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Rigidity of certain solvable actions on the torus

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Abstract.

An analog of the Baumslag–Solitar group $BS(1,k)$ acts on the torus naturally. The action is not locally rigid in higher dimension, but any perturbation of the action should be homogeneous.

§1. Introduction

For integers $n \geq 1$ and $k \geq 2$, let $\Gamma_{n,k}$ be the finitely presented group given by

$$\Gamma_{n,k} = \langle a, b_1, \dots, b_n \mid ab_ia^{-1} = b_i^k,\ b_ib_j = b_jb_i \text{ for any } i, j = 1, \dots, n\rangle.$$

The group $\Gamma_{1,k}$ is just the Baumslag–Solitar group $BS(1,k) = \langle a, b \mid aba^{-1} = b^k\rangle$. It acts on the projective line $\mathbb{R}P^1 = \mathbb{R} \cup \{\infty\}$ by $a \cdot x = kx$ and $b \cdot x = x + 1$, where we set $c \cdot \infty = \infty$ and $\infty + t = \infty$ for any $c \neq 0$ and $t \in \mathbb{R}$. This action preserves the standard projective structure on $\mathbb{R}P^1$. In [2], Burslem and Wilkinson proved a classification theorem of smooth[1] $BS(1,k)$-action on $\mathbb{R}P^1$. As a corollary, they obtained the following rigidity result.

Theorem 1.1 (Burslem and Wilkinson [2]). *Any real analytic $BS(1,k)$-action on $\mathbb{R}P^1$ is locally rigid. In particular, the above projective action is locally rigid.*

Recall the definition of local rigidity of a smooth action of a discrete group. Let Γ be a discrete group and M a smooth closed manifold. The

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[1]The term 'smooth' means 'of C^∞' in this paper.

group $\mathrm{Diff}(M)$ of smooth diffeomorphisms is endowed with the C^∞-topology. A Γ-*action* is a homomorphism from Γ to $\mathrm{Diff}(M)$. For a Γ-action ρ and $\gamma \in \Gamma$, we write ρ^γ for the diffeomorphism $\rho(\gamma)$. By $\mathcal{A}(\Gamma, M)$, we denote the set of smooth Γ-actions on M. This set is endowed with the topology generated by the open basis

$$\{\mathcal{O}_{\gamma,U} = \{\rho \in \mathcal{A}(\Gamma, M) \mid \rho^\gamma \in U\}\},$$

where γ and U run over Γ and all open subsets of $\mathrm{Diff}(M)$. We say two Γ-actions ρ_1 and ρ_2 are *smoothly conjugate* if there exists a diffeomorphism h of M such that $\rho_2^\gamma = h \circ \rho_1^\gamma \circ h^{-1}$ for any $\gamma \in \Gamma$. An Γ-action ρ_0 is *locally rigid* if it admits a neighborhood in $\mathcal{A}(\Gamma, M)$ such that any action in it is smoothly conjugate to ρ_0.

The above projective $BS(1,k)$-action on $\mathbb{R}P^1$ can be generalized to $\Gamma_{n,k}$-actions on the sphere S^n. Let $B = (v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$. We define an $BS(n,k)$-action $\bar{\rho}_B$ on $S^n = \mathbb{R}^n \cup \{\infty\}$ by $\bar{\rho}_B^a(x) = k \cdot x$ and $\bar{\rho}_B^{b_i}(x) = x + v_i$ for $x \in \mathbb{R}^n$, where $c \cdot \infty = \infty$ and $\infty + v = \infty$ for any $c \neq 0$ and $v \in \mathbb{R}^n$. The sphere S^n admits a natural conformal structure and the action ρ_B preserves it. In [1], the author of this paper proved that the action $\bar{\rho}_B$ is not locally rigid but it exhibits a weak form of rigidity.

Proposition 1.2 ([1])**.** *$\bar{\rho}_B$ and $\bar{\rho}_{B'}$ are smoothly conjugate if and only if there exists a conformal linear transformation T of $\mathbb{R}^n$ such that $TB = B'$. In particular, $\bar{\rho}_B$ is not locally rigid if $n \geq 2$.*

Theorem 1.3 ([1])**.** *There exists a neighborhood of $\bar{\rho}_B$ in $\mathcal{A}(\Gamma_{n,k}, S^n)$ such that any action in it is smoothly conjugate to $\bar{\rho}_{B'}$ with some basis B'. In particular, any $\Gamma_{n,k}$-action close to $\bar{\rho}_B$ preserves a smooth conformal structure on S^n.*

In this paper, we prove analogous results for another generalization of the projective $BS(1,k)$-action on $\mathbb{R}P^1$. Let $B = (v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ with $v_j = (v_{ij})_{i=1}^n$. We define a $\Gamma_{n,k}$-action ρ_B on the n-dimensional torus $\mathbb{T}^n = (\mathbb{R} \cup \{\infty\})^n$ by

$$\begin{aligned}
\rho_B^a(x_1, \ldots, x_n) &= (k \cdot x_1, \ldots, k \cdot x_n), \\
\rho_B^{b_j}(x_1, \ldots, x_n) &= (x_1 + v_{1j}, \ldots, x_n + v_{nj}).
\end{aligned}$$

Remark that the point $x_\infty = (\infty, \ldots, \infty) \in \mathbb{T}^n$ is a global fixed point of the action ρ_B.

The aim of this paper is to show that the action ρ_B is not locally rigid if $n \geq 2$, but it exhibits rigidity like the above $\Gamma_{n,k}$-action on S^n. Let G be the subgroup of $GL_n\mathbb{R}$ consisting of linear transformations

f which have the form $f(x_1, \ldots, x_n) = (a_1 x_{\sigma(1)}, \ldots, a_n x_{\sigma(n)})$ with real numbers $a_1, \ldots, a_n \neq 0$ and a permutation σ on $\{1, \ldots, n\}$.

Proposition 1.4. *Two actions ρ_B and $\rho_{B'}$ are smoothly conjugate if and only if $B' = gB$ for some $g \in G$. In particular, ρ_B is not locally rigid if $n \geq 2$.*

Theorem 1.5. *There exists a neighborhood of ρ_B in $\mathcal{A}(\Gamma_{n,k}, \mathbb{T}^n)$ such that any action in it is smoothly conjugate to ρ_B for some basis B of $\mathbb{R}^n$.*

The theorem is proved by an application of the method used in [1]. Firstly, we show persistence of the global fixed point x_∞. Next, we reduce the theorem to the corresponding theorem for local actions at the global fixed point. The same argument as in [1], we can see that the theorem for local actions follows from exactness of a finite dimensional linear complex. The exactness can be checked by an elementary computation.

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§2. Proof of Theorem 1.5

2.1. Reduction from global to local

Let Γ be a discrete group and M a smooth closed manifold. We say that a point $x_* \in M$ is a *global fixed point* of a Γ-action ρ on M if $\rho^\gamma(x) = x$ for any $\gamma \in \Gamma$. We can apply the following general result on persistence of a global fixed point of $\Gamma_{n,k}$-action to the action ρ_B.

Lemma 2.1 ([1, Lemma 2.10])**.** *Let M be a manifold and ρ_* be a $\Gamma_{n,k}$-action on M. Suppose that ρ_* has a global fixed point p_0 such that $(D\rho_*^a)_{p_0} = k^{-1}I$ and $(D\rho_*^{b_i})_{p_0} = I$ for any $i = 1, \ldots, n$. Then, there exists a neighborhood $\mathcal{U} \subset \mathcal{A}(\Gamma_{n,k}, M)$ of ρ_* and a continuous map $\hat{p} \colon \mathcal{U} \to M$ such that $\hat{p}(\rho_*) = p_0$ and that $\hat{p}(\rho)$ is a global fixed point of ρ for any $\rho \in \mathcal{U}$.*

The action ρ_B and its global fixed point x_∞ satisfy the assumption of the lemma. Hence, any action ρ close to ρ_B admits a global fixed point x_ρ close to x_∞.

A Γ-action with a global fixed point induces a local Γ-action. We define the space of local actions on $\mathbb{R}^n$ as follows. Let $\mathcal{D}$ be the group of germs of local diffeomorphisms of $\mathbb{R}^n$ fixing the origin. For $F \in \mathcal{D}$ and $r \geq 1$, we denote the r-th derivative of F at the origin by $D_0^{(r)} F$. It is an element of the vector space $\mathcal{S}^{r,n}$ of symmetric r-multilinear maps from $(\mathbb{R}^n)^r$ to $\mathbb{R}^n$. We define a norm $\|\cdot\|^{(r)}$ on $\mathcal{S}^{r,n}$ by

$$\|L\|^{(r)} = \sup\{\|L(\xi_1, \ldots, \xi_r)\| \mid \xi_1, \ldots, \xi_r \in \mathbb{R}^n, \|\xi_i\| \leq 1 \text{ for any } i\},$$

for $L \in \mathcal{S}^{r,n}$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. We also define a pseudo-distance d_r on $\mathcal{D}$ by

$$d_r(G_1, G_2) = \sum_{i=1}^{r} \|D_0^{(i)} G_1 - D_0^{(i)} G_2\|^{(i)}$$

for $G_1, G_2 \in \mathcal{D}$. The pseudo-distance on $\mathcal{D}$ induces a non-Hausdorff topology on $\mathcal{D}$. We call it *the C^r_{loc}-topology*. Let $\mathrm{Hom}(\Gamma, \mathcal{D})$ the set of homomorphisms from Γ to $\mathcal{D}$, which can be regarded as the set of local Γ-actions on $(\mathbb{R}^n, 0)$. The C^r_{loc}-topology on $\mathcal{D}$ induces a topology on $\mathrm{Hom}(\Gamma, \mathcal{D})$ like $\mathcal{A}(\Gamma, M)$. We also call this topology on $\mathrm{Hom}(\Gamma, \mathcal{D})$ the C^r_{loc}-topology. We say that two local Γ-actions P_1 and P_2 are *smoothly conjugate* if there exists $H \in \mathcal{D}$ such that $P_2^\gamma = H \circ P_1^\gamma \circ H^{-1}$ for any $\gamma \in \Gamma$.

Let φ be the local coordinate of $\mathbb{T}^n$ at x_∞ given by

$$\varphi(x_1, \ldots, x_n) = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right),$$

where $1/\infty = 0$. For a basis B of $\mathbb{R}^n$, we define a local $\Gamma_{n,k}$-action P_B by $P_B^\gamma = \varphi \circ \rho_B^\gamma \circ \varphi^{-1}$. For each $\Gamma_{n,k}$-action ρ close to ρ_B, we can take a local coordinate φ_ρ close to φ with $\varphi_\rho(x_\rho) = 0$ so that a local $\Gamma_{n,k}$-action given by $P_\rho^\gamma = \varphi_\rho \circ \rho_B^\gamma \circ \varphi_\rho^{-1}$ is C^3_{loc}-close to ρ_B.

The following proposition reduces Theorem 1.5 to the corresponding result for local actions.

Proposition 2.2. *Let ρ be a $\Gamma_{n,k}$-action on $\mathbb{T}^n$ close to ρ_B. Suppose that the induced local action P_ρ is smoothly conjugate to $P_{B'}$ for some basis B' of $\mathbb{R}^n$. Then, the action ρ is smoothly conjugate to $\rho_{B'}$.*

The rest of this subsection is devoted to the proof of the proposition. Let $B' = (v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ such that P_ρ is smoothly conjugate to $P_{B'}$. For each $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{\pm 1\}^n$, there exist integers $m_1^\sigma, \ldots, m_n^\sigma$ such that $\sigma_i \cdot \sum_{j=1}^n m_j^\sigma v_{ij} > 0$ for any $i = 1, \ldots, n$. Set $b_\sigma = b_1^{m_1^\sigma} \cdots b_n^{m_n^\sigma}$ and $v_i^\sigma = \sum_{j=1}^n m_j^\sigma v_{ij}$. Then, we have

$$\rho_{B'}^{b_\sigma}(x_1, \ldots, x_n) = (x + v_1^\sigma, \ldots, x_n + v_n^\sigma). \tag{1}$$

Let $\bar{m}$ be the maximum of $\{|m_i|^\sigma \mid \sigma \in \{\pm 1^n\}, i = 1, \ldots, n\}$ and put $S = \{a^{\pm 1}\} \cup \{b_1^{l_1} \cdots b_n^{l_n} \mid |l_i| \leq \bar{m}\}$. By the assumption of the proposition, there exists a diffeomorphism h from a neighborhood V of x_∞ to a neighborhood V' of x_ρ and a family $(V_\gamma)_{\gamma\in\Gamma_{n,k}}$ of neighborhoods of x_∞ such that $V_\gamma \subset V \cap (\rho^\gamma_{B'})^{-1}(V)$ and $h \circ \rho^\gamma_{B'}(x) = \rho^\gamma \circ h(x)$ for any $\gamma \in \Gamma_{n,k}$ and any $x \in V_\gamma$. Since S is a finite set, we can take an open interval $I \subset \mathbb{R}P^1 \setminus \{0\}$ such that $\infty \in I$ and $I^n \subset \bigcap_{\gamma\in S} V_\gamma$. The set I^n is a neighborhood of x_∞ and $h \circ \rho^\gamma_{B'}(x) = \rho^\gamma \circ h(x)$ for any $x \in I^n$ and $\gamma \in S$.

Put $I_1 = \{x \in I \mid x = \infty \text{ or } x > 0\}$, $I_{-1} = \{x \in I \mid x = \infty \text{ or } x < 0\}$, and $U_\sigma = I_{\sigma_1} \times \cdots \times I_{\sigma_n}$ for $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{\pm 1\}^n$. Equation (1) implies that $\rho^{b_\sigma}_{B'}(U_\sigma) \subset U_\sigma$, $\bigcap_{n\geq 0}(\rho^{b_\sigma}_{B'})^n(\overline{U_\sigma}) = \{x_\infty\}$, and $\bigcup_{n\geq 0}(\rho^{b_\sigma}_{B'})^{-n}(U_\sigma) = \mathbb{T}^n$ for any $\sigma \in \{\pm 1\}^n$, where $\overline{U_\sigma}$ is the closure of U_σ. For $\sigma \in \{\pm 1\}^n$, let $m(x, \sigma)$ be the minimal integer m such that $(\rho^{b_\sigma}_{B'})^m(x)$ is contained in U_σ. We define a map $h_\sigma : \mathbb{T}^n \to \mathbb{T}^n$ by

$$h_\sigma(x) = (\rho^{b_\sigma})^{-m(x,\sigma)} \circ h \circ (\rho^{b_\sigma}_{B'})^{m(x,\sigma)}(x).$$

We prove Proposition 2.2 by showing that h_σ does not depend on the choice of σ and it is a smooth conjugacy between $\rho_{B'}$ and ρ.

Lemma 2.3. $h_\sigma(x) = (\rho^{b_\sigma})^{-m} \circ h \circ (\rho^{b_\sigma}_{B'})^m(x)$ *for any* $m \geq m(x, \sigma)$.

Proof. The lemma is shown by induction of m. Suppose that the equation holds for some $m \geq m(x, \sigma)$. Since $(\rho^{b_\sigma}_{B'})^m(U_\sigma) \subset U_\sigma$, we have

$$\begin{aligned}(\rho^{b_\sigma})^{-(m+1)} \circ h \circ (\rho^{b_\sigma}_{B'})^{m+1}(x) &= (\rho^{b_\sigma})^{-(m+1)} \circ (h \circ \rho^{b_\sigma}_{B'}) \circ (\rho^{b_\sigma}_{B'})^m(x)\\ &= (\rho^{b_\sigma})^{-(m+1)} \circ (\rho^{b_\sigma} \circ h) \circ (\rho^{b_\sigma}_{B'})^m(x)\\ &= (\rho^{b_\sigma})^{-m} \circ h \circ (\rho^{b_\sigma}_{B'})^m(x).\end{aligned}$$

Hence, the required equation holds for $m + 1$. Q.E.D.

Lemma 2.4. *The map* h_σ *is injective.*

Proof. Take $x_1, x_2 \in \mathbb{T}^2$ and $m = \max\{m(x_1, \sigma), m(x_2, \sigma)\}$. Then, we have

$$h_\sigma(x_i) = (\rho^{b_\sigma})^{-m} \circ h \circ (\rho^{b_\sigma}_{B'})^m(x_i).$$

for $i = 1, 2$. The map in the right-hand side is injective. Q.E.D.

Lemma 2.5. $h_\sigma \circ \rho^\gamma_{B'} = \rho^\gamma \circ h_\sigma$ *for any* $\gamma \in \Gamma$.

Proof. Fix $x \in \mathbb{T}^n$ and take $m \geq m(x, \sigma)$ such that $m \geq m(\rho^\gamma_{B'}(x), \sigma)$ for any $\gamma \in S$. It is sufficient to show that $h_\sigma \circ \rho^\gamma_{B'} = \rho^\gamma \circ h_\sigma$ for

$\gamma \in \{a, b_1, \dots, b_n\}$. For any $i = 1, \dots, n$, the identity $b_i b_j = b_j b_i$ implies that

$$\begin{aligned} h_\sigma \circ \rho_{B'}^{b_i}(x) &= (\rho^{b_\sigma})^{-m} \circ h \circ (\rho_{B'}^{b_\sigma})^m \circ \rho_{B'}^{b_i}(x) \\ &= (\rho^{b_\sigma})^{-m} \circ (h \circ \rho_{B'}^{b_i}) \circ (\rho_{B'}^{b_\sigma})^m(x) \\ &= (\rho^{b_\sigma})^{-m} \circ (\rho^{b_i} \circ h) \circ (\rho_{B'}^{b_\sigma})^m(x) \\ &= \rho^{b_i} \circ (\rho^{b_\sigma})^{-m} \circ h \circ (\rho_{B'}^{b_\sigma})^m(x) \\ &= \rho^{b_i} \circ h_\sigma(x). \end{aligned}$$

The identity $ab_i = b_i^k a$ also implies

$$\begin{aligned} h_\sigma \circ \rho_{B'}^{a}(x) &= (\rho^{b_\sigma})^{-km} \circ h \circ (\rho^{b_\sigma})^{km} \circ \rho_{B'}^{a}(x) \\ &= (\rho^{b_\sigma})^{-km} \circ (h \circ \rho_{B'}^{a}) \circ (\rho_{B'}^{b_\sigma})^m(x) \\ &= (\rho^{b_\sigma})^{-km} \circ (\rho^{a} \circ h) \circ (\rho_{B'}^{b_\sigma})^m(x) \\ &= \rho^{a} \circ (\rho^{b_\sigma})^{-m} \circ h \circ (\rho_{B'}^{b_\sigma})^m(x) \\ &= \rho^{a} \circ h_\sigma(x). \end{aligned}$$

Q.E.D.

For a diffeomorphism f on a manifold M and a hyperbolic fixed point p of f, we denote the unstable manifold of p by $W^u(p, f)$ (see e.g., [3] for the definitions and basic results on hyperbolic dynamics). By $\mathrm{Fix}(f)$, we also denote the set of fixed points of f. For $l \geq 0$, let $\mathrm{Fix}_l(f)$ be the set of hyperbolic fixed point of f whose unstable manifold is l-dimensional.

The diffeomorphisms ρ_B^a and $\rho_{B'}^a$ are Morse–Smale diffeomorphisms with the fixed point set $\{0, \infty\}^n$. For each fixed point $p = (p_1, \dots, p_n) \in \{0, \infty\}^n$, $W^u(p, \rho_{B'}^a) = W_1 \times \cdots \times W_n$ with $W_j = \mathbb{R}$ if $p_j = 0$ and $W_j = \{\infty\}$ if $p_j = \infty$. If ρ is sufficiently close to ρ_B, then ρ^a is a Morse–Smale diffeomorphism and $\mathrm{Fix}_l(\rho^a)$ has the same cardinality as $\mathrm{Fix}_l(\rho_B^a)$, and hence, as $\mathrm{Fix}_l(\rho_{B'}^a)$ for any $l = 0, \dots, n$. By Lemma 2.4 and Lemma 2.5, h_σ maps $\mathrm{Fix}(\rho_{B'}^a)$ to $\mathrm{Fix}(\rho^a)$ bijectively.

Lemma 2.6. *For any $l = 0, \dots, n$ and $p \in \mathrm{Fix}_l(\rho_{B'}^a)$, $h_\sigma(p)$ is a point in $\mathrm{Fix}_l(\rho^a)$. Moreover, the restriction of h_σ to $W^u(p, \rho_{B'}^a)$ is a diffeomorphism onto $W^u(h_\sigma(p), \rho^a)$.*

Remark that $W^u(q, \rho^a)$ is an (embedded) submanifold diffeomorphic to $\mathbb{R}^l$ for $q \in \mathrm{Fix}_l(\rho^a)$ since ρ^a is Morse–Smale.

Proof. Take $l = 0, \dots, n$ and $p \in \mathrm{Fix}_l(\rho_{B'}^a)$. Notice that $W^u(p, \rho_{B'}^a) \cap U_\sigma$ is a non-empty open subset of $W^u(p, \rho_{B'}^a)$. Thus, there exists a neighborhood V^u of p in $W^u(p, \rho^a_{B'})$ such that $(\rho^{b_\sigma}_{B'})^{m(p,\sigma)}(V^u) \subset U_\sigma$. We have $m(y,\sigma) \leq m(p,\sigma)$ for any $y \in V^u$. This implies that $h_\sigma = (\rho^{b_\sigma})^{-m(p,\sigma)} \circ h \circ (\rho^{b_\sigma}_{B'})^{m(p,\sigma)}$ on V^u. In particular, the restriction of h_σ to V^u is a diffeomorphism onto $h_\sigma(V^u)$. Since $W^u(p, \rho^a_{B'}) = \bigcup_{m \geq 0} (\rho^a_{B'})^m(V^u)$, $h_\sigma \circ \rho^a_{B'} = \rho^a \circ h_\sigma$, and h_σ is injective, the restriction of h_σ to $W^u(p, \rho^a_{B'})$ is a diffeomorphism onto $h_\sigma(W^u(p, \rho^a_{B'}))$.

For $x \in W^u(p, \rho^a_{B'})$, we have

$$(\rho^a)^{-m}(h_\sigma(x)) = h_\sigma \circ (\rho^a_{B'})^{-m}(x) \xrightarrow{m\to\infty} h_\sigma(p).$$

This implies that $h_\sigma(W^u(p, \rho^a_{B'})$ is a subset of $W^u(h_\sigma(p), \rho^a)$. In particular, the dimension of $W^u(h_\sigma(p), \rho^a)$ is at least l. Since h_σ maps the finite set $\mathrm{Fix}(\rho^a_{B'})$ to $\mathrm{Fix}(\rho^a)$ bijectively and the sets $\mathrm{Fix}_j(\rho^a_{B'})$ and $\mathrm{Fix}_j(\rho^a)$ have the same cardinality for each $j = 0, \ldots, n$, the map h_σ is a bijection from $\mathrm{Fix}_l(\rho^a_{B'})$ to $\mathrm{Fix}_l(\rho^a)$. The set $h_\sigma(W^u(p, \rho^a_{B'}))$ is a ρ^a-invariant open subset of $W^u(h_\sigma(p), \rho^a)$ which contains $h_\sigma(p)$. It should coincide with $W^u(h_\sigma(p), \rho^a)$, and hence, the restriction of h_σ to $W^u(p, \rho^a_{B'})$ is a diffeomorphism onto $W^u(h_\sigma(p), \rho^a)$. Q.E.D.

Lemma 2.7. *$h_\sigma(p)$ does not depend on the choice of σ for any $p \in \mathrm{Fix}(\rho^a_{B'})$.*

Proof. Take $l = 0, \ldots, n$ and $p = (p_1, \ldots, p_n) \in \mathrm{Fix}_l(\rho_{B'})$. Put $b_p = \prod_{p_i = \infty} b_i$. Then, p is the unique element in $\mathrm{Fix}_l(\rho^a_{B'})$ which is fixed by $\rho^{b_p}_{B'}$. By the identity $\rho^{b_p} \circ h_\sigma = h_\sigma \circ \rho^{b_p}_{B'}$, $h_\sigma(p)$ is the unique element in $\mathrm{Fix}_l(\rho^a) = h_\sigma(\mathrm{Fix}_l(\rho^a_{B'}))$ which is fixed by ρ^{b_p}. Q.E.D.

Lemma 2.8. *The map h_σ does not depend on the choice of σ.*

Proof. Take $\sigma, \sigma' \in \{\pm 1\}^n$ and put $g = h^{-1}_{\sigma'} \circ h_\sigma$. It is sufficient to show that the restriction g_p of g to $W^u(p, \rho^a_{B'})$ is the identity map for each $p = (p_1, \ldots, p_j) \in \mathrm{Fix}(\rho^a_{B'}) = \{0, \infty\}^n$. By the above lemmas, $g_p(p) = p$ and the restriction of g_p is a diffeomorphism of $W^u(p, \rho^a_{B'})$ which commutes with $\rho^a_{B'}$. Recall that $\rho^a_{B'}(x) = kx$ and $W^u(p, \rho^a_{B'})$ is naturally identified with a vector space $\bigoplus_{p_i = 0} \mathbb{R}$. Under the identification, we have

$$(Dg_p)_0 \cdot x = \lim_{m \to +\infty} \frac{g_p(k^{-m}x)}{k^{-m}} = \rho^a_{B'} \circ g_p \circ (\rho^a_{B'})^{-1}(x) = g_p(x).$$

In particular, the map g_p is an linear isomorphism. The linear map g_p commutes with $\rho^{b_j}_{B'}$ for any $j = 1, \ldots, n$. This implies that $g_p(\pi_p(v_j)) = \pi_p(v_j)$, where $\pi_p \colon \mathbb{R}^n \to \bigoplus_{p_i = 0} \mathbb{R}$ is the natural projection. Since $(\pi_p(v_j))_{j=1}^n$ spans $\bigoplus_{p_i = 0} \mathbb{R}$, the map g_p is the identity map on $W^u(p, \rho^a_{B'})$ for each $p \in \mathrm{Fix}(\rho^a_{B'})$. Q.E.D.

Since $I^n = \bigcup_{\sigma \in \{\pm 1\}^n} U_\sigma$ and $h_\sigma = h$ on U_σ, the above lemma implies that $h_\sigma = h$ on I^n. For any $x \in \mathbb{T}^n$ and $\sigma \in \{\pm 1\}^n$, the point $(\rho_{B'}^{b_\sigma})^{m(x,\sigma)}(x)$ is contained in I^n. Take a neighborhood N_x of x such that $(\rho_{B'}^{b_\sigma})^{m(x,\sigma)}(N_x) \subset I^n$. By Lemma 2.5,

$$\begin{aligned} h_\sigma(y) &= (\rho^{b_\sigma})^{-m(x,\sigma)} \circ h_\sigma \circ (\rho_{B'}^{a})^{m(x,\sigma)}(y) \\ &= (\rho^{b_\sigma})^{-m(x,\sigma)} \circ h \circ (\rho_{B'}^{a})^{m(x,\sigma)}(y) \end{aligned}$$

for any $y \in N_x$. Hence, h_σ is a local diffeomorphism. Since h_σ is injective, it is a diffeomorphism of $\mathbb{T}^n$. By Lemma 2.5, it is a smooth conjugacy between two actions $\rho_{B'}$ and ρ.

2.2. Rigidity of local actions

Fix a basis $B = (v_1, \ldots, v_n)$ of $\mathbb{R}^n$ with $v_j = (v_{ij})_{i=1}^n$. Let P_B be the local $\Gamma_{n,k}$-action defined in the previous section. In this subsection, we show the local version of Theorem 1.5.

Theorem 2.9. *If a local action $P \in \operatorname{Hom}(\Gamma_{n,k}, \mathcal{D})$ is sufficiently close to P_B in C^3_{loc}-topology, then it is smoothly conjugate to $P_{B'}$ for some basis B' of $\mathbb{R}^n$.*

Combined with Proposition 2.2, the theorem implies Theorem 1.5.

The above theorem follows from the same argument as in [1]. Firstly, we prove the stability of the linear part of the local action. Secondly, we show exactness of a linear complex and see that existence of B' follow from it.

For $w = (w_i)_{i=1}^n \in \mathbb{R}^n$, we define a map $Q_w \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$Q_w(x, y) = \sum_{j=1}^{n} (w_j x_j y_j) e_j$$

for $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n$, where $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$. Then, the local action P_B satisfies that

$$\begin{aligned} P_B^a(x_1, \ldots, x_n) &= k^{-1} \cdot x, \\ P_B^{b_j}(x_1, \ldots, x_n) &= x - Q_{v_j}(x, x) + O(\|x\|^3). \end{aligned}$$

Let I be the identity map of $\mathbb{R}^n$. We recall a lemma in [1] concerning stability of the linear part of P^{b_i}.

Lemma 2.10 ([1, Lemma 2.2])**.** *Let P_* be a local action in $\operatorname{Hom}(\Gamma_{n,k}, \mathcal{D})$ such that $D_0^{(1)} P_*^a = k^{-1} I$ and $D_0^{(1)} P_*^{b_i} = I$ for any $i = 1, \ldots, n$. Then, there exists a C^1_{loc}-neighborhood $\mathcal{U}$ of P_* in $\operatorname{Hom}(\Gamma_{n,k}, \mathcal{D})$ such that $D_0^{(1)} P^{b_i} = I$ for any $P \in \mathcal{U}$ and $i = 1, \ldots, n$.*

Hence, $D_0^{(1)} P^{b_j} = I$ for any $j = 1, \dots, n$ if P is sufficiently C^1_{loc}-close to P_B. The following lemma is essentially same as Lemma 2.3 of [1].

Lemma 2.11. *Let P_* be a local action in* $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D})$ *such that $D_0^{(1)} P_*^a = k^{-1} I$ and $D_0^{(1)} P_*^{b_j} = I$ for any $j = 1, \dots, n$. Suppose that there exists $\delta > 0$ such that*

$$\max_{j=1,\dots,n} \|A \circ D_0^{(2)} P_*^{b_j} - 2 D_0^{(2)} P_*^{b_j} \circ (A, I)\|^{(2)} \geq \delta \|A\|^{(1)},$$

for any linear map $A \colon \mathbb{R}^n \to \mathbb{R}^n$. Then, $P^a = k^{-1} I$ for any P which is sufficiently C^2_{loc}-close to P_*.

Proof. Let $\mathcal{U}$ be a C^2_{loc}-open neighborhood of P_* consisting of $P \in \mathrm{Hom}(\Gamma_{n,k}, \mathcal{D})$ such that

$$3\|D_0^{(2)} P^{b_j} - D_0^{(2)} P_*^{b_j}\|^{(2)} + \|D_0^{(1)} P^a - k^{-1} I\| \cdot \|D_0^{(2)} P^{b_j}\|^{(2)} < \delta/2$$

for any $j = 1, \dots, n$. Fix $P \in \mathcal{U}$ and put

$$\begin{aligned} A &= D_0^{(1)} P^a - k^{-1} I, \\ B_j &= D_0^{(2)} P^{b_j} - D_0^{(2)} P_*^{b_j}, \\ C_j &= A \circ D_0^{(2)} P_*^{b_j} - 2 D_0^{(2)} P_*^{b_j} \circ (A, I). \end{aligned}$$

We will show that $A = 0$. The identity $P^a \circ P^{b_j} = P^{b_j^k} \circ P^a$ implies that

$$(k^{-1} I + A) \circ (D_0^{(2)} P_*^{b_j} + B_j) = k \cdot (D_0^{(2)} P_*^{b_j} + B_j) \circ (k^{-1} I + A, k^{-1} I + A).$$

Thus, we have that

$$\begin{aligned} \|C_j\|^{(2)} &= \|A \circ B_j - 2 B_j \circ (A, I) - (D_0^{(2)} P_*^{b_j} + B_j) \circ (A, A)\|^{(2)} \\ &\leq \|A\|^{(1)} \cdot \Big(3\|B_j\|^{(2)} + \|A\|^{(1)} \cdot \|D_0^{(2)} P^{b_j}\|^{(2)} \Big) \\ &\leq (\delta/2) \|A\|^{(1)} \end{aligned}$$

for any $j = 1, \dots, n$. By assumption, $A = 0$. Q.E.D.

We apply the lemma for P_B.

Lemma 2.12. *The local action P_B satisfies the assumption of* Lemma 2.11. *In particular, $P^a = k^{-1} I$ for any $P \in$* $\mathrm{Hom}(\Gamma_{n,k}, \mathcal{D})$ *which is sufficiently C^2_{loc}-close to P_B.*

Proof. Take a square matrix $A = (a_{ij})$ of size n and put

$$C_j = A \circ D_0^{(2)} P_B^{b_j} - 2D_0^{(2)} P_B^{b_j} \circ (A, I) = -2\{A \circ Q_{v_j} - 2Q_{v_j}(A, I)\}.$$

Then,

$$\begin{aligned} C_j(e_i, e_i) &= -2\{A \circ Q_{v_j}(e_i, e_i) - 2Q_{v_j}(Ae_i, e_i)\} \\ &= -2\{A(-v_{ij}e_i) - 2(-v_{ij}a_{ii}e_i)\} \\ &= -2v_{ij}\left\{a_{ii}e_i - \sum_{k\neq i} a_{ki}e_k\right\}. \end{aligned}$$

This implies that $\|C_j(e_i, e_i)\| = 2|v_{ij}| \cdot \|(a_{ki})_{k=1}^n\|$, and hence,

$$\max_{j=1,\dots,n} \|C_j\|^{(2)} \geq 2 \max_{j=1,\dots,n} |v_{ij}| \cdot \|(a_{ki})_{k=1}^n\|$$

for any $i = 1, \dots, n$. Since $(v_1, \dots, v_n)$ is a basis of $\mathbb{R}^n$, there exists $\delta > 0$ such that $\max_{j=1,\dots,n}|v_{ij}| \geq \delta$ for any $i = 1, \dots, n$. We also have $\|A\|^{(1)} \leq n \max_{i=1,\dots,n}\|(a_{ki})_{k=1}^n\|$. This implies that $\max_{j=1,\dots,n}\|C_j\|^{(2)} \geq (2\delta/n)\|A\|^{(1)}$. Q.E.D.

Recall that $\mathcal{S}^{r,n}$ is the vector space of symmetric r-multilinear maps from $(\mathbb{R}^n)^r$ to $\mathbb{R}^n$. Elements of $\mathcal{S}^{1,n}$ are just linear endomorphisms of $\mathbb{R}^n$. For $Q, Q' \in \mathcal{S}^{2,n}$, we define $[Q, Q'] \in \mathcal{S}^{3,n}$ by

$$[Q, Q'](\xi_0, \xi_1, \xi_2) = \sum_{k=0}^{2} Q(\xi_k, Q'(\xi_{k+1}, \xi_{k+2})) - Q'(\xi_k, Q(\xi_{k+1}, \xi_{k+2})),$$

where we set $\xi_3 = \xi_0$ and $\xi_4 = \xi_1$. We also define linear maps $L_B^0 \colon (\mathcal{S}^{1,n})^2 \to (\mathcal{S}^{2,n})^n$ and $L_B^1 \colon (\mathcal{S}^{2,n})^n \to (\mathcal{S}^{3,n})^{n(n-1)/2}$ by

$$\begin{aligned} &L_B^0(A', B') = (A' \circ Q_{v_i} - Q_{v_i} \circ (A', I) - Q_{v_i} \circ (I, A') + Q_{B'e_i})_{i=1}^n, \\ &L_B^1(q_1, \dots, q_n) = ([q_i, Q_{v_j}] - [q_j, Q_{v_i}])_{1\leq i\leq j\leq n}. \end{aligned}$$

By the exactly same argument as in p. 1841–1844 of [1], Theorem 2.9 follows from the following

Proposition 2.13. $\operatorname{Ker} L_B^1 = \operatorname{Im} L_B^0$.

We show this proposition in the next subsection.

2.3. Proof of Proposition 2.13

It is not hard to check that $\operatorname{Im} L_B^0 \subset \operatorname{Ker} L_B^1$. We will show $\operatorname{Ker} L_B^1 \subset \operatorname{Im} L_B^0$. Recall that $I = (e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. As shown in Lemma 2.11 of [1], it is enough to prove Proposition 2.13 for the case $B = I$. Set $L^0 = L_I^0$ and $L^1 = L_I^1$.

For $v, w \in \mathbb{R}^n$, let $\langle v, w \rangle$ be the standard inner product of v and w, i.e., $\langle v, w \rangle = \sum_{i=1}^n v_i w_i$. Let W be the subspace of $(\mathcal{S}^{2,n})^n$ consisting of the elements $(q_i)_{i=1}^n$ such that $q_i(e_i, e_i) = 0$ and $\langle q_i(e_j, e_j), e_j \rangle = 0$ for any $i, j = 1, \ldots, n$.

The following formula on Q_i is useful for computation;

$$Q_v(e_i, e_j) = Q_{e_i}(v, e_i) = \begin{cases} \langle v, e_i \rangle e_i & (i = j), \\ 0 & (i \neq j) \end{cases}$$

for $v \in \mathbb{R}^n$ and $i, j = 1, \ldots, n$. In particular, $Q_{e_i}(e_j, e_k) = e_i$ if $i = j = k$ and $Q_{e_i}(e_j, e_k) = 0$ otherwise.

Lemma 2.14. $W + \operatorname{Im} L^0 = (\mathcal{S}^{2,n})^n$.

Proof. Take $(q_i)_{i=1}^n \in (\mathcal{S}^{2,n})^n$. We put $a_{ii} = -\langle q_i(e_i, e_i), e_i \rangle$, $a_{ji} = \langle q_i(e_i, e_i), e_j \rangle$, and $b_{ii} = 0$, $b_{ji} = \langle q_i(e_j, e_j), e_j \rangle$ for distinct $i, j = 1, \ldots, n$. Let A and B be square matrices of size n whose (i, j)-entries are a_{ij} and b_{ij}, respectively. Then, $L^0(A, B) = (q_i^{A,B})_{i=1}^n$ satisfies that

$$\begin{aligned} q_i^{A,B}(e_i, e_i) &= A \cdot Q_{e_i}(e_i, e_i) - 2Q_{e_i}(Ae_i, e_i) + Q_{Be_i}(e_i, e_i) \\ &= Ae_i - 2a_{ii}e_i + b_{ii}e_i \\ &= q_i(e_i, e_i), \\ q_i^{A,B}(e_j, e_j) &= A \cdot Q_{e_i}(e_j, e_j) - 2Q_{e_i}(Ae_j, e_j) + Q_{Be_i}(e_j, e_j) \\ &= b_{ji}e_j \\ &= \langle q_i(e_j, e_k), e_j \rangle e_j. \end{aligned}$$

Hence, $q_i - q_i^{A,B}$ is an element of W. Q.E.D.

Lemma 2.15. $\operatorname{Ker} L^1 \cap W = \{0\}$.

Proof. Take $(q_i)_{i=1}^n \in \operatorname{Ker} L^1 \cap W$. Since $(q_i)_{i=1}^n \in W$, we have $q_i(e_i, e_i) = 0$ and $\langle q_i(e_j, e_j), e_j \rangle = 0$ for any $i, j = 1, \ldots, n$. If $i \neq j$,

$$\begin{aligned} [q_i, Q_{e_j}](e_j, e_j, e_j) &= 3\{q_i(e_j, e_j) - Q_{e_j}(e_j, q_i(e_j, e_j))\} \\ &= 3\{q_i(e_j, e_j) - \langle q_i(e_j, e_j), e_j \rangle e_j\}, \\ &= 3q_i(e_j, e_j), \\ [q_j, Q_{e_i}](e_j, e_j, e_j) &= 3\{q_j(e_j, e_j) - Q_{e_i}(e_j, q_j(e_j, e_j))\} \\ &= 0. \end{aligned}$$

Since $[q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0$, we obtain that $q_i(e_j, e_j) = 0$.

If $i \neq j$,

$$\begin{aligned}
[q_i, Q_{e_j}](e_i, e_j, e_j) &= q_i(e_i, e_j) - 2Q_{e_j}(e_j, q_i(e_i, e_j)) \\
&= q_i(e_i, e_j) - 2\langle q_i(e_i, e_j), e_j\rangle e_j \\
[q_j, Q_{e_i}](e_i, e_j, e_j) &= -Q_{e_i}(e_i, q_j(e_j, e_j)) \\
&= -Q_{e_i}(e_i, 0) \\
&= 0.
\end{aligned}$$

Since $[q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0$, we obtain that $q_i(e_i, e_j) = 0$.

For distinct $i, j, k = 1, \ldots, n$,

$$\begin{aligned}
[q_i, Q_{e_j}](e_j, e_j, e_k) &= q_i(e_k, e_j) - 2Q_{e_j}(e_j, q_i(e_j, e_k)) \\
&= q_i(e_j, e_k) - 2\langle q_i(e_j, e_k), e_j\rangle e_j, \\
[q_j, Q_{e_i}](e_j, e_j, e_k) &= 0.
\end{aligned}$$

Since $[q_i, Q_{e_j}] - [q_j, Q_{e_i}] = 0$, we obtain that $q_i(e_j, e_k) = 0$. Now, we have $q_i(e_j, e_k) = 0$ for any $(q_i)_{i=1}^n \in \operatorname{Ker} L^1 \cap W$ and any $i, j, k = 1, \ldots, n$.
Q.E.D.

Now, we prove Proposition 2.13. Since $\operatorname{Im} L^0$ is a subspace of $\operatorname{Ker} L^1$, we have $(\mathcal{S}^{2,n})^n = W \oplus \operatorname{Im} L^0$ by the above lemmas. By $\operatorname{Im} L^0 \subset \operatorname{Ker} L^1$ and $\operatorname{Ker} L^1 \cap W = \{0\}$ again, we obtain that $\operatorname{Ker} L^1 = \operatorname{Im} L^0$.

§3. Proof of Proposition 1.4

It is easy to see that any linear isomorphism $g \in G$ of $\mathbb{R}^n$ can be extended uniquely to a diffeomorphism h_g of $\mathbb{T}^n = (\mathbb{R} \cup \{\infty\})^n$ and the diffeomorphism h_g is a conjugacy between ρ_B and ρ_{gB}.

Suppose that ρ_B and $\rho_{B'}$ are smoothly conjugate by a diffeomorphism h. We will show that $h = h_g$ for some $g \in G$. The conjugacy h preserves the unique repelling fixed point $(0, \ldots, 0)$ of ρ_B^a and $\rho_{B'}^a$ and their unstable manifold $\mathbb{R}^n \subset \mathbb{T}^n = (\mathbb{R} \cup \{\infty\})^n$. The restriction $h_{\mathbb{R}}$ of h to $\mathbb{R}^n$ commutes with the linear map $x \mapsto kx$. By the same argument as in the proof of Lemma 2.8, the map $h_{\mathbb{R}}$ is linear. Take $(a_{ij})_{i,j=1}^n$ such that $h_{\mathbb{R}}(e_j) = \sum_{i=1}^n a_{ij} e_i$.

We set

$$V_j = \{(x_1, \ldots, x_n) \in \mathbb{T}^n \mid x_j = \infty, x_i \neq \infty \text{ if } i \neq j\}$$

for $i = 1, \ldots, n$. Each V_i is a submanifold of V_j which is diffeomorphic to $\mathbb{R}^{n-1}$. Since h is continuous, we have

$$h(V_j) \subset \bigcap_{a_{ij} \neq 0} V_i.$$

Since h is a diffeomorphism of $\mathbb{T}^n$, there exists a unique $\sigma(j) \in \{1, \ldots, n\}$ such that $a_{ij} \neq 0$ for each $j = 1, \ldots n$. Since the linear transformation $h|_{\mathbb{R}}$ is invertible, σ is a permutation of $\{1, \ldots, n\}$. Therefore, $h_{\mathbb{R}}$ is an element of G.

References

[1] M. Asaoka, Rigidity of certain solvable actions on the sphere, *Geom. Topol.* **16** (2012), 1835–1857.

[2] L. Burslem and A. Wilkinson, Global rigidity of solvable group actions on S^1, *Geom. Topol.* **8** (2004), 877–924.

[3] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge Univ. Press, Cambridge, 1995.

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