,n}$ (5) for the symmetric Wilson polynomials associated with the multiplication of $P_\lambda^W(x_1,\ldots,x_n;g,g_l)$ by

$$\begin{aligned}E_r(x_1,\ldots,x_n) &= E_r^W(x_1,\ldots,x_n;g,g_0)\\ &:= (-1)^r\sum_{0\le m\le r}\boldsymbol{e}_m(x_1^2,\ldots,x_n^2)\boldsymbol{h}_{r-m}(\rho_r^2,\ldots,\rho_n^2)\end{aligned} \tag{15}$$

are given by

$$C_{\lambda,r}^{\mu,n} = C_{\lambda,r}^{W,\mu,n}(g,g_l) := \frac{p_\lambda^W(g,g_l)}{p_\mu^W(g,g_l)}V_{J_+,J_-}^{W,n}(\lambda;g,g_l)U_{J^c,r-|J|}^{W,n}(\lambda;g,g_l), \tag{16}$$

where

$$p_\lambda^W(g,g_l) :=$$
$$(-1)^{|\lambda|}\prod_{1\le j\le n}\frac{\prod_{0\le l\le 3}(\hat g_l+\hat\rho_j)_{\lambda_j}}{(2\hat\rho_j)_{2\lambda_j}}\prod_{1\le j<k\le n}\frac{(g+\hat\rho_j+\hat\rho_k)_{\lambda_j+\lambda_k}(g+\hat\rho_j-\hat\rho_k)_{\lambda_j-\lambda_k}}{(\hat\rho_j+\hat\rho_k)_{\lambda_j+\lambda_k}(\hat\rho_j-\hat\rho_k)_{\lambda_j-\lambda_k}},$$

$$V^{W,n}_{J_+,J_-}(\lambda;g,g_l) := \prod_{j\in J} \frac{\prod_{0\leq l\leq 3}(\hat{g}_l+\epsilon_j(\hat{\rho}_j+\lambda_j))}{2\epsilon_j(\hat{\rho}_j+\lambda_j)(1+2\epsilon_j(\hat{\rho}_j+\lambda_j))}$$
$$\times \prod_{\substack{j,j'\in J\\ j<j'}} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))}$$
$$\times \prod_{j\in J,k\in J^c} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(g+\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)}$$

with $\epsilon_j \equiv \epsilon_j(J_+,J_-)$, and

$$U^{W,n}_{K,p}(\lambda;g,g_l) := (-1)^p \sum_{\substack{I_+,I_-\subset K\\ I_+\cap I_-=\emptyset\\ |I_+|+|I_-|=p}} \Biggl(\prod_{j\in I} \frac{\prod_{0\leq l\leq 3}(\hat{g}_l+\epsilon_j(\hat{\rho}_j+\lambda_j))}{2\epsilon_j(\hat{\rho}_j+\lambda_j)(1+2\epsilon_j(\hat{\rho}_j+\lambda_j))}$$
$$\times \prod_{\substack{j,j'\in I\\ j<j'}} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1-g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))}$$
$$\times \prod_{j\in I,k\in K\setminus I} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(g+\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)} \Biggr)$$

with $\epsilon_j \equiv \epsilon_j(I_+,I_-)$, $I = I_+\cup I_-$ and $p=0,\ldots,|K|$. Here ρ_j and $\hat{\rho}_j$ $(j=1,\ldots,n)$ are defined in accordance with the conventions at the end of the introduction.

3.3. Branching polynomials

Upon performing a rescaling of the trigonometric variables $x_j\to\alpha x_j$ and picking parameters of the form

$$q=e^{-\alpha},\quad t=e^{-\alpha g},\quad t_l=e^{-\alpha g_l}\quad (l=0,1,2,3), \tag{17a}$$

the Macdonald-Koornwinder polynomials degenerate into the symmetric Wilson polynomials in the rational limit $\alpha\to 0$ [D5, Sec. 4.1]:

$$P^W_\lambda(x_1,\ldots,x_n;g,g_l) = \lim_{\alpha\to 0}(-\alpha^{-2})^{|\lambda|}P_\lambda(\alpha x_1,\ldots,\alpha x_n;q,t,t_l). \tag{17b}$$

The corresponding rescaled branching polynomials are of the form $(-\alpha^2)^{|\lambda|-|\mu|}P_{\lambda/\mu}(\alpha x;q,t,t_l)$ (10a), (10b), and degenerate in this limit into the following branching polynomials for the symmetric Wilson polynomials:

$$\boxed{P^W_{\lambda/\mu}(x;g,g_l) = \sum_{0\leq k\leq d} B^{W,k}_{\lambda/\mu}(g,g_l)(g_0+ix,g_0-ix)_k} \tag{18a}$$

with

$$\text{(18b)} \quad \boxed{B^{W,k}_{\lambda/\mu}(g, g_l) = (-1)^{|\lambda|-|\mu|+m} g^{2(|\lambda|-|\mu|-k)} C^{W,(n+1)^m-\lambda',m}_{n^m-\mu',m-k}(\tfrac{1}{g}, \tfrac{g_l}{g})}$$

for $k = 0, \ldots, d = d(\lambda, \mu)$. Here one uses that

$$\lim_{\alpha\to 0} (-\alpha^{-2})^k \langle \alpha x, t_0 \rangle_{q,k} = (g_0 + ix)_k (g_0 - ix)_k$$

and that

$$\lim_{\alpha\to 0} (-\alpha^{-2})^{|\lambda|-|\mu|} \alpha^{-2r} C^{\mu,n}_{\lambda,r}(q, t, t_l) = C^{W,\mu,n}_{\lambda,r}(g, g_l).$$

§4. Continuous Hahn level

4.1. Symmetric continuous Hahn polynomials [D2, D5]

The symmetric continuous Hahn polynomials

$$\text{(19)} \qquad P_\lambda(x_1, \ldots, x_n) = P^{cH}_\lambda(x_1, \ldots, x_n; g, g_0, g_1) \qquad (\lambda \in \Lambda_n)$$

are of the form in Eqs. (1a), (1b), with

$$M_\lambda(x_1, \ldots, x_n) = m_\lambda(x_1, \ldots, x_n)$$

and

$$\text{(20)} \quad \Delta(x_1, \ldots, x_n) = \Delta^{cH}(x_1, \ldots, x_n; g, g_0, g_1)$$
$$:= \prod_{1\le j\le n} |\Gamma(g_0 + ix_j)\Gamma(g_1 + ix_j)|^2 \prod_{1\le j<k\le n} \left| \frac{\Gamma(g + i(x_j - x_k))}{\Gamma(i(x_j - x_k))} \right|^2$$

supported on $\mathcal{D} = \mathbb{R}^n$. Here it is assumed that g and $\mathrm{Re}(g_0)$, $\mathrm{Re}(g_1)$ are all positive.

4.2. Pieri coefficients [D5]

Let

$$\hat{g}_0 = -\frac{1}{2} + \mathrm{Re}(g_0) + \mathrm{Re}(g_1), \qquad \hat{g}_1 = \frac{1}{2} + \mathrm{Re}(g_0) - \mathrm{Re}(g_1)$$

and

$$\hat{g}_2 = \frac{1}{2} + i(\mathrm{Im}(g_0) - \mathrm{Im}(g_1)).$$

The Pieri coefficients $C^{\mu,n}_{\lambda,r}$ (5) for the symmetric continuous Hahn polynomials associated with the multiplication of $P^{cH}_{\lambda}(x_1,\ldots,x_n;g,g_0,g_1)$ by

$$E_r(x_1,\ldots,x_n) = E_r^{cH}(x_1,\ldots,x_n;g,g_0) := (-1)^r \sum_{0\leq m\leq r} \boldsymbol{e}_m(ix_1,\ldots,ix_n)\boldsymbol{h}_{r-m}(\rho_r,\ldots,\rho_n) \tag{21}$$

are given by

$$C^{\mu,n}_{\lambda,r} = C^{cH,\mu,n}_{\lambda,r}(g,g_0,g_1) := \frac{p^{cH}_{\lambda}(g,g_0,g_1)}{p^{cH}_{\mu}(g,g_0,g_1)} V^{cH,n}_{J_+,J_-}(\lambda;g,g_0,g_1) U^{cH,n}_{J^c,r-|J|}(\lambda;g,g_0,g_1), \tag{22}$$

where

$$p^{cH}_{\lambda}(g,g_0,g_1) := i^{|\lambda|} \prod_{1\leq j\leq n} \frac{\prod_{0\leq l\leq 2}(\hat{g}_l+\hat{\rho}_j)_{\lambda_j}}{(2\hat{\rho}_j)_{2\lambda_j}} \prod_{1\leq j<k\leq n} \frac{(g+\hat{\rho}_j+\hat{\rho}_k)_{\lambda_j+\lambda_k}(g+\hat{\rho}_j-\hat{\rho}_k)_{\lambda_j-\lambda_k}}{(\hat{\rho}_j+\hat{\rho}_k)_{\lambda_j+\lambda_k}(\hat{\rho}_j-\hat{\rho}_k)_{\lambda_j-\lambda_k}},$$

$$\begin{aligned} V^{cH,n}_{J_+,J_-}(\lambda;g,g_0,g_1) &:= \prod_{j\in J} \frac{\prod_{0\leq l\leq 2}(\hat{g}_l+\epsilon_j(\hat{\rho}_j+\lambda_j))}{2\epsilon_j(\hat{\rho}_j+\lambda_j)(1+2\epsilon_j(\hat{\rho}_j+\lambda_j))} \\ &\times \prod_{\substack{j,j'\in J\\ j<j'}} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))} \\ &\times \prod_{j\in J,k\in J^c} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(g+\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)} \end{aligned}$$

with $\epsilon_j \equiv \epsilon_j(I_+,I_-)$, and

$$\begin{aligned} U^{cH,n}_{K,p}(\lambda;g,g_0,g_1) &:= (-1)^p \sum_{\substack{I_+,I_-\subset K\\ I_+\cap I_-=\emptyset\\ |I_+|+|I_-|=p}} \Biggl(\prod_{j\in I} \frac{\prod_{0\leq l\leq 2}(\hat{g}_l+\epsilon_j(\hat{\rho}_j+\lambda_j))}{2\epsilon_j(\hat{\rho}_j+\lambda_j)(1+2\epsilon_j(\hat{\rho}_j+\lambda_j))} \\ &\times \prod_{\substack{j,j'\in I\\ j<j'}} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1-g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))} \\ &\times \prod_{j\in I,k\in K\setminus I} \frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(g+\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)} \Biggr) \end{aligned}$$

with $\epsilon_j \equiv \epsilon_j(I_+,I_-)$, $I = I_+\cup I_-$ and $p = 0,\ldots,|K|$.

4.3. Branching polynomials

Upon picking parameters of the form

$$q = e^{-\alpha}, \quad t = e^{-\alpha g}, \quad t_l = \bar{t}_{l+2} = -ie^{-\alpha g_l} \ (l = 0, 1), \tag{23a}$$

the Macdonald-Koornwinder polynomials degenerate into the symmetric continuous Hahn polynomials in the shifted rational limit [D5, Sec. 4.2]:

$$P_\lambda^{cH}(x_1, \ldots, x_n; g, g_0, g_1) = \lim_{\alpha \to 0} (2\alpha)^{-|\lambda|} P_\lambda(\alpha x_1 - \frac{\pi}{2}, \ldots, \alpha x_n - \frac{\pi}{2}; q, t, t_l). \tag{23b}$$

The corresponding branching polynomials $(2\alpha)^{-|\lambda|+|\mu|} P_{\lambda/\mu}(\alpha x - \frac{\pi}{2}; q, t, t_l)$ (10a), (10b) are seen to degenerate in this limit into the following branching polynomials for the symmetric continuous Hahn polynomials:

$$\boxed{P_{\lambda/\mu}^{cH}(x; g, g_0, g_1) = \sum_{0 \le k \le d} B_{\lambda/\mu}^{cH,k}(g, g_0, g_1)(g_0 + ix)_k} \tag{24a}$$

with

$$\boxed{B_{\lambda/\mu}^{cH,k}(g, g_0, g_1) = i^m (-1)^{|\lambda|-|\mu|} g^{|\lambda|-|\mu|-k} C_{n^m-\mu', m-k}^{cH,(n+1)^m-\lambda', m}(\tfrac{1}{g}, \tfrac{g_0}{g}, \tfrac{g_1}{g})} \tag{24b}$$

for $k = 0, \ldots, d = d(\lambda, \mu)$. Here one uses that

$$\lim_{\alpha \to 0} (-2i\alpha)^{-k} \langle \alpha x - \frac{\pi}{2}, t_0 \rangle_{q,k} = (g_0 + ix)_k$$

and that

$$\lim_{\alpha \to 0} (2\alpha)^{-|\lambda|+|\mu|} (2\alpha i)^{-r} C_{\lambda,r}^{\mu,n}(q, t, t_l) = C_{\lambda,r}^{cH,\mu,n}(g, g_0, g_1).$$

§5. Jacobi level

5.1. Symmetric Jacobi polynomials [V, HO, H, M1, D, L1, BO, BF, DLM, DES, HL, K2]

The symmetric Jacobi polynomials are trigonometric polynomials

$$P_\lambda(x_1, \ldots, x_n) = P_\lambda^J(x_1, \ldots, x_n; g, g_0, g_1) \qquad (\lambda \in \Lambda_n) \tag{25}$$

of the form in Eqs. (1a), (1b), with

$$M_\lambda(x_1, \ldots, x_n) = m_\lambda(e^{ix_1} + e^{-ix_1}, \ldots, e^{ix_n} + e^{-ix_n})$$

and

$$\Delta(x_1,\ldots,x_n) = \Delta^J(x_1,\ldots,x_n;g,g_0,g_1) \tag{26}$$

$$:= \prod_{1\le j\le n}\Big|\sin\Big(\frac{x_j}{2}\Big)\Big|^{2g_0-1}\Big|\cos\Big(\frac{x_j}{2}\Big)\Big|^{2g_1-1}\prod_{1\le j<k\le n}\Big|\sin\Big(\frac{x_j+x_k}{2}\Big)\sin\Big(\frac{x_j-x_k}{2}\Big)\Big|^{2g}$$

supported on $\mathcal{D}=[-\pi,\pi]^n$. Here it is assumed that the parameters g, g_0 and g_1 are all positive.

5.2. Pieri coefficients [D5]

Let

$$\hat{g}_0 = \frac{1}{2}(g_0+g_1-1) \quad \text{and} \quad \hat{g}_1 = \frac{1}{2}(g_0-g_1+1).$$

The Pieri coefficients $C^{\mu,n}_{\lambda,r}$ (5) for the symmetric Jacobi polynomials associated with the multiplication of $P^J_\lambda(x_1,\ldots,x_n;g,g_0,g_1)$ by

$$\begin{aligned} E_r(x_1,\ldots,x_n) &= E^J_r(x_1,\ldots,x_n) \\ &:= (-1)^r \boldsymbol{e}_r\left(\sin^2\left(\frac{x_1}{2}\right),\ldots,\sin^2\left(\frac{x_n}{2}\right)\right) \end{aligned} \tag{27}$$

are given by

$$\begin{aligned} C^{\mu,n}_{\lambda,r} &= C^{J,\mu,n}_{\lambda,r}(g,g_0,g_1) \\ &:= \frac{p^J_\lambda(g,g_0,g_1)}{p^J_\mu(g,g_0,g_1)} V^{J,n}_{J_+,J_-}(\lambda;g,g_0,g_1) U^{J,n}_{J^c,r-|J|}(\lambda;g,g_0,g_1), \end{aligned} \tag{28}$$

where

$$p^J_\lambda(g,g_0,g_1) := 4^{|\lambda|}\prod_{1\le j\le n}\frac{(\hat{g}_0+\hat{\rho}_j,\hat{g}_1+\hat{\rho}_j)_{\lambda_j}}{(2\hat{\rho}_j)_{2\lambda_j}}\prod_{1\le j<k\le n}\frac{(g+\hat{\rho}_j+\hat{\rho}_k)_{\lambda_j+\lambda_k}(g+\hat{\rho}_j-\hat{\rho}_k)_{\lambda_j-\lambda_k}}{(\hat{\rho}_j+\hat{\rho}_k)_{\lambda_j+\lambda_k}(\hat{\rho}_j-\hat{\rho}_k)_{\lambda_j-\lambda_k}},$$

$$\begin{aligned} V^{J,n}_{J_+,J_-}(\lambda;g,g_0,g_1) &:= \prod_{j\in J}\frac{(\hat{g}_0+\epsilon_j(\hat{\rho}_j+\lambda_j),\hat{g}_1+\epsilon_j(\hat{\rho}_j+\lambda_j))}{2\epsilon_j(\hat{\rho}_j+\lambda_j)(1+2\epsilon_j(\hat{\rho}_j+\lambda_j))} \\ &\times \prod_{\substack{j,j'\in J\\ j<j'}}\frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))(1+\epsilon_j(\hat{\rho}_j+\lambda_j)+\epsilon_{j'}(\hat{\rho}_{j'}+\lambda_{j'}))} \\ &\times \prod_{j\in J,k\in J^c}\frac{(g+\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(g+\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)}{(\epsilon_j(\hat{\rho}_j+\lambda_j)+\hat{\rho}_k+\lambda_k)(\epsilon_j(\hat{\rho}_j+\lambda_j)-\hat{\rho}_k-\lambda_k)} \end{aligned}$$

with $\epsilon_j \equiv \epsilon_j(J_+, J_-)$,

$$U^{J,n}_{K,p}(\lambda; g, g_0, g_1) := (-1)^p \sum_{\substack{I_+, I_- \subset K \\ I_+ \cap I_- = \emptyset \\ |I_+| + |I_-| = p}} \Biggl(\prod_{j \in I} \frac{(\hat{g}_0 + \epsilon_j(\hat{\rho}_j + \lambda_j), \hat{g}_1 + \epsilon_j(\hat{\rho}_j + \lambda_j))}{2\epsilon_j(\hat{\rho}_j + \lambda_j)(1 + 2\epsilon_j(\hat{\rho}_j + \lambda_j))}$$

$$\times \prod_{\substack{j, j' \in I \\ j < j'}} \frac{(g + \epsilon_j(\hat{\rho}_j + \lambda_j) + \epsilon_{j'}(\hat{\rho}_{j'} + \lambda_{j'}))(1 - g + \epsilon_j(\hat{\rho}_j + \lambda_j) + \epsilon_{j'}(\hat{\rho}_{j'} + \lambda_{j'}))}{(\epsilon_j(\hat{\rho}_j + \lambda_j) + \epsilon_{j'}(\hat{\rho}_{j'} + \lambda_{j'}))(1 + \epsilon_j(\hat{\rho}_j + \lambda_j) + \epsilon_{j'}(\hat{\rho}_{j'} + \lambda_{j'}))}$$

$$\times \prod_{j \in I, k \in K \setminus I} \frac{(g + \epsilon_j(\hat{\rho}_j + \lambda_j) + \hat{\rho}_k + \lambda_k)(g + \epsilon_j(\hat{\rho}_j + \lambda_j) - \hat{\rho}_k - \lambda_k)}{(\epsilon_j(\hat{\rho}_j + \lambda_j) + \hat{\rho}_k + \lambda_k)(\epsilon_j(\hat{\rho}_j + \lambda_j) - \hat{\rho}_k - \lambda_k)} \Biggr)$$

with $\epsilon_j \equiv \epsilon_j(I_+, I_-)$, $I = I_+ \cup I_-$ and $p = 0, \ldots, |K|$.

5.3. Branching polynomials

Upon picking parameters of the form

$$t = q^g, \qquad t_l = (-1)^l q^{h_l} \quad (l = 0, 1, 2, 3) \tag{29a}$$

such that $h_0 + h_2 = g_0$ and $h_1 + h_3 = g_1$, the Macdonald-Koornwinder polynomials degenerate into the symmetric Jacobi polynomials for $q \to 1$ [M3, §11], [D1, Sec. 4.1], [D5, Sec. 4.3]:

$$P^J_\lambda(x_1, \ldots, x_n; g, g_0, g_1) = \lim_{q \to 1} P_\lambda(x_1, \ldots, x_n; q, t, t_l). \tag{29b}$$

The corresponding branching polynomials $P_{\lambda/\mu}(x; q, t, t_l)$ of the form in Eqs. (10a), (10b) are seen to degenerate in this limit into the following branching polynomials for the symmetric Jacobi polynomials:

$$\boxed{P^J_{\lambda/\mu}(x; g, g_0, g_1) = \sum_{0 \le k \le d} B^{J,k}_{\lambda/\mu}(g, g_0, g_1) \sin^{2k}\left(\frac{x}{2}\right)} \tag{30a}$$

with

$$\boxed{B^{J,k}_{\lambda/\mu}(g, g_0, g_1) = 4^m (-1)^{|\lambda| - |\mu|} C^{J,(n+1)^m - \lambda', m}_{n^m - \mu', m-k}\left(\tfrac{1}{g}, \tfrac{g_0}{g}, \tfrac{g_1}{g}\right)} \tag{30b}$$

for $k = 0, \ldots, d = d(\lambda, \mu)$. Here one uses that

$$\lim_{q \to 1} (-4)^{-k} \langle x, t_0 \rangle_{q,k} = \sin^{2k}\left(\frac{x}{2}\right)$$

and that

$$\lim_{q \to 1} 4^{-r} C^{\mu,n}_{\lambda,r}(q, t, t_l) = C^{J,\mu,n}_{\lambda,r}(g, g_0, g_1).$$

§6. Laguerre level

6.1. Symmetric Laguerre polynomials [M1, L2, BF, D4, X, ADO, DES, HL]

The symmetric Laguerre polynomials are even polynomials

$$P_\lambda(x_1,\ldots,x_n) = P_\lambda^L(x_1,\ldots,x_n;g,h,\omega) \qquad (\lambda\in\Lambda_n) \tag{31}$$

of the form in Eqs. (1a), (1b), with

$$M_\lambda(x_1,\ldots,x_n) = m_\lambda(x_1^2,\ldots,x_n^2)$$

and

$$\begin{aligned}\Delta(x_1,\ldots,x_n) &= \Delta^L(x_1,\ldots,x_n;g,h,\omega)\\ &:= \prod_{1\le j\le n} e^{-\omega x_j^2}\left|x_j\right|^{2h-1}\prod_{1\le j<k\le n}\left|x_j^2-x_k^2\right|^{2g}\end{aligned} \tag{32}$$

supported on $\mathcal{D}=\mathbb{R}^n$. Here it is assumed that the parameters g,h and the scaling parameter ω are all positive.

6.2. Pieri coefficients [D4]

The Pieri coefficients $C_{\lambda,r}^{\mu,n}$ (5) for the symmetric Laguerre polynomials associated with the multiplication of $P_\lambda^L(x_1,\ldots,x_n;g,h,\omega)$ by

$$E_r(x_1,\ldots,x_n) = E_r^L(x_1,\ldots,x_n) := \boldsymbol{e}_r(x_1^2,\ldots,x_n^2) \tag{33}$$

are given by

$$\begin{aligned}C_{\lambda,r}^{\mu,n} &= C_{\lambda,r}^{L,\mu,n}(g,h,\omega)\\ &:= (-\omega)^{-r}\frac{p_\lambda^L(g,h,\omega)}{p_\mu^L(g,h,\omega)}V_{J_+,J_-}^{L,n}(\lambda;g,h)U_{J^c,r-|J|}^{L,n}(\lambda;g,h),\end{aligned} \tag{34}$$

where

$$p_\lambda^L(g,h,\omega) := (-\omega)^{-|\lambda|}\prod_{1\le j\le n}((n-j)g+h)_{\lambda_j}\prod_{1\le j<k\le n}\frac{(1+(k-j)g)_{\lambda_j-\lambda_k}}{((k-j)g)_{\lambda_j-\lambda_k}},$$

$$\begin{aligned}V_{J_+,J_-}^{L,n}(\lambda;g,h) &:= \prod_{j\in J_+}((n-j)g+h+\lambda_j)\prod_{j\in J_-}((n-j)g+\lambda_j)\\ &\times \prod_{\substack{j\in J_+\\ j'\in J_-}}\left(1+\frac{g}{(j'-j)g+\lambda_j-\lambda_{j'}}\right)\left(1+\frac{g}{(j'-j)g+\lambda_j-\lambda_{j'}+1}\right)\\ &\times \prod_{\substack{j\in J_+\\ k\notin J_+\cup J_-}}\left(1+\frac{g}{(k-j)g+\lambda_j-\lambda_k}\right)\prod_{\substack{j\in J_-\\ k\notin J_+\cup J_-}}\left(1-\frac{g}{(k-j)g+\lambda_j-\lambda_k}\right),\end{aligned}$$

and

$$U_{K,p}^{L,n}(\lambda;g,h) := (-1)^p \sum_{\substack{I_+,I_-\subset K\\ I_+\cap I_-=\emptyset\\ |I_+|+|I_-|=p}} \prod_{j\in I_+}((n-j)g+h+\lambda_j)\prod_{j\in I_-}((n-j)g+\lambda_j)$$
$$\times \prod_{j\in I_+,j'\in I_-}\Bigl(1+\frac{g}{(j'-j)g+\lambda_j-\lambda_{j'}}\Bigr)\Bigl(1-\frac{g}{(j'-j)g+\lambda_j-\lambda_{j'}+1}\Bigr)$$
$$\times \prod_{\substack{j\in I_+\\ k\in K\setminus(I_+\cup I_-)}}\Bigl(1+\frac{g}{(k-j)g+\lambda_j-\lambda_k})\Bigr)\prod_{\substack{j\in I_-\\ k\in K\setminus(I_+\cup I_-)}}\Bigl(1-\frac{g}{(k-j)g+\lambda_j-\lambda_k}\Bigr)$$

for $p=0,\ldots,|K|$.

6.3. Branching polynomials

Upon picking parameters such that

$$g_0+g_1=h \quad \text{and} \quad g_{l+2}=\frac{1}{\omega_l\beta^2} \quad (l=0,1), \tag{35a}$$

with $\omega_0+\omega_1=\omega$, the symmetric Wilson polynomials degenerate into the symmetric Laguerre polynomials via the following limit [D4, Sec. 4.1]:

$$P_\lambda^L(x_1,\ldots,x_n;g,h,\omega)=\lim_{\beta\to 0}\beta^{2|\lambda|}P_\lambda^W(\tfrac{x_1}{\beta},\ldots,\tfrac{x_n}{\beta};g,g_l). \tag{35b}$$

The corresponding rescaled branching polynomials are of the form $\beta^{2(|\lambda|-|\mu|)}P^W_{\lambda/\mu}(\frac{x}{\beta};g,g_l)$ (18a), (18b) and converge correspondingly to the following branching polynomials for the symmetric Laguerre polynomials:

$$\boxed{P^L_{\lambda/\mu}(x;g,h,\omega)=\sum_{0\le k\le d}B^{L,k}_{\lambda/\mu}(g,h,\omega)x^{2k}} \tag{36a}$$

with

$$\boxed{B^{L,k}_{\lambda/\mu}(g,h,\omega)=(-1)^{k+|\lambda|-|\mu|}C^{L,(n+1)^m-\lambda',m}_{n^m-\mu',m-k}(\tfrac{1}{g},\tfrac{h}{g},\tfrac{\omega}{g})} \tag{36b}$$

for $k=0,\ldots,d=d(\lambda,\mu)$. Here one uses that

$$\lim_{\beta\to 0}\beta^{2k}(g_0+ix\beta^{-1})_k(g_0-ix\beta^{-1})_k=x^{2k}$$

and that

$$\lim_{\beta\to 0}(-1)^r\beta^{2(|\lambda|-|\mu|+r)}C^{W,\mu,n}_{\lambda,r}(g,g_l)=C^{L,\mu,n}_{\lambda,r}(g,h,\omega).$$

§7. Hermite level

7.1. Symmetric Hermite polynomials [M1, L3, BF, D4, X, DES, HL]

The symmetric Hermite polynomials

$$P_\lambda(x_1,\ldots,x_n) = P_\lambda^H(x_1,\ldots,x_n;g,\omega) \qquad (\lambda \in \Lambda_n) \tag{37}$$

are of the form in Eqs. (1a), (1b), with

$$M_\lambda(x_1,\ldots,x_n) = m_\lambda(x_1,\ldots,x_n)$$

and

$$\begin{aligned}\Delta(x_1,\ldots,x_n) &= \Delta^H(x_1,\ldots,x_n;g,\omega) \\ &:= \prod_{1\le j\le n} e^{-\omega x_j^2} \prod_{1\le j<k\le n} |x_j-x_k|^{2g}\end{aligned} \tag{38}$$

supported on $\mathcal{D}=\mathbb{R}^n$. Here it is assumed that the parameter g and the scaling parameter ω are both positive.

7.2. Pieri coefficients [D4]

Let us denote by $C_{\lambda,r}^{H,\mu,n}(g,\omega)$ the Pieri coefficients $C_{\lambda,r}^{\mu,n}$ (5) associated with the multiplication of the symmetric Hermite polynomial $P_\lambda^H(x_1,\ldots,x_n;g,\omega)$ by the elementary symmetric polynomial

$$E_r(x_1,\ldots,x_n) = E_r^H(x_1,\ldots,x_n) := \boldsymbol{e}_r(x_1,\ldots,x_n). \tag{39}$$

It is known that $C_{\lambda,r}^{H,\mu,n}(g,\omega)=0$ unless $\mu \sim_r \lambda$, but unlike in the preceding cases above, a general explicit expression for this Pieri coefficient is not available except when the cardinality of $J=J(\lambda,\mu)$ is equal to r:

$$C_{\lambda,r}^{H,\mu,n}(g,\omega) = \frac{p_\lambda^H(g)}{p_\mu^H(g)} V_{J_+,J_-}^{H,n}(\lambda;g,\omega) \qquad (\text{when } |J_+|+|J_-|=r), \tag{40}$$

where

$$p_\lambda^H(g) := \prod_{1\le j<k\le n} \frac{(1+(k-j)g)_{\lambda_j-\lambda_k}}{((k-j)g)_{\lambda_j-\lambda_k}}$$

and

$$V^{H,n}_{J_+,J_-}(\lambda;g,\omega) := \prod_{j\in J_-} \frac{(n-j)g+\lambda_j}{2\omega}$$
$$\times \prod_{j\in J_+,j'\in J_-}\Bigl(1+\frac{g}{(j'-j)g+\lambda_j-\lambda_{j'}}\Bigr)\Bigl(1+\frac{g}{(j'-j)g+\lambda_j-\lambda_{j'}+1}\Bigr)$$
$$\times \prod_{\substack{j\in J_+\\ k\notin J_+\cup J_-}}\Bigl(1+\frac{g}{(k-j)g+\lambda_j-\lambda_k}\Bigr) \prod_{\substack{j\in J_-\\ k\notin J_+\cup J_-}}\Bigl(1-\frac{g}{(k-j)g+\lambda_j-\lambda_k}\Bigr).$$

7.3. Branching polynomials

Upon picking parameters such that

$$g_l = \frac{1}{\omega_l\beta^2} \quad (l=0,1) \tag{41a}$$

with $\omega_0+\omega_1=\omega$, the symmetric continuous Hahn polynomials degenerate into the symmetric Hermite polynomials via the following limit [D4, Sec. 4.1]:

$$P^H_\lambda(x_1,\ldots,x_n;g,\omega) = \lim_{\beta\to 0}\beta^{|\lambda|}P^{cH}_\lambda(\tfrac{x_1}{\beta},\ldots,\tfrac{x_n}{\beta};g,g_0,g_1). \tag{41b}$$

Since it turns out to be very cumbersome to perform the limit (41a), (41b) at the level of the Pieri coefficients, we have done so instead at the level of the Cauchy identity (see Appendix A). This allows to deduce the branching formula for the symmetric Hermite polynomials directly following the approach in [DE3, Sec. 4] for the Macdonald-Koornwinder polynomials.

Specifically, by expanding the first two factors of the trivial identity

$$\prod_{\substack{1\leq j\leq m\\ 1\leq k\leq n+1}}(x_j-z_k) = \prod_{\substack{1\leq j\leq m\\ 1\leq k\leq n}}(x_j-z_k)\prod_{1\leq j\leq m}(x_j-z_{n+1})$$

with the aid of the Cauchy identity for the symmetric Hermite polynomials in Eq. (52), and employing the elementary expansion

$$\prod_{1\leq j\leq m}(x_j-z) = \sum_{0\leq r\leq m}(-1)^{m-r}\boldsymbol{e}_r(x_1,\ldots,x_m)z^{m-r}, \qquad z=z_{n+1}, \tag{42}$$

for the last factor, one arrives at the equality

$$\sum_{\lambda\subset(n+1)^m}(-1)^{m(n+1)-|\lambda|}P^H_\lambda(x_1,\ldots,x_m;g,\omega)P^H_{m^{n+1}-\lambda'}(z_1,\ldots,z_{n+1};\tfrac{1}{g},\tfrac{\omega}{g})$$

$$= \sum_{\substack{\mu\subset n^m\\ 0\leq r\leq m}} (-1)^{m(n+1)-|\mu|-r}\Big(P^H_{m^n-\mu'}(z_1,\ldots,z_n;\tfrac{1}{g},\tfrac{\omega}{g})z_{n+1}^{m-r} \times \boldsymbol{e}_r(x_1,\ldots,x_m)P^H_\mu(x_1,\ldots,x_m;g,\omega)\Big).$$

After invoking the Pieri formula for the symmetric Hermite polynomials and reordering of the summations, the RHS is rewritten as

$$= \sum_{\lambda\subset(n+1)^m} (-1)^{m(n+1)-|\lambda|}\Big(P^H_\lambda(x_1,\ldots,x_m;g,\omega) \times \sum_{\substack{\mu\subset n^m,\,0\leq r\leq m\\ \mu\sim_r\lambda}} (-1)^{r+|\lambda|-|\mu|}C^{H,\lambda,m}_{\mu,r}(g,\omega)P^H_{m^n-\mu'}(z_1,\ldots,z_n;\tfrac{1}{g},\tfrac{\omega}{g})z_{n+1}^{m-r}\Big).$$

Hence, it is seen by comparing with the LHS that for any $\lambda\subset(n+1)^m$:

$$P^H_{m^{n+1}-\lambda'}(z_1,\ldots,z_{n+1};\tfrac{1}{g},\tfrac{\omega}{g}) = \sum_{\substack{\mu\subset n^m,\,0\leq r\leq m\\ \mu\sim_r\lambda}} (-1)^{r+|\lambda|-|\mu|}C^{H,\lambda,m}_{\mu,r}(g,\omega)P^H_{m^n-\mu'}(z_1,\ldots,z_n;\tfrac{1}{g},\tfrac{\omega}{g})z_{n+1}^{m-r},$$

i.e. for any $\lambda\in m^{n+1}$ (cf. [DE3, Lem. 5]):

$$P^H_\lambda(z_1,\ldots,z_{n+1};g,\omega) = \sum_{\substack{\mu\subset m^n,\,\mu\preceq\lambda\\ m-d(\lambda,\mu)\leq r\leq m}} (-1)^{m-r+|\lambda|-|\mu|}C^{H,(n+1)^m-\lambda',m}_{n^m-\mu',r}(\tfrac{1}{g},\tfrac{\omega}{g})P^H_\mu(z_1,\ldots,z_n;g,\omega)z_{n+1}^{m-r}.$$

The upshot is that the branching rule for the symmetric Hermite polynomials is of the form in Eq. (2) with the one-variable branching polynomial given by

$$\boxed{P^H_{\lambda/\mu}(x;g,\omega) = \sum_{0\leq k\leq d} B^{H,k}_{\lambda/\mu}(g,\omega)x^k} \tag{43a}$$

where

$$\boxed{B^{H,k}_{\lambda/\mu}(g,\omega) = (-1)^{k+|\mu|-|\lambda|}C^{H,(n+1)^m-\lambda',m}_{n^m-\mu',m-k}(\tfrac{1}{g},\tfrac{\omega}{g})} \tag{43b}$$

for $k=0,\ldots,d=d(\lambda,\mu)$.

Remark 4. *Except for the Hermite case, the explicit formulas for branching polynomials in this paper were all obtained via straightforward limit transitions as degenerations of the branching polynomials for the*

Macdonald-Koorwinder polynomials derived in [DE3]*. Given the pertinent Pieri coefficients, however, one may alternatively derive the branching rules in question directly from the corresponding Cauchy identities by adapting the above proof for the Hermite case. At the Askey-Wilson level, this boils down to replacing in the proof at issue: (i) the degenerate Cauchy identity* (52) *by Mimachi's Cauchy identity* (47) *and (ii) the elementary expansion* (42) *by the special 'column-row' case of Okounkov's Cauchy identity in* [KNS, Lem. 5.1]*:*

$$\prod_{1\le j\le m}(e^{ix_j}+e^{-ix_j}-e^{iz}-e^{-iz}) = \sum_{0\le r\le m}(-1)^{m-r}E_r(x_1,\ldots,x_m;t,t_0)\,\langle z;t_0\rangle_{t,m-r}. \tag{44}$$

Indeed, this way one precisely reproduces the proof of the branching rule for the Macdonald-Koornwinder polynomials in [DE3, Sec. 4]*. The branching rules for the remaining hypergeometric families follow in turn with the aid of the degenerate Cauchy identities in Appendix A and the corresponding degenerations of the 'column-row' Cauchy identity* (44)*. At the Wilson level and the continuous Hahn level the degenerations of the latter 'column-row' Cauchy identity become of the form*

$$\prod_{1\le j\le m}(x_j^2-z^2) = (-1)^m\sum_{0\le r\le m}E_r^W(x_1,\ldots,x_m;g,g_0)\,g^{2(m-r)}\,\left(\tfrac{g_0+iz}{g},\tfrac{g_0-iz}{g}\right)_{m-r} \tag{45}$$

and

$$\prod_{1\le j\le m}(x_j-z) = i^m\sum_{0\le r\le m}E_r^{cH}(x_1,\ldots,x_m;g,g_0)\,g^{m-r}\,\left(\tfrac{g_0+iz}{g}\right)_{m-r}, \tag{46}$$

whereas at the Jacobi level and the Laguerre level the degenerate Cauchy identity is of the elementary form in (42)*, up to a trigonometric change of variables* $x_j\to\sin^2\left(\frac{x_j}{2}\right)$*,* $z\to\sin^2\left(\frac{z}{2}\right)$ *and a quadratic change of variables* $x_j\to x_j^2$*,* $z\to z^2$*, respectively.*

Remark 5. *It is known* (*cf. e.g.* [M1, L3, BF, D4]) *that the highest-degree leading terms of* $P_\lambda^H(x_1,\ldots,x_n;g,\omega)$ *consist of the* (*monic*) *Jack polynomial* $P_\lambda(x_1,\ldots,x_n;g)$*. By filtering the highest-degree terms on both sides of the branching formula* (2) *for the symmetric Hermite polynomial* $P_\lambda^H(x_1,\ldots,x_n;g,\omega)$*, one recovers in turn a celebrated branching* *rule for the Jack polynomials* [St, M2, OO1]*:*

$$P_\lambda(x_1,\ldots,x_n,x;g)=\sum_{\substack{\mu\in\Lambda_n,\,\mu\subset\lambda\\ \lambda/\mu\ \text{horizontal strip}}} P_\mu(x_1,\ldots,x_n;g)P_{\lambda/\mu}(x;g)$$

$(\lambda\in\Lambda_{n+1})$*, where*

$$P_{\lambda/\mu}(x;g)=x^{|\lambda|-|\mu|}B_{\lambda/\mu}(g)$$

and

$$\begin{aligned}
B_{\lambda/\mu}(g)=&B^{H,d}_{\lambda/\mu}(g,\omega)=C^{H,(n+1)^m-\lambda',m}_{n^m-\mu',m-d}(\tfrac{1}{g},\tfrac{\omega}{g})\quad \text{with } d=|\lambda|-|\mu|\\
=&\frac{p^H_{n^m-\mu'}(\frac{1}{g})}{p^H_{(n+1)^m-\lambda'}(\frac{1}{g})}V^{H,m}_{J(n^m-\mu',(n+1)^m-\lambda'),\emptyset}(n^m-\mu';\tfrac{1}{g},\tfrac{\omega}{g})\\
=&\prod_{1\le j<k\le m}\frac{(1+(k-j)g^{-1})_{\mu'_j-\mu'_k}}{((k-j)g^{-1})_{\mu'_j-\mu'_k}}\frac{((k-j)g^{-1})_{\lambda'_j-\lambda'_k}}{(1+(k-j)g^{-1})_{\lambda'_j-\lambda'_k}}\\
&\times\prod_{\substack{1\le j,k\le m\\ \mu'_j\neq\lambda'_j\\ \mu'_k=\lambda'_k}}\left(1+\frac{g^{-1}}{(k-j)g^{-1}+\mu'_j-\mu'_k}\right)\\
=&\prod_{\substack{1\le j<k\le m\\ \mu'_j=\lambda'_j\\ \mu'_k\neq\lambda'_k}}\left(1+\frac{1}{k-j+g(\lambda'_j-\lambda'_k)}\right)\left(1-\frac{1}{k-j+g(\mu'_j-\mu'_k)}\right)\\
=&\prod_{1\le j\le k\le\ell(\mu)}\frac{(\mu_j-\mu_k+g(1+k-j),1+\mu_j-\lambda_{k+1}+g(k-j))_{\lambda_j-\mu_j}}{(1+\mu_j-\mu_k+g(k-j),\mu_j-\lambda_{k+1}+g(1+k-j))_{\lambda_j-\mu_j}}.
\end{aligned}$$

This branching formula for the Jack polynomials amounts to the $t=q^g$, $q\to 1$ *degeneration of the branching rule for the Macdonald polynomials, cf. Remark 2 of Section 2 above.*

§Appendix A. Hypergeometric Cauchy identities

This appendix collects the degenerations of Mimachi's (dual) Cauchy identity for all families of symmetric hypergeometric polynomials considered above. At the lowest level of the symmetric Hermite polynomials, we relied on the pertinent Cauchy identity for a direct verifcation of our branching rule in the absence of explicit limiting expressions for the corresponding Pieri coefficients.

A.1. Askey-Wilson level [Mi, Thm. 2.1]

$$\prod_{\substack{1\le j\le m\\1\le k\le n}} (e^{ix_j}+e^{-ix_j}-e^{iz_k}-e^{-iz_k}) = \sum_{\lambda\subset n^m} (-1)^{mn-|\lambda|} P_\lambda(x_1,\ldots,x_m;q,t,t_l) P_{m^n-\lambda'}(z_1,\ldots,z_n;t,q,t_l) \tag{47}$$

When $t=q$, $t_0=-t_1=q^{1/2}$ and $t_2=-t_3=q$, Mimachi's Cauchy identity (47) recovers a well-known Cauchy identity for the symplectic Schur functions [Mo, K, T, HK].

A.2. Wilson level

$$\prod_{\substack{1\le j\le m\\1\le k\le n}} (x_j^2-z_k^2) = \sum_{\lambda\subset n^m} (-g^2)^{mn-|\lambda|} P^W_\lambda(x_1,\ldots,x_m;g,g_l) P^W_{m^n-\lambda'}(\tfrac{z_1}{g},\ldots,\tfrac{z_n}{g};\tfrac{1}{g},\tfrac{g_l}{g}) \tag{48}$$

Eq. (48) is obtained from Eq. (47) via the limit transition (17a), (17b).

A.3. Continuous Hahn level

$$\prod_{\substack{1\le j\le m\\1\le k\le n}} (x_j-z_k) = \sum_{\lambda\subset n^m} (-g)^{mn-|\lambda|} P^{cH}_\lambda(x_1,\ldots,x_m;g,g_0,g_1) P^{cH}_{m^n-\lambda'}(\tfrac{z_1}{g},\ldots,\tfrac{z_n}{g};\tfrac{1}{g},\tfrac{g_0}{g},\tfrac{g_1}{g}) \tag{49}$$

Eq. (49) is obtained from Eq. (47) via the limit transition (23a), (23b).

A.4. Jacobi level [Se, Sec. 6], [Mi, Thm. 4.1]

$$\prod_{\substack{1\le j\le m\\1\le k\le n}} (e^{ix_j}+e^{-ix_j}-e^{iz_k}-e^{-iz_k}) = \sum_{\lambda\subset n^m} (-1)^{mn-|\lambda|} P^J_\lambda(x_1,\ldots,x_m;g,g_0,g_1) P^J_{m^n-\lambda'}(z_1,\ldots,z_n;\tfrac{1}{g},\tfrac{g_0}{g},\tfrac{g_1}{g}) \tag{50}$$

Eq. (50) is obtained from Eq. (47) via the limit transition (29a), (29b) [Mi].

A.5. Laguerre level

(51)
$$\prod_{\substack{1\le j\le m\\ 1\le k\le n}} (x_j^2 - z_k^2) = \sum_{\lambda\subset n^m} (-1)^{mn-|\lambda|} P_\lambda^L(x_1,\ldots,x_m;g,h,\omega) P^L_{m^n-\lambda'}(z_1,\ldots,z_n;\tfrac{1}{g},\tfrac{h}{g},\tfrac{\omega}{g})$$

Eq. (51) is obtained from Eq. (48) via the limit transition (35a), (35b).

A.6. Hermite level

(52)
$$\prod_{\substack{1\le j\le m\\ 1\le k\le n}} (x_j - z_k) = \sum_{\lambda\subset n^m} (-1)^{mn-|\lambda|} P_\lambda^H(x_1,\ldots,x_m;g,\omega) P^H_{m^n-\lambda'}(z_1,\ldots,z_n;\tfrac{1}{g},\tfrac{\omega}{g})$$

Eq. (52) is obtained from Eq. (49) via the limit transition (41a), (41b).

References

[ADO] M. Aristidou, M. Davidson and G. Ólafsson, Differential recursion relations for Laguerre functions on symmetric cones, Bull. Sci. Math., **130** (2006), 246–263.

[BF] T. H. Baker and P. J. Forrester, The Calogero-Sutherland model and generalized classical polynomials, Comm. Math. Phys., **188** (1997), 175–216.

[BB] P. Banerjee and B. Basu-Mallick, Exact solution of D_N-type quantum Calogero model through a mapping to free harmonic oscillators, J. Math. Phys., **52** (2011), no. 5, 052106, pp. 11.

[BO] R. J. Beerends and E. M. Opdam, Certain hypergeometric series related to the root system BC, Trans. Amer. Math. Soc., **339** (1993), 581–609.

[D] A. Debiard, Système différentiel hypergéométrique et parties radiales des opérateurs invariants des espaces symétriques de type BC_p, In: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin (ed. M.-P. Malliavin), Lecture Notes in Math., **1296**, Springer-Verlag, Berlin, 1988, pp. 42–124.

[DH] P. Desrosiers and M. Hallnäs, Hermite and Laguerre symmetric functions associated with operators of Calogero-Moser-Sutherland type, SIGMA Symmetry Integrability Geom. Methods Appl., **8** (2012), Paper 049, pp. 51.

[D1] J. F. van Diejen, Commuting difference operators with polynomial eigenfunctions, Compositio Math., **95** (1995), 183–233.

[D2] J. F. van Diejen, Multivariable continuous Hahn and Wilson polynomials related to integrable difference systems, J. Phys. A, **28** (1995), L369–L374.

[D3] J. F. van Diejen, Self-dual Koornwinder-Macdonald polynomials, Invent. Math., **126** (1996), 319–339.

[D4] J. F. van Diejen, Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement, Comm. Math. Phys., **188** (1997), 467–497.

[D5] J. F. van Diejen, Properties of some families of hypergeometric orthogonal polynomials in several variables, Trans. Amer. Math. Soc., **351** (1999), 233–270.

[DE1] J. F. van Diejen and E. Emsiz, Integrable boundary interactions for Ruijsenaars' difference Toda chain, Comm. Math. Phys., **337** (2015), 171–189.

[DE2] J. F. van Diejen and E. Emsiz, Quantum integrals for a semi-infinite q-boson system with boundary interactions, SIGMA Symmetry Integrability Geom. Methods Appl., **11** (2015), Paper 037, pp. 9.

[DE3] J. F. van Diejen and E. Emsiz, Branching formula for Macdonald-Koornwinder polynomials, J. Algebra, **444** (2015), 606–614.

[DLM] J. F. van Diejen, L. Lapointe and J. Morse, Determinantal construction of orthogonal polynomials associated with root systems, Compos. Math., **140** (2004), 255–273.

[DES] I. Dumitriu, A. Edelman and G. Shuman, MOPS: multivariate orthogonal polynomials (symbolically), J. Symbolic Comput., **42** (2007), 587–620.

[G] W. Groenevelt, Multivariable Wilson polynomials and degenerate Hecke algebras, Selecta Math. (N.S.), **15** (2009), 377–418.

[HL] M. Hallnäs and E. Langmann, A unified construction of generalized classical polynomials associated with operators of Calogero-Sutherland type, Constr. Approx., **31** (2010), 309–342.

[HK] A. M. Hamel and R. C. King, Bijective proof of a symplectic dual pair identity, SIAM J. Discrete Math., **25** (2011), 539–560.

[H] G. J. Heckman, Root systems and hypergeometric functions. II, Compos. Math., **64** (1987), 353–373.

[HO] G. J. Heckman and E. M. Opdam, Root systems and hypergeometric functions. I, Compos. Math., **64** (1987), 329–352.

[I] M. Ito, q-difference shift for a BC_n-type Jackson integral arising from 'elementary' symmetric polynomials, Adv. Math., **204** (2006), 619–646.

[K] R. C. King, Branching rules for classical Lie groups using tensor and spinor methods, J. Phys. A, **8** (1975), 429–449.

[KLS] R. Koekoek, P. A. Lesky and R. F. Swarttouw, Hypergeometric Orthogonal Polynomials and their q-Analogues, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

[KNS] Y. Komori, M. Noumi and J. Shiraishi, Kernel functions for difference operators of Ruijsenaars type and their applications, SIGMA Symmetry Integrability Geom. Methods Appl., **5** (2009), Paper 054, pp. 40.

[K1] T. H. Koornwinder, Askey-Wilson polynomials for root systems of type BC, In: Hypergeometric Functions on Domains of Positivity, Jack Polynomials, and Applications (ed. D. St. P. Richards), Contemp. Math., **138**, Amer. Math. Soc., Providence, RI, 1992, pp. 189–204.

[K2] T. H. Koornwinder, Okounkov's BC-type interpolation Macdonald polynomials and their $q = 1$ limit, Sém. Lothar. Combin., **72** (2015), Art. B72a, pp. 27.

[LW] A. Lascoux and S. O. Warnaar, Branching rules for symmetric functions and $\mathfrak{sl}_n$ basic hypergeometric series, Adv. in Appl. Math., **46** (2011), 424–456.

[L1] M. Lassalle, Polynômes de Jacobi généralisés, C. R. Acad. Sci. Paris Sér. I Math., **312** (1991), 425–428.

[L2] M. Lassalle, Polynômes de Laguerre généralisés, C. R. Acad. Sci. Paris Sér. I Math., **312** (1991), 725–728.

[L3] M. Lassalle, Polynômes de Hermite généralisés, C. R. Acad. Sci. Paris Sér. I Math., **313** (1991), 579–582.

[M1] I. G. Macdonald, Hypergeometric functions I, preprint (1987/8), arXiv:1309.4568, pp. 63.

[M2] I. G. Macdonald, Symmetric Functions and Hall Polynomials, Second Edition, Clarendon Press, Oxford, 1995.

[M3] I. G. Macdonald, Orthogonal polynomials associated with root systems, Sém. Lothar. Combin., **45** (2000/01), Art. B45a, pp. 40.

[Mi] K. Mimachi, A duality of Macdonald-Koornwinder polynomials and its application to integral representations, Duke Math. J., **107** (2001), 265–281.

[Mo] A.O. Morris, Spin representations of a direct sum and a direct product, J. Lond. Math. Soc. (2), **33** (1958), 326–333.

[O] A. Okounkov, BC-type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials, Transform. Groups, **3** (1998), 181–207.

[OO1] A. Okounkov and G. Olshanski, Shifted Jack polynomials, binomial formula, and applications, Math. Res. Lett., **4** (1997), 69–78.

[OO2] A. Okounkov and G. Olshanski, Limits of BC-type orthogonal polynomials as the number of variables goes to infinity, In: Jack, Hall-Littlewood and Macdonald polynomials (eds. V.B. Kuznetsov and S. Sahi), Contemp. Math., **417**, Amer. Math. Soc., Providence, RI, 2006, pp. 281–318.

[Ol] G. Olshanski, Laguerre and Meixner orthogonal bases in the algebra of symmetric functions, Int. Math. Res. Not. IMRN, **2012**, no. 16, 3615–3679.
[R] E. M. Rains, BC_n-symmetric polynomials, Transform. Groups, **10** (2005), 63–132.
[Sa] S. Sahi, Nonsymmetric Koornwinder polynomials and duality, Ann. of Math. (2), **150** (1999), 267–282.
[Se] D. Serban, Some properties of the Calogero-Sutherland model with reflections, J. Phys. A, **30** (1997), 4215–4225.
[SV] A. N. Sergeev and A. P. Veselov, BC_∞ Calogero-Moser operator and super Jacobi polynomials, Adv. Math., **222** (2009),1687–1726.
[St] R. P. Stanley, Some combinatorial properties of Jack symmetric functions, Adv. Math., **77** (1989), 76–115.
[Su] Y. Sun, A representation-theoretic proof of the branching rule for Macdonald polynomials, Math. Res. Lett., **23** (2016), 887–927.
[T] I. Terada, A Robinson-Schensted-type correspondence for dual pairs on spinors, J. Combin. Theory Ser. A, **63** (1993), 90–109.
[V] L. Vretare, Formulas for elementary spherical functions and generalized Jacobi polynomials, SIAM J. Math. Anal., **15** (1984), 805–833.
[WNU] M. Wadati, A. Nishino, H. Ujino and Y. Komori, Symmetric and nonsymmetric bases of quantum integrable particle systems with long-range interactions, J. Statist. Phys., **102** (2001), 1049–1064.
[X] Y. Xu, Orthogonal polynomials on the ball and the simplex for weight functions with reflection symmetries, Constr. Approx., **17** (2001), 38–412.
[Z] G. Zhang, Spherical transform and Jacobi polynomials on root systems of type BC, Int. Math. Res. Not., **2005**, no. 15, 3169–3189.

Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile
E-mail address: `diejen@inst-mat.utalca.cl`

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Correo 22, Santiago, Chile
E-mail address: `eemsiz@mat.uc.cl`