# A note on acylindrical hyperbolicity of Mapping Class Groups

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**Abstract.**

The aim of this note is to give the simplest possible proof that Mapping Class Groups of closed hyperbolic surfaces are acylindrically hyperbolic, and more specifically that their curve graphs are hyperbolic and that pseudo-Anosovs act on them as loxodromic WPDs.

## §1.  Introduction

Following Osin [Osi15], we say that a group is *acylindrically hyperbolic* if it is not virtually cyclic and it acts on a (Gromov-)hyperbolic space non-trivially in the sense that there is a loxodromic WPD (weakly proper discontinuous) element. Recall that an element $g$ of a group acting on a metric space $X$ is *loxodromic* if for each $x \in X$ there exists $\epsilon > 0$ such that for any integer $n$ we have $d_X(x, g^n x) \geq \epsilon|n|$. Also, $g$ is *WPD* if, given any $x \in X$ and any $R \geq 0$, for any sufficiently large $n$ there are only finitely many elements $h \in G$ satisfying $d_X(x, hx) \leq R$ and $d_X(g^n x, hg^n x) \leq R$. The notion of WPD element is due to Bestvina–Fujiwara [BF02].

Acylindrical hyperbolicity has strong consequences: All acylindrically hyperbolic groups are SQ-universal, contain free normal subgroups [DGO11], contain Morse elements and hence have cut-points in all asymptotic cones [Sis15], and have infinite dimensional bounded cohomology in degrees 2 [HO13] and 3 [FPS13]. Moreover, acylindrically hyperbolic groups without finite normal subgroups have simple reduced $C^*$-algebra [DGO11] and their commensurating endomorphisms are inner automorphisms [AMS13].

Mapping Class Groups of closed surfaces of genus at least 2 are among the motivating examples of acylindrically hyperbolic groups. A natural hyperbolic space on which the Mapping Class Group MCG($S$)

of the surface $S$ as above acts is the *curve graph* $\mathcal{C}(S)$ of $S$. The vertices of $\mathcal{C}(S)$ are isotopy classes of essential simple closed curves on $S$, and two isotopy classes are joined by an edge (of length 1) if they contain disjoint representatives. From now on a *curve* means the isotopy class of an essential simple closed curve on $S$, and we say that two curves are *disjoint* (resp. *intersecting $\leq M$ times*) if they can be represented disjointly (resp. intersecting $\leq M$ times).

The fact that $\mathcal{C}(S)$ is hyperbolic was proved by Masur–Minsky [MM99], who also proved that any pseudo-Anosov element acts loxodromically, while the fact that any pseudo-Anosov is WPD is due to Bestvina–Fujiwara [BF02]. (Bowditch proved a stronger property, namely acylindricity [Bow08].)

As it turns out, there are relatively simple proofs of all these facts, which we present in this note. We think that a result as important as the acylindrical hyperbolicity of Mapping Class Groups deserves such a(n almost) self-contained account.

In Section 2 we show that curve graphs are hyperbolic (Theorem 2.1), while in Section 3 we show that pseudo-Anosovs are loxodromic (Corollary 3.7) and WPD (Corollary 3.9).

## §2. Curve graphs are hyperbolic

Here is the main theorem of the section.

**Theorem 2.1.** *Let $S$ be a closed orientable surface of genus at least 2. Then its curve graph $\mathcal{C}(S)$ is hyperbolic.*

Our proof is inspired by the one in [HPW15]. It also yields uniform hyperbolicity (since in Proposition 2.2 the hyperbolicity constant of $X$ is bounded in terms of $D$ only), but we do not record it in this note.

We will actually prove hyperbolicity of the *augmented curve graph* $\mathcal{C}_{\mathrm{aug}}(S)$, the metric graph with the same vertex set as $\mathcal{C}(S)$ and edges connecting pairs of curves that intersect at most twice. Notice that $\mathcal{C}_{\mathrm{aug}}(S)$ is quasi-isometric to $\mathcal{C}(S)$. This follows from the fact that two curves that intersect at most twice have a common neighbour in $\mathcal{C}(S)$: the neighbourhood of their union is a non-closed surface of Euler characteristic $\geq -2$, whence of genus $\leq 1$, and thus has an essential boundary component.

We will use the following criterion for hyperbolicity due to Masur–Schleimer [MS13]. A one page proof is available in [Bow14]. Here $N_D(\eta)$ denotes the $D$-neighbourhood of a subset $\eta$ of a metric space.

**Proposition 2.2.** *Let $X$ be a metric graph, and let $D \geq 0$. Suppose that to every pair of vertices $x, y \in X^{(0)}$ we have assigned a connected subgraph $\eta(x, y)$ containing $x$ and $y$ in such a way that*

(1)  *for all $x, y \in X^{(0)}$ with $d_X(x, y) \leq 1$ we have $\mathrm{diam}_X(\eta(x, y)) \leq D$,*

(2)  *for all $x, y, z \in X^{(0)}$ we have $\eta(x, y) \subseteq N_D\big(\eta(x, z) \cup \eta(z, y)\big)$.*

*Then $X$ is hyperbolic.*

In our proof of Theorem 2.1, the sets $\eta(\cdot, \cdot)$ will be spanned by the following curves.

**Definition 2.3.** (Bicorn curves) Let $a$ and $b$ be curves on $S$. Consider their representatives on $S$ in minimal position (i.e. intersecting a minimal number of times); we will call them also $a$ and $b$ slightly abusing the notation. Recall that minimal position is unique up to isotopy.

A curve $c$ is a *bicorn curve between $a$ and $b$* if either $c = a$ or $c = b$ or $c$ is represented by the union of an arc $a'$ of $a$ and an arc $b'$ of $b$, which we call the *a-arc* and the *b-arc* of $c$. If $c = a$, then its $a$-arc is $a$ and its $b$-arc is empty, similarly if $c = b$, then its $b$-arc is $b$ and its $a$-arc is empty.

Notice that whenever we have arcs $a'$ of $a$ and $b'$ of $b$ that only intersect at their endpoints, such arcs define a (bicorn) curve which is essential, because $a$ and $b$ are in minimal position.

Also, notice that, given $a$ and $b$, there are only finitely many bicorn curves between $a$ and $b$.

**Lemma 2.4.** *Let $a$ and $b$ be curves on $S$, and let $\eta(a, b)$ be the full subgraph in $\mathcal{C}_{\mathrm{aug}}(S)$ spanned by all bicorn curves between $a$ and $b$. Then $\eta(a, b)$ is connected.*

In the proof, we will use the following order on the set of bicorn curves.

**Definition 2.5.** Fix curves $a, b$. For $c$ and $c'$ bicorn curves between $a$ and $b$, we write $c < c'$ if the $b$-arc of $c'$ strictly contains the $b$-arc of $c$.

*Proof of Lemma 2.4.* All bicorn curves in this proof are bicorn curves between $a$ and $b$.

We claim that if $c$ is a bicorn curve and $c \neq b$, then there exists a bicorn curve $c'$ such that $c < c'$ and $c'$ is a neighbour of $c$ in $\mathcal{C}_{\mathrm{aug}}(S)$. Since the set of bicorn curves is finite, the claim implies that $\eta(a, b)$ is connected.

We now justify the claim. Let $c \neq b$ be a bicorn curve. First, if $c$ and $b$ intersect at most twice, then we can take $c' = b$, so let us assume that this is not the case.
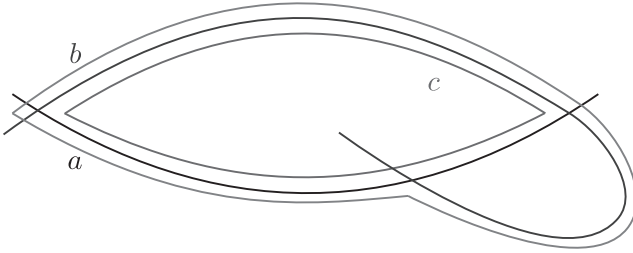
If $c = a$, then let $b'$ be a minimal arc of $b$ whose both endpoints lie in $a$. Let $c'$ be any of the two bicorn curves defined by $b'$ and an arc of $a$ with the same endpoints. It is easy to check that the intersection number of $a$ and $c'$ is at most 1, and clearly $c < c'$.

If $c \neq a$ with $a$-arc $a'$ and $b$-arc $b'$, then consider a minimal arc $b'' \supsetneq b'$ with both endpoints in $a'$. The arc $b''$ and the subarc $a''$ of $a'$ with the same endpoints as $b''$ define a new bicorn curve $c'$ with $c < c'$ and intersection number at most 1 with $c$. This justifies the claim.                                                                Q.E.D.

The following lemma says that bicorn curve triangles are 1-slim.

**Lemma 2.6.** *Let $a, b$ and $d$ be curves and let $c$ be a bicorn curve between $a$ and $b$. Then there is a bicorn curve $c'$ between $a$ and $d$ or between $b$ and $d$ that intersects $c$ at most twice.*

*Proof.* If the intersection number of $c$ and $d$ is at most 2, then we can take $c' = d$. Otherwise, again slightly abusing the notation, we consider representatives $a, b$ and $d$ on $S$ pairwise in minimal position.

We claim that there is an arc $d'$ of $d$ intersecting $a'$ (or $b'$) only at its endpoints and either intersecting $b'$ (resp. $a'$) at most once, or intersecting it exactly twice: at the endpoints. Indeed, to justify the claim it suffices to take the minimal arc $d'$ of $d$ with both endpoints in $a'$ or both endpoints in $b'$; such an arc exists since $d$ intersects $c$ at least 3 times.

Let $d'$ be the arc guaranteed by the claim and assume without loss of generality that its endpoints lie in $a'$. Then $d'$ and the subarc of $a'$ with the same endpoints as $d'$ define a bicorn curve $c'$ between $a$ and $d$, and $c'$ intersects $c$ at most twice, as desired.                    Q.E.D.

*Proof of Theorem 2.1.* For vertices $a, b$ of $\mathcal{C}_{\mathrm{aug}}(S)$, we define $\eta(a, b)$ to be the full subgraph spanned in $\mathcal{C}_{\mathrm{aug}}(S)$ by the bicorn curves between $a$ and $b$, as in Lemma 2.4. Clearly, $\eta(a, b)$ contains $a$ and $b$. By Lemma 2.4, the subgraph $\eta(a, b)$ is connected, as required in Proposition 2.2.
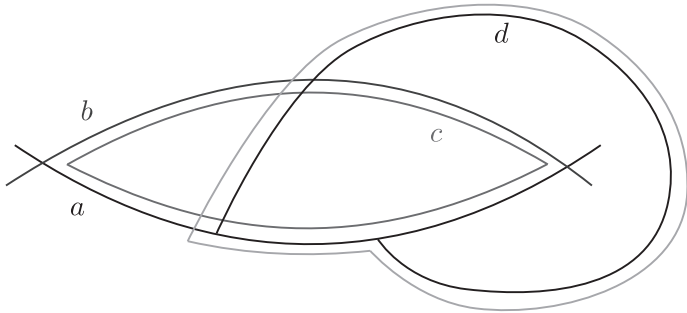
Hypothesis (1) of Proposition 2.2 is obviously satisfied with $D = 1$. Hypothesis (2) of Proposition 2.2 is satisfied with $D = 1$ by Lemma 2.6. Thus by Proposition 2.2, the graph $\mathcal{C}_{\mathrm{aug}}(S)$ is hyperbolic.        Q.E.D.

## §3.   Pseudo-Anosovs are loxodromic WPD

### 3.1.   Notation

Throughout the section we fix a pseudo-Anosov homeomorphism $\phi : S \to S$, where $S$ is a closed surface. We will use some well-known facts about pseudo-Anosovs discovered by Thurston [Thu88]. Recall that $\phi$ comes with a one-parameter family of locally CAT(0) Euclidean metrics with singularities $\{d_t\}_{t \in \mathbb{R}}$ on $S$. We denote the length of the path $\alpha$ with respect to the metric $d_t$ by $l_t(\alpha)$. The metrics $\{d_t\}$ have the following properties:

- the finitely many singularities of the metrics coincide and they are invariant under $\phi$.
- the metrics all have the same geodesics, up to reparametrisation.
- the push-forward of $d_t$ by $\phi$ is $d_{t+1}$.
- $S$ has two transverse singular foliations, called the horizontal and the vertical foliation, that form an angle of $\pi/2$ at every non-singular point with respect to any Euclidean structure $d_t$.
- the singularities have cone angle larger than $2\pi$ with respect to any Euclidean structure $d_t$, and in particular the metrics $d_t$ are locally CAT(0).
- $\phi$ preserves the horizontal and the vertical foliation, and there exists $\lambda > 1$ so that if $\alpha$ is a subpath of the horizontal (resp. vertical) foliation then $l_t(\alpha) = \lambda^{t-t'} l_{t'}(\alpha)$ (resp. $l_t(\alpha) = (1/\lambda)^{t-t'} l_{t'}(\alpha)$).
- any half-leaf of either foliation is dense in $S$.

A *saddle connection* is either a closed geodesic or a geodesic on $S$ whose intersection with the singular set consists of its endpoints. Note that a saddle connection cannot be contained in the horizontal or the vertical foliation. The *balanced time* $\beta(\gamma)$ of a saddle connection $\gamma$ is the only $t \in \mathbb{R}$ such that $l_t(\gamma)$ is minimal, or, equivalently, the only $t$ such that $\gamma$ forms an angle of $\pi/4$ with the horizontal and vertical foliations with respect to the metric $d_t$ at any point in its interior. It should be mentioned that the balanced time also plays a role in [MM99], where arguments of a similar nature to the ones below are used.

**Definition 3.1.** Given a curve $c$, we denote by $\mathcal{S}(c)$ the set of all saddle connections contained in the geodesic representative of $c$ with respect to a (hence any) metric $d_t$, each counted with multiplicity.

Moreover, we denote by $\beta(c)$ the average over all $\gamma \in \mathcal{S}(c)$ (counted with multiplicity) of the balanced time of $\gamma$.

**Remark 3.2.** The map $\beta$ is clearly $\phi$-equivariant, meaning that for every curve $c$ we have $\beta(\phi(c)) = \beta(c) + 1$.

In fact, due to Lemmas 3.3 and 3.4, any function $\beta(c)$ with values between $\min_{\gamma \in \mathcal{S}(c)} \beta(\gamma)$ and $\max_{\gamma \in \mathcal{S}(c)} \beta(\gamma)$ satisfying Remark 3.2 would work for our purposes.

We will be interested in points of transverse intersection of pairs of saddle connections. Note that an intersection point of the saddle connections $\gamma_1$ and $\gamma_2$ is not a point of transverse intersection if and only if it is either a common endpoint of $\gamma_1$ and $\gamma_2$ or an interior point of $\gamma_1 = \gamma_2$.

### 3.2. Preliminary lemmas

In both lemmas below, part 2) will be a (technical) generalisation of part 1). We will use parts 1) to show that $\phi$ is loxodromic and part 2) to show that it is WPD. Hence, on first reading, the reader may wish to read parts 1), then the proof that $\phi$ is loxodromic and only afterwards move on to parts 2) and the proof that $\phi$ is WPD.

**Lemma 3.3.** *There exists a constant $C$ so that*

(1) *if the saddle connections $\gamma_1$ and $\gamma_2$ do not have points of transverse intersection, then $|\beta(\gamma_1) - \beta(\gamma_2)| \leq C$.*

(2) *for each $M$ there exists $D$ with the following property. If the saddle connections $\gamma_1$ and $\gamma_2$ satisfy $|\beta(\gamma_1) - \beta(\gamma_2)| \geq C$ and $l_{\beta(\gamma_1)}(\gamma_2) \geq D$, then $\gamma_1$ and $\gamma_2$ have at least $M$ points of transverse intersection.*

*Proof.* We will say that a geodesic in a (hence any) metric $d_t$ is *Euclidean* if it intersects the singular set at most at the endpoints. Notice

that such a geodesic forms a well-defined angle with the horizontal and vertical foliation with respect to any metric $d_t$.

Let $\gamma_1$ and $\gamma_2$ be saddle connections. From now on, lengths and angles will be measured in the metric $d_0$.

Up to using the action of $\phi$, we can assume that the balanced time of $\gamma_1$ is within 0 and 1, so that (in the metric $d_0$) the angle that $\gamma_1$ forms with both the horizontal and the vertical foliation is bounded below by some $\epsilon > 0$ depending only on the pseudo-Anosov $\phi$. Also, by the discreteness of the singular set, up to decreasing $\epsilon$ we can assume that the length of $\gamma_1$ is $\geq \epsilon$.

Now, by the compactness of $S$ and since half-leaves of both foliations are dense, there exists $\delta > 0$ so that any Euclidean geodesic $\gamma$ with length $\geq 1/\delta$ forming an angle $\leq \delta$ with either the horizontal or vertical foliation intersects transversally every Euclidean geodesic of length $\geq \epsilon$ that forms an angle $\geq \epsilon$ with both foliations.

We can now prove 1). Whenever $|\beta(\gamma_2)|$ is large enough, the angle that $\gamma_2$ forms with one of the foliations is $\leq \delta$. Moreover, notice that there is a uniform (in $t$) lower bound on the length of a saddle connection in a metric $d_t$, since each $d_t$ is isometric to some $d_{t'}$ with $t' \in [0, 1)$. In particular, for $|\beta(\gamma_2)|$ large enough, the length of $\gamma_2$ is $\geq 1/\delta$. We can thus apply the above statement with $\gamma = \gamma_2$ and conclude that $\gamma_2$ intersects $\gamma_1$ transversally, as desired.

In order to prove 2), notice that whenever $|\beta(\gamma_2)|$ is large enough and $\gamma_2$ is sufficiently long, then we can split it into several $\gamma$'s as above.

Q.E.D.

**Lemma 3.4.**     (1)    *If the curves $c_1$ and $c_2$ are disjoint (including the case $c_1 = c_2$) then no saddle connection of $\mathcal{S}(c_1)$ intersects transversally a saddle connection of $\mathcal{S}(c_2)$.*

(2)    *Let $c_1$ and $c_2$ be curves and let $\gamma_1^1, \ldots, \gamma_1^k \in \mathcal{S}(c_1), \gamma_2 \in \mathcal{S}(c_2)$ be saddle connections so that $\gamma_1^i$ and $\gamma_2$ intersect transversally at $M_i$ points. Then the intersection number of $c_1$ and $c_2$ is at least $\sum M_i$.*

*Proof.* Let us prove 1) first. Let $\alpha_1$ and $\alpha_2$ be the geodesic representatives of $c_1$ and $c_2$. Suppose that $\alpha_1$ contains a saddle connection that intersects transversally a saddle connection contained in $\alpha_2$. We can lift $\alpha_1$ and $\alpha_2$ to the universal cover of $(S, d_0)$ to two bi-infinite geodesics $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ that intersect transversally at a point. The universal cover of $(S, d_0)$ is a CAT(0) space quasi-isometric to $\mathbb{H}^2$. Thus $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ intersect exactly once and moreover the points at infinity of $\tilde{\alpha}_1$ separate the points at infinity of $\tilde{\alpha}_2$. Thus any representatives of $c_1$ and $c_2$ intersect, a contradiction.

To prove 2), we can proceed similarly and this time find $M = \sum M_i$ lifts $\tilde{\alpha}_1^1, \ldots, \tilde{\alpha}_1^M$ each intersecting transversally $\tilde{\alpha}_2$, in distinct orbits of the stabiliser of $\tilde{\alpha}_2$ in $\pi_1(S)$ (regarded as the group of deck transformations). The conclusion follows.                    Q.E.D.

### 3.3.  Pseudo-Anosovs are loxodromic

**Proposition 3.5.** *The balanced time map* $\beta : \mathcal{C}(S)^{(0)} \to \mathbb{R}$ *from Definition 3.1 is coarsely Lipschitz, namely there exists* $L \geq 1$ *so that for all curves* $c_1$ *and* $c_2$ *we have*

$$|\beta(c_1) - \beta(c_2)| \leq L d_{\mathcal{C}(S)}(c_1, c_2).$$

*Proof.*  It is enough to show that there exists $L$ so that $|\beta(c_1) - \beta(c_2)| \leq L$ when $c_1$ and $c_2$ are disjoint. By Lemma 3.4, no $\gamma \in \mathcal{S}(c_i)$ intersects transversally a $\gamma' \in \mathcal{S}(c_j)$, for $i, j \in \{1, 2\}$. Hence, we get $|\beta(\gamma) - \beta(\gamma')| \leq C$ for $C$ as in Lemma 3.3. Thus we can take $L = C$.                    Q.E.D.

**Remark 3.6.** Notice that we have also just proved that for any curve $c$ and any $\gamma \in \mathcal{S}(c)$ we have $|\beta(\gamma) - \beta(c)| \leq C$ for $C$ as in Lemma 3.3.

**Corollary 3.7.** $\phi$ *acts loxodromically on the curve graph.*

*Proof.*  Recall from Remark 3.2 that $\beta$ is $\phi$-equivariant, meaning that for every curve $c$ we have $\beta(\phi(c)) = \beta(c) + 1$. Hence, in view of Proposition 3.5, for any fixed curve $c$ and integer $n$ we have

$$d_{\mathcal{C}(S)}(c, \phi^n(c)) \geq |\beta(\phi^n(c)) - \beta(c)|/L = |n|/L,$$

so that $\phi$ is loxodromic as required.                    Q.E.D.

### 3.4.  Pseudo-Anosovs are WPD

To simplify notation, we will denote $c_n = \phi^n(c)$ for any curve $c$.

**Proposition 3.8.** *For every curve* $c$ *and* $R \geq 0$ *the following holds. For every sufficiently large* $n$ *there are only finitely many curves of the form* $hc$ *or* $hc_n$ *where* $h \in MCG(S)$ *satisfies* $d_{\mathcal{C}(S)}(c, hc) \leq R$ *and* $d_{\mathcal{C}(S)}(c_n, hc_n) \leq R$.

*Proof.*  Let us show finiteness of the set of curves of the form $hc_n$, since finiteness of the set of curves of the form $hc$ can be proven symmetrically.

Our aim is to show that whenever $n$ is large enough we can bound $l_t(hc_n)$ (meaning the length of the geodesic representative) with $t = \beta(c)$. Once we do this, we get that there are finitely many possible $hc_n$'s since

there are only finitely many geodesics whose length with respect to $d_t$ is bounded above by any given constant.

We now have to choose constants carefully. Let $C$ be as in Lemma 3.3 and $L$ be the coarsely Lipschitz constant for $\beta$ as in Proposition 3.5. Suppose that $n \geq 2LR + 3C$, and let $M - 1$ be the intersection number of $c$ and $c_n$. Finally, let $D$ be as in Lemma 3.3 for the given $M$, and let $D' = \lambda^{LR+C} MD$.

Now, assume by contradiction $l_t(hc_n) \geq D'$. Notice that by Proposition 3.5 we have $|\beta(hc) - \beta(c)| \leq LR$ and $|\beta(hc_n) - \beta(c_n)| \leq LR$, so that

$$|\beta(hc) - \beta(hc_n)| \geq |\beta(c) - \beta(c_n)| - 2LR \geq n - 2LR \geq 3C.$$

In particular, for every $\gamma \in \mathcal{S}(hc)$ and $\gamma_n \in \mathcal{S}(hc_n)$ we have $|\beta(\gamma) - \beta(\gamma_n)| \geq C$ by Remark 3.6.

First, suppose that $\mathcal{S}(hc_n)$ contains at least $M$ elements. Notice that each saddle connection in $\mathcal{S}(hc_n)$ intersects transversally any saddle connection in $\mathcal{S}(hc)$ by Lemma 3.3-(1). Hence, by Lemma 3.4-(2) the intersection number of $hc$ and $hc_n$ is at least $M$, a contradiction.

Otherwise, there exists $\gamma_n \in \mathcal{S}(hc_n)$ such that $l_t(\gamma_n) \geq D'/M$. Pick any $\gamma \in \mathcal{S}(hc)$. Then, since $|t - \beta(\gamma)| \leq |\beta(c) - \beta(hc)| + |\beta(hc) - \beta(\gamma)| \leq LR + C$ (by Proposition 3.5 and Remark 3.6), we have

$$l_{\beta(\gamma)}(\gamma_n) \geq l_t(\gamma_n)\lambda^{-LR-C} \geq D.$$

Hence, by Lemma 3.3-(2), $\gamma$ and $\gamma_n$ intersect transversally at least $M$ times, so that by Lemma 3.4-(2) the intersection number of $hc$ and $hc_n$ is at least $M$, a contradiction.                                    Q.E.D.

Recall that $\phi \in \mathrm{MCG}(S)$ is *WPD* (for the action on $\mathcal{C}(S)$) if, given any curve $c$ and any $R \geq 0$, for any sufficiently large $n$ there are only finitely many elements $h \in \mathrm{MCG}(S)$ satisfying $d_{\mathcal{C}(S)}(c, hc) \leq R$ and $d_{\mathcal{C}(S)}(c_n, hc_n) \leq R$.

**Corollary 3.9.** *$\phi$ is WPD.*

*Proof.* Fix any curve $c$ and any $R \geq 0$. For $n$ sufficiently large the curves $c$ and $c_n$ form a filling pair, that is, the complementary regions of the union of the representatives of $c$ and $c_n$ in minimal position are discs. By Proposition 3.8, there are finitely many possibilities for $b = hc$ and $b_n = hc_n$. Since the complementary regions of $c \cup c_n$ are disks, for fixed $b$ and $b_n$ there are only finitely many $h$ satisfying $hc = b, hc_n = b_n$. Hence, there are finitely many possibilities for $h$, as required.     Q.E.D.

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