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Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface.

Kōta Yoshioka

Dedicated to Shigeru Mukai for his sixtieth birthday

Abstract.

We shall study the chamber structure of positive cone of the albanese fiber of the moduli spaces of stable objects on an abelian surface via the chamber structure of stability conditions.

§0. Introduction

The space of stability conditions on an abelian surfaces X is studied by Bridgeland in [8]. In particular, he completely described a connected component $\mathrm{Stab}(X)^*$ consisting of stability conditions σ such that the structure sheaves of points k_x $(x \in X)$ are stable of a fixed phase ϕ. In the space of stability conditions, there is a natural action of the universal cover $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ of $\mathrm{GL}^+(2,\mathbb{R})$. In our situation, $\mathrm{Stab}(X)^*/\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is isomorphic to $\mathrm{NS}(X)_\mathbb{R} \times \mathrm{Amp}(X)_\mathbb{R}$ as stated in [8, sect. 15]. In particular, if $\mathrm{NS}(X) = \mathbb{Z}H$, then $\mathrm{Stab}(X)^*/\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is isomorphic to the upper half plane $\mathbb{H}$. For the stability conditions $\sigma_{(\beta,\omega)}$ corresponding to $(\beta,\omega) \in \mathrm{NS}(X)_\mathbb{R} \times \mathrm{Amp}(X)_\mathbb{R}$, moduli spaces of $\sigma_{(\beta,\omega)}$-semi-stable objects are extensively studied in [18], [19] and [28]. In particular, the projectivity of the moduli spaces are proved for a general $\sigma_{(\beta,\omega)}$. We also constructed ample line bundles on the moduli spaces. As a consequence of these results, we also got some results on the moduli spaces of Gieseker semi-stable sheaves. Indeed for a parameter $(\beta,\omega) = (\beta, tH)$ $(t \gg 0)$ called the large volume limit, Bridgeland stability coincides with Gieseker stability. For the study of Gieseker stability on abelian surfaces, Fourier-Mukai transforms are very important tool, though

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Gieseker stability is not preserved in general. For the proof of projectivity of the moduli space of Bridgeland semi-stable objects, we constructed a Fourier-Mukai transform which induces an isomorphism to a moduli space of Gieseker semi-stable objects. In this sense, Bridgeland stability is regarded as a minimal generalization of Gieseker stability preserved by Fourier-Mukai transforms.

In this note, we continue to study the moduli spaces of Bridgeland semi-stable objects. In particular, we shall study the birational geometry of the moduli spaces. Before explaining our main results, we prepare some notation and explain some results in [19]. For the algebraic cohomology groups $H^*(X,\mathbb{Z})_{\mathrm{alg}} := \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\varrho_X$, let $\langle\ ,\ \rangle$ be the Mukai pairing, where ϱ_X is the fundamental class of X. For $x = x_0 + x_1 + x_2\varrho_X$ with $x_0, x_2 \in \mathbb{Z}$ and $x_1 \in \mathrm{NS}(X)$, we also write $x = (x_0, x_1, x_2)$. For $E \in \mathbf{D}(X)$, $v(E) = \mathrm{ch}(E)$ denotes the Mukai vector of E. For $v \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$, $M_{(\beta,\omega)}(v)$ denotes the moduli space of $\sigma_{(\beta,\omega)}$-semi-stable objects E with $v(E) = v$. $M_{(\beta,\omega)}(v)$ is a projective scheme if (β,ω) is general ([19, Thm. 1.4]). If v is primitive and $\langle v^2\rangle \geq 6$, then as a Bogomolov factor, we have an irreducible symplectic manifold $K_{(\beta,\omega)}(v)$ which is deformation equivalent to the generalized Kummer variety constructed by Beauville [5]. $K_{(\beta,\omega)}(v)$ is a fiber of the albanese map of $M_{(\beta,\omega)}(v)$. We also have an isometry

$$\theta_{v,\beta,\omega} : v^\perp \cap H^*(X,\mathbb{Z})_{\mathrm{alg}} \to \mathrm{NS}(K_{(\beta,\omega)}(v))$$

where $\mathrm{NS}(K_{(\beta,\omega)}(v))$ is equipped with the Beauville-Fujiki form. For a Mukai vector v of a coherent sheaf (i.e., $v = v(F), F \in \mathrm{Coh}(X)$), $M_H^\beta(v)$ denotes the moduli space of β-twisted semi-stable sheaves E with $v(E) = v$. If $\beta = 0$, then we denote it by $M_H(v)$. Since $M_H^\beta(v) = M_{(\beta,tH)}(v)$ $(t \gg 0)$, a fiber $K_H^\beta(v)$ of the albanese map is $K_{(\beta,tH)}(v)$.

In [19, sect. 5.3], we relate the ample cone of $K_{(\beta,\omega)}(v)$ to a chamber structure of $\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$. In this note, we refine this correspondence. For a Mukai vector $v \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$, we shall construct a map from our space of stability conditions $\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$ to the positive cone $P^+(v^\perp)_{\mathbb{R}}$ of $v^\perp$. This map is surjective up to the action of $\mathbb{R}_{>0}$ on $P^+(v^\perp)_{\mathbb{R}}$. More precisely, we slightly extend the map in order to treat the boundary of positive cone. In order to state the precise statement (Proposition 0.1), we need more notation.

We fix a norm $||\ \ ||$ on $\mathrm{NS}(X)_{\mathbb{R}}$. For the closure $\overline{\mathrm{Amp}(X)_{\mathbb{R}}}$ of the ample cone of X, we set

$$\begin{aligned} C(\overline{\mathrm{Amp}(X)_{\mathbb{R}}}) &:= \{x \in \overline{\mathrm{Amp}(X)_{\mathbb{R}}} \mid ||x|| = 1\}, \\ \overline{\mathfrak{H}} &:= \mathrm{NS}(X)_{\mathbb{R}} \times C(\overline{\mathrm{Amp}(X)_{\mathbb{R}}}) \times \mathbb{R}_{\geq 0}. \end{aligned}$$

Then we have an embedding $\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}} \to \overline{\mathfrak{H}}$ by sending (β,ω) to $(\beta, \omega/||\omega||, ||\omega||)$.

For $v = (r, c_1, a) \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$, we set

$$\overline{P^+(v^\perp)_{\mathbb{R}}} := \left\{ x \in H^*(X,\mathbb{Z})_{\mathrm{alg}} \otimes \mathbb{R} \;\middle|\; \begin{array}{r} x \in v^\perp, \langle v^2 \rangle \geq 0, \\ \langle x, rH_0 + (H_0, c_1)\varrho_X \rangle > 0 \end{array} \right\},$$

where H_0 is an ample divisor on X. $\overline{P^+(v^\perp)_{\mathbb{R}}}$ is the closure of the positive cone $P^+(v^\perp)_{\mathbb{R}}$ of $v^\perp$.

For $(\beta, H, t) \in \overline{\mathfrak{H}}$, we set

$$\begin{aligned} \xi(\beta, H, t) := & \left(r\frac{t^2(H^2)}{2} + \langle e^\beta, v \rangle \right) (H + (\beta, H)\varrho_X) \\ & - (c_1 - r\beta, H) \left(e^\beta - \frac{t^2(H^2)}{2}\varrho_X \right). \end{aligned}$$

Then $\xi(\beta, H, t) \in \overline{P^+(v^\perp)_{\mathbb{R}}}$.

Proposition 0.1 (Proposition. 3.11)**.** *We have a surjective map*

$$\begin{array}{cccc} \Xi : & \overline{\mathfrak{H}} & \to & \overline{P^+(v^\perp)_{\mathbb{R}}}/\mathbb{R}_{>0} \\ & (\beta, H, t) & \mapsto & \mathbb{R}_{>0}\xi(\beta, H, t). \end{array}$$

Moreover if tH is ample, then $\xi(\beta, H, t)$ belongs to the positive cone of $v^\perp$.

We introduce the wall and chamber structures on $\overline{\mathfrak{H}}$ and $\overline{P^+(v^\perp)_{\mathbb{R}}}$ and show that they correspond each other. By using these descriptions, we also study the movable cone of $K_{(\beta,\omega)}(v)$.

Let $\mathfrak{W}$ be the set of Mukai vectors v_1 such that

$$\langle v_1, v - v_1 \rangle > 0, \ \langle v_1^2 \rangle \geq 0, \ \langle (v - v_1)^2 \rangle \geq 0. \tag{0.1}$$

Then we have a chamber structure on $P^+(v^\perp)_{\mathbb{R}}$ by the set of walls

$$\{ v_1^\perp \mid v_1 \in \mathfrak{W} \}.$$

Theorem 0.2 (Theorem 3.31)**.** *Assume that $(\beta, H, t) \in \overline{\mathfrak{H}}$ satisfies $\xi(\beta, H, t) \notin \cup_{v_1 \in \mathfrak{W}} v_1^\perp$. Let $\mathfrak{I}$ be the set of primitive and isotropic Mukai vectors u with $\langle u, v \rangle = 0, 1, 2$. Let $\mathcal{D}(\beta, tH)$ be the connected component of $P^+(v^\perp)_{\mathbb{R}} \setminus \cup_{u \in \mathfrak{I}} u^\perp$ containing $\xi(\beta, H, t)$. Then*

$$\overline{\mathrm{Mov}(K_{(\beta,tH)}(v))_{\mathbb{R}}} = \theta_{v,\beta,tH}(\overline{\mathcal{D}(\beta, tH)}).$$

Moreover

$$\theta_{v,\beta,tH}(H^*(X,\mathbb{Z})_{\mathrm{alg}} \cap \overline{\mathcal{D}(\beta, tH)}) \subset \mathrm{Mov}(K_{(\beta,tH)}(v)).$$

In the movable cone of $K_{(\beta,tH)}(v)$, Hassett and Tschinkel [9, Thm. 7, Prop. 17] introduced the chamber structure. The chamber structure of $\mathcal{D}(\beta, tH)$ by $\{v_1^\perp \mid v_1 \in \mathfrak{W}\}$ corresponds to the chamber structure of the interior of $\mathrm{Mov}(K_{(\beta,tH)}(v))$ via $\theta_{v,\beta,tH}$.

As an application of our results, we get a result on the birational structure of $M_H^\beta(v)$.

Proposition 0.3 (Proposition 3.39)**.** *Let (X, H) be a polarized abelian surface and v a Mukai vector such that $2\ell := \langle v^2 \rangle \geq 6$. Then $M_H^\beta(v)$ is birationally equivalent to $\mathrm{Pic}^0(Y) \times \mathrm{Hilb}_Y^\ell$ if and only if there is an isotropic Mukai vector $w \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ with $\langle v, w \rangle = 1$, where Y is an abelian surface.*

This result also follows from a characterization of the generalized Kummer variety by Markman and Mehrotra [16]. Proposition 0.3 gives an affirmative solution of a conjecture of Mukai [24].

Corollary 0.4 (Corollary 3.42)**.** *Let (X, H) be a principally polarized abelian surface with $\mathrm{NS}(X) = \mathbb{Z}H$. Let $v = (r, dH, a)$ be a Mukai vector with $\ell := d^2 - ra \geq 3$. Then $M_H^\beta(v)$ is birationally equivalent to $X \times \mathrm{Hilb}_X^\ell$ if and only if the quadratic equation*

$$rx^2 + 2dxy + ay^2 = \pm 1$$

has an integer valued solution.

We also study the location of walls. If $\mathrm{rk}\,\mathrm{NS}(X) \geq 2$, we show that the stabilizer of v in the group of autoequivalences is infinite. Hence if there is a wall, then we can generate infinitely many walls by the action of autoequivalences. We also show that there is an example of X and v such that there is no wall, which implies that the ample cone of $K_H^\beta(v)$ is the same as the positive cone and the autoequivalences act as automorphisms of $M_H^\beta(v)$.

The study of the movable cone is motivated by recent works [1] and [3]. They studied the movable cones of the moduli spaces for the projective plane and a $K3$ surface by analyzing the chamber structure of Bridgeland's stability. For an irreducible symplectic manifold, Markman [14] studied the movable cone extensively. In particular, he obtained a numerical characterization of the movable cone. In this sense, our result (Theorem 3.31) gives concrete examples of his results. In particular, we give a moduli-theoretic explanation of birational models of $K_H^\beta(v)$.

Let us briefly explain the contents of this note. In section 1, we introduce some notations and recall known results on irreducible symplectic manifolds. In particular, we define our parameter space of stability condition and the wall for stability conditions. We also give a characterization of the walls in terms of Mukai lattice (Proposition 1.3). In section 2, we shall study the cohomological action of the autoequivalences of $\mathbf{D}(X)$, which will be used to study the set of walls. We first treat the case where $\operatorname{rk}\operatorname{NS}(X) = 1$. In this case, we can use the 2 by 2 matrices description of the cohomological action of the Fourier-Mukai transforms in [27]. We then describe the stabilizer group $\operatorname{Stab}(v)$ of a Mukai vector v. By using it, we shall construct many autoequivalences fixing v for all abelian surfaces.

In section 3, we relate our space of stability condition with the positive cone of the moduli spaces. We first construct a map from the space of stability conditions to the positive cone. Then we describe the nef cone of the moduli spaces. In subsection 3.3, we study the divisorial contractions of the moduli spaces. Then we get the description of movable cones (Theorem 3.31).

In section 4, as an example, we treat the case where $\operatorname{NS}(X) = \mathbb{Z}H$. In this case, the boundaries of $\overline{P^+(v^\perp)_\mathbb{R}}$ are spanned by two isotropic vectors $v_\pm$. For a Mukai vector $v = (r, dH, a)$, we show that $v_\pm$ are not defined over $\mathbb{Q}$ if and only if $\sqrt{\langle v^2\rangle/(H^2)} \notin \mathbb{Q}$. For the rank 1 case, this condition is equivalent to the existence of infinitely many walls [28]. According to Markman's solution [14] of the movable cone conjecture of Kawamata and Morrison ([11], [21]), we have infinitely many walls under this condition. By our correspondence of the space of stability conditions and the positive cone, we see that the accumulation points correspond to the two boundaries $\mathbb{R}_{>0}v_\pm$ which are the accumulation points set of walls. Thus we get an explanation of the existence of accumulation points in terms of the positive cone. For the general cases, if $\sqrt{\langle v^2\rangle/(H^2)} \notin \mathbb{Q}$, then we show that infinitely many Fourier-Mukai transforms preserve v as in the rank 1 case. So there are infinitely many walls if there is a wall. However as in the case where $\operatorname{rk}\operatorname{NS}(X) \geq 2$, we have an abelian surface and a Mukai vector v such that there is no wall for v. In section 5, we shall explain how our result on the movable cone follows from Markman's general theory. In appendix, we shall study the base of Lagrangian fibrations.

After we wrote the first version of this note, Bayer and Macri [4] completed their study of the birational geometry of moduli spaces over $K3$ surfaces. In particular, they completely described the nef cone and the movable cone of the moduli spaces. Moreover the results are generalized to deformations of the moduli spaces [2], [20].

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§1. Preliminaries

1.1. Notation.

We denote the category of coherent sheaves on X by $\mathrm{Coh}(X)$ and the bounded derived category of $\mathrm{Coh}(X)$ by $\mathbf{D}(X)$. A Mukai lattice of X consists of $H^{2*}(X,\mathbb{Z}) := \bigoplus_{i=0}^{2} H^{2i}(X,\mathbb{Z})$ and an integral bilinear form $\langle\ ,\ \rangle$ on $H^{2*}(X,\mathbb{Z})$:

$$\langle x_0 + x_1 + x_2\varrho_X, y_0 + y_1 + y_2\varrho_X\rangle := (x_1, y_1) - x_0y_2 - x_2y_0 \in \mathbb{Z},$$

where $x_1, y_1 \in H^2(X,\mathbb{Z})$, $x_0, x_2, y_0, y_2 \in \mathbb{Z}$ and $\varrho_X \in H^4(X,\mathbb{Z})$ is the fundamental class of X. We also introduce the algebraic Mukai lattice as the pair of $H^*(X,\mathbb{Z})_{\mathrm{alg}} := \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$ and $\langle\ ,\ \rangle$ on $H^*(X,\mathbb{Z})_{\mathrm{alg}}$. For $x = x_0 + x_1 + x_2\varrho_X$ with $x_0, x_2 \in \mathbb{Z}$ and $x_1 \in H^2(X,\mathbb{Z})$, we also write $x = (x_0, x_1, x_2)$. For $E \in \mathbf{D}(X)$, $v(E) := \mathrm{ch}(E)$ denotes the Mukai vector of E.

For $\mathbf{E} \in \mathbf{D}(X \times Y)$, we set

$$\Phi^{\mathbf{E}}_{X\to Y}(x) := \mathbf{R}p_{Y*}(\mathbf{E} \otimes p_X^*(x)),\ x \in \mathbf{D}(X),$$

where p_X, p_Y are projections from $X \times Y$ to X and Y respectively. Let $\mathrm{Eq}(\mathbf{D}(X), \mathbf{D}(Y))$ be the set of equivalences between $\mathbf{D}(X)$ and $\mathbf{D}(Y)$. We set

$$\begin{aligned}
&\mathrm{Eq}_0(\mathbf{D}(Y), \mathbf{D}(Z)) \\
:= &\left\{ \Phi^{\mathbf{E}[2k]}_{Y\to Z} \in \mathrm{Eq}(\mathbf{D}(Y), \mathbf{D}(Z)) \,\middle|\, \mathbf{E} \in \mathrm{Coh}(Y \times Z),\ k \in \mathbb{Z} \right\}, \\
&\mathcal{E}(Z) := \bigcup_Y \mathrm{Eq}_0(\mathbf{D}(Y), \mathbf{D}(Z)), \\
&\mathcal{E} := \bigcup_Z \mathcal{E}(Z) = \bigcup_{Y,Z} \mathrm{Eq}_0(\mathbf{D}(Y), \mathbf{D}(Z)).
\end{aligned}$$

Note that $\mathcal{E}$ is a groupoid with respect to the composition of the equivalences.

As we explained in the introduction, $\mathrm{Stab}(X)^*/\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is isomorphic to $\mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$. Let us briefly explain a stability condition $\sigma_{(\beta,\omega)}$ associated to $(\beta,\omega) \in \mathrm{NS}(X)_{\mathbb{Q}} \times \mathrm{Amp}(X)_{\mathbb{Q}}$. Let $\mathfrak{T}_{(\beta,\omega)}$ be a full subcategory of $\mathrm{Coh}(X)$ generated by torsion sheaves and μ-stable torsion free sheaves E with $(c_1(E) - \mathrm{rk}\, E\beta, \omega) > 0$, and let $\mathfrak{F}_{(\beta,\omega)}$ be a full subcategory of $\mathrm{Coh}(X)$ generated by μ-stable torsion free sheaves E

with $(c_1(E) - \operatorname{rk} E\beta, \omega) \leq 0$. $(\mathfrak{T}_{(\beta,\omega)}, \mathfrak{F}_{(\beta,\omega)})$ is a torsion pair of $\operatorname{Coh}(X)$. Let $\mathfrak{A}_{(\beta,\omega)}$ be its tilting. Thus,

$$\mathfrak{A}_{(\beta,\omega)} := \left\{ E \in \mathbf{D}(X) \;\middle|\; \begin{array}{l} H^i(E) = 0,\ i \neq -1, 0, \\ H^{-1}(E) \in \mathfrak{F}_{(\beta,\omega)},\ H^0(E) \in \mathfrak{T}_{(\beta,\omega)} \end{array} \right\}.$$

Let $Z_{(\beta,\omega)} : \mathbf{D}(X) \to \mathbb{C}$ is a group homomorphism called the stability function. In terms of the Mukai lattice $(H^*(X,\mathbb{Z})_{\mathrm{alg}}, \langle\ ,\ \rangle)$, $Z_{(\beta,\omega)}$ is given by

$$Z_{(\beta,\omega)}(E) = \langle e^{\beta+\sqrt{-1}\omega}, v(E) \rangle, \quad E \in \mathbf{D}(X).$$

Then $Z_{(\beta,\omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ for $0 \neq E \in \mathfrak{A}_{(\beta,\omega)}$. We define the phase $\phi_{(\beta,\omega)}(E) \in (0,1]$ of $0 \neq E \in \mathfrak{A}_{(\beta,\omega)}$ by

$$Z_{(\beta,\omega)}(E) = |Z_{(\beta,\omega)}(E)| e^{\pi\sqrt{-1}\phi_{(\beta,\omega)}(E)}.$$

Then $(\mathfrak{A}_{(\beta,\omega)}, Z_{(\beta,\omega)})$ is the stability condition $\sigma_{(\beta,\omega)}$. In particular, k_x is a stable object of the phase $\phi_{(\beta,\omega)}(k_x) = 1$.

Definition 1.1. (1) An object $0 \neq E \in \mathfrak{A}_{(\beta,\omega)}$ is $\sigma_{(\beta,\omega)}$-semi-stable if

$$\phi_{(\beta,\omega)}(F) \leq \phi_{(\beta,\omega)}(E)$$

for all proper subobject $F \neq 0$ of E. If the inequality is strict, then E is $\sigma_{(\beta,\omega)}$-stable.

(2) An object $0 \neq E \in \mathbf{D}(X)$ is $\sigma_{(\beta,\omega)}$-semi-stable (resp. $\sigma_{(\beta,\omega)}$-stable), if there is an integer n such that $E[-n] \in \mathfrak{A}_{(\beta,\omega)}$ and $E[-n]$ is $\sigma_{(\beta,\omega)}$-semi-stable (resp. $\sigma_{(\beta,\omega)}$-stable).

1.2. A parameter space of stability conditions.

For an abelian surface X, the ample cone $\operatorname{Amp}(X)$ is described as

$$\operatorname{Amp}(X) = \{x \in \operatorname{NS}(X) \mid (x^2) > 0, (x,h) > 0\},$$

where $h \in \operatorname{NS}(X)$ is an ample class of X. We set

$$\overline{\operatorname{Amp}(X)}_k := \{x \in \operatorname{NS}(X)_k \mid (x^2) \geq 0, (x,h) > 0\},$$

where $k = \mathbb{Q}, \mathbb{R}$. For a cone $V \subset \mathbb{R}^m$, we set $C(V) := (V \setminus \{0\})/\mathbb{R}_{>0}$. We fix a norm $||\ \ ||$ on $\mathbb{R}^m$ and identify $C(V)$ with $\{x \in V \mid ||x|| = 1\}$. Then we have a bijection $V \setminus \{0\} \to C(V) \times \mathbb{R}_{>0}$ by sending $x \in V \setminus \{0\}$ to $(x/||x||, ||x||)$.

We have a map

$$\begin{array}{ccc} C(\overline{\operatorname{Amp}(X)}_{\mathbb{R}}) \times \mathbb{R}_{\geq 0} & \to & \overline{\operatorname{Amp}(X)}_{\mathbb{R}} \cup \{0\} \\ (L,t) & \mapsto & tL \end{array}$$

which is bijective over $\overline{\operatorname{Amp}(X)}_{\mathbb{R}}$ and the fiber over 0 is $\overline{\operatorname{Amp}(X)}_{\mathbb{R}} \times \{0\}$. Thus $C(\overline{\operatorname{Amp}(X)}_{\mathbb{R}}) \times \mathbb{R}_{\geq 0}$ is a partial compactification of $\overline{\operatorname{Amp}(X)}_{\mathbb{R}}$.

We set

$$\begin{aligned} \mathfrak{H} &:= \operatorname{NS}(X)_{\mathbb{R}} \times C(\operatorname{Amp}(X)_{\mathbb{R}}) \times \mathbb{R}_{>0}, \\ \overline{\mathfrak{H}} &:= \operatorname{NS}(X)_{\mathbb{R}} \times C(\overline{\operatorname{Amp}(X)}_{\mathbb{R}}) \times \mathbb{R}_{\geq 0}. \end{aligned}$$

We have an identification

$$\begin{array}{ccc} \mathfrak{H} & \to & \operatorname{NS}(X)_{\mathbb{R}} \times \operatorname{Amp}(X)_{\mathbb{R}} \\ (\beta,H,t) & \mapsto & (\beta,tH) \end{array} \tag{1.1}$$

and these spaces are our parameter space of stability conditions and its partial compactification.

Let us introduce a wall and chamber structure on $\overline{\mathfrak{H}}$.

Definition 1.2 (cf. [28, Defn. 2.7])**.** Let v be a Mukai vector.

(1) For a Mukai vector v_1 satisfying

$$\langle v_1, v-v_1\rangle > 0,\ \langle v_1^2\rangle \geq 0,\ \langle (v-v_1)^2\rangle \geq 0, \tag{1.2}$$

we define the wall W_{v_1} as

$$W_{v_1} := \{(\beta,H,t) \in \overline{\mathfrak{H}} \mid \mathbb{R}Z_{(\beta,tH)}(v_1) = \mathbb{R}Z_{(\beta,tH)}(v)\}. \tag{1.3}$$

(2) $\mathfrak{W}$ denotes the set of Mukai vectors v_1 satisfying (1.2).

(3) A chamber for stabilities is a connected component of $\overline{\mathfrak{H}} \setminus \cup_{v_1 \in \mathfrak{W}} W_{v_1}$.

(4) We also have a wall and chamber structure on $\operatorname{NS}(X)_{\mathbb{R}} \times \operatorname{Amp}(X)_{\mathbb{R}}$ via (1.1), which is the same as was introduced in [28], [19].

(5) We say that $(\beta,H,t) \in \mathfrak{H}$ (resp. $(\beta,\omega) \in \operatorname{NS}(X)_{\mathbb{R}} \times \operatorname{Amp}(X)_{\mathbb{R}}$) is general, if it is in a chamber.

As we explained in [28], [19, Prop. 5.7] implies that if $(\beta,H,t) \in W_{v_1}$, there is a properly $\sigma_{(\beta,tH)}$-semi-stable object E with $v(E)=v$. In general, W_{v_1} may be an empty set. We have the following characterization for the non-emptiness of the wall whose proof is given in subsection 3.2.

Proposition 1.3. *Let v_1 be a Mukai vector satisfying* (1.2)*. Then $W_{v_1} \cap \mathfrak{H} \neq \emptyset$ if and only if*

$$\langle v, v_1 \rangle^2 > \langle v^2 \rangle \langle v_1^2 \rangle. \tag{1.4}$$

Corollary 1.4. *Let w be an isotropic Mukai vector. If $\langle v^2 \rangle/2 > \langle w, v \rangle > 0$, then w satisfies* (1.2) *and $W_w \cap \mathfrak{H}$ is non-empty. In particular, if $\langle w, v \rangle = 1, 2$ and $\langle v^2 \rangle \geq 6$, then w satisfies* (1.2) *and $W_w \cap \mathfrak{H} \neq \emptyset$.*

We set $v_1 := (r_1, \xi_1, a_1)$. Then the defining equation of W_{v_1} is

(1.5)

$$\begin{aligned} &\det \begin{pmatrix} a - (\xi, \beta) + r\frac{(\beta^2) - t^2(H^2)}{2} & a_1 - (\xi_1, \beta) + r_1 \frac{(\beta^2) - t^2(H^2)}{2} \\ -(\xi - r\beta, H) & -(\xi_1 - r_1\beta, H) \end{pmatrix} \\ =&(\xi_1 - r_1\beta, H)a - (\xi - r\beta, H)a_1 + (r_1\xi - r\xi_1, \beta)(\beta, H) \\ &+ (\xi, \beta)(\xi_1, H) - (\xi_1, \beta)(\xi, H) - (r_1\xi - r\xi_1, H)\frac{(\beta^2) - t^2(H^2)}{2} = 0. \end{aligned}$$

Lemma 1.5. (1) *If $r_1\xi - r\xi_1 \neq 0$, then*

$$W_{v_1} \not\supset \{(\beta, H, t) \in \overline{\mathfrak{H}} \mid (\xi - r\beta, H) = 0\}.$$

(2) *If $r_1\xi - r\xi_1 = 0$, then*

$$W_{v_1} = \{(\beta, H, t) \in \overline{\mathfrak{H}} \mid (\xi - r\beta, H) = 0\}.$$

Proof. (1) Assume that $r_1\xi - r\xi_1 \neq 0$. Then we can take $H \in \operatorname{Amp}(X)_{\mathbb{Q}}$ with $(r_1\xi - r\xi_1, H) \neq 0$. We take $\beta \in \operatorname{NS}(X)_{\mathbb{Q}}$ with $(\xi - r\beta, H) = 0$. Then we have $(\xi_1 - r_1\beta, H) \neq 0$. Since $Z_{(\beta, tH)}(v) \neq 0$, (1.5) implies that $(\beta, H, t) \notin W_{v_1}$. Since the hypersurface $(\xi - r\beta, H) = 0$ is irreducible, we get the claim.

(2) If $r_1\xi - r\xi_1 = 0$, then (1.5) implies that

$$\left(\frac{r_1}{r}a - a_1\right)(\xi - r\beta, H) = 0.$$

Since $v_1 \notin \mathbb{Q}v$, W_{v_1} is defined by $(\xi - r\beta, H) = 0$. Q.E.D.

Remark 1.6. The assumption of Lemma 1.5 (2) is equivalent to $\varrho_X^{\perp} \cap v^{\perp} = v_1^{\perp} \cap v^{\perp}$. Indeed $\varrho_X^{\perp} \cap v^{\perp} = v_1^{\perp} \cap v^{\perp}$ is equivalent to $\mathbb{Q}v + \mathbb{Q}v_1 = \mathbb{Q}v + \mathbb{Q}\varrho_X$. Since $v_1 \notin \mathbb{Q}v$, it is equivalent to $v_1 \in \mathbb{Q}v + \mathbb{Q}\varrho_X$.

1.3. Facts on irreducible symplectic manifolds.

For a smooth projective manifold M, $\mathrm{Amp}(M)_k \subset \mathrm{NS}(M)_k$ denotes the ample cone of M and $\mathrm{Nef}(M)_k \subset \mathrm{NS}(M)_k$ denotes the nef cone of numerically effective divisors on M, where $k = \mathbb{Q}, \mathbb{R}$.

Definition 1.7. Let M be a smooth projective manifold.

(1) (a) A divisor D on M is movable, if the base locus of $|D|$ has codimension ≥ 2.

(b) $\mathrm{Mov}(M)_k \subset \mathrm{NS}(X)_k$ $(k = \mathbb{Q}, \mathbb{R})$ denotes the cone generated by movable divisors and $\overline{\mathrm{Mov}(M)_{\mathbb{R}}}$ the closure in $\mathrm{NS}(X)_{\mathbb{R}}$.

(2) For an irreducible symplectic manifold M, q_M denotes the Beauville-Fujiki form on $H^2(M, \mathbb{Z})$. Then the positive cone is defined as

$$P^+(M)_k := \{x \in \mathrm{NS}(X)_k \mid q_M(x,x) > 0, q_M(x,h) > 0\}$$

where $k = \mathbb{Q}, \mathbb{R}$ and h is an ample divisor on M. We also set

$$\overline{P^+(M)}_k := \{x \in \mathrm{NS}(X)_k \mid q_M(x,x) \geq 0, q_M(x,h) > 0\}.$$

Remark 1.8. By the definition, $\mathrm{Mov}(M)_{\mathbb{Q}} = \mathrm{Mov}(M)_{\mathbb{R}} \cap \mathrm{NS}(M)_{\mathbb{Q}}$.

We note that $\mathrm{Mov}(M)_{\mathbb{Q}}$ is contained in $\overline{P^+(M)}_{\mathbb{Q}}$ by works of Huybrechts ([10], [9, Thm. 7]). There is a different argument in [14, Lem. 6.22] based on results of Boucksom [7].

1.4. Moduli spaces

Definition 1.9. A Mukai vector $v := (r, \xi, a) \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ is positive, if

(i) $r > 0$ or
(ii) $r = 0$ and ξ is effective or
(iii) $r = \xi = 0$ and $a > 0$.

Definition 1.10. Let $v \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ be a Mukai vector.

(1) If v is positive, then let $M_H^\beta(v)$ be the moduli space of β-twisted semi-stable sheaves E on X with $v(E) = v$. If $\beta = 0$, then we also denote $M_H^\beta(v)$ by $M_H(v)$.

(2) $M_{(\beta,\omega)}(v)$ denotes the moduli space of $\sigma_{(\beta,\omega)}$-semi-stable objects E with $v(E) = v$.

Remark 1.11. (1) If H is general in $\mathrm{Amp}(X)$, then $M_H^\beta(v)$ does not depend on the choice of β.

(2) If v is positive, then $M_{(\beta+sH,tH)}(v) = M_H^\beta(v)$ for some (s,t). Thus twisted semi-stability is a special case of Bridgeland semi-stability.

Assume that v is primitive and (β,ω) is general with respect to v. We fix $E_0 \in M_{(\beta,\omega)}(v)$. Let

$$\Phi^{\mathbf{P}}_{X\to\widehat{X}} : \mathbf{D}(X) \to \mathbf{D}(\widehat{X})$$

be the Fourier-Mukai transform by the Poincare line bundle $\mathbf{P}$ on $X\times\widehat{X}$, where $\widehat{X} := \mathrm{Pic}^0(X)$ is the dual of X. Then we have an albanese map $\mathfrak{a} : M_{(\beta,\omega)}(v) \to X \times \widehat{X}$ by

$$\mathfrak{a}(E) := (\det(\Phi^{\mathbf{P}}_{X\to\widehat{X}}(E-E_0)), \det(E-E_0)) \in X \times \widehat{X}$$

([19, Rem. 4.10]). $\mathfrak{a}$ is an étale locally trivial fibration.

Definition 1.12. Assume that v is primitive and $\langle v^2\rangle \geq 6$.

(1) $K_{(\beta,\omega)}(v)$ denotes a fiber of the albanese map $M_{(\beta,\omega)}(v) \to X \times \widehat{X}$. If v is positive, then we also denote a fiber of $\mathfrak{a} : M_H^\beta(v) \to X \times \widehat{X}$ by $K_H^\beta(v)$.

(2)

$$\theta_{v,\beta,\omega} : v^\perp \to H^2(M_{(\beta,\omega)}(v),\mathbb{Z}) \to H^2(K_{(\beta,\omega)}(v),\mathbb{Z}) \tag{1.6}$$

denotes the Mukai's homomorphism. If there is a universal family $\mathbf{E}$ on $M_{(\beta,\omega)}(v)$, e.g., there is a Mukai vector w with $\langle v,w\rangle = 1$, then

$$\theta_{v,\beta,\omega}(x) = c_1(p_{M_{(\beta,\omega)}(v)*}(\mathrm{ch}(\mathbf{E})p_X^*(x^\vee)))_{|K_{(\beta,\omega)}(v)},$$

where $p_X, p_{M_{(\beta,\omega)}(v)}$ are projections from $X \times M_{(\beta,\omega)}(v)$ to X and $M_{(\beta,\omega)}(v)$ respectively.

Theorem 1.13 ([19, Prop. 5.16])**.** *For $v \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$, $M_{(\beta,\omega)}(v)$ is a smooth projective symplectic manifold which is deformation equivalent to* $\mathrm{Hilb}_X^{\langle v^2\rangle/2} \times X$. *Assume that $\langle v^2\rangle \geq 6$.*

(1) *$K_{(\beta,\omega)}(v)$ is an irreducible symplectic manifold of*

$$\dim K_{(\beta,\omega)}(v) = \langle v^2\rangle - 2$$

which is deformation equivalent to the generalized Kummer variety constructed by Beauville [5].

(2)

$$\theta_{v,\beta,\omega} : (v^\perp, \langle\ ,\ \rangle) \to (H^2(K_{(\beta,\omega)}(v),\mathbb{Z}), q_{K_{(\beta,\omega)}(v)})$$

is an isometry of Hodge structure.

§2. Fourier-Mukai transforms on abelian surfaces.

2.1. Cohomological Fourier-Mukai transforms

We collect some results on the Fourier-Mukai transforms on abelian surfaces X with $\operatorname{rk}\operatorname{NS}(X) = 1$. Let H_X be the ample generator of $\operatorname{NS}(X)$. We shall describe the action of Fourier-Mukai transforms on the cohomology lattices in [27]. For $Y \in \operatorname{FM}(X)$, we have $(H_Y^2) = (H_X^2)$. We set $n := (H_X^2)/2$. In [27, sect. 6.4], we constructed an isomorphism of lattices

$$\begin{array}{cccc} \iota_X : & (H^*(X,\mathbb{Z})_{\mathrm{alg}}, \langle\ ,\ \rangle) & \xrightarrow{\sim} & (\operatorname{Sym}_2(\mathbb{Z},n), B), \\ & (r, dH_X, a) & \mapsto & \begin{pmatrix} r & d\sqrt{n} \\ d\sqrt{n} & a \end{pmatrix}, \end{array}$$

where $\operatorname{Sym}_2(\mathbb{Z},n)$ is given by

$$\operatorname{Sym}_2(\mathbb{Z},n) := \left\{ \begin{pmatrix} x & y\sqrt{n} \\ y\sqrt{n} & z \end{pmatrix} \middle| \, x,y,z \in \mathbb{Z} \right\},$$

and the bilinear form B on $\operatorname{Sym}_2(\mathbb{Z},n)$ is given by

$$B(X_1, X_2) := 2ny_1y_2 - (x_1z_2 + z_1x_2)$$

for $X_i = \begin{pmatrix} x_i & y_i\sqrt{n} \\ y_i\sqrt{n} & z_i \end{pmatrix} \in \operatorname{Sym}_2(\mathbb{Z},n)$ $(i = 1,2)$.

Each $\Phi_{X\to Y}$ gives an isometry

$$\iota_Y \circ \Phi^H_{X\to Y} \circ \iota_X^{-1} \in \operatorname{O}(\operatorname{Sym}_2(\mathbb{Z},n)), \tag{2.1}$$

where $\operatorname{O}(\operatorname{Sym}_2(\mathbb{Z},n))$ is the isometry group of the lattice $(\operatorname{Sym}_2(\mathbb{Z},n), B)$. Thus we have a map

$$\eta : \mathcal{E} \to \operatorname{O}(\operatorname{Sym}_2(\mathbb{Z},n))$$

which preserves the structures of multiplications.

Definition 2.1. We set

$$\begin{aligned} \widehat{G} &:= \left\{ \begin{pmatrix} a\sqrt{r} & b\sqrt{s} \\ c\sqrt{s} & d\sqrt{r} \end{pmatrix} \middle| \begin{array}{r} a,b,c,d,r,s \in \mathbb{Z},\ r,s > 0 \\ rs = n,\ adr - bcs = \pm 1 \end{array} \right\}, \\ G &:= \widehat{G} \cap \operatorname{SL}(2,\mathbb{R}). \end{aligned}$$

We have a right action $\cdot$ of $\widehat{G}$ on the lattice $(\operatorname{Sym}_2(\mathbb{Z},n), B)$:

$$(2.2)\qquad \begin{pmatrix} r & d\sqrt{n} \\ d\sqrt{n} & a \end{pmatrix} \cdot g := {}^t g \begin{pmatrix} r & d\sqrt{n} \\ d\sqrt{n} & a \end{pmatrix} g,\ g \in \widehat{G}.$$

Thus we have an anti-homomorphism:

$$\alpha : \widehat{G}/\{\pm 1\} \to \mathrm{O}(\mathrm{Sym}_2(n, \mathbb{Z})).$$

Theorem 2.2 ([27, Thm. 6.16, Prop. 6.19])**.** *Let*

$$\Phi \in \mathrm{Eq}_0(\mathbf{D}(Y), \mathbf{D}(X))$$

be an equivalence.

(1) $v_1 := v(\Phi(\mathcal{O}_Y))$ *and* $v_2 := \Phi(\varrho_Y)$ *are positive isotropic Mukai vectors with* $\langle v_1, v_2 \rangle = -1$ *and we can write*

$$\begin{aligned} &v_1 = (p_1^2 r_1, p_1 q_1 H_Y, q_1^2 r_2), \quad v_2 = (p_2^2 r_2, p_2 q_2 H_Y, q_2^2 r_1), \\ &p_1, q_1, p_2, q_2, r_1, r_2 \in \mathbb{Z},\ \ p_1, r_1, r_2 > 0, \\ &r_1 r_2 = n,\ \ p_1 q_2 r_1 - p_2 q_1 r_2 = 1. \end{aligned}$$

(2) *We set*

$$\theta(\Phi) := \pm \begin{pmatrix} p_1\sqrt{r_1} & q_1\sqrt{r_2} \\ p_2\sqrt{r_2} & q_2\sqrt{r_1} \end{pmatrix} \in G/\{\pm 1\}.$$

Then $\theta(\Phi)$ *is uniquely determined by* Φ *and we have a map*

$$\theta : \mathcal{E} \to G/\{\pm 1\}.$$

(3) *The action of* $\theta(\Phi)$ *on* $\mathrm{Sym}_2(n, \mathbb{Z})$ *is the action of* Φ *on* $H^*(X, \mathbb{Z})_{\mathrm{alg}}$*:*

$$\iota_X \circ \Phi(v) = \iota_Y(v) \cdot \theta(\Phi).$$

Thus we have the following commutative diagram:

$$(2.3)\qquad \begin{tikzcd} \mathcal{E} \arrow[d, "\theta"'] \arrow[dr, "\eta"] & \\ \widehat{G}/\{\pm 1\} \arrow[r, "\alpha"'] & \mathrm{O}(\mathrm{Sym}_2(n, \mathbb{Z})) \end{tikzcd}$$

From now on, we identify the Mukai lattice $H^*(X, \mathbb{Z})_{\mathrm{alg}}$ with $\mathrm{Sym}_2(n, \mathbb{Z})$ via ι_X. Then for $g \in \widehat{G}$ and $v \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$, $v \cdot g$ means $\iota_X(v \cdot g) = \iota_X(v) \cdot g$.

For an isotropic Mukai vector $v = (x^2, \frac{xy}{\sqrt{n}} H_X, y^2) = x^2 e^{\frac{y}{x\sqrt{n}} H}$, $v \cdot g = (x'^2, \frac{x'y'}{\sqrt{n}} H_X, y'^2)$, where $(x', y') = (x, y)g$.

We also need to treat the composition of a Fourier-Mukai transform and the dualizing functor $\mathcal{D}_X$. For a Fourier-Mukai transform $\Phi \in \mathrm{Eq}_0(\mathbf{D}(X), \mathbf{D}(Y))$, we set

$$\theta(\Phi \circ \mathcal{D}_X) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \theta(\Phi) \in \widehat{G}/\{\pm 1\}.$$

Then the action of $\theta(\Phi \circ \mathcal{D}_X)$ on $\mathrm{Sym}_2(\mathbb{Z}, n)$ is the same as the action of $\Phi \circ \mathcal{D}_X$.

2.2. A stabilizer subgroup.

We keep the notation in subsection 2.1. In particular, we assume that $\mathrm{rk}\,\mathrm{NS}(X) = 1$. Let $v := (r, dH, a)$ be a primitive Mukai vector with $r \neq 0$. We shall study the stabilizer of $\pm v \in H^*(X, \mathbb{Z})_{\mathrm{alg}}/\{\pm 1\}$ in $\widehat{G}$. Assume that

$$\begin{pmatrix} x & z \\ y & w \end{pmatrix} \begin{pmatrix} r & d\sqrt{n} \\ d\sqrt{n} & a \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \epsilon \begin{pmatrix} r & d\sqrt{n} \\ d\sqrt{n} & a \end{pmatrix}$$

and $xw - yz = \epsilon$. Then we have

$$\begin{aligned} &rx^2 + 2d\sqrt{n}xz + az^2 = \epsilon r, \\ &ry^2 + 2d\sqrt{n}yw + aw^2 = \epsilon a, \\ &rxy + d\sqrt{n}(xw + zy) + azw = \epsilon d\sqrt{n}, \\ &xw - yz = \epsilon. \end{aligned} \tag{2.4}$$

Hence

$$y(rx + 2d\sqrt{n}z) + (az)w = rxy + 2d\sqrt{n}zy + azw = 0.$$

We note that

$$(rx + 2d\sqrt{n}z, az) \neq (0, 0)$$

by

$$x(rx + 2d\sqrt{n}z) + z(az) = \epsilon r \neq 0.$$

We set $y := -\lambda az$ and $w := \lambda(rx + 2d\sqrt{n}z)$. Then

$$\epsilon = xw - yz = \lambda(rx^2 + 2d\sqrt{n}xz + az^2) = \lambda\epsilon r.$$

Hence $\lambda = 1/r$. Therefore

$$y = -\frac{a}{r}z,\ w = x + 2d\sqrt{n}\frac{z}{r}. \tag{2.5}$$

Conversely for x, z with

$$rx^2 + 2d\sqrt{n}xz + az^2 = \epsilon r, \tag{2.6}$$

we define y, w by (2.5). Then (2.4) are satisfied. We note that (2.6) is written as

$$(x + d\sqrt{n}\tfrac{z}{r})^2 - \ell(\tfrac{z}{r})^2 = \epsilon.$$

We set $X := x + d\sqrt{n}\frac{z}{r}$ and $Z := \frac{z}{r}$. Then

$$X^2 - \ell Z^2 = \epsilon \tag{2.7}$$

and

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} X - d\sqrt{n}Z & -aZ \\ rZ & X + d\sqrt{n}Z. \end{pmatrix} = XI_2 + ZF,$$

where

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; F := \begin{pmatrix} -d\sqrt{n} & -a \\ r & d\sqrt{n} \end{pmatrix}. \tag{2.8}$$

We have $F^2 = \ell I_2$. We set

$$\begin{aligned} \mathrm{Stab}_0(v) :=& \{g \in \widehat{G} \mid g(v) = (\det g)v\} \\ =& \{g \in \widehat{G} \mid g = XI_2 + ZF\}. \end{aligned}$$

$\mathrm{Stab}_0(v)$ is a normal subgroup of $\mathrm{Stab}(v)$ of index 2. Indeed for $g \in \mathrm{Stab}(v)$, we have $g(v) = \eta(g)(\det g)v$ and $\eta(gg') = \eta(g)\eta(g')$, $g, g' \in \mathrm{Stab}(v)$. Thus $\ker \eta = \mathrm{Stab}_0(v)$. We get a homomorphism

$$\begin{array}{cccc} \varphi : & \mathrm{Stab}_0(v) & \to & \mathbb{R} \\ & XI_2 + ZF & \mapsto & X + Z\sqrt{\ell}. \end{array}$$

Proposition 2.3. *Assume that $\sqrt{n\ell} \notin \mathbb{Q}$.*

(1) *For $XI_2 + ZF \in \mathrm{Stab}_0(v)$, $X + Z\sqrt{\ell}$ is an algebraic integer such that $(X + Z\sqrt{\ell})^2 \in \mathbb{Q}(\sqrt{n\ell})$.*

(2) $\mathrm{im}\,\varphi \cong \mathbb{Z} \oplus \mathbb{Z}_2$.

(3) *φ is injective or* $\ker \varphi = \{\sqrt{\ell}^{-1} F\}$ *if $\sqrt{\ell}^{-1} F \in \mathrm{Stab}_0(v)$. In particular, if $n = 1$ and $\ell > 1$, then φ is injective.*

Proof. We set $\alpha := X + Z\sqrt{\ell}$. Assume that $XI_2 + ZF \in \widehat{G}$. Then

$$x^2, y^2, xw, yz, \frac{xy}{\sqrt{n}}, \frac{xz}{\sqrt{n}}, \frac{yw}{\sqrt{n}}, \frac{zw}{\sqrt{n}}, \in \mathbb{Z}.$$

Hence

$$\begin{aligned} 2(X^2+d^2nZ^2) =& x^2+w^2 \in \mathbb{Z}, \\ X^2-d^2nZ^2 =& xw \in \mathbb{Z}, \\ r^2\frac{XZ}{\sqrt{n}} =& r\frac{xz}{\sqrt{n}}+dz^2 \in \mathbb{Z}, \end{aligned}$$

which impy that

$$X^2+\ell Z^2, \frac{XZ}{\sqrt{n}} \in \mathbb{Q}. \tag{2.9}$$

We note that α satisfies the equation

$$\alpha^2 - 2X\alpha + \epsilon = 0.$$

Since $2X = x + w$ is an algebraic integer, α is an algebraic integer. By (2.9),

$$\alpha^2 = (X^2+\ell Z^2) + 2\frac{XY}{\sqrt{n}}\sqrt{n\ell} \in \mathbb{Q}(\sqrt{n\ell}).$$

Thus (1) holds.

(2) We first prove that $\operatorname{im}\varphi \neq \{\pm 1\}$. We take a solution $(p,q) \in \mathbb{Z}^{\oplus 2}$ of $p^2 - n\ell q^2 = 1$ such that $q \neq 0$. We set $X := p$ and $Z := q\sqrt{n}$. Then

$$XI_2 + ZF = \begin{pmatrix} p-dnq & -aq\sqrt{n} \\ rq\sqrt{n} & p+dnq \end{pmatrix} \tag{2.10}$$

satisfies all the requirements. Therefore $\operatorname{im}\varphi \neq \{\pm 1\}$. By the Dirichlet unit theorem, the torsion part of $\operatorname{im}\varphi$ is $\{\pm 1\}$. Since α^2 is a unit of the ring of integers of $\mathbb{Q}(\sqrt{n\ell})$, the rank of $\operatorname{im}\varphi$ is 1, which implies the claim.

If $\alpha = X + Z\sqrt{\ell} = 1$, then $\alpha^2 = 1$ implies that $XZ = 0$. If $Z = 0$, then $X = 1$. If $X = 0$, then $Z\sqrt{\ell} = 1$. Therefore the first part of the claims holds.

Assume that $n = 1$ and $\ell > 1$. Then

$$\frac{1}{\sqrt{\ell}}F = \frac{1}{\sqrt{\ell}}\begin{pmatrix} -d\sqrt{n} & -a \\ r & d\sqrt{n} \end{pmatrix}.$$

Hence $\ell \mid a^2, \ell \mid r^2, \ell \mid d^2 n$. Since v is primitive, a^2, r^2, d^2 are relatively prime. Hence $\ell \mid n$, which is a contradiction. Therefore the second part also holds. Q.E.D.

We set

$$\operatorname{Stab}_0(v)^* := \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \operatorname{Stab}_0(v) \middle| xw - yz = 1,\ y \in \sqrt{n}\mathbb{Z} \right\}.$$

All elements of $\operatorname{Stab}_0(v)^*$ come from autoequivalences of $\mathbf{D}(X)$ (see Lemma 2.5 below). $\operatorname{Stab}_0(v)/\operatorname{Stab}_0(v)^*$ is a finite group of type $(\mathbb{Z}/2\mathbb{Z})^{\oplus k}$.

If

$$A := \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \operatorname{Stab}(v) \setminus \operatorname{Stab}_0(v),$$

then

$$A = \begin{pmatrix} x & \frac{1}{r}(az + 2d\sqrt{n}x) \\ z & -x \end{pmatrix}.$$

In particular, $A^2 = \pm I_2$.

Example 2.4. Assume that $n = 1$. Then

$$XI_2 + ZF \in \mathrm{GL}(2, \mathbb{Z}) = \widehat{G} \iff X \pm dZ, aZ, rZ \in \mathbb{Z}.$$

Assume that $2 \mid r$ and $2 \mid a$. Then the primitivity of v implies that $2 \nmid d$. Hence $\ell = d^2 - ra \equiv 1 \mod 4$. Then $\mathfrak{O} := \mathbb{Z}[\frac{1+\sqrt{\ell}}{2}]$ is the ring of integers. We note that $X \pm dZ, aZ, rZ \in \mathbb{Z}$ imply that $2dZ, aZ, rZ \in \mathbb{Z}$. Since $\gcd(r/2, a/2, d) = 1$, we have $2Z \in \mathbb{Z}$. Then $X - dZ \in \mathbb{Z}$ implies that $X - Z \in \mathbb{Z}$. Therefore $X + Z\sqrt{\ell} \in \mathfrak{O}$ with (2.7). Conversely for $X + Z\sqrt{\ell} \in \mathfrak{O}$ with (2.7), we have $X \pm dZ, aZ, rZ \in \mathbb{Z}$. Therefore the fundamental unit of $\mathfrak{O}$ is the generator of $\operatorname{im}\varphi$.

2.3. The case where $\operatorname{rk}\operatorname{NS}(X) \geq 2$.

Assume that $\operatorname{rk}\operatorname{NS}(X) \geq 2$. Let $v = (r, \xi, a)$ be a Mukai vector with $\langle v^2 \rangle = 2\ell$. By using Proposition 2.3, we shall construct many autoequivalences preserving v. Assume that $\xi \in \operatorname{Amp}(X)$ and set $\xi = dH$, where H is a primitive and ample divisor. Let $L := \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\varrho_X$ be a sublattice of $H^*(X, \mathbb{Z})_{\mathrm{alg}}$.

Lemma 2.5. *Let $v_0 = (p^2 n, pqH, q^2)$ be a primitive and isotropic Mukai vector with $n = (H^2)/2$ and $\gcd(pn, q) = 1$.*

(1) $M_H(v_0) \cong X$.
(2) *For an isotropic vector $v_1 \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ with $\langle v_0, v_1 \rangle = 1$, there is a Fourier-Mukai transform $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ such that $\Phi(-\varrho_X) = v_0$ and $\Phi(v(\mathcal{O}_X)) = v_1$. Moreover we have the following.*

(a) Φ *is unique up to the action of* $\mathrm{Aut}(X) \times \mathrm{Pic}^0(X) \times 2\mathbb{Z}$, *where* $2k \in 2\mathbb{Z}$ *acts as the shift functor* $[2k] : \mathbf{D}(X) \to \mathbf{D}(X)$.

(b) *If* $v_1 \in L$, *then we can take* Φ *such that* $\Phi(L) = L$ *and* $\Phi_{|L^\perp}$ *is the identity.*

Proof. (1) We fix a stable sheaf E with $v(E) = v_0$. Let $\mathbf{P}$ be the Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$. Then we have a surjective homomorphism $\mathrm{Pic}^0(X) \to M_H(v_0)$ by sending $y \in \mathrm{Pic}^0(X)$ to $E \otimes \mathbf{P}_{|X \times \{y\}} \in M_H(v_0)$ ([22]). So we get an isomorphism $\mathrm{Pic}^0(X)/\Sigma(E) \to M_H(v_0)$, where

$$\Sigma(E) := \{y \in \mathrm{Pic}^0(X) \mid E \otimes \mathbf{P}_{|X \times \{y\}} \cong E\}.$$

Let $T_x : X \to X$ be the translation by x. For a divisor D on X, $\phi_D : X \to \mathrm{Pic}^0(X)$ denotes the homomorphism such that $\phi_D(x) = T_x^* \mathcal{O}_X(D) \otimes \mathcal{O}_X(-D)$. We set $K(D) := \ker \phi_D$. If $(D^2) > 0$, then ϕ_D is finite and $\# K(D) = d^2$, where $d := (D^2)/2$. For $D = pqH$, [22, Thm. 7.11] implies that $\phi_{pqH}(X_{p^2 n}) = \Sigma(E)$, where X_m denotes the set of m-torsion points of X, which is the kernel of $m : X \to X$ the multiplication by $m \in \mathbb{Z}_{>0}$. Hence we have a morphism $\phi : X \to M_H(v_0)$ such that

$$\phi(x) = E \otimes T_x^*(\mathcal{O}_X(pqH)) \otimes \mathcal{O}_X(-pqH), \ x \in X$$

and ϕ induces an isomorphism

$$X/(X_{p^2 n} + K(pqH)) \cong \widehat{X}/\Sigma(E) \cong M_H(v_0).$$

Since $K(pqH) = (pq)^{-1}(K(H))$, $n(K(H)) = 0$ and $(pn, q) = 1$, we have a sequence of isomorphisms

$$X/(X_{p^2 n} + K(pqH)) \overset{p^2 n}{\to} X/p^2 n K(pqH) = X/q^{-1}(0) \overset{q}{\to} X.$$

Therefore $M_H(v_0) \cong X$.

(2) By our assumption, we have a universal family $\mathbf{E}$ on $X \times M_H(v_0)$. By (1), we have an isomorphism

$$X \times M_H(v_0) \to X \times X.$$

Thus we may assume that $\mathbf{E} \in \mathrm{Coh}(X \times X)$. Then we have an equivalence

$$\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$$

such that $\Phi(\mathbb{C}_x) = \mathbf{E}_{|X \times \{x\}}[1]$ for $x \in X$. Since $\langle \Phi^{-1}(v_1), \Phi^{-1}(v_0) \rangle = 1$, $\mathrm{rk}\, \Phi^{-1}(v_1) = 1$. Let $p_2 : X \times X \to X$ be the second projection.

Replacing $\mathbf{E}$ by $\mathbf{E} \otimes p_2^*(L)$ $(L \in \operatorname{Pic}(X))$, we have $\Phi^{-1}(v_1) = v(\mathcal{O}_X)$. Thus the first claim holds.

(a) If $\Phi' : \mathbf{D}(X) \to \mathbf{D}(X)$ also satisfies the same properties, then $\Phi^{-1}\Phi'(\varrho_X) = \varrho_X$ and $\Phi^{-1}\Phi'(v(\mathcal{O}_X)) = v(\mathcal{O}_X)$. Hence $\Phi^{-1}\Phi'$ is the Fourier-Mukai transform whose kernel is $\mathcal{O}_\Gamma \otimes p_2^*(N)[2k]$, where $N \in \operatorname{Pic}^0(X)$ and Γ is the graph of $g \in \operatorname{Aut}(X)$. Hence

$$\Phi'(E) = \Phi(g_*(E \otimes N))[2k], \ E \in \mathbf{D}(X).$$

(b) We take a complex $E_1 \in \mathbf{D}(X)$ with $v(E_1) = v_1$. For a S-flat coherent sheaf $\mathbf{F}$ on $X \times S$ with $\mathbf{F}_{|X \times \{s\}} \in M_H(v_0)$ $(s \in S)$, we set

$$\mathcal{L}_{\mathbf{F}} := \det p_{S_*}(\mathbf{F}^\vee \otimes E_1).$$

If we replace $\mathbf{F}$ by $\mathbf{F} \otimes \mathcal{L}_{\mathbf{F}}^\vee$, then we have $\mathcal{L}_{\mathbf{F}} = \mathcal{O}_S$. We set $\mathbf{E}_1 := E \otimes \mathbf{P}$. Then by the identification $X \cong M_H(v_0)$ in the proof of (1), we have

$$(1 \times \phi_{pqH})^*(\mathbf{E}_1 \otimes \mathcal{L}_{\mathbf{E}_1}^\vee) \cong (1 \times p^2nq)^*(\mathbf{E}),$$

where $\mathbf{E}$ is the universal family in (2) which is normalized to satisfy $\mathcal{L}_{\mathbf{E}} \cong \mathcal{O}_X$. Indeed $\Phi(v(\mathcal{O}_X)) = v(E_1)$ implies that

$$c_1(\mathcal{L}_{\mathbf{E}}) = -c_1(\Phi_{X \to X}^{\mathbf{E}^\vee[1]}(E_1)) = -c_1(v(\mathcal{O}_X)) = 0.$$

Hence $\mathbf{E}' := \mathbf{E} \otimes \mathcal{L}_{\mathbf{E}}^\vee$ also satisfies $v(\Phi_{X \to X}^{\mathbf{E}'[1]}(\mathcal{O}_X)) = v_1$.

We set $\beta := \frac{q}{pn}H$. For $G \in \mathbf{D}(X)$ with

$$v(G) = re^\beta + a\varrho_X + dH + D + (dH + D, \beta)\varrho_X, \ D \in H^\perp,$$

Lemma 2.6 below implies that

$$\begin{aligned}
&(p^2qn)^*(v(\Phi_{X \to X}^{\mathbf{E}^\vee}(G))) \\
=&\phi_{pqH}^*(v(\mathbf{R}p_{\operatorname{Pic}^0(X)*}(\mathbf{E}_1^\vee \otimes G) \otimes \mathcal{L}_{\mathbf{E}_1})) \\
=&\phi_{pqH}^*(v(\Phi_{X \to \operatorname{Pic}^0(X)}^{\mathbf{P}^\vee}(E^\vee \otimes G) \otimes \mathcal{L}_{\mathbf{E}_1})) \\
=&(p^2qn)^*(p^2na + \tfrac{r}{p^2n}\varrho_X - dH + D) \otimes \phi_{pqH}^*(v(\mathcal{L}_{\mathbf{E}_1})).
\end{aligned} \tag{2.11}$$

In order to complete the proof of the claim, we need to show that $\phi_{pqH}^*(\mathcal{L}_{\mathbf{E}_1}) \in (p^2qn)^*(L)$. We take integers x, y with $ypn - xq = \pm 1$. Then we have $v_1 = (x^2, xyH, y^2n)$. Hence

$$v_1 = x^2e^\beta \pm \frac{x}{pn}(H + (H, \beta)\varrho_X) + \frac{1}{p^2n}\varrho_X.$$

Applying (2.11) to $v(\mathcal{O}_X) = \Phi_{X \to X}^{\mathbf{E}^\vee}(v_1)$, we have

$$\begin{aligned}(p^2qn)^*(v(\mathcal{O}_X)) =&(p^2qn)^*(1 + \tfrac{x^2}{p^2n}\varrho_X \mp \frac{x}{pn}H) \otimes \phi_{pqH}^*(v(\mathcal{L}_{\mathbf{E}_1}))\\ =&(p^2qn)^*(v(\mathcal{O}_X(\mp\frac{x}{pn}H))) \otimes \phi_{pqH}^*(v(\mathcal{L}_{\mathbf{E}_1})).\end{aligned}$$

Therefore the claim holds.

Q.E.D.

Let $m : X \times X \to X$ be the addition map. Then

$$m^*(\mathcal{O}_X(pqH)) \otimes p_1^*(\mathcal{O}_X(-pqH)) \otimes p_2^*(\mathcal{O}_X(-pqH))) \cong (1 \times \phi_{pqH})^*(\mathbf{P}).$$

We shall compute $\phi_{pqH}^*(\Phi_{X \to \mathrm{Pic}^0(X)}^{\mathbf{P}^\vee}(w))$ for $w \in H^*(X, \mathbb{Z})$.

Lemma 2.6. *We set $\beta := \frac{q}{pn}H$. For*

$$u := re^\beta + a\varrho_X + (dH + D) + (dH + D, \beta)\varrho_X,\ D \in \mathrm{NS}(X)_\mathbb{Q} \cap H^\perp,$$

we have

$$\phi_{pqH}^*(\Phi_{X \to X}^{\mathbf{P}^\vee}(v_0^\vee u)) = (p^2qn)^*(p^2na + \tfrac{r}{p^2n}\varrho_X - dH + D).$$

Proof. Let e_1, e_2, e_3, e_4 be a basis of $H^1(X, \mathbb{Z})$ such that $c_1(H) = e_1 \wedge e_2 + ne_3 \wedge e_4$. In

$$H^*(X \times X, \mathbb{Z}) = H^*(X, \mathbb{Z}) \otimes H^*(X, \mathbb{Z}),$$

we identify $e_i \otimes 1$ with e_i and denote $1 \otimes e_i$ by f_i. Then

$$\begin{aligned}&c_1(m^*(\mathcal{O}_X(H)) \otimes p_1^*(\mathcal{O}_X(-H)) \otimes p_2^*(\mathcal{O}_X(-H)))\\ =&(e_1 + f_1) \wedge (e_2 + f_2) + n(e_3 + f_3) \wedge (e_4 + f_4)\\ &\quad - (e_1 \wedge e_2 + ne_3 \wedge e_4) - (f_1 \wedge f_2 + nf_3 \wedge f_4)\\ =&e_1 \wedge f_2 + f_1 \wedge e_2 + n(e_3 \wedge f_4 + f_3 \wedge e_4) =: \eta.\end{aligned}$$

We denote the class by η. Then we see that

$$\begin{aligned}\frac{\eta^2}{2!} =& - (e_3 \wedge e_4)^* \wedge (f_1 \wedge f_2) - n^2(e_1 \wedge e_2)^* \wedge (f_3 \wedge f_4)\\ &+ n((e_2 \wedge e_4)^* \wedge (f_2 \wedge f_4) + (e_1 \wedge e_4)^* \wedge (f_1 \wedge f_4)\\ &+ (e_2 \wedge e_3)^* \wedge (f_2 \wedge f_3) + (e_1 \wedge e_3)^* \wedge (f_1 \wedge f_3)),\\ \frac{\eta^4}{4!} =&n^2(e_1 \wedge e_2 \wedge e_3 \wedge e_4) \wedge (f_1 \wedge f_2 \wedge f_3 \wedge f_4),\end{aligned}$$

where $\{(e_i \wedge e_j)^* \mid i,j\}$ is the dual basis of $\{e_i \wedge e_j \mid i,j\}$ via the intersection pairing. We note that $H^\perp$ is generated by

$$e_1 \wedge e_2 - ne_3 \wedge e_4,\ e_2 \wedge e_4,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ e_1 \wedge e_3.$$

Then we see that

$$\begin{aligned}
&\phi_{pqH}^*(\Phi_{X \to \mathrm{Pic}^0(X)}^{\mathbf{P}^\vee}(v_0^\vee u)) \\
=&p_{2*}(p_1^*(p^2 n e^{-\beta}(re^\beta + a\varrho_X + (dH + D + (dH+D,\beta)\varrho_X))e^{pq\eta}) \\
=&p_{2*}(p_1^*(p^2 n(r + a\varrho_X + dH + D))e^{pq\eta}) \\
=&p^2 n^2 (pq)^2(-dH + D) + p^2 na + p^6 n^3 q^4 r\varrho_X.
\end{aligned}$$

Since $(p^2qn)^*(x_i) = (p^2qn)^{2i} x_i$ for $x_i \in H^{2i}(X,\mathbb{Q})$, we get the claim. Q.E.D.

By Lemma 2.5, every member of $\mathrm{Stab}_0(v)^*$ comes from an autoequivalence of $\mathbf{D}(X)$ and we have a homomorphism

$$\mathrm{Stab}_0(v)^* \to \mathrm{O}(H^*(X,\mathbb{Z})_{\mathrm{alg}}).$$

Lemma 2.7. *For a Mukai vector $v=(r,\xi,a)$, assume that $(\xi^2)>0$. We set $\xi := dH$, where H is a primitive ample divisor and $d \in \mathbb{Z}$. If $\sqrt{\langle v^2\rangle/(\xi^2)} \notin \mathbb{Q}$, then there is an autoequivalence $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ such that Φ acts on $L := \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\varrho_X$ as an isometry of infinite order and $\Phi(v) = v$.*

Proof. By the proof of Proposition 2.3 (2), $\mathrm{Stab}_0(v)^*$ contains an element g of infinite order. By Lemma 2.5, we have a Fourier-Mukai transform $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ inducing the action of g on L. Q.E.D.

§3. The space of stability conditions and the positive cone of the moduli spaces.

3.1. A polarization of $M_{(\beta,tH)}(v)$.

We shall explain a natural ample line bundle on the moduli space $M_{(\beta,tH)}(v)$, which is introduced in [19] and [3].

For $v=(r,\xi,a)$, we set

$$d_\beta := \frac{(\xi - r\beta, H)}{(H^2)},\ a_\beta := -\langle e^\beta, v\rangle.$$

Then

$$v = re^\beta + a_\beta\varrho_X + d_\beta H + D_\beta + (d_\beta H + D_\beta, \beta)\varrho_X,\ D_\beta \in \mathrm{NS}(X)_{\mathbb{Q}} \cap H^\perp.$$

Let

$$\begin{aligned}\xi_\omega :=&\frac{(\omega^2)}{2d_\beta}\left(r(H+(H,\beta)\varrho_X)+d_\beta(H^2)\varrho_X\right)\\ &-\frac{1}{d_\beta}\left(a_\beta(H+(H,\beta)\varrho_X)+d_\beta(H^2)e^\beta\right)\in H^*(X,\mathbb{Z})_{\rm alg}\otimes\mathbb{R}\end{aligned}$$

be a vector in [19, Defn. 5.12], where $\omega=tH$.

Definition 3.1. For $(\beta,H,t)\in \mathrm{NS}(X)_{\mathbb{R}}\times\overline{\mathrm{Amp}(X)}_{\mathbb{R}}\times\mathbb{R}_{\geq 0}$, we set

$$\begin{aligned}\xi(\beta,H,t):=&d_\beta\xi_\omega\\ =&\left(r\frac{t^2(H^2)}{2}+\langle e^\beta,v\rangle\right)(H+(\beta,H)\varrho_X)\\ &-(\xi-r\beta,H)\left(e^\beta-\frac{t^2(H^2)}{2}\varrho_X\right).\end{aligned}$$

Assume that $r\neq 0$. We set $\delta:=\frac{\xi}{r}$. Then $v=re^\delta+a_\delta\varrho_X$. For $\beta=\delta+sH+D$ with $D\in\mathrm{NS}(X)_{\mathbb{R}}\cap H^\perp$, we set

$$d_\beta=\frac{r(\delta-\beta,H)}{(H^2)},\ a_\beta=a_\delta+\frac{((\beta-\delta)^2)}{2}r.$$

$$\begin{aligned}\xi(\beta,H,t)=&r\left(\frac{t^2(H^2)-((\beta-\delta)^2)}{2}+\frac{\langle v^2\rangle}{2r^2}\right)(H+(\delta,H)\varrho_X)\\ &-r(\delta-\beta,H)\left(e^\beta-\frac{a_\beta}{r}\varrho_X\right)\\ =&r\left(\frac{(\omega^2)-((\beta-\delta)^2)}{2}+\frac{\langle v^2\rangle}{2r^2}\right)(H+(\delta,H)\varrho_X)\\ &-r(\delta-\beta,H)\left((\beta-\delta+(\beta-\delta,\delta)\varrho_X)+\left(e^\delta-\frac{a_\delta}{r}\varrho_X\right)\right).\end{aligned}$$

Assume that $r=0$. Then we also have

$$\begin{aligned}\xi(\beta,H,t)=&-a_\beta(H+(H,\beta)\varrho_X)-d_\beta(H^2)\left(e^\beta-\frac{t^2(H^2)}{2}\varrho_X\right)\\ =&((\xi,\beta)-a)(H+(H,\beta)\varrho_X)-(\xi,H)\left(e^\beta-\frac{t^2(H^2)}{2}\varrho_X\right).\end{aligned}$$

Lemma 3.2.

$$\mathbb{R}_{>0}\xi(\beta,H,t)=\mathbb{R}_{>0}\mathrm{Im}(Z_{(\beta,tH)}(v)^{-1}e^{\beta+\sqrt{-1}tH}).$$

Proof.

$$Z_{(\beta,tH)}(v) = \frac{r}{2}t^2(H^2) - a_\beta + (\xi - r\beta, tH)\sqrt{-1}$$

For the complex conjugate $\overline{Z_{(\beta,tH)}(v)}$ of $Z_{(\beta,tH)}(v)$, we have

$$\begin{aligned}
&\operatorname{Im}(\overline{Z_{(\beta,tH)}(v)} e^{\beta+\sqrt{-1}tH}) \\
=&\left(r\frac{t^2(H^2)}{2} - a_\beta\right) t(H + (\beta,H)\varrho_X) - (\xi - r\beta, tH)\left(e^\beta - \frac{t^2(H^2)}{2}\varrho_X\right) \\
=&t\xi(\beta,H,t).
\end{aligned}$$

Hence the claim holds. Q.E.D.

Remark 3.3. The expression of $\xi(\beta,H,t)$ in Lemma 3.2 appeared in [3]. The difference of the sign comes from the difference of θ_v and $\theta_{v,\beta,tH}$.

Remark 3.4. Assume that $r = 0$ and $t > 0$. Then

$$\{\xi(\beta,H,st) \mid s \geq 1\} = \xi(\beta,H,t) + \mathbb{R}_{\geq 0}\varrho_X$$

and

$$\lim_{t\to\infty} \xi(\beta,H,t)/t^2 = (\xi,H)\frac{(H^2)}{2}\varrho_X.$$

If ξ is effective, then we have a morphism $M_H^\beta(v) \to \operatorname{Hilb}_X^\xi$ by sending E to the scheme-theoretic support $\operatorname{Div}(E)$ and $\theta_v(\varrho_X)$ comes from $\operatorname{Hilb}_X^\xi$. Since $M_{(\beta,tH)}(v) = M_H^\beta(v)$ for $t \gg 0$ and $(\xi,H) > 0$, $\theta_{v,\beta,tH}((\xi,H)\varrho_X)$ is base point free for $t \gg 0$.

We shall remark the behavior of $\xi(\beta,H,t)$ under the Fourier-Mukai transforms. Let

$$\Phi : \mathbf{D}(X) \to \mathbf{D}^{\alpha_1}(X_1)$$

be a twisted Fourier-Mukai transform such that

$$\Phi(r_1 e^\gamma) = -\varrho_{X_1},\ \Phi(\varrho_X) = -r_1 e^{\gamma'},$$

where α_1 is a representative of a suitable Brauer class. Then we can describe the cohomological Fourier-Mukai transform as

$$\Phi(re^\gamma + a\varrho_X + \xi + (\xi,\gamma)\varrho_X) = -\frac{r}{r_1}\varrho_{X_1} - r_1 a e^{\gamma'} + \frac{r_1}{|r_1|}(\widehat{\xi} + (\widehat{\xi},\gamma')\varrho_{X_1}),$$

where

$$\xi \in \operatorname{NS}(X)_{\mathbb{Q}},\ \widehat{\xi} := \frac{r_1}{|r_1|}c_1(\Phi(\xi + (\xi,\gamma)\varrho_X)) \in \operatorname{NS}(X_1)_{\mathbb{Q}}.$$

Remark 3.5. By taking a locally free α_1-twisted stable sheaf G with $\chi(G,G)=0$, we have a notion of Mukai vector, thus, we have a map ([18, Rem. 1.2.10]):

$$v_G : \mathbf{D}^{\alpha_1}(X_1) \to H^*(X_1,\mathbb{Q})_{\mathrm{alg}}.$$

For $(\beta,\omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$, we set

$$\begin{aligned}
\widetilde{\omega} :=& -\frac{1}{|r_1|}\frac{\frac{((\beta-\gamma)^2)-(\omega^2)}{2}\widehat{\omega}-(\beta-\gamma,\omega)(\widehat{\beta}-\widehat{\gamma})}{\left(\frac{((\beta-\gamma)^2)-(\omega^2)}{2}\right)^2+(\beta-\gamma,\omega)^2},\\
\widetilde{\beta} :=& \gamma' -\frac{1}{|r_1|}\frac{\frac{((\beta-\gamma)^2)-(\omega^2)}{2}(\widehat{\beta}-\widehat{\gamma})-(\beta-\gamma,\omega)\widehat{\omega}}{\left(\frac{((\beta-\gamma)^2)-(\omega^2)}{2}\right)^2+(\beta-\gamma,\omega)^2}.
\end{aligned} \tag{3.1}$$

Then $(\widetilde{\beta},\widetilde{\omega}) \in \mathrm{NS}(X_1)_{\mathbb{R}} \times \mathrm{Amp}(X_1)_{\mathbb{R}}$.

By [19, sect. A.1], we get the following commutative diagram:

$$\begin{tikzcd}
\mathbf{D}(X) \arrow[r] \arrow[d, "Z_{(\beta,\omega)}"'] & \mathbf{D}^{\alpha_1}(X_1) \arrow[d, "Z_{(\widetilde{\beta},\widetilde{\omega})}"] \\
\mathbb{C} \arrow[r, "\zeta^{-1}"'] & \mathbb{C}
\end{tikzcd}$$

where

$$\zeta = -r_1\left(\frac{((\gamma-\beta)^2)-(\omega^2)}{2}+\sqrt{-1}(\beta-\gamma,\omega)\right).$$

Proposition 3.6. *For $(\beta,H,t) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}} \times \mathbb{R}_{>0}$, we have $\mathbb{R}_{>0}\Phi(\xi(\beta,H,t)) = \mathbb{R}_{>0}\xi(\widetilde{\beta},H_1,t_1)$, where $t_1H_1=\widetilde{tH}$.*

Proof.

$$\begin{aligned}
\mathbb{R}_{>0}\Phi(\xi(\beta,H,t)) =& \mathbb{R}_{>0}\Phi(\mathrm{Im}(Z_{(\beta,tH)}(v)^{-1}e^{\beta+\sqrt{-1}tH}))\\
=& \mathbb{R}_{>0}\mathrm{Im}(Z_{(\beta,tH)}(v)^{-1}\Phi(e^{\beta+\sqrt{-1}tH}))\\
=& \mathbb{R}_{>0}\mathrm{Im}(Z_{(\widetilde{\beta},\widetilde{tH})}(\Phi(v))^{-1}\zeta^{-1}\zeta e^{\widetilde{\beta}+\sqrt{-1}\widetilde{tH}})\\
=& \mathbb{R}_{>0}\xi(\widetilde{\beta},H_1,t_1).
\end{aligned}$$

Q.E.D.

Remark 3.7. We have a commutative diagram

$$\begin{tikzcd} v^{\perp} \arrow[r, "\Phi"] \arrow[d, "\theta_{v,\beta,\omega}"'] & \Phi(v)^{\perp} \arrow[d, "\theta_{\Phi(v),\widetilde{\beta},\widetilde{\omega}}"] \\ \mathrm{NS}(K_{(\beta,\omega)}(v)) \arrow[r, "\Phi_*"'] & \mathrm{NS}(K^{\alpha}_{(\widetilde{\beta},\widetilde{\omega})}(\Phi(v))) \end{tikzcd}$$

where $K^{\alpha}_{(\widetilde{\beta},\widetilde{\omega})}(\Phi(v))$ is the Bogomolov factor of the moduli of $\sigma_{(\widetilde{\beta},\widetilde{\omega})}$-stable objects $M^{\alpha}_{(\widetilde{\beta},\widetilde{\omega})}(\Phi(v))$. Then we have

$$\mathbb{R}_{>0}\Phi_*(\theta_{v,\beta,\omega}(\xi(\beta',H',t'))) = \mathbb{R}_{>0}\theta_{\Phi(v),\widetilde{\beta},\widetilde{\omega}}(\xi(\widetilde{\beta'},H_1,t_1)),$$

where $t_1H_1 = \widetilde{t'H'}$.

3.2. Stability conditions and the positive cone.

Assume that $r \neq 0$. We note that $\xi(\beta,H,t) \in \varrho_X^{\perp}$ if and only if $\delta-\beta \in H^{\perp}$. In this case, we have $((\delta-\beta)^2) \leq 0$, which implies that

$$\frac{(\omega^2)-((\beta-\delta)^2)}{2}+\frac{\langle v^2\rangle}{2r^2} > 0.$$

Therefore we get the following claim.

Lemma 3.8.

$$\xi(\beta,H,t) \in \varrho_X^{\perp} \iff \xi(\beta,H,t) = r(\eta+(\eta,\delta)\varrho_X),\ \eta \in \overline{\mathrm{Amp}(X)}_{\mathbb{R}}$$

Moreover

$$\eta \in \mathrm{Amp}(X)_{\mathbb{R}} \iff H \in \mathrm{Amp}(X)_{\mathbb{R}}.$$

Lemma 3.9. *Assume that* $(\beta-\delta,H) \neq 0$ *and set*

$$\eta := \beta-\delta+\frac{1}{(\beta-\delta,H)}\left(\frac{t^2(H^2)-((\beta-\delta)^2)}{2}+\frac{\langle v^2\rangle}{2r^2}\right)H.$$

Then

(1) $(\eta^2) \geq \frac{\langle v^2\rangle}{r^2}$. *Moreover*

$$(\eta^2) = \frac{\langle v^2\rangle}{r^2} \iff \begin{cases} (H^2)=0 \ \textit{or} \\ t=0 \ \textit{and} \ ((\beta-\delta)^2)=\frac{\langle v^2\rangle}{r^2}. \end{cases}$$

(2) $(\beta-\delta,H)(\eta,H') > 0$ *for* $H' \in \mathrm{Amp}(X)_{\mathbb{R}}$ *which is sufficiently close to* H.

Proof. (1)

$$
\begin{aligned}
(\eta^2) =& ((\beta-\delta)^2) + 2\left(\frac{t^2(H^2)-((\beta-\delta)^2)}{2}\frac{\langle v^2\rangle}{2r^2}\right) \\
&+ \frac{(H^2)}{(\beta-\delta,H)^2}\left(\frac{t^2(H^2)-((\beta-\delta)^2)}{2}+\frac{\langle v^2\rangle}{2r^2}\right)^2 \\
=& t^2(H^2)+\frac{\langle v^2\rangle}{r^2}+\frac{(H^2)}{(\beta-\delta,H)^2}\left(\frac{t^2(H^2)-((\beta-\delta)^2)}{2}+\frac{\langle v^2\rangle}{2r^2}\right)^2 \\
\geq& \frac{\langle v^2\rangle}{r^2}.
\end{aligned}
$$

Moreover the equality holds if and only if (β, H, t) satisfy (i) or (ii).

(2) It is sufficient to prove $(\beta-\delta, H)(\eta, H) > 0$. If $(H^2) = 0$, then $(\beta-\delta, H)(\eta, H) = (\beta-\delta, H)^2 > 0$. Assume that $(H^2) > 0$. We set $\beta - \delta = sH + D$ ($s \in \mathbb{R}$, $D \in H^{\perp}$). Then

$$
\begin{aligned}
&(\beta-\delta,H)(\eta,H) \\
=& \frac{1}{2}\left((\beta-\delta,H)^2-(H^2)(D^2)+t^2(H^2)^2+\frac{(H^2)\langle v^2\rangle}{r^2}\right) > 0.
\end{aligned}
$$

Q.E.D.

Definition 3.10. We set

$$
\begin{aligned}
P^+(v^{\perp})_k &:= \left\{ x \in H^*(X,\mathbb{Z})_{\mathrm{alg}} \otimes k \;\middle|\; \begin{array}{l} x \in v^{\perp}, \langle x^2\rangle > 0, \\ \langle x, rH_0 + (rH_0,\delta)\varrho_X\rangle > 0 \end{array} \right\}, \\
\overline{P^+(v^{\perp})_k} &:= \left\{ x \in H^*(X,\mathbb{Z})_{\mathrm{alg}} \otimes k \;\middle|\; \begin{array}{l} x \in v^{\perp}, \langle v^2\rangle \geq 0, \\ \langle x, rH_0 + (rH_0,\delta)\varrho_X\rangle > 0 \end{array} \right\},
\end{aligned}
$$

where $k = \mathbb{Q}, \mathbb{R}$ and $H_0 \in \mathrm{Amp}(X)_{\mathbb{Q}}$.

We take a norm $||\ \ ||$ on $\mathrm{NS}(X)_{\mathbb{R}}$ defined over $\mathbb{Q}$ and regard the cone $C(\overline{\mathrm{Amp}(X)_{\mathbb{R}}})$ in subsection 1.2 as a subset of $\overline{\mathrm{Amp}(X)_{\mathbb{R}}}$ (subsection 1.2):

$$
C(\overline{\mathrm{Amp}(X)_{\mathbb{R}}}) = \{L \in \overline{\mathrm{Amp}(X)_{\mathbb{R}}} \mid ||L|| = 1\}.
$$

Under this identification, $\xi(\beta, H, t)$ is defined for

$$
(\beta, H, t) \in \overline{\mathfrak{H}} = \mathrm{NS}(X)_{\mathbb{R}} \times C(\overline{\mathrm{Amp}(X)_{\mathbb{R}}}) \times \mathbb{R}_{\geq 0}.
$$

For the embedding

$$
\begin{array}{ccc}
\mathrm{NS}(X)_{\mathbb{R}} \times \overline{\mathrm{Amp}(X)_{\mathbb{R}}} & \to & \overline{\mathfrak{H}} \\
(\beta, H) & \mapsto & (\beta, H/||H||, ||H||),
\end{array}
$$

we have

$$\begin{aligned}\mathbb{R}_{>0}\xi(\beta,H,t) =&\mathbb{R}_{>0}\xi(\beta,H/||H||,||H||t),\\ &(\beta,H,t)\in \mathrm{NS}(X)_{\mathbb{R}}\times \overline{\mathrm{Amp}(X)}_{\mathbb{R}}\times \mathbb{R}_{\geq 0}.\end{aligned}$$

Proposition 3.11. *We have a map*

$$\begin{array}{rccc}\Xi: & \overline{\mathfrak{H}} & \to & C(H^*(X,\mathbb{Z})_{\mathrm{alg}}\otimes\mathbb{R})\\ & (\beta,H,t) & \mapsto & \mathbb{R}_{>0}\xi(\beta,H,t)\end{array}$$

whose image is the positive cone $C^+ := C(\overline{P^+(v^\perp)_{\mathbb{R}}})$ *of* $v^\perp$*. Moreover if* tH *is ample, then* $\xi(\beta,H,t)$ *belongs to the interior of* C^+*.*

Proof. By Proposition 3.6, it is sufficient to prove the claim for $r\neq 0$. An element

$$u=\zeta+(\zeta,\delta)\varrho_X+y(e^\delta+\tfrac{\langle v^2\rangle}{2r^2}\varrho_X)\in v^\perp\cap H^*(X,\mathbb{Z})_{\mathrm{alg}}\otimes\mathbb{R} \tag{3.2}$$

satisfies $\langle u^2\rangle\geq 0$ and $\langle u, rH+(rH,\delta)\varrho_X\rangle>0$ if and only if $(\zeta^2)\geq y^2\frac{\langle v^2\rangle}{r^2}$ and $(\zeta,rH)>0$. We first assume that $y\neq 0$. We set $\frac{\zeta}{y}:=xH+D$ ($x\in\mathbb{R}$, $D\in H^\perp$) and

$$\sigma_\pm:=\pm\sqrt{\frac{\langle v^2\rangle-r^2(D^2)}{r^2(H^2)}}.$$

Then the conditions are $x\geq\sigma_+$ if $ry>0$ and $x\leq\sigma_-$ if $ry<0$. If $y=0$, then the condition is $(\zeta^2)\geq 0$ and $(\zeta,rH)>0$, that is, $r\zeta\in\overline{\mathrm{Amp}(X)}_{\mathbb{R}}$.

We shall first prove that $\mathrm{im}(\Xi)$ contains C^+. We shall find (β,H,t) such that $\beta=\delta+sH+D$, ($s\in\mathbb{R}, D\in H^\perp$) and

$$\mathbb{R}_{>0}\xi(\beta,H,t)=\mathbb{R}_{>0}u.$$

We set

$$g(s,t):=\frac{(H^2)(s^2+t^2)+\frac{\langle v^2\rangle}{r^2}-(D^2)}{2s(H^2)}.$$

Then $g(s,0)$ define continuous functions from $(0,\infty)$ to $[\sigma_+,\infty)$ and from $(-\infty,0)$ to $(-\infty,\sigma_-]$. Hence we can take $s\in\mathbb{R}$ with $x=g(s,0)$. For

$\beta := \delta + sH + D$, we have

$$\begin{aligned}
&\frac{\xi(\delta+sH+D,H,0)}{r(\beta-\delta,H)}\\
=&\frac{1}{(\beta-\delta,H)}\left(-\frac{((\beta-\delta)^2)}{2}+\frac{\langle v^2\rangle}{2r^2}\right)(H+(H,\delta)\varrho_X)\\
&\quad+(\beta-\delta+(\beta-\delta,\delta)\varrho_X)+\left(e^{\delta}-\frac{a_\delta}{r}\varrho_X\right)\\
=&g(s,0)(H+(H,\delta)\varrho_X)+(D+(D,\delta)\varrho_X)+\left(e^{\delta}-\frac{a_\delta}{r}\varrho_X\right)=\frac{u}{y}.
\end{aligned}$$

Since $r(\beta-\delta,H)y = rsy(H^2)$ and $sx = sg(s,0) > 0$,

$$rxy = (\zeta, rH)/(H^2) > 0$$

implies that $r(\beta-\delta,H)$ and y have the same sign. Thus

$$\mathbb{R}_{>0}\xi(\delta+sH+D,H,0) = \mathbb{R}_{>0}u.$$

If $y = 0$, then Lemma 3.8 implies that

$$\xi(\delta, r\zeta, t) = \frac{r^4t^2(\zeta^2)+\langle v^2\rangle}{2}(\zeta+(\zeta,\delta)\varrho_X) \in \mathbb{R}_{>0}u.$$

Hence $u \in \operatorname{im}\Xi$. Conversely for $\xi(\beta,H,t)$, Lemma 3.8 or Lemma 3.9 implies that $\xi(\beta,H,t) \in C^+$. Therefore the claim holds. Q.E.D.

Remark 3.12. If u in (3.2) belongs to $H^*(X,\mathbb{Z})_{\mathrm{alg}} \otimes \mathbb{Q}$ and satisfies $\langle u^2\rangle > 0$, then there is (β,H,t) such that $\Xi(\beta,H,t) = u$, $\beta, H \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $t^2 \in \mathbb{Q}$: Indeed if $u \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$, then we may assume that $H \in \mathrm{NS}(X)_{\mathbb{Q}}$. For $g(s,0) \in \mathbb{Q}$, we can take (s',t') such that $s', {t'}^2 \in \mathbb{Q}$ and $g(s',t') = g(s,0)$. Hence the claim holds.

Proposition 3.13. *$(\beta,H,t) \in W_{v_1}$ (see Definition 1.2) if and only if $\xi(\beta,H,t) \in v^\perp \cap v_1^\perp$.*

Proof. We note that (1.5) is written as

$$\begin{aligned}
&\det\begin{pmatrix} a-(\xi,\beta)+r\frac{(\beta^2)-t^2(H^2)}{2} & a_1-(\xi_1,\beta)+r_1\frac{(\beta^2)-t^2(H^2)}{2} \\ -(\xi-r\beta,H) & -(\xi_1-r_1\beta,H)\end{pmatrix}\\
=&\begin{pmatrix} \frac{((\delta-\beta)^2)-t^2(H^2)}{2}-\frac{\langle v^2\rangle}{2r^2} & \langle -e^{\beta}+\frac{t^2(H^2)}{2}\varrho_X, v_1\rangle \\ -(\xi-r\beta,H) & -\langle H+(H,\beta)\varrho_X, v_1\rangle\end{pmatrix}\\
=&\langle \xi(\beta,H,t), v_1\rangle.
\end{aligned} \tag{3.3}$$

Hence the claim holds. Q.E.D.

Proof of Proposition 1.3. By Proposition 3.13 and Proposition 3.11, it is sufficient to find the condition $\overline{P^+(v^\perp)_{\mathbb{R}}} \cap v_1^\perp \neq \emptyset$. We set $\eta := \langle v^2 \rangle v_1 - \langle v_1, v \rangle v \in v^\perp$. Then $v_1^\perp \cap v^\perp = \eta^\perp \cap v^\perp$. Since the signature of $v^\perp$ is $(1, \operatorname{rk} \mathrm{NS}(X))$, the condition is $(\eta^2) < 0$. Since

$$\langle \eta^2 \rangle = \langle v^2 \rangle (\langle v_1^2 \rangle \langle v^2 \rangle - \langle v_1, v \rangle^2),$$

we get the claim. Q.E.D.

Definition 3.14. (1) A connected component D of $P^+(v^\perp)_{\mathbb{R}} \setminus \cup_{v_1 \in \mathfrak{W}} v_1^\perp$ is a chamber.

(2) $D(\beta, H, t)$ is a chamber such that $\xi(\beta, H, t) \in D(\beta, H, t)$.

$\overline{D(\beta, H, t)}$ consists of $x \in \overline{P^+(v^\perp)_{\mathbb{R}}}$ such that $\langle \xi(\beta, H, t), \pm v_1 \rangle > 0$ implies $\langle x, \pm v_1 \rangle \geq 0$, that is, $\langle \xi(\beta, H, t), v_1 \rangle \langle x, v_1 \rangle \geq 0$.

Proposition 3.15 ([19, Thm. 1.6])**.** *We fix a general (β, H, t).*

(1) *Assume that (β', H', t') belongs to a chamber and $H' \in \mathrm{Amp}(X)_{\mathbb{Q}}$, ${t'}^2 \in \mathbb{Q}$, $\beta' \in \mathrm{NS}(X)_{\mathbb{Q}}$. Then $\theta_{v,\beta',t'H'}(\xi(\beta', H', t'))$ is an ample $\mathbb{Q}$-divisor of $K_{(\beta',t'H')}(v)$.*

(2) *We have a surjective map*

$$\varphi_{(\beta,H,t)} : \overline{\mathfrak{H}} \to C(\overline{P^+(K_{(\beta,tH)}(v))_{\mathbb{R}}})$$

such that

$$\varphi_{(\beta,H,t)}((\beta', H', t')) := \mathbb{R}_{>0} \theta_{v,\beta,tH}(\xi(\beta', H', t')).$$

(3) *Let $\mathcal{C}$ be a chamber in $\overline{\mathfrak{H}}$. Then $\Xi(\mathcal{C})$ is the chamber $D(\beta', H', t')$ $((\beta', H', t') \in \mathcal{C})$ in C^+ and*

$$\theta_{v,\beta,tH}(\overline{D(\beta, H, t)}) = \mathrm{Nef}(K_{(\beta,tH)}(v))_{\mathbb{R}}.$$

Proof. (1) Since (β', H', t') is general, we may assume that $d_{\beta'}(v) \neq 0$. If $d_{\beta'}(v) > 0$, then the claim is a consequence of [19, Thm. 1.6]. If $d_{\beta'}(v) < 0$, then we apply [19, Thm. 1.6] to $M_{(\beta',t'H')}(-v)$. Since ξ_ω for v is the same as that for $-v$ and $\theta_{-v,\beta',t'H'} = -\theta_{v,\beta',t'H'}$, $-\theta_{v,\beta',t'H'}(\xi_{t'H'})$ is ample, which implies that $d_{\beta'} \theta_{v,\beta',t'H'}(\xi_{t'H'})$ is ample.

(2) Since $\theta_{v,\beta,tH} : \overline{P^+(v^\perp)_{\mathbb{R}}} \to \overline{P^+(K_{(\beta,tH)}(v))_{\mathbb{R}}}$ is an isomorphism, the claim follows from Proposition 3.11.

(3) By (1) and (2), $\theta_{\beta,H,t}(D(\beta, H, t))$ is contained in $\mathrm{Nef}(K_{(\beta,tH)}(v))_{\mathbb{R}}$. Then the claim follows from [19, Cor. 5.17 (2)]. More precisely, we proved that $\xi(\beta', H', t') \in P^+(v^\perp)_{\mathbb{R}} \cap v_1^\perp \cap \overline{D(\beta, H, t)}$ gives a contraction under the assumption $d_{\beta'} > 0$. For the case where $d_{\beta'} < 0$, we get the claim by the same reduction in (1). We next treat the case

where $d_{\beta'} = 0$. If $r_1\xi - r\xi_1 \neq 0$, then W_{v_1} does not contain the set $d_{\beta'} = 0$ by Lemma 1.5 (1). For a general point of W_{v_1}, we can apply [19, Cor. 5.17 (2)]. Hence for any $\xi(\beta', H', t') \in P^+(v^\perp)_{\mathbb{R}} \cap v_1^\perp \cap \overline{D(\beta, H, t)}$, $\theta_{v,\beta,tH}(\xi(\beta', H', t'))$ gives a contraction.

If $r_1\xi - r\xi_1 = 0$, then W_{v_1} is the set $d_{\beta'} = 0$ by Lemma 1.5 (1). In this case, $\theta_{v,\beta,tH}(\xi(\beta', H', t'))$ gives a contraction. Q.E.D.

Corollary 3.16. *For $(\beta', H', t') \in \overline{D(\beta, H, t)}$ with $\langle \xi(\beta', H', t')^2 \rangle > 0$, $\theta_{v,\beta,tH}(\xi(\beta', H', t'))$ gives a birational contraction of $K_{(\beta,tH)}(v)$.*

Proof. We first note that the canonical bundle of $K_{(\beta,tH)}(v)$ is trivial. By Proposition 3.15 (3) and $\langle \xi(\beta', H', t')^2 \rangle > 0$, we can apply the base point free theorem to get the claim. Q.E.D.

The following result describe the exceptional locus of the contraction.

Proposition 3.17. *Assume that $(\beta', H', t') \in \overline{D(\beta, H, t)} \setminus D(\beta, H, t)$. We set*

$$\begin{aligned} M_{(\beta,tH)}(v)^* &:= \{E \in M_{(\beta,tH)}(v) \mid E \text{ is } \sigma_{(\beta',t'H')}\text{-stable}\},\\ K_{(\beta,tH)}(v)^* &:= M_{(\beta,tH)}(v)^* \cap K_{(\beta,tH)}(v). \end{aligned}$$

If $\langle \xi(\beta', H', t')^2 \rangle > 0$, then $\theta_{v,\beta,tH}(\xi(\beta', H', t'))$ is ample over $K_{(\beta,tH)}(v)^*$. Thus the exceptional locus of the contraction by $\theta_{v,\beta,tH}(\xi(\beta', H', t'))$ is contained in $K_{(\beta,tH)}(v) \setminus K_{(\beta,tH)}(v)^*$.

Proof. Since $\langle \xi(\beta', H', t')^2 \rangle > 0$, we can take a general (β_1, H_1, t_1) such that $\beta_1, H_1 \in \mathrm{NS}(X)_{\mathbb{Q}}$, $t_1^2 \in \mathbb{Q}$ and

$$\xi(\beta', H', t') = x\xi(\beta, H, t) + (1-x)\xi(\beta_1, H_1, t_1),\ x \in (0,1).$$

Since

$$M_{(\beta,tH)}(v)^* \subset M_{(\beta,tH)}(v) \cap M_{(\beta_1,t_1H_1)}(v),$$

we have

$$\theta_{v,\beta,tH}(x)_{|M_{(\beta,tH)}(v)^*} = \theta_{v,\beta_1,t_1H_1}(x)_{|M_{(\beta,tH)}(v)^*},\ x \in v^\perp,$$

where

$$\theta_{v,\beta,tH}(x) := c_1(p_{M_{(\beta,tH)}(v)*}(\mathrm{ch}(\mathbf{E})p_X^*(x^\vee)) \in \mathrm{NS}(M_{(\beta,tH)}(v)),$$

(cf. Definition 1.12). Since $\theta_{v,\beta,tH}(\xi(\beta, H, t))$ and $\theta_{v,\beta_1,t_1H_1}(\xi(\beta_1, H_1, t_1))$ are ample divisors on $M_{(\beta,tH)}(v)$ and $M_{(\beta_1,t_1H_1)}(v)$ respectively, $\theta_{v,\beta,tH}(\xi(\beta', H', t'))$ is ample over $M_{(\beta,tH)}(v)^*$. Q.E.D.

3.3. The movable cone of $K_{(\beta,tH)}(v)$.

Definition 3.18. We set

$$\mathfrak{I}_k := \left\{ w \,\middle|\, \begin{array}{l} w \in H^*(X,\mathbb{Z})_{\mathrm{alg}} \text{ is primitive,} \\ \langle w^2 \rangle = 0, \langle v, w \rangle = k \end{array} \right\},$$

$$\mathfrak{I} := \bigcup_{k=0}^{2} \mathfrak{I}_k.$$

By the classification of walls in [18], the following is obvious.

Lemma 3.19. *For $v_1 \in \mathfrak{I}_1$ and $w_1 \in \mathfrak{W}$ with*

$$w_1 \notin \{iv_1, v - iv_1 \mid 1 \leq i \leq \langle v^2 \rangle/2\},$$

$$v_1^{\perp} \cap w_1^{\perp} \cap P^+(v^{\perp})_{\mathbb{R}} = \emptyset.$$

Proof. If $v_1^{\perp} \cap w_1^{\perp} \cap P^+(v^{\perp})_{\mathbb{R}} \neq \emptyset$, then we have $\xi \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$ such that $\xi \in v_1^{\perp} \cap w_1^{\perp} \cap v^{\perp}$ and $\langle \xi^2 \rangle > 0$. Since $v_1 \in \mathfrak{I}_1$, we have a decomposition $v = \ell v_1 + v_2$ where $\langle v_1^2 \rangle = 0$, $\langle v_1, v_2 \rangle = 1$ and $\ell = \langle v^2 \rangle/2$. Then we have a decomposition

$$\xi^{\perp} = (\mathbb{Z}v_1 \oplus \mathbb{Z}v_2) \perp L.$$

Since $\langle \xi^2 \rangle > 0$, L is negative definite. We set

$$\begin{aligned} w_1 &:= xv_1 + yv_2 + \eta, \\ w_2 &:= (\ell - x)v_1 + (1-y)v_2 - \eta, \end{aligned}$$

where $x, y \in \mathbb{Z}$ and $\eta \in L$. By replacing w_1 by w_2 if necessary, we may assume that $y \geq 1$. Since $w_1 \in \mathfrak{W}$, $\langle w_1^2 \rangle \geq 0, \langle w_2^2 \rangle \geq 0$ and $\langle w_1, w_2 \rangle > 0$. Thus we have $2xy + (\eta^2) \geq 0$, $2(\ell - x)(1-y) + (\eta^2) \geq 0$ and $y(\ell - x) + x(1-y) - (\eta^2) > 0$. On the other hand,

$$\begin{aligned} y(\ell - x) + x(1-y) - (\eta^2) &\leq y(\ell - x) + x(1-y) + 2(\ell - x)(1-y) \\ &= \ell(2-y) - x. \end{aligned}$$

If $y \geq 2$, then $xy \geq -(\eta^2)/2 \geq 0$ implies that $x \geq 0$. Hence $\langle w_1, w_2 \rangle \leq 0$. If $y = 1$, then $\langle w_2^2 \rangle \geq 0$ implies that $\eta = 0$. Hence $w_2 = (\ell - x)v_1$ with $0 \leq x < \ell$, which is a contradiction. Therefore $v_1^{\perp} \cap w_1^{\perp} \cap P^+(v^{\perp})_{\mathbb{R}} = \emptyset$. Q.E.D.

Remark 3.20. For $w \in \mathfrak{I}_0$, we have $W_w = \mathbb{R}w \cap \overline{P^+(v^{\perp})}_{\mathbb{R}}$.

We also have the following result.

Proposition 3.21. *Assume that v is a primitive Mukai vector with $\ell := \langle v^2 \rangle/2 \geq 4$. For $v_1, w_1 \in \mathfrak{I}_2$ with $w_1 \notin \mathbb{Z}v_1$,*

$$v_1^\perp \cap w_1^\perp \cap P^+(v^\perp)_\mathbb{R} = \emptyset.$$

Proof. If $v_1^\perp \cap w_1^\perp \cap P^+(v^\perp)_\mathbb{R} \neq \emptyset$, we can take $\xi \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$ such that $\xi \in v_1^\perp \cap w_1^\perp \cap v^\perp$ and $\langle \xi^2 \rangle > 0$. Then $v - \ell v_1, v - \ell w_1 \in \xi^\perp \cap v^\perp \cap H^*(X,\mathbb{Z})_{\mathrm{alg}}$. Since $\xi^\perp \cap v^\perp \cap H^*(X,\mathbb{Z})_{\mathrm{alg}}$ is negative definite, we have $\langle v-\ell v_1, v-\ell w_1 \rangle^2 \leq \langle (v-\ell v_1)^2 \rangle \langle (v-\ell w_1)^2 \rangle$. Moreover if the equality holds, then $v - \ell v_1 = \pm(v - \ell w_1)$ by $\langle (v - \ell v_1)^2 \rangle = \langle (v - \ell w_1)^2 \rangle = -2\ell$. In this case, we have $v_1 = w_1$ or $2v = \ell(v_1 + w_1)$. By the primitivity of v and $\ell \geq 4$, the latter case does not occur. Hence we have

$$\begin{aligned} 4\ell^2 =& \langle (v - \ell v_1)^2 \rangle \langle (v - \ell w_1)^2 \rangle \\ >& \langle v - \ell v_1, v - \ell w_1 \rangle^2 = (\ell^2 \langle v_1, w_1 \rangle - 2\ell)^2. \end{aligned}$$

Thus $-2\ell < \ell^2 \langle v_1, w_1 \rangle - 2\ell < 2\ell$, which implies that $0 < \ell \langle v_1, w_1 \rangle < 4$. Since $\ell \geq 4$, this does not occur. Therefore the claim holds. Q.E.D.

Remark 3.22. If $\operatorname{rk} \mathrm{NS}(X) \geq 2$, then $\mathfrak{I}_0 \neq \emptyset$ if and only if $\#\mathfrak{I}_0 = \infty$.

Definition 3.23. Let v be a primitive Mukai vector. For $u \in \mathfrak{I}_1 \cup \mathfrak{I}_2$, we set

$$d_u := v - \frac{\langle v^2 \rangle}{\langle v, u \rangle} u.$$

Lemma 3.24. *Let u be an isotropic Mukai vector with $\langle v, u \rangle = 1, 2$.*

(1) *d_u is a primitive vector with $\langle d_u^2 \rangle = -\langle v^2 \rangle$.*

(2) *d_u defines a reflection of the lattice $v^\perp$:*

$$\begin{array}{cccc} R_u: & v^\perp & \to & v^\perp \\ & x & \mapsto & x - \frac{2\langle d_u, x \rangle}{\langle d_u, d_u \rangle} d_u. \end{array}$$

Proof. (1) We set $d_u = k d_1$, $k \in \mathbb{Z}$. Then $\langle v^2 \rangle = k^2 \langle d_1^2 \rangle + 2k\ell \frac{2}{\langle v, u \rangle} \langle d_1, u \rangle \in 2k\mathbb{Z}$. Hence $v = k d_1 + \frac{\langle v^2 \rangle}{\langle v, u \rangle} u$ is divisible by k. By the primitivity of v, $k = 1$. $\langle d_u^2 \rangle = -\langle v^2 \rangle$ is easy.

(2) For $x \in v^\perp$, we have $\langle d_u, x \rangle = -\frac{\langle v^2 \rangle}{\langle v, u \rangle} \langle x, u \rangle$. Hence

$$R_u(x) = x - \frac{2}{\langle v, u \rangle} \langle x, u \rangle d_u \in H^*(X,\mathbb{Z})_{\mathrm{alg}}.$$

Obviously R_u preserves the bilinear form. Therefore R_u is an isometry of $v^\perp$. Q.E.D.

Proposition 3.25. *Let v_1 be an isotropic Mukai vector such that $\langle v, v_1 \rangle = 1$. Then*

(1) $v = \ell v_1 + v_2$, *where* $\ell := \langle v^2 \rangle / 2$, $\langle v_2^2 \rangle = 0$ *and* $\langle v_1, v_2 \rangle = 1$.

(2) *We set* $Y := M_H(v_1)$ *and let*

$$\Phi : \mathbf{D}(X) \to \mathbf{D}(Y)$$

be a Fourier-Mukai transform such that $\Phi(v_1) = (0,0,-1)$ *and* $\Phi(v_2) = (1,0,0)$. *Then the contravariant Fourier-Mukai transform* $\Psi := [1] \circ \Phi^{-1} \circ \mathcal{D}_Y \circ \Phi$ *gives an isometry* Ψ *of* $H^*(X, \mathbb{Z})_{\mathrm{alg}}$ *such that*

$$\Psi_{|v^\perp} = R_{v_1}.$$

Proof. We have a decomposition

$$H^*(X, \mathbb{Z})_{\mathrm{alg}} = (\mathbb{Z}v_1 + \mathbb{Z}v_2) \perp L,$$

where L is a lattice with $\Phi(L) = \mathrm{NS}(Y)$. Hence we see that

$$\begin{aligned} \Psi(w) =& w, \ w \in L, \\ \Psi(v_1) =& -v_1, \\ \Psi(v_2) =& -v_2. \end{aligned}$$

Since $v, d_{v_1} \in (\mathbb{Z}v_1 + \mathbb{Z}v_2)$, we get the claim. Q.E.D.

Definition 3.26. Let W be a wall for v. Let (β, H, t) be a point of W and (β', H', t') be a point in an adjacent chamber. Then we define *the codimension of the wall W* by

$$\operatorname{codim} W := \min_{v = \sum_i v_i} \left\{ \sum_{i<j} \langle v_i, v_j \rangle - \left(\sum_i (\dim \mathcal{M}_H^{\beta'}(v_i)^{ss} - \langle v_i^2 \rangle) \right) + 1 \right\},$$

where $v = \sum_i v_i$ are decompositions of v such that $\phi_{(\beta, tH)}(v) = \phi_{(\beta, tH)}(v_i)$ and $\phi_{(\beta', t'H')}(v_i) > \phi_{(\beta', t'H')}(v_j)$, $i < j$.

If $\operatorname{codim} W \geq 2$, then the proof of [18, Prop. 4.3.5] implies that

$$\dim\{E \in M_{(\beta', t'H')}(v) \mid E \text{ is not } \sigma_{(\beta, tH)}\text{-stable}\} \leq \langle v^2 \rangle.$$

Proposition 3.27. *Let v be a primitive Mukai vector with $\langle v^2 \rangle \geq 6$. Let W be a wall for v and take $(\beta, H, t) \in W$ such that $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $H \in \mathrm{Amp}(X)_{\mathbb{Q}}$.*

(1) *W is a codimension 0 wall if and only if W is defined by v_1 such that*

$$v = \ell v_1 + v_2,\ \langle v_1^2 \rangle = \langle v_2^2 \rangle = 0,\ \langle v_1, v_2 \rangle = 1,\ \ell = \langle v^2 \rangle /2.$$

(2) *We take $(\beta_\pm, \omega_\pm)$ from chambers separated by the codimension 0 wall W in (1). We may assume that $\phi_{(\beta_+,\omega_+)}(E_1) < \phi_{(\beta_+,\omega_+)}(E)$ for $E_1 \in M_{(\beta,\omega)}(v_1)$ and $E \in M_{(\beta_+,\omega_+)}(v)$. Then*
  (a) *$\theta_{v,\beta_\pm,\omega_\pm}(\xi(\beta, H, t))$ give divisorial contractions.*
  (b) *Let $D_\pm \subset M_{(\beta_\pm,\omega_\pm)}(v)$ be the exceptional divisors of the contractions. Then $D_\pm$ are irreducible divisors such that*

$$(D_\pm)_{|K_{(\beta_\pm,\omega_\pm)}(v)} = \pm 2\theta_{v,\beta_\pm,\omega_\pm}(d_{v_1}) \in \mathrm{NS}(K_{(\beta_\pm,\omega_\pm)}(v)).$$

Proof. (1) is a consequence of [18, Lem. 4.3.4 (2)].

(2) Let $\Phi : \mathbf{D}(X) \to \mathbf{D}(Y)$ be the Fourier-Mukai transform in Proposition 3.25. By [18, Prop. 4.1.4], Φ induces an isomorphism

$$M_{(\beta_+,\omega_+)}(v) \to \mathrm{Pic}^0(Y) \times \mathrm{Hilb}_Y^\ell .$$

By using the Hilbert-Chow morphism $\mathrm{Hilb}_Y^\ell \to S^\ell Y$, we have a divisorial contraction

$$M_{(\beta_+,\omega_+)}(v) \to \mathrm{Pic}^0(Y) \times \mathrm{Hilb}_Y^\ell \to \mathrm{Pic}^0(Y) \times S^\ell Y.$$

By using Lemma 3.28, we get the claim (a). Since $\mathcal{D}_Y \circ \Phi$ gives an isomorphism

$$M_{(\beta_-,\omega_-)}(v) \to \mathrm{Pic}^0(Y) \times \mathrm{Hilb}_Y^\ell$$

by [18, Prop. 4.1.4], we also get the claim (a). (b) is a consequence of Lemma 3.28 below. Q.E.D.

Lemma 3.28. *Let $v = (1, 0, -\ell)$ be a primitive Mukai vector with $\ell \geq 3$. We set $v_1 = (0, 0, -1)$ and $v_2 = (1, 0, 0)$. Then*

$$d_{v_1} = v - 2\ell v_1 = (1, 0, \ell).$$

For the Hilbert-Chow morphism $\mathrm{Hilb}_X^\ell \to S^\ell X$, the exceptional divisor D is divisible by 2 and satisfies

$$D_{|K_H(v)} = 2\theta_v((1, 0, \ell)) = 2\theta_v(d_{v_1}).$$

Proposition 3.29. *Let v be a primitive Mukai vector with $\langle v^2 \rangle \geq 6$. Let W be a wall for v and take $(\beta, H, t) \in W$ such that $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $H \in \mathrm{Amp}(X)_{\mathbb{Q}}$.*

(1) *W is a codimension 1 wall if and only if W is defined by v_1 such that*
 (i) *$v = v_1 + v_2$, $\langle v_1^2 \rangle = 0$, $\langle v_2^2 \rangle \geq 0$, $\langle v, v_1 \rangle = 2$ and v_1 is primitive or*
 (ii) *$v = v_1 + v_2 + v_3$, $\langle v_i^2 \rangle = 0$ ($i = 1, 2, 3$), $\langle v_1, v_2 \rangle = \langle v_2, v_3 \rangle = \langle v_3, v_1 \rangle = 1$.*

For the second case, $\langle v^2 \rangle = 6$, W_{v_1} and W_{v_2} intersect transversely and $(\beta, H, t) \in W_{v_1} \cap W_{v_2}$.

(2) *Assume that (β, H, t) belongs to exactly one wall W. We take $(\beta_\pm, \omega_\pm)$ from chambers separated by the codimension 1 wall W in (1). In the notation of (1), we may assume that $\phi_{(\beta_+,\omega_+)}(E_1) < \phi_{(\beta_+,\omega_+)}(E)$ for $E_1 \in M_{(\beta,\omega)}(v_1)$ and $E \in M_{(\beta_+,\omega_+)}(v)$. We set*

$$D_+ := \{E \in M_{(\beta_+,\omega_+)}(v) \mid \mathrm{Hom}(E_1, E) \neq 0, E_1 \in M_{(\beta,\omega)}(v_1)\},$$
$$D_- := \{E \in M_{(\beta_-,\omega_-)}(v) \mid \mathrm{Hom}(E, E_1) \neq 0, E_1 \in M_{(\beta,\omega)}(v_1)\}.$$

Then
 (a) *$D_\pm$ are non-empty and irreducible divisors.*
 (b) *$(D_\pm)_{|K_{(\beta_\pm,\omega_\pm)}(v)} = \pm\theta_{v,\beta_\pm,\omega_\pm}(d_{v_1}) \in \mathrm{NS}(K_{(\beta_\pm,\omega_\pm)}(v))$. In particular, $D_\pm$ are primitive.*
 (c) *$\theta_{v,\beta_\pm,\omega_\pm}(\xi(\beta, H, t))$ gives contractions of $D_\pm$.*

Proof. (1) The classification of codimension 1 walls in [18, Lem. 4.3.4 (2), Prop. 4.3.5] imply that W is defined by v_1 with the required properties. It is easy to see that v_1, v_2, v_3 spans a lattice of rank 3 and $v_1^\perp \cap v_2^\perp = \mathbb{Z}(v_3 - v_1 - v_2)$. Hence W_{v_1} and W_{v_2} intersect transversely.

(2) $D_+ = \theta_{v,\beta_+,\omega_+}(d_{v_1})$ is a consequence of [18, Lem. 4.5.1] and $D_- = -\theta_{v,\beta_-,\omega_-}(d_{v_1})$ follows from [18, Prop. 4.5.2]. By Lemma 3.24 (1), $D_\pm$ are primitive.

The non-emptiness and the contractibility of $D_\pm$ are showed in the proof of [19, Cor. 5.17]. Q.E.D.

Remark 3.30. For the wall W in Proposition 3.29, we have an isomorphism

$$f : M_{(\beta_+,\omega_+)}(v) \to M_{(\beta_-,\omega_-)}(v)$$

such that $f_* \circ \theta_{v,\beta_+,\omega_+} = \theta_{v,\beta_-,\omega_-} \circ R_{v_1}$: Indeed we have a map f as a birational map. We set $\omega_+ := tH_+$. Then $\theta_{v,\beta_+,\omega_+}(\xi(\beta_+, H_+, t)) \in \mathrm{NS}(M_{(\beta_+,\omega_+)}(v))$ is relatively ample over $X \times \widehat{X}$. Since

$$\theta_{v,\beta_-,\omega_-}(R_{v_1}(\xi(\beta_+, H_+, t))) \in \mathrm{NS}(M_{(\beta_-,\omega_-)}(v))$$

is relatively ample over $X \times \widehat{X}$, f is an isomorphism. For the wall W in Proposition 3.27, we also have a similar isomorphism f by using Fourier-Mukai transform Ψ in Proposition 3.25.

The following results fit in the general result of Markman [13] on the movable cone of irreducible symplectic manifolds.

Theorem 3.31. *Let (X,H) be a polarized abelian surface X. Let v be a primitive Mukai vector with $\langle v^2\rangle \geq 6$. We take $(\beta,H,t) \in \mathfrak{H} = \mathrm{NS}(X)_{\mathbb{R}} \times C(\mathrm{Amp}(X)_{\mathbb{R}}) \times \mathbb{R}_{>0}$ such that $\xi(\beta,H,t) \in P^+(v^\perp)_{\mathbb{R}} \setminus \cup_{u\in\mathfrak{W}} u^\perp$, where $\mathfrak{W}$ is the set of Mukai vectors satisfying (1.2). Thus (β,tH) is general (see Definition 1.2).*

(1) *Let $\mathcal{D}(\beta,tH)$ be a connected component of $P^+(v^\perp)_{\mathbb{R}} \setminus \cup_{u\in\mathfrak{I}} u^\perp$ containing $\xi(\beta,H,t)$ (cf. Definition 3.18). Then*

$$\overline{\mathrm{Mov}(K_{(\beta,tH)}(v))_{\mathbb{R}}} = \theta_{v,\beta,tH}(\overline{\mathcal{D}(\beta,tH)}).$$

Moreover

$$\theta_{v,\beta,tH}(H^*(X,\mathbb{Z})_{\mathrm{alg}} \cap \overline{\mathcal{D}(\beta,tH)}) \subset \mathrm{Mov}(K_{(\beta,tH)}(v)).$$

(2) *We choose $u \in \mathfrak{I}_i$ $(i=1,2)$. Let $x \in H^*(X,\mathbb{Z})_{\mathrm{alg}} \cap \overline{\mathcal{D}(\beta,tH)}$ be a general element of the boundary defined by $\langle x,u\rangle = 0$. Then*
   - (a) *$\theta_v(x)$ defines a divisorial contraction from a birational model $K_{(\beta',t'H')}(v)$ of $K_{(\beta,tH)}(v)$.*
   - (b) *The exceptional divisor of the contraction is primitive in $\mathrm{NS}(K_{(\beta,tH)}(v))$ if $u \in \mathfrak{I}_2$, and the exceptional divisor is divisible by 2 in $\mathrm{NS}(K_{(\beta,tH)}(v))$ if $u \in \mathfrak{I}_1$.*

Proof. (1) We note that

$$\mathrm{Mov}(K_{(\beta,tH)}(v))_{\mathbb{R}} \subset C(\overline{P^+(K_{(\beta,tH)}(v))_{\mathbb{R}}}) = \theta_{v,\beta,tH}(\overline{P^+(v^\perp)_{\mathbb{R}}})$$

and

$$\bigcup_{\xi(\beta',H',t')\in\mathcal{D}(\beta,tH)} \overline{D(\beta',H',t')} = \overline{\mathcal{D}(\beta,tH)}. \tag{3.4}$$

Assume that $\xi(\beta',H',t') \in \mathcal{D}(\beta,tH)$. Since there is no wall of codimension 0,1, we have a natural birational identification $K_{(\beta,tH)}(v) \cdots\to K_{(\beta',t'H')}(v)$ with an identification $\mathrm{NS}(K_{(\beta,tH)}(v)) \to \mathrm{NS}(K_{(\beta',t'H'')}(v))$. Hence

$$\bigcup_{\xi(\beta',H',t')\in\mathcal{D}(\beta,tH)} \theta_{v,\beta,tH}(D(\beta',H',t')) \subset \mathrm{Mov}(K_{(\beta,tH)}(v))_{\mathbb{R}}$$

and

$$\begin{aligned}&\theta_{v,\beta,tH}(\overline{\mathcal{D}(\beta,tH)})\\ =&\bigcup_{\xi(\beta',H',t')\in\mathcal{D}(\beta,tH)}\theta_{v,\beta,tH}(\overline{D(\beta',H',t')})\subset\overline{\mathrm{Mov}(K_{(\beta,tH)}(v))}_{\mathbb{R}}.\end{aligned}$$

Assume that $\xi(\beta',H',t')\notin\overline{\mathcal{D}(\beta,tH)}$. We set

$$\eta_x:=x\xi(\beta',H',t')+(1-x)\xi(\beta,H,t),\ x\in[0,1].$$

If $\theta_{v,\beta,tH}(\xi(\beta',H',t'))$ is movable, then $L_x:=\theta_{v,\beta,tH}(\eta_x)$ is movable for $0\le x\le 1$. We take an adjacent wall $u^\perp$ ($u\in\mathfrak{I}$) of $\mathcal{D}(\beta,tH)$ separating $\xi(\beta',H',t')$ and $\xi(\beta,H,t)$. Then we find $x_0\in\mathbb{Q}\cap(0,1)$ such that $\langle\eta_{x_0},u\rangle=0$. Let $D(\beta_1,H_1,t_1)\subset\mathcal{D}(\beta,tH)$ be the chamber such that $\eta_x\in\xi(D(\beta_1,H_1,t_1))$ for $x'<x<x_0$ with $x'<x_0$. Since all walls W between η_0 and η_{x_0} satisfy $\operatorname{codim}W\ge 2$, we have a natural birational map $\varphi: K_{(\beta,tH)}(v)\cdots\to K_{(\beta_1,t_1H_1)}(v)$ which induces a commutative diagram

$$\begin{array}{ccc} v^\perp & = & v^\perp\\ {\scriptstyle\theta_{v,\beta,tH}}\downarrow & & \downarrow{\scriptstyle\theta_{v,\beta_1,t_1H_1}}\\ \mathrm{NS}(K_{(\beta,tH)}(v)) & \xrightarrow{\varphi_*} & \mathrm{NS}(K_{(\beta_1,t_1H_1)}(v))\end{array}$$

Since $(\eta_{x_0}^2)>0$, $\varphi_*(L_{x_0})$ gives a divisorial contraction of $K_{(\beta_1,t_1H_1)}(v)$. Let $C(\subset K_{(\beta_1,t_1H_1)}(v))$ be a general curve contracted by $\varphi_*(L_{x_0})$. Since $L_1=\theta_{v,\beta,tH}(\xi(\beta',H',t'))$ is movable, $\varphi_*(L_1)$ is also movable. Hence we may assume that C is not contained in its base locus. Then $(\varphi_*(L_1),C)\ge 0$. Since $(\varphi_*(L_x),C)>0$ for $x'<x<x_0$, we have $(\varphi_*(L_{x_0}),C)>0$, which is a contradiction. Hence $\theta_{v,\beta,tH}(\xi(\beta',H',t'))$ is not movable.

Assume that $\xi\in H^*(X,\mathbb{Z})_{\mathrm{alg}}$ belongs to $\overline{\mathcal{D}(\beta,tH)}$. We take $\overline{D(\beta',H',t')}$ containing ξ by (3.4). If $(\xi^2)>0$, then Corollary 3.16 implies that $\theta_{v,\beta,tH}(\xi)$ gives a birational contraction of $K_{(\beta',t'H')}(v)$. Hence it is movable.

If $(\xi^2)=0$, then see Proposition 3.38. Therefore (1) holds.

(2) is a consequence of Proposition 3.27 and 3.29. Q.E.D.

Remark 3.32. For $u\in\mathfrak{I}_0$, $u^\perp$ is a tangent of $\overline{P^+(v^\perp)}_{\mathbb{R}}$.

Corollary 3.33. *Keep notations in Theorem 3.31.*

(1) *Let (K,L) be a pair of a smooth manifold K with a trivial canonical bundle and an ample divisor L on K. If K is birationally equivalent to $K_{(\beta,tH)}(v)$, then there is $\xi(\beta',H',t')\in\mathcal{D}(\beta,tH)$ such that $K\cong K_{(\beta',t'H')}(v)$ and $\mathbb{R}_{>0}L$ corresponds to $\mathbb{R}_{>0}\theta_{v,\beta,\omega}(\xi(\beta',H',t'))$.*

(2) *Let (M, L) be a pair of smooth manifold M with a trivial canonical bundle and an ample divisor L on M. If M is birationally equivalent to $M_{(\beta,tH)}(v)$, then there is $\xi(\beta', H', t') \in \mathcal{D}(\beta, tH)$ such that $M \cong M_{(\beta',t'H')}(v)$ and $\mathbb{R}_{>0}L$ corresponds to $\mathbb{R}_{>0}\theta_{v,\beta,\omega}(\xi(\beta', H', t'))$ up to line bundles coming from the albanese variety* $\mathrm{Alb}(M_{(\beta',t'H')}(v))$.

Proof. (1) Since the canonical bundles of K and $K_{(\beta,tH)}(v)$ are trivial, we have a birational map

$$f : K \cdots \to K_{(\beta,tH)}(v)$$

with an isomorphism $f_* : \mathrm{NS}(K) \to \mathrm{NS}(K_{(\beta,tH)}(v))$. Then $f_*(L) \in \mathrm{Mov}(K_{(\beta,tH)}(v))_{\mathbb{Q}}$. We take $(\beta', H', t') \in \mathfrak{H}$ such that

$$f_*(L) \in \mathbb{R}_{>0}\theta_{v,\beta,tH}(\xi(\beta', H', t'))$$

and $\beta', H' \in \mathrm{NS}(X)_{\mathbb{Q}}$ (see Remark 3.12). Assume that $\xi(\beta', H', t') \in \overline{D(\beta_1, H_1, t_1)} (\subset \overline{\mathcal{D}(\beta, tH)})$. We take an ample divisor L_0 on K such that $f_*(L_0) \in \theta_{v,\beta,tH}(D(\beta_1, H_1, t_1))$. For the birational map $g : K_{(\beta,tH)}(v) \to K_{(\beta_1,t_1H_1)}(v)$, $L_1 := (g \circ f)_*(L_0)$ is ample. We note that $g \circ f$ induces an isomorphism

$$\bigoplus_{n \geq 0} H^0(K, \mathcal{O}_K(nL_0)) \cong \bigoplus_{n \geq 0} H^0(K_{(\beta_1,t_1H_1)}(v), \mathcal{O}_{K_{(\beta_1,t_1H_1)}(v)}(nL_1)).$$

Then the ampleness of L_0 and L_1 imply that $g \circ f : K \to K_{(\beta_1,t_1H_1)}(v)$ is an isomorphism. Since L is ample, $\theta_{v,\beta,tH}(\xi(\beta', H', t'))$ is also ample on $K_{(\beta_1,t_1H_1)}(v)$, which implies that $\xi(\beta', H', t') \in D(\beta_1, H_1, t_1)$. Hence $K \cong K_{(\beta_1,t_1H_1)}(v) \cong K_{(\beta',t'H')}(v)$.

(2) For a birational map $\varphi : M \cdots \to M_{(\beta,tH)}(v)$, we have a commutative diagram

$$\begin{array}{ccc} M & \cdots\xrightarrow{\varphi} & M_{(\beta,tH)}(v) \\ \downarrow & & \downarrow \\ \mathrm{Alb}(M) & \xrightarrow{\eta} & \mathrm{Alb}(M_{(\beta,tH)}(v)) \end{array}$$

where η is an isomorphism.

Let K be a smooth fiber of the albanese map of M. Then the canonical bundle of K is trivial. For a general smooth fiber K, φ induces a birational map $K \to K_{(\beta,tH)}(v)$. There is $\xi(\beta', H', t') \in D(\beta, tH)$ such that for the birational map $\psi : M_{(\beta,tH)}(v) \cdots \to M_{(\beta',t'H')}(v)$, $\psi \circ \varphi$ indices an isomorphism $K \to K_{(\beta',t'H')}(v)$. Thus $(\psi \circ \varphi)_*(L)_{|K_{(\beta',t'H')}(v)}$

is ample. Since the albanese map of $M_{(\beta',t'H')}(v)$ is isotrivial, $(\psi\circ\varphi)_*(L)$ is relatively ample over $\mathrm{Alb}(M_{(\beta',t'H')}(v))$. As in the proof of (1), by looking at relative global sections, we see the $\psi\circ\varphi$ is an isomorphism. Q.E.D.

Corollary 3.34. *Keep notations in Theorem 3.31*

(1) *Assume that* $\mathfrak{I}_1=\mathfrak{I}_2=\emptyset$, *that is,*

$$\min\{\langle v,w\rangle>0 \mid \langle w^2\rangle=0\}\geq 3.$$

Then

$$\mathcal{D}(\beta,tH)=P^+(v^\perp)_{\mathbb{R}}.$$

In particular, $\mathrm{Mov}(K_{(\beta,tH)}(v))_{\mathbb{Q}}=\overline{P^+(K_{(\beta,tH)}(v))}_{\mathbb{Q}}$.

(2) *Assume that* $\mathfrak{I}_1=\emptyset$ *and* $\mathfrak{I}_2\neq\emptyset$, *that is,*

$$\min\{\langle v,w\rangle>0 \mid \langle w^2\rangle=0\}=2.$$

For every divisorial contraction from a birational model of $K_{(\beta,tH)}(v)$, *the exceptional divisor is primitive in* $\mathrm{NS}(K_{(\beta,tH)}(v))$.

(3) *Assume that* $\mathfrak{I}_1\neq\emptyset$, *that is,*

$$\min\{\langle v,w\rangle>0 \mid \langle w^2\rangle=0\}=1.$$

Then there is a divisorial contraction from a birational model of $K_{(\beta,tH)}(v)$ *such that the exceptional divisor is a prime divisor and divisible by 2 in* $\mathrm{NS}(K_{(\beta,tH)}(v))$.

Let us study the structure of walls in a neighborhood of a rational point of $\overline{P^+(v^\perp)}_{\mathbb{R}}\setminus P^+(v^\perp)_{\mathbb{R}}$. Let A be a compact subset of $P^+(v^\perp)_{\mathbb{R}}$ and u an isotropic Mukai vector in the boundary of $\overline{P^+(v^\perp)}_{\mathbb{R}}$. Let $\overline{uA}$ be the cone spanned by u and A. We note that $\mathfrak{W}_A:=\{v_1\in\mathfrak{W}\mid A\cap v_1^\perp\neq\emptyset\}$ is a finite set. We fix a point $a\in A$ which does not lie on any wall and assume that there is no wall between u and a (cf. Remark 3.37).

Lemma 3.35. $v_1\in\mathfrak{W}$ *satisfies* $v_1^\perp\cap\overline{uA}\neq\emptyset$ *if and only if* $v_1^\perp\cap(A\cup\{u\})\neq\emptyset$.

Proof. Assume that $w\in v_1^\perp\cap\overline{uA}$. We take $w'\in A$ such that w belongs to the segment $\overline{uw'}$ connecting u and w'. Assume that $\langle v_1,u\rangle\neq 0$. If $v_1^\perp\cap A=\emptyset$, then $\langle v_1,u\rangle\langle v_1,w'\rangle<0$. Since there is no wall between u and a, we have $\langle v_1,u\rangle\langle v_1,a\rangle>0$. Then w' and a are separated by the hyperplane $v_1^\perp$. Therefore there is a point $x\in A$ with $x\in v_1^\perp$, which is a contradiction. Hence $v_1^\perp\cap A\neq\emptyset$. Q.E.D.

Proposition 3.36. (1) *$\{v_1 \in \mathfrak{W} \mid v_1^\perp \cap (\overline{uA} \setminus \{u\}) \neq \emptyset\}$ is a finite set.*

(2) *There is an open neighborhood of u such that*

$$\{v_1 \in \mathfrak{W} \mid v_1^\perp \cap U \cap (\overline{uA} \setminus \{u\}) \neq \emptyset\} \subset u^\perp.$$

In particular the set of walls is finite in $U \cap \overline{uA}$ and all walls pass the point u.

Proof. By Lemma 3.35,

$$\{v_1 \in \mathfrak{W} \mid v_1^\perp \cap (\overline{uA} \setminus \{u\}) \neq \emptyset\} \subset \mathfrak{W}_A.$$

Hence (1) holds. (2) easily follow from (1). Q.E.D.

Remark 3.37. Assume that e^β satisfies $\langle e^\beta, v\rangle = 0$. Then we can set $v = re^\beta + \xi + (\xi, \beta)\varrho_X$. We set

$$B := \{v_1 \in \mathfrak{W} \mid e^\beta \in v_1^\perp\}.$$

For $v_1 \in B$ and $v_2 := v - v_1 \in B$, we can set

$$\begin{aligned} v_1 :=& r_1 e^\beta + \xi_1 + (\xi_1, \beta)\varrho_X, \\ v_2 :=& r_2 e^\beta + \xi_2 + (\xi_2, \beta)\varrho_X. \end{aligned}$$

Since $0 < \langle v_1, v_2\rangle = (\xi_1, \xi_2)$, $\xi_1 \neq 0$ and $\xi_2 \neq 0$. If (β, ω') belongs to the wall defined by v_1, then $(\xi_1^2), (\xi_2^2) \geq 0$ implies that both of $n\xi_1$ and $n\xi_2$ are effective, or both of $-n\xi_1$ and $-n\xi_2$ are effective, where n is the denominator of β. In particular, $|(\xi_1, H)| < |(\xi, H)|$ for all ample divisor H. Then we also see that the set of ξ_1 is finite.

If $r_1 = r\frac{(\xi_1,H)}{(\xi,H)}$, then $|r_1| < |r|$. Therefore

$$B' := \{v_1 \in B \mid (r_1\xi - r\xi_1, H) = 0 \text{ for some } H \in \operatorname{Amp}(X)_{\mathbb{Q}}\ \}$$

is a finite set. We take $H \in \operatorname{Amp}(X)_{\mathbb{Q}}$ such that $(H, r_1\xi - r\xi_1) \neq 0$ for all $v_1 \in B'$. Then (β, H, t) $(t \ll 1)$ belongs to a chamber.

Proposition 3.38. *Let u be a primitive and isotropic Mukai vector with $u \in \overline{P^+(v^\perp)_{\mathbb{R}}}$. We take $(\beta, H, t) \in \mathfrak{H} \setminus \cup_{v_1 \in \mathfrak{W}} W_{v_1}$ such that $\xi(\beta, H, t)$ and u are not separated by a wall. Then $\theta_{v,\beta,tH}(u)$ gives a Lagrangian fibration $K_{(\beta,tH)}(v) \to \mathbb{P}^{\langle v^2\rangle/2-1}$.*

Proof. We take a Fourier-Mukai transform $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ such that $\operatorname{rk}\Phi(u) \neq 0$. Since Φ induces an isomorphism $M_{(\beta,tH)}(v) \to M_{(\beta_1,t_1H_1)}(\Phi(v))$ with $\mathbb{R}_{>0}\Phi(\xi(\beta_1, H_1, t_1)) = \mathbb{R}_{>0}\xi(\beta, H, t)$, we may assume that $\operatorname{rk} u \neq 0$. Then we have $u = re^{\beta'}$ with $r \neq 0$. We take

$w = \pm u$ with $\operatorname{rk} w > 0$. We set $Y := M_H(w)$ and let $\mathbf{E}$ be the universal family as twisted objects. Then we have (twisted) Fourier-Mukai transforms $\Phi_{X\to Y}^{\mathbf{E}^\vee[k]} : \mathbf{D}(X) \to \mathbf{D}^\alpha(Y)$, where α are suitable 2-cocycles of $\mathcal{O}_Y^\times$ defining $\mathbf{E}^\vee$ and $k = 1$ for $(\delta - \beta', H) > 0$, $k = 2$ for $(\delta - \beta', H) < 0$. We set $v' = \Phi_{X\to Y}^{\mathbf{E}^\vee[k]}(v)$. Since $(\widetilde{\beta}, \widetilde{tH}, s)$ does not lie on any wall for $s \geq 1$, we have $M_{(\widetilde{\beta}, s\widetilde{tH})}(v') \cong M_{\widetilde{tH}}^\alpha(v')$ for $s \geq 1$, where $M_{\widetilde{tH}}^\alpha(v')$ is the moduli space of semi-stable α-twisted sheaves on Y ([30]). Thus we get an isomorphism $M_{(\beta,tH)}(v) \to M_{\widetilde{tH}}^\alpha(v')$ with $\operatorname{rk} v' = 0$. We note that the scheme-theoretic support $\operatorname{Div}(E)$ of a purely 1-dimensional sheaf is well-defined even for a twisted sheaf. Hence we have a morphism $f : M_{\widetilde{tH}}^\alpha(v') \to \operatorname{Hilb}_Y^\eta$ by sending $E \in M_{\widetilde{tH}}^\alpha(v')$ to $\operatorname{Div}(E)$, where $\operatorname{Hilb}_Y^\eta$ is the Hilbert scheme of effective divisors D on Y with $\eta = c_1(D)$. For a smooth divisor D, $f^{-1}(D) \cong \operatorname{Pic}^0(D)$. Hence f is dominant, which implies f is surjective. Therefore we get a surjective morphism $M_{(\beta,tH)}(v) \to \operatorname{Hilb}_Y^\eta$. Combining with the properties of the albanese map, we have a commutative diagram:

$$\begin{array}{ccc} M_{(\beta,tH)}(v) & \longrightarrow & \operatorname{Hilb}_Y^\eta \\ {\scriptstyle \mathfrak{a}}\big\downarrow & & \big\downarrow \\ X \times \widehat{X} & \longrightarrow & \operatorname{Pic}^0(Y) \end{array} .$$

Hence we get a morphism $K_{(\beta,tH)}(v) \to \mathbb{P}(H^0(Y, \mathcal{O}(D)))$, where $D \in \operatorname{Hilb}_Y^\eta$. Then we see that $\theta_{v,\beta,tH}(u)$ comes from $\mathbb{P}(H^0(Y, \mathcal{O}(D)))$. Thus $\theta_{v,\beta,tH}(u)$ gives a Lagrangian fibration. Q.E.D.

As we shall see in appendix, the fiber of $K_{(\beta,tH)}(v) \to \mathbb{P}(H^0(Y, \mathcal{O}(D)))$ is connected.

3.4. The birational classes of the moduli spaces of rank 1 sheaves.

Proposition 3.39. *Let (X, H) be a polarized abelian surface. Let $v = (r, \xi, a)$ be a Mukai vector such that $2\ell := \langle v^2 \rangle \geq 6$. Then $M_H^\beta(v)$ is birationally equivalent to $\operatorname{Pic}^0(Y) \times \operatorname{Hilb}_Y^\ell$ if and only if there is an isotropic Mukai vector $w \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ with $\langle v, w \rangle = 1$, where Y is an abelian surface.*

Proof. By using a Fourier-Mukai transform, we may assume that $r > 0$. If there is an isotropic Mukai vector w with $\langle v, w \rangle = 1$, then $M_H^\beta(w)$ is a fine moduli space and the claim follows by [31, Cor. 0.3].

Conversely if $M_H^\beta(v)$ is birationally equivalent to $\operatorname{Pic}^0(Y) \times \operatorname{Hilb}_Y^\ell$, then we have a birational map $f : K_H^\beta(v) \to K_{H'}(1, 0, -\ell)$, where H' is an ample divisor on Y. Then we have an isomorphism $f_*: \mathrm{NS}(K_H(v)) \to \mathrm{NS}(K_{H'}(1,0,-\ell))$. By the isomorphism f_*, the movable cones are isomorphic. By Thoerem 3.31, $\mathfrak{I}_1 \neq \emptyset$. Q.E.D.

Remark 3.40. Proposition 3.39 also follows from [16, Lem. 4.9]. Indeed they characterize the generalized Kummer variety in terms of the class of the stably prime exceptional divisor δ such that 2δ should corresponds to the diagonal divisor. It implies the existence of an isotropic vector w in Proposition 3.39.

Remark 3.41. If

$$\min\{\langle v, w\rangle > 0 \mid \langle w^2\rangle = 0\} \geq 3,$$

then $M_H^\beta(v)$ is not birationally equivalent to any moduli space $M_{H'}^{\beta'}(v')$ on an abelian surface Y with $\operatorname{rk} v' = 2$.

The following result was conjecture by Mukai [24].

Corollary 3.42. *Let (X, H) be a principally polarized abelian surface with $\mathrm{NS}(X) = \mathbb{Z}H$. Let $v = (r, dH, a)$ be a Mukai vector with $\ell := d^2 - ra \geq 3$. Then $M_H^\beta(v)$ is birationally equivalent to $X \times \mathrm{Hilb}_X^\ell$ if and only if the quadratic equation*

$$rx^2 + 2dxy + ay^2 = \pm 1$$

has an integer valued solution.

Proof. Primitive isotropic Mukai vectors are described as $w = \pm(p^2, -pqH, q^2)$, $p, q \in \mathbb{Z}$ and

$$\langle v, w\rangle = \mp(rq^2 + 2rpq + ap^2).$$

Hence the claim follows from Proposition 3.39. Q.E.D.

Remark 3.43. We assume that (X, H) is a principally polarized abelian surface with $\mathrm{NS}(X) = \mathbb{Z}H$. If $\ell = 1, 2$, then Mukai proved $M_H^\beta(v) \cong X \times \mathrm{Hilb}_X^\ell$ (see [28, section 7]).

3.5. Walls for X with $\operatorname{rk}\mathrm{NS}(X) \geq 2$.

We shall show that there are many walls by using Fourier-Mukai transforms. We set

$$Q_\ell := \{\xi \in \mathrm{NS}(X)_{\mathbb{R}} \mid (\xi^2) = 2\ell\}.$$

Lemma 3.44. *We set $v = (r, \xi_0, a)$ $(r \neq 0)$ and $\ell := \langle v^2\rangle/2$. An isotropic vector $w \in H^*(X, \mathbb{Z})_{\mathrm{alg}} \otimes \mathbb{R}$ satisfies $\langle w, v\rangle = 0$ if and only if $w = (\operatorname{rk} w)e^{\xi_0/r+\xi}$ with $r\xi \in Q_\ell$ or $w = (0, \xi, (\xi, \xi_0)/r)$ with $(\xi^2) = 0$.*

Proof. Assume that $\operatorname{rk} w \neq 0$ and set $w = (\operatorname{rk} w)e^{\xi_0/r+\xi}$. Since $v = re^{\xi_0/r} - \frac{\ell}{r}\varrho_X$, the condition is

$$0 = \langle e^{\xi_0/r+\xi}, v\rangle = -r\frac{(\xi^2)}{2} + \frac{\ell}{r}.$$

Thus $r\xi \in Q_\ell$.

If $\operatorname{rk} w = 0$, then we set $w = e^{\xi_0/r}(0,\xi,a)$ with $(\xi^2) = 0$. Then the condition is $a = 0$, which implies $w = (0,\xi,(\xi,\xi_0)/r)$. Q.E.D.

Lemma 3.45. *Assume that* $\operatorname{rk}\operatorname{NS}(X) \geq 2$. *For* $\xi_0 \in \operatorname{NS}(X)$ *and an ample divisor* ξ, *there is* $\xi_1 \in \operatorname{NS}(X)$ *such that* $\xi_1 = \xi_0 + r(k\xi + \eta)$ *(*$k \gg -(\eta^2)$, $\eta \in \operatorname{NS}(X)$*) and* $\sqrt{\frac{2\ell}{(\xi_1^2)}} \notin \mathbb{Q}$.

Proof. We take an integer $k_1 \gg 0$ such that $((\xi_0 + rk_1\xi)^2) > 0$. If $\sqrt{2\ell((\xi_0 + rk_1\xi)^2)} \notin \mathbb{Z}$, then $\xi_0 + rk_1\xi$ satisfies the claim. Assume that $a := \sqrt{2\ell((\xi_0 + rk_1\xi)^2)} \in \mathbb{Z}$. We take $\eta \in \operatorname{NS}(X)$ with $\langle \eta, (\xi_0 + rk_1\xi)\rangle = 0$. Then $\xi_1 := (rk_2+1)(\xi_0 + rk_1\xi) - r\eta$ satisfies $2\ell(\xi_1^2) = (rk_2+1)^2a^2 + 2\ell r^2(\eta^2)$. If $2\ell(\xi_1^2) = x^2$, $x \in \mathbb{Z}$, then

$$-2\ell r^2(\eta^2) = ((rk_2+1)a - x)((rk_2+1)a + x) > (rk_2+1)a.$$

Hence for $k_2 \gg 0$, $\sqrt{\frac{2\ell}{(\xi_1^2)}} = \frac{\sqrt{2\ell(\xi_1^2)}}{(\xi_1^2)} \notin \mathbb{Q}$. Q.E.D.

Proposition 3.46. *Assume that* $\operatorname{rk}\operatorname{NS}(X) \geq 2$. *For* $v = (r,\xi_0,a)$ *with* $\langle v^2\rangle = 2\ell$, *if* $\ell \geq r > 0$, *then* $\overline{\cup_{u\in\mathfrak{W}} u^\perp}$ *contains* $\overline{P^+(v^\perp)_\mathbb{R}} \setminus P^+(v^\perp)_\mathbb{R}$.

Proof. Since $\ell \geq r$, $u := (0,0,-1)$ satisfies (1.2) and (1.4). Hence u defines a non-empty wall $u^\perp$.

We shall use Lemma 3.44 to show the claim. For $r\xi \in Q_\ell$, we take $r\xi_1 \in \operatorname{NS}(X)_\mathbb{Q}$ in a neighborhood of $r\xi$. Then $\sqrt{\frac{2\ell}{(\xi_1^2)}}\xi_1 \in Q_\ell$ is also sufficiently close to $r\xi$. Replacing ξ by $-\xi$ if necessary, we assume that $r\xi_1$ is ample. We take a primitive and ample divisor H such that $dH = \xi_0 + r(k\xi_1 + \eta)$ ($k \gg 0$) and $\sqrt{\frac{2\ell}{(H^2)}} \notin \mathbb{Q}$. Then $\lim_{k\to\infty}\sqrt{\frac{2\ell}{(H^2)}}H = \sqrt{\frac{2\ell}{(\xi_1^2)}}\xi_1$. In particular, for any open neighborhood U of $r\xi$, we can take an ample divisor H such that $\sqrt{\frac{2\ell}{(H^2)}}H \in U$. We set $L := \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z}\varrho_X$. For $v' := ve^{k\xi_1+\eta} = (r, dH, a') \in L$, Lemma 2.7 implies we have an autoequivalence Φ of $\mathbf{D}(X)$ such that $\Phi(v') = v'$ and $\Phi_{|L}$ is of infinite order. We set $\zeta_\pm := e^{\frac{1}{r}\left(d\pm\sqrt{\frac{2\ell}{(H^2)}}\right)H}$. Then $\mathbb{R}_{>0}\zeta_\pm$ are the fixed rays of Φ in $L_\mathbb{R} \cap \overline{P^+(v^\perp)_\mathbb{R}}$ and the rays defined by $\Phi^n(u)^\perp$ converge to the fixed rays. Hence $\zeta_\pm \in \overline{\cup_{n\in\mathbb{Z}}\Phi^n(u)^\perp}$. We set $\Psi := e^{-(k\xi_1+\eta)}\Phi e^{k\xi_1+\eta} \in$

$\mathrm{Eq}(\mathbf{D}(X), v)$, where $\mathrm{Eq}(\mathbf{D}(X), v)$ is the set of autoequivalences of $\mathbf{D}(X)$ fixing v. Since

$$\left(d \pm \sqrt{\frac{2\ell}{(H^2)}}\right) H = \xi_0 \pm \sqrt{\frac{2\ell}{(H^2)}} H + r(k\xi_1 + \eta),$$

$e^{\frac{1}{r}\left(\xi_0 \pm \sqrt{\frac{2\ell}{(H^2)}} H\right)} \in \overline{\cup_{n\in\mathbb{Z}} \Psi^n(u)^\perp}$. Hence $e^{\xi_0/r \pm \xi} \in \overline{\cup_{\Phi\in \mathrm{Eq}(\mathbf{D}(X),v)} \Phi(u)^\perp}$.

For $\xi \in \overline{\mathrm{Amp}(X)}$ with $(\xi^2) = 0$, we have $(0, \xi, (\xi, \xi_0)/r) \in (0, 0, -1)^\perp$. Therefore the claim holds. Q.E.D.

Lemma 3.47. *Assume that v_1 defines a wall for v. Then $(\langle v^2\rangle + \langle v_1^2\rangle)/2 \geq \langle v, v_1\rangle > \langle v_1^2\rangle$. In particular, $\langle v^2\rangle > \langle v, v_1\rangle > \langle v_1^2\rangle$.*

Proof. Since $\langle v - v_1, v_1\rangle > 0$, we have $\langle v, v_1\rangle > \langle v_1^2\rangle$. Since $0 \leq \langle (v-v_1)^2\rangle = \langle v^2\rangle + \langle v_1^2\rangle - 2\langle v, v_1\rangle$, we get the first claim. Then we have $\langle v^2\rangle > \langle v_1^2\rangle$, which implies the second claim. Q.E.D.

Lemma 3.48. *Let X be an abelian surface with* $\mathrm{NS}(X) = \mathbb{Z}H \perp L$, *where H is an ample divisor and L is a negative definite lattice. Assume that $(H^2) = 2\ell(4\ell ra + 1)$, $r, a \in \mathbb{Z}_{>0}$. We set $v := (2\ell r, H, 2\ell a)$. Then $\langle v^2\rangle = (H^2) - 8\ell^2 ra = 2\ell$ and there is no wall for v. In particular,* $\mathrm{Amp}(K_H(v))$ *coincides with its positive cone.*

Proof. We note that $\langle v, w\rangle \in 2\ell\mathbb{Z}$ for all $w \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$. By Lemma 3.47, there is no wall for v. Q.E.D.

Remark 3.49. If $\sqrt{\langle v^2\rangle (H^2)} = 2\ell\sqrt{4\ell ra + 1} \notin \mathbb{Q}$ or $\mathrm{rk}\,\mathrm{NS}(X) \geq 2$, then there are infinitely many autoequivalences of $\mathbf{D}(X)$ preserving v.

Proposition 3.50. *Assume that* $\mathrm{rk}\,\mathrm{NS}(X) \geq 2$ *or* $\sqrt{\langle v^2\rangle/(H^2)} \notin \mathbb{Q}$. *If there is no wall for v, then* $\mathrm{Aut}(M_H(v))$ *contains an automorphism of infinite order.*

Proof. We may assume that $v = (r, \xi, a)$, where ξ is ample. We take $g \in \mathrm{Stab}_0(v)^*$ of infinite order. Then there is an autoequivalence Φ of $\mathbf{D}(X)$ which induces g. Then Φ induces an isomorphism $M_H(v) \to M_H(v)$. Q.E.D.

If $M_{(\beta,tH)}(v)$ is birationally equivalent to $M_H(1, 0, -\ell)$, we can get a more precise description of the stabilizer group. Since there is an autoequivalence $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ with $\Phi(v) = (1, 0, -\ell)$, it is sufficient to treat the case $v = (1, 0, -\ell)$.

Proposition 3.51. *We set $v = (1, 0, -\ell)$.*

(1) *A Fourier-Mukai transform* $\Phi : \mathbf{D}(Y) \to \mathbf{D}(X)$ *with the condition* $\Phi((1,0,-\ell)) = (1,0,-\ell)$ *corresponds to a decomposition*

$$v = \ell u_1 + u_2,\ \langle u_1, u_2\rangle = 1,\ \langle u_1^2\rangle = \langle u_2^2\rangle = 0, \tag{3.5}$$

where $u_1 = \Phi(-\varrho_Y)$ *and* $u_2 = \Phi(v(\mathcal{O}_Y))$.

(2) *For the decomposition* (3.5), *one of the following holds.*

(i)

$$u_1 = (p^2 s, pq\xi, q^2 t),\ u_2 = -(q^2 t, \ell pq\xi, \ell^2 p^2 s),$$

where $p, q, s, t \in \mathbb{Z}$ *satisfy* $\ell sp^2 - tq^2 = 1$ *and* ξ *is a primitive divisor with* $(\xi^2) = 2st > 0$.

(ii)

$$\ell = 1,\ u_1 = (0, \xi, -1),\ u_2 = (1, -\xi, 0),$$

where $(\xi^2) = 0$.

(3) *For the case (i) of (2), if* $t = 1$, *then* $Y \cong X$.

Proof. (1) Since $v = \ell(-\varrho_Y) + v(\mathcal{O}_Y)$, we have

$$v = \Phi(v) = \ell\Phi(-\varrho_Y) + v(\mathcal{O}_Y).$$

Since ϱ_Y and $v(\mathcal{O}_Y)$ are isotropic vector with $\langle -\varrho_Y, v(\mathcal{O}_Y)\rangle = 1$, we get the decomposition (3.5). Conversely for the decomposition (3.5), we have an equivalence $\Phi : \mathbf{D}(Y) \to \mathbf{D}(X)$ such that $u_1 = \Phi(-\varrho_Y)$ and $u_2 = \Phi(v(\mathcal{O}_Y))$, where $Y = M_H(\pm u_1)$ and H is an ample divisor on X.

(2) We first assume that $\operatorname{rk} u_1 \neq 0$. We set $u_1 = (px, pq\xi, y)$, where $\xi \in \mathrm{NS}(X)$ is primitive and $\gcd(x, q\xi) = 1$. Since u_1 is primitive, $\gcd(p, y) = 1$. Assume that $q^2(\xi^2) \neq 0$. Then $p^2 q^2(\xi^2) = 2pxy$ implies that $p \mid x$ and $q^2 \mid y$. So we write $u_1 = (p^2 s, pq\xi, q^2 t)$ with $(\xi^2) = 2st$. Since $1 = \langle v, u_1\rangle$, we have $p^2 s\ell - q^2 t = 1$. We set $u_2 := v - \ell u_1$. Then

$$u_2 = (1 - \ell sp^2, -\ell pq\xi, -\ell(1 + q^2 t)) = -(q^2 t, \ell pq\xi, \ell^2 p^2 s).$$

If $st < 0$, then we have $(p^2 s\ell, q^2 t) = (1, 0)$ or $(p^2 s\ell, q^2 t) = (0, -1)$. Since $\operatorname{rk} u_1 \neq 0$, we have $p^2 = s = \ell = 1$ and $q = 0$. Since $q^2(\xi^2) \neq 0$, this case does not occur. Therefore $st > 0$.

If $q^2(\xi^2) = 0$, then $y = 0$ and $p = \pm 1$. Hence $u_1 = \pm(x, q\xi, 0)$. Since $1 = \langle v, u_1\rangle = \pm\ell x$, $x = \pm 1$ and $\ell = 1$. Thus $u_1 = (1, q\xi, 0)$ and $u_2 = (0, -q\xi, -1)$. By exchanging u_1 by u_2, we have (ii).

We next assume that $\operatorname{rk} u_1 = 0$. We set $u_1 = (0, D, y)$. Then $1 = \langle v, u_1\rangle = -y$. Hence $u_1 = (0, D, -1)$ with $(D^2) = 0$. Then (i) holds, where $t = -1$, $q^2 = 1$ and pq is the multiplicity of D. (3) If $(\xi^2) = 2s$, then Lemma 2.5 implies the claim. Q.E.D.

Remark 3.52. In [27], the condition $v = \ell u_1 + u_2$ with $\langle u_1, u_2\rangle = 1$, $\langle u_1^2\rangle = \langle u_2^2\rangle = 0$ is called *numerical equation* and plays important role for the study of stable sheaves.

For $m \in \mathbb{Q}$, we set

$$Q_{\ell,m} := \{\xi \in \mathrm{NS}(X)_{\mathbb{R}} \mid \xi \in \sqrt{m}\,\mathrm{NS}(X)_{\mathbb{Q}}, (\xi^2) = 2\ell\}.$$

$Q_{\ell,m} = Q_{\ell,m'}$ if and only if $\sqrt{m} \in \mathbb{Q}\sqrt{m'}$. For $\xi \in Q_{\ell,m}$, we take an ample divisor H with $\mathbb{Q}\frac{\xi}{\sqrt{m}} \cap \mathrm{NS}(X) = \mathbb{Z}H$. If $p^2\ell(H^2)/2 - q^2 = 1$ has an integral solution (p, q), then there is an autoequivalence Φ in Proposition 3.51. In particular, if $\sqrt{m} \notin \mathbb{Q}$, then there are infinitely many Φ in Proposition 3.51.

Lemma 3.53. (1) *Each $Q_{\ell,m}$ is a dense or empty subset of Q_ℓ.*

(2) *$\cup_{\sqrt{m}\notin\mathbb{Q}} Q_{\ell,m}$ is dense in Q_ℓ. In particular, there are many autoequivalences Φ in Proposition 3.51.*

Proof. (1) We set $Q_\ell^+ := Q_\ell \cap \mathrm{Amp}(X)_{\mathbb{R}}$ and $Q_{\ell,m}^+ = Q_{\ell,m} \cap Q_\ell^+$. Then $Q_\ell = Q_\ell^+ \cup -Q_\ell^+$ and $Q_{\ell,m} = Q_{\ell,m}^+ \cup -Q_{\ell,m}^+$. Assume that $Q_{\ell,m} \neq \emptyset$. We take $\xi_0 \in Q_{\ell,m}^+$ and set

$$B_{\xi_0} := \{\eta \in \xi_0^\perp \mid -(\eta^2) < 2\ell\}.$$

Then we have a bijective correspondence

$$\begin{array}{ccc} Q_\ell^+ & \to & B_{\xi_0} \\ \xi & \mapsto & \eta, \end{array}$$

where

$$\begin{aligned} \eta =& \frac{2\ell\xi - (\xi, \xi_0)\xi_0}{2\ell + (\xi_0, \xi)}, \\ \xi =& \frac{4\ell}{(\eta^2) + 2\ell}\eta + \frac{2\ell - (\eta^2)}{2\ell + (\eta^2)}\xi_0. \end{aligned}$$

Then $Q_{\ell,m}^+$ corresponds to $\eta \in \sqrt{m}\,\mathrm{NS}(X)_{\mathbb{Q}}$. Therefore $Q_{\ell,m}$ is dense in Q_ℓ.

(2) is a consequence of Lemma 3.45 and (1). Q.E.D.

§4. The case where $\mathrm{rk}\,\mathrm{NS}(X) = 1$.

4.1. The walls and chambers on the (s, t)-plane.

From now on, let X be an abelian surface such that $\mathrm{NS}(X) = \mathbb{Z}H$, where H is an ample generator. We set $(H^2) = 2n$. Let $\mathbb{H} := \{(s, t) \mid$

$t > 0\}$ be the upper half plane and $\overline{\mathbb{H}} := \{(s,t) \mid t \geq 0\}$ its closure. We identify $\mathrm{NS}(X)_{\mathbb{R}} \times C(\overline{\mathrm{Amp}(X)_{\mathbb{R}}}) \times \mathbb{R}_{\geq 0}$ with $\overline{\mathbb{H}}$ via $(sH, H, t) \mapsto (s,t)$. We shall study the set of walls.

Lemma 4.1 ([12], [28, Cor. 5.10])**.** *If $\sqrt{\ell/n} \in \mathbb{Q}$, there are finitely many chamber.*

By this lemma, it is sufficient to treat the case where $\sqrt{\ell/n} \notin \mathbb{Q}$.

Proposition 4.2. *Assume that $n \leq 4$. Let $v := (r, dH, a)$ be a primitive Mukai vector with $\langle v^2 \rangle > 0$. Then there is an isometry Φ of $H^*(X,\mathbb{Z})_{\mathrm{alg}} = \mathbb{Z}^{\oplus 3}$ such that $\Phi(v) = (r', d'H, a')$ with $r'a' \leq 0$.*

Proof. We may assume that $r \neq 0$. Replacing v by $-v$, we may assume that $r > 0$. By the action of e^{mH}, we may assume that $|d| \leq r/2$. Since $d^2n - ra > 0$, we have $n/4 \geq (d/r)^2 n > a/r$. By our assumption, we have $a < r$. If $a > 0$, then we apply the isometry $\varphi : (r, dH, a) \mapsto (a, -dH, r)$. Applying the same arguments to $\varphi(v)$, we finally get an isometry Φ such that $\Phi(v) = (r', d'H, a')$ with $r'a' \leq 0$. Q.E.D.

Corollary 4.3. *Assume that $n \leq 4$. Let $v := (r, dH, a)$ be a primitive Mukai vector with $\langle v^2 \rangle > 0$. If $\sqrt{\ell/n} \notin \mathbb{Q}$, then there are infinitely many isotropic Mukai vectors u such that $\langle u, v \rangle > 0$ and $\langle (v-u)^2 \rangle \geq 0$. In particular, there are infinitely many walls for v.*

Proof. We first show that there is a Mukai vector u satisfying the requirements. We may assume that $ra \leq 0$. If $ra = 0$, then $\langle v^2 \rangle = d^2(H^2)$. Hence $ra < 0$. Then $u = (0,0,1)$ or $(0,0,-1)$ satisfies the requirements.

Since $\sqrt{\ell/n} \notin \mathbb{Q}$, $\mathrm{Stab}_0(v)^*$ contains an element g of infinite order. Then $g^n(u)$ $(n \in \mathbb{Z})$ also satisfies the requirements. Q.E.D.

Remark 4.4. The condition $(H^2) \leq 8$ in Corollary 4.3 is necessary by Lemma 3.48.

4.2. An example

Assume that $n := (H^2)/2 = 1$. We set $v := (2, H, -2)$. Then $\ell := \langle v^2 \rangle/2 = 5$. We set

$$g := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \ h := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In this case, we have

$$\begin{aligned} \mathrm{Stab}_0(v) =& \{\pm g^n \mid n \in \mathbb{Z}\}, \\ \mathrm{Stab}(v) =& \mathrm{Stab}_0(v) \rtimes \langle h \rangle, \\ h^{-1}gh =& -g^{-1}. \end{aligned}$$

Let C_0 is a line defined by

$$s = \frac{1}{2}.$$

It is the wall defined by $v_0 := (0, 0, -1)$. Let C_{-1} is a circle defined by

$$t^2 + s(s+2) = 0.$$

It is the wall defined by $v_{-1} := (1, 0, 0) = v_0 \cdot g^{-1}$. We set $v_n := v_0 \cdot g^n$ and let C_n be the wall defined by v_n.

Lemma 4.5. *$\{C_n \mid n \in \mathbb{Z}\}$ is the set of walls for v.*

Proof. It is sufficient to prove that there is no wall between C_{-1} and C_0. If a Mukai vector $w := (r', d'H, a')$ defines a wall for v and the wall W_w lies between C_{-1} and C_0. Since C_{-1} passes $(0, 0)$, W_w intersects with the line $s = 0$. Hence we get $d', 1 - d' > 0$, which is a contradiction. Q.E.D.

Proposition 4.6. (1) *For $(s, t) \notin \cup_{n \in \mathbb{Z}} C_n$,*

$$M_{(sH,tH)}(2, H, -2) \cong M_H(2, H, -2).$$

(2) *Let (r, dH, a) be a primitive Mukai vector such that $2 \mid r, 2 \mid a$ and $d^2 - ra = 5$. Then*

$$M_H(r, dH, a) \cong M_H(2, H, -2).$$

Proof. (1) Let $\mathcal{C}$ be the chamber between C_0 and C_{-1}. Then $\cup_{n \in \mathbb{Z}} g^n(\mathcal{C}) = \mathbb{H} \setminus \cup_{n \in \mathbb{Z}} C_n$. Let $\Psi : \mathbf{D}(X) \to \mathbf{D}(X)$ be a contravariant Fourier-Mukai transform inducing g. Since Ψ preserves the stability, the claim holds.

(2) By the action of $\mathrm{GL}(2, \mathbb{Z})$, $w := (r, dH, a)$ is transformed to $(1, 0, -5)$ or $(2, H, -2)$ ([27, Prop. 7.12]). Since $2 \mid \langle w, u \rangle$ for all $u \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$, $w := (r, dH, a)$ is transformed to $(2, H, -2)$. Then the claim follows from (1). Q.E.D.

4.3. Divisors on the moduli spaces $M_{(sH,tH)}(v)$.

Definition 4.7. Let $v = (r, dH, a)$ be a Mukai vector with $r > 0$ and set $\ell := \langle v^2 \rangle / 2 = d^2 n - ra$. We set

$$s_\pm := \frac{d}{r} \pm \frac{1}{r}\sqrt{\frac{\ell}{n}}.$$

Lemma 4.8. *We take $g \in \mathrm{Stab}_0(v)^*$ such that the order is infinite. Then $(s_\pm, 0)$ are the fixed points of the action of g on (s,t)-plane. In particular, if there is a wall, then $(s_\pm, 0)$ are the accumulation points of the set of walls.*

We set $(\beta, \omega) := (sH, tH)$. Then $v = (r, dH, a)$ is written as

$$v = re^\beta + d_\beta(H + (\beta, H)\varrho_X) + a_\beta \varrho_X,$$

where

$$\begin{aligned} d_\beta =& d - rs, \\ a_\beta =& -\langle e^\beta, v\rangle = a - (dH, \beta) + \frac{(\beta^2)}{2} r. \end{aligned}$$

Definition 4.9. We set

$$\begin{aligned} \xi(s,t) :=& \xi(sH, H, t) \\ =& \big(r(s^2+t^2)n - a\big)\left(H + \frac{2dn}{r}\varrho_X\right) - 2n(d - rs)\left(1 - \frac{a}{r}\varrho_X\right). \end{aligned} \tag{4.1}$$

We consider the circle

$$C_{v,\lambda} : t^2 + (s-\lambda)\left(s - \frac{1}{r\lambda - d}\left(\lambda d - \frac{a}{n}\right)\right) = 0, \ \lambda \in \mathbb{R},$$

that is, $\mathbb{R}Z_{(sH,tH)}(v) = \mathbb{R}Z_{(sH,tH)}(e^{\lambda H})$, where $\lambda \neq d/r$. We note that $(\lambda, 0) \in C_{v,\lambda}$ and $(d - rs)(d - r\lambda) > 0$ for $(s,t) \in C_{v,\lambda}$. For $(s,t) \in C_{v,\lambda}$, we see that

$$\xi(s,t) = (d - rs)\left(\frac{r\lambda^2 n - a}{d - r\lambda}\left(H + \frac{2dn}{r}\varrho_X\right) - 2n\left(1 - \frac{a}{r}\varrho_X\right)\right).$$

Hence $\mathbb{R}_{>0}\xi(s,t) = \mathbb{R}_{>0}\xi(\lambda, 0)$ and is determined by λ. If

$$C_{v,\lambda} = \{(s,t) \mid \mathbb{R}Z_{(sH,tH)}(v) = \mathbb{R}Z_{(sH,tH)}(v_1)\},$$

that is, $C_{v,\lambda}$ is the wall defined by a Mukai vector $v_1 := (r_1, d_1 H, a_1)$, then $\frac{r\lambda^2 n - a}{d - r\lambda} = \frac{ra_1 - r_1 a}{r_1 d - r d_1} \in \mathbb{Q}$. Thus $\xi(s,t)/(d - rs) \in H^*(X, \mathbb{Q})_{\mathrm{alg}}$.

Lemma 4.10.

$$\xi(s_\pm, 0) = 2\left(\frac{\ell}{r} \pm n\frac{d}{r}\sqrt{\frac{\ell}{n}}\right)\left(H + \frac{(dH,H)}{r}\varrho_X\right) \pm 2n\sqrt{\frac{\ell}{n}}\left(1 - \frac{a}{r}\varrho_X\right)$$

and satisfy $\langle \xi(s_\pm, 0)^2\rangle = 0$. Thus $\xi(s_\pm, 0)$ define isotropic vectors in $v^\perp \subset H^*(X, \mathbb{Z})_{\mathrm{alg}} \otimes \mathbb{R}$.

Proof.

$$
\begin{aligned}
\langle \xi(s_{\pm},0)^2\rangle =& 4\left(\frac{\ell}{r} \pm n\frac{d}{r}\sqrt{\frac{\ell}{n}}\right)^2 (2n) + 4\ell n\frac{2a}{r} \mp 8\left(\frac{\ell}{r} \pm n\frac{d}{r}\sqrt{\frac{\ell}{n}}\right)\sqrt{n\ell}\frac{d}{r}(2n) \\
=& 4\left(\frac{\ell}{r} \pm n\frac{d}{r}\sqrt{\frac{\ell}{n}}\right)\left(\frac{\ell}{r} \mp n\frac{d}{r}\sqrt{\frac{\ell}{n}}\right)(2n) + 4\ell n\frac{2a}{r} \\
=& 4\frac{\ell^2}{r^2}(2n) - 4n\frac{d^2}{r^2}\ell(2n) + 4\ell n\frac{2a}{r} \\
=& 4\frac{\ell}{r^2}(\ell - d^2 n)(2n) + 4\frac{\ell}{r^2}(ra)(2n) = 0.
\end{aligned}
$$

Therefore we get the claim. Q.E.D.

Proposition 3.15 is written as follows.

Proposition 4.11. (1) *If (s,t) belongs to a chamber and $s, t^2 \in \mathbb{Q}$, then $\theta_{v,sH,tH}(\xi(s,t))$ is an ample $\mathbb{Q}$-divisor of $K_{(sH,tH)}(v)$.*

(2) *We have a bijective map*

$$\varphi : [s_-, s_+] \to C(\overline{P^+(K_{(sH,tH)}(v))_{\mathbb{R}}})$$

such that

$$\varphi(\lambda) := \mathbb{R}_{>0}\theta_{v,sH,tH}(\xi(\lambda,0)).$$

(3) $\mathrm{Nef}(K_{(sH,tH)}(v))_{\mathbb{R}} = \varphi(\overline{D(sH,H,t)} \cap [s_-, s_+])$.

Proof. (1) is obvious. (2) We note that $f(\lambda) := (r\lambda^2 n - a)/(d - r\lambda)$ gives a bijective map

$$f : [s_-, s_+] \to [\tfrac{2\sqrt{n\ell}}{r} - \tfrac{2dn}{r}, \infty] \cup [-\infty, -\tfrac{2\sqrt{n\ell}}{r} - \tfrac{2dn}{r}]$$

and

$$\bigcup_{\lambda \in [s_-, s_+]} C_{v,\lambda} = \mathbb{R}^2 \setminus \{(s_-,0), (s_+,0)\},$$

where we identify ∞ with $-\infty$, $\lambda = d/r$ corresponds to $\pm\infty$ and $C_{v,d/r}$ is the line $s = d/r$. Since $C_{v,\lambda} \cap [s_-, s_+] = \{\lambda\}$, $[s_-, s_+]$ is the parameter space of $C_{v,\lambda}$. Since $\xi(s,t)$ is determined by $f(\lambda)$, Proposition 3.15 implies φ is bijective.

(3) is a consequence of Proposition 3.15 (1) and (2). Q.E.D.

4.4. The movable cone of $K_{(sH,tH)}(v)$.

Lemma 4.12. *Let v be a Mukai vector with $\langle v^2\rangle = 2\ell$.*

(1) $\mathfrak{I}_0 \neq \emptyset$ *if and only if* $\sqrt{\ell/n} \in \mathbb{Q}$.

(2) *Assume that* $\sqrt{\ell/n} \notin \mathbb{Q}$. *Then* $\mathfrak{I}_k \neq \emptyset$ *if and only if* $\#\mathfrak{I}_k = \infty$. *In this case,* $(s_+, 0)$ *and* $(s_-, 0)$ *are the accumulation points of* $\cup_{w \in \mathfrak{I}_k} W_w$.

Proof. Since $x \in H^*(X,\mathbb{Z})_{\mathrm{alg}} \otimes \mathbb{R}$ satisfies $\langle x^2\rangle = \langle x, v\rangle = 0$ if and only if $x \in \mathbb{R}\xi(s_\pm, 0)$. Hence (1) holds. By Proposition 2.3, $\mathrm{Stab}_0(v)$ contains an element g of infinite order. Hence (2) is obvious. Q.E.D.

Proposition 4.13. *Let v be a primitive Mukai vector with $\langle v^2\rangle \geq 6$. Let $W \subset \overline{\mathbb{H}}$ be a wall for v and take $(s,t) \in W$ such that $s \in \mathbb{Q}$. Then W is a codimension 1 wall if and only if W is defined by v_1 such that $v = v_1 + v_2$, $\langle v_1^2\rangle = 0$, $\langle v, v_1\rangle = 2$, v_1 is primitive and there are $\sigma_{(\beta,\omega)}$-stable objects E_i with $v(E_i) = v_i$ for $i = 1, 2$.*

Proof. Since $\mathrm{NS}(X) = \mathbb{Z}H$, there is no decomposition $v = u_1+u_2+u_3$ such that $\langle u_i^2\rangle = 0$ $(i = 1,2,3)$ and $\langle u_i, u_j\rangle = 1$ $(i \neq j)$. Then the classification of codimension 1 walls in [18, Lem. 4.3.4 (2), Prop. 4.3.5] imply that W is defined by v_1 with the required properties. Q.E.D.

Theorem 4.14. *Let (X,H) be a polarized abelian surface X with $\mathrm{NS}(X) = \mathbb{Z}H$. Let v be a primitive Mukai vector with $\langle v^2\rangle \geq 6$. Assume that $\sqrt{\ell/n} \notin \mathbb{Q}$.*

(1) *Assume that* $\mathfrak{I}_1 = \mathfrak{I}_2 = \emptyset$, *that is,*

$$\min\{\langle v, w\rangle > 0 \mid \langle w^2\rangle = 0\} \geq 3.$$

Then the movable cone of $K_{(sH,tH)}(v)$ is the same as the positive cone of $K_{(sH,tH)}(v)$. In this case, there is an action of birational automorphisms such that a fundamental domain is a cone spanned by rational vectors.

(2) *Assume that* $\mathfrak{I}_1 = \emptyset$ *and* $\mathfrak{I}_2 \neq \emptyset$, *that is,*

$$\min\{\langle v, w\rangle > 0 \mid \langle w^2\rangle = 0\} = 2.$$

Then the movable cone of $K_{(sH,tH)}(v)$ is spanned by two vectors, which give divisorial contractions. Moreover the exceptional divisors are primitive in $\mathrm{NS}(K_{(sH,tH)}(v))$.

(3) *Assume that* $\mathfrak{I}_1 \neq \emptyset$, *that is,*

$$\min\{\langle v, w\rangle > 0 \mid \langle w^2\rangle = 0\} = 1.$$

Then the movable cone of $K_{(sH,tH)}(v)$ is spanned by two vectors, which give divisorial contractions. Moreover one of the exceptional divisors is divisible by 2 in $\mathrm{NS}(K_{(sH,tH)}(v))$.

Proof. (1) Let $\Phi : \mathbf{D}(X) \to \mathbf{D}(X)$ be a Fourier-Mukai transform preserving $\pm v$. We set $(s'H, t'H) := \Phi((sH, tH))$. Then we have an isomorphism $\Phi : M_{(sH,tH)}(v) \to M_{(s'H,t'H)}(v)$, which induces a birational map

$$M_{(sH,tH)}(v) \overset{\Phi}{\to} M_{(s'H,t'H)}(v) \cdots \to M_{(sH,tH)}(v).$$

Since $\theta_v : v^\perp \to \mathrm{NS}(K_{(sH,tH)}(v))$ is compatible with respect to the action of Φ, we have an action of $\mathrm{Stab}_0(v)^*$ on $\mathrm{NS}(K_{(sH,tH)}(v))$. Since $\sqrt{\ell/n} \notin \mathbb{Q}$, Lemma 2.7 implies that $\mathrm{Stab}_0(v)^*$ contains an element g of infinite order. Hence the claim holds.

(2) and (3) are consequence of Corollary 3.34. Q.E.D.

Remark 4.15. By Lemma 3.48, there are v satisfying (1). For $v = (2, H, -2k)$, we have $\langle v^2 \rangle = 2(n + 4k)$ and case (2) holds, if $\sqrt{\ell/n} \notin \mathbb{Q}$. If $\mathrm{rk}\, v = 1$, then case (3) holds.

Proposition 4.16. *Let v be a primitive Mukai vector with $\langle v^2 \rangle \geq 6$. Assume that $\sqrt{\ell/n} \in \mathbb{Q}$.*

(1) *There is at most one isotropic Mukai vector v_1 with $\langle v, v_1 \rangle = 1, 2$.*

(2) *If there is a vector v_1 of (1), then*

$$\overline{P^+(v^\perp)_\mathbb{R}} = \mathrm{Mov}(K_{(sH,tH)}(v))_\mathbb{R} \cup R_{v_1}(\mathrm{Mov}(K_{(sH,tH)}(v)_\mathbb{R})$$

and the two chambers are separated by $d_{v_1}^\perp$.

(3) *If there is no v_1 of (1), then $\overline{P^+(v^\perp)_\mathbb{R}} = \mathrm{Mov}(K_{(sH,tH)}(v))_\mathbb{R}$.*

Proof. (1) Since $\sqrt{\ell/n} \in \mathbb{Q}$, there are two isotropic Mukai vectors w_1, w_2 such that

$$\{x \in H^2(X, \mathbb{Z})_{\mathrm{alg}} \mid \langle x, v \rangle = \langle x^2 \rangle = 0\} = \mathbb{Z}w_1 \cup \mathbb{Z}w_2.$$

Then $v^\perp \otimes \mathbb{Q} = \mathbb{Q}w_1 + \mathbb{Q}w_2$. We may assume that $\langle w_1, w_2 \rangle < 0$. Let v_1 be an isotropic Mukai vectors such that $\langle v, v_1 \rangle = 1, 2$. Since $\langle v, d_{v_1} \rangle = 0$, we set $d_{v_1} := aw_1 + bw_2$ $(a, b \in \mathbb{Q})$. Then we have $2ab\langle w_1, w_2 \rangle = -\langle v^2 \rangle < 0$. By Lemma 3.24 (2), R_{v_1} preserves $\{\pm w_1, \pm w_2\}$. Since

$$R_{v_1}(w_1) = w_1 - \frac{1}{a}(aw_1 + bw_2) = -\frac{b}{a}w_2,$$

$R_{v_1}(w_1) = \pm w_2$. If $R_{v_1}(w_1) = w_2$, then we have $R_{v_1}(w_1 + w_2) = w_1 + w_2$. Hence $\langle d_{v_1}, w_1 + w_2 \rangle = 0$. Since $\langle (w_1 + w_2)^2 \rangle = 2\langle w_1, w_2 \rangle < 0$ and

$\langle d_{v_1}^2 \rangle < 0$, $v^\perp$ is negative definite, which is a contradiction. Therefore $R_{v_1}(w_1) = -w_2$. Then we see that $w_1 + w_2 \in \mathbb{Z} d_{v_1}$. If $\langle v, v_2 \rangle = 1, 2$, then the primitivities of d_{v_1} and d_{v_2} imply that $d_{v_1} = \pm d_{v_2}$. If $d_{v_1} = -d_{v_2}$, then we see that

$$v = \frac{\langle v^2 \rangle}{4} \left(\frac{2}{\langle v, v_1 \rangle} v_1 + \frac{2}{\langle v, v_2 \rangle} v_2 \right).$$

Since $\langle v^2 \rangle \geq 6$ and v is primitive, this case does not occur. Therefore $d_{v_1} = d_{v_2}$, which implies that $v_1 = v_2$.

(2) and (3) are obvious. Q.E.D.

Remark 4.17. Theorem 4.14 and Proposition 4.16 are compatible with Oguiso's general results [26, Thm. 1.3].

§5. Relations with Markman's results.

In this section, we shall explain the relation between our results and Markman's general results [13], [14], [15].

Definition 5.1. Let M be an irreducible symplectic manifold and h an ample class.

(1) An effective divisor E is prime exceptional, if E is reduced and irreducible of $q_M(E^2) < 0$.

(2) Let $e \in \mathrm{NS}(M)$ be a primitive class. e is stably prime exceptional, if $q_M(e, h) > 0$ and there is a projective irreducible symplectic manifold M', a parallel-transport operator

$$g : H^2(M, \mathbb{Z}) \to H^2(M', \mathbb{Z}),$$

and an integer k, such that $kg(e)$ is the class of a prime exceptional divisor $E \subset M'$.

Let Spe_M be the set of stably prime exceptional divisors of M. Then Markman described the interior of the movable cone in terms of Spe_M.

Theorem 5.2 ([14, Prop. 1.8, Lem. 6.22]). *Let M be an irreducible symplectic manifold and $\mathrm{Mov}(M)^0$ the interior of $\mathrm{Mov}(v)_{\mathbb{R}}$. Then*

$$\mathrm{Mov}(M)^0 = \{x \in P^+(M) \mid q_M(x, e) > 0 \text{ for all } e \in \mathrm{Spe}_M\}.$$

Let M be an irreducible symplectic manifold which is deformation equivalent to the generalized Kummer variety $K_H(1, 0, -\ell)$ of dimension $2\ell - 2$. We shall explain the description of Spe_M. For $e \in H^2(M, \mathbb{Z})$ with $q_M(e^2) = -2\ell$, we set

$$\operatorname{div} q_M(e, \bullet) := \min\{q_M(e, x) > 0 \mid x \in H^2(M, \mathbb{Z})\}.$$

As an abstract lattice, the cohomological Mukai lattice $H^{2*}(X,\mathbb{Z}) = \bigoplus_{i=0}^{2} H^{2i}(X,\mathbb{Z})$ is independent of the choice of an abelian surface X. We denote this lattice by $\widetilde{\Lambda}$. It is a direct sum of 4 copies of the hyperbolic lattice. Since M is deformation equivalent to $K_H(1,0,-\ell)$, by using a parallel-transport, we have a primitive embedding $H^2(M,\mathbb{Z}) \to \widetilde{\Lambda}$. The $\mathrm{O}(\widetilde{\Lambda})$-orbit of the embedding is independent of the choice of a parallel-transport by similar claims to [14, Thm. 9.3, Thm. 9.8]. For $M = K_H(v)$, it is the embedding

$$H^2(K_H(v),\mathbb{Z}) \overset{\theta_v^{-1}}{\to} v^\perp \subset H^{2*}(X,\mathbb{Z})$$

([14, Example 9.6]). We fix an embedding and regard $H^2(M,\mathbb{Z})$ as a sublattice of $\widetilde{\Lambda}$. Let $\mathbb{Z}v$ be the orthogonal compliment of $H^2(M,\mathbb{Z})$ in $\widetilde{\Lambda}$. Then $\langle v^2\rangle = 2\ell$. Since $\langle v,e\rangle = 0$, $e \pm v$ are isotropic. We define $(\rho,\sigma) \in \mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ by requiring that $(e+v)/\rho$ and $(e-v)/\sigma$ are primitive and isotropic. We also set $r := \rho/\gcd(\rho,\sigma)$ and $s := \sigma/\gcd(\rho,\sigma)$. If $\ell \mid \operatorname{div} q_M(e,\bullet)$, then r and s are relatively prime integers with $rs = \ell, \ell/2, \ell/4$. We set $rs(e) := \{r,s\}$.

Proposition 5.3 ([15])**.** *Let M be an irreducible symplectic manifold of $\dim M = 2\ell-2$ which is deformation equivalent to $K_H(1,0,-\ell)$.*

(1) *For $e \in \mathrm{Spe}_M$, $q_M(e^2) = -2\ell$ and $\ell \mid \operatorname{div} q_M(e,\bullet)$.*
(2) *For $e \in H^2(M,\mathbb{Z})$ with $q_M(e^2) = -2\ell$ and $\ell \mid \operatorname{div} q_M(e,\bullet)$, the orbit of e of the monodromy group action is classified by $rs(e) := \{r,s\}$ and $\operatorname{div} q_M(e,\bullet)$.*

For each value of $\{r,s\}$ with $rs \in \{\ell,\ell/2,\ell/4\}$, the same examples in [13, sect. 10, 11] show that there are $K_H(v)$ with $\dim K_H(v) = 2\ell-2$ and $e \in \mathrm{NS}(K_H(v))$ such that $\operatorname{div} q_M(e,\bullet) = \ell, 2\ell$ and $rs(e) = \{r,s\}$. Then we also get the following description of Spe_M.

Proposition 5.4. *Let M be an irreducible symplectic manifold of $\dim M = 2\ell-2$ which is deformation equivalent to $K_H(1,0,-\ell)$ and h an ample divisor on M. A divisor e with $q_M(e,e) = -2\ell$ and $q_M(e,h) > 0$ is stably prime exceptional if and only if*

(1) *$\operatorname{div} q_M(e,\bullet) = 2\ell$ and $\{r,s\} = \{1,\ell\}$ or*
(2) *$\operatorname{div} q_M(e,\bullet)$ and $\{r,s\}$ are one of the following.*
    (a) *$\operatorname{div} q_M(e,\bullet) = 2\ell$ and $\{r,s\} = \{2,\ell/2\}$, $\ell \geq 6$, $\ell \equiv 2 \mod 4$.*
    (b) *$\operatorname{div} q_M(e,\bullet) = \ell$ and $\{r,s\} = \{1,\ell\}$, $\ell \geq 3$, $2 \nmid \ell$.*
    (c) *$\operatorname{div} q_M(e,\bullet) = \ell$ and $\{r,s\} = \{1,\ell/2\}$, $\ell \geq 2$, $2 \mid \ell$.*

If $M = K_{(\beta,\omega)}(v)$ for some v, then we shall explain that the case (1) corresponds to the codimension 0 wall $u^\perp$ ($u \in \mathfrak{I}_1$) and the case (2) corresponds to the codimension 1 wall $u^\perp$ ($u \in \mathfrak{I}_2$).

(1) We first assume that $u \in \mathfrak{I}_1$. Since $d_u = v - 2\ell u$, $v + d_u = 2(v - \ell u)$ and $v - d_u = 2\ell u$. Since $\langle d_u, \ \rangle = -2\ell\langle u, \ \rangle$ on $v^\perp$, $\operatorname{div} q_{K_H(v)}(d_u, \bullet) = 2\ell$. Since $\langle v + d_u, u\rangle = 2$, $v + d_u$ is primitive, which implies that $\{\rho, \sigma\} = \{2, 2\ell\}$. Therefore $\{r, s\} = \{1, \ell\}$.

(2) We next assume that $u \in \mathfrak{I}_2$. In this case, we have $d_u = v - \ell u$. Thus $v + d_u = 2v - \ell u$ and $v - d_u = \ell u$. We shall compute $\operatorname{div} q_{K_H(v)}(d_u, \bullet)$. We first note that $\langle d_u, \ \rangle = -\ell\langle u, \ \rangle$ on $v^\perp$. Since $H^*(X, \mathbb{Z})$ is a 4 copies of hyperbolic lattice, there is an isotropic Mukai vector $\lambda \in H^{2*}(X, \mathbb{Z})$ with $\langle u, \lambda\rangle = 1$. Then

$$H^{2*}(X, \mathbb{Z}) = (\mathbb{Z}u + \mathbb{Z}\lambda) \oplus (\mathbb{Z}u + \mathbb{Z}\lambda)^\perp.$$

We set

$$v := 2\lambda + au + \xi, \ a \in \mathbb{Z}, \xi \in (\mathbb{Z}u + \mathbb{Z}\lambda)^\perp. \tag{5.1}$$

Then $x\lambda + yu + z\eta$ ($x, y, z \in \mathbb{Z}, \eta \in (\mathbb{Z}u + \mathbb{Z}\lambda)^\perp$) belongs to $v^\perp$ if and only if $xa + 2y + z\langle \xi, \eta\rangle = 0$. If $2 \nmid \xi$, then the unimodularity of $(\mathbb{Z}u + \mathbb{Z}\lambda)^\perp$ implies that we can take η with $2 \nmid \langle \xi, \eta\rangle$. We take $z \in \mathbb{Z}$ such that $y = -(a + z\langle \xi, \eta\rangle)/2 \in \mathbb{Z}$. Then $\lambda + yu + z\eta \in v^\perp$ and $\langle d_u, \lambda + yu + z\eta\rangle = -\ell\langle u, \lambda + yu + z\eta\rangle = -\ell$. Therefore $\operatorname{div} q_{K_H(v)}(d_u, \bullet) = \ell$. If $2 \mid \xi$, then the primitivity of v implies that $2 \nmid a$. Hence $x\lambda + yu + z\eta \in v^\perp$ satisfies $2 \mid x$. Then we have

$$\langle d_u, x\lambda + yu + z\eta\rangle = -\ell\langle u, x\lambda + yu + z\eta\rangle = -\ell x\langle u, \lambda\rangle \in 2\ell\mathbb{Z}.$$

Hence $\operatorname{div} q_{K_H(v)}(d_u, \bullet) = 2\ell$. Therefore $\operatorname{div} q_{K_H(v)}(d_u, \bullet) = 2\ell$ if and only if there is $a \in \mathbb{Z}$ such that $2 \mid (v - au)$.

We next compute $\{r, s\}$. We take w such that $\mathbb{Z}u + \mathbb{Z}w$ is a saturated sublattice of $H^*(X, \mathbb{Z})_{\text{alg}}$ containing v. For the notation of (5.1), $w = 2\lambda + \xi$ or $2w = 2\lambda + \xi$. Thus they are distinguished by $\operatorname{div} q_{K_H(v)}(d_u, \bullet)$. We set $v = au + bw$ ($b = 1, 2$). Then $v + d_u = (2a - \ell)u + 2bw$. We note that $\ell = 2a + b^2\langle w^2\rangle/2$ by $b\langle u, w\rangle = 2$. If $2 \nmid \ell$, then $v + d_u$ is primitive, which implies that $\{\rho, \sigma\} = \{1, \ell\}$. In this case, $b = 1$ and $\operatorname{div} q_{K_H(v)}(d_u, \bullet) = \ell$. Assume that $2 \mid \ell$. Then $(v + d_u)/2 = (a - \ell/2)u + bw \in H^*(X, \mathbb{Z})_{\text{alg}}$. If $b = 1$, then $(v + d_u)/2$ is primitive, which implies $\{\rho, \sigma\} = \{2, \ell\}$. If $b = 2$, then the primitivity of v implies that $2 \nmid a$. Then $\ell \equiv 2 \mod 4$ and $(v + d_u)/4 \in H^*(X, \mathbb{Z})_{\text{alg}}$ is primitive, which implies $\{\rho, \sigma\} = \{4, \ell\}$. Therefore $\{r, s\}$ satisfies (a), (b) or (c).

§6. Appendix

6.1. The base of Lagrangian fibrations

Let $\Phi_{X\to Y}^{\mathbf{E}^{\vee}[k]}:\mathbf{D}(X)\to\mathbf{D}^{\alpha}(Y)$ be the Fourier-Mukai transform in the proof of Proposition 3.38. Then we have an isomorphism

$$\phi: M_{(\beta,tH)}(v)\to M_{\widetilde{tH}}^{\alpha}(v')$$

and a morphism

$$f: M_{\widetilde{tH}}^{\alpha}(v')\to \mathrm{Hilb}_Y^{\eta}.$$

We set $v':=(0,\eta,b), b\in\mathbb{Z}$ and $H':=\widetilde{tH}$. Since $\mathfrak{a}: M_{(\beta,tH)}(v)\to X\times\widehat{X}$ is the albanese map,

$$\mathfrak{a}': M_{H'}^{\alpha}(0,\eta,b)\to M_{(\beta,tH)}(v)\xrightarrow{\mathfrak{a}} X\times\widehat{X}$$

is the albanese map. Then

$$M_{H'}^{\alpha}(0,\eta,b)\to \mathrm{Hilb}_Y^{\eta}\to \mathrm{Pic}^0(Y)$$

induces a morphism $g: X\times\widehat{X}\to\mathrm{Pic}^0(Y)$ with a commutative diagram

$$\begin{array}{ccc} M_{H'}^{\alpha}(0,\eta,b) & \xrightarrow{f} & \mathrm{Hilb}_Y^{\eta} \\ {\scriptstyle\mathfrak{a}'}\downarrow & & \downarrow \\ X\times\widehat{X} & \xrightarrow{g} & \mathrm{Pic}^0(Y). \end{array}$$

Let $K_{H'}^{\alpha}(0,\eta,b)$ be a fiber of $M_{H'}^{\alpha}(0,\eta,b)\to X\times\widehat{X}$. Since $\mathrm{Hilb}_Y^{\eta}\to\mathrm{Pic}^0(Y)$ is a $\mathbb{P}^{(\eta^2)/2-1}$-bundle, we have a morphism

$$K_{H'}^{\alpha}(0,\eta,b)\to\mathbb{P}^{(\eta^2)/2-1}.$$

We shall prove the following.

Proposition 6.1. *The fiber of $K_{H'}^{\alpha}(0,\eta,b)\to\mathbb{P}^{(\eta^2)/2-1}$ is connected.*

For $D\in\mathrm{Hilb}_Y^{\eta}$, $f^{-1}(D)$ consists of α-twisted stable sheaves E such that E is an $\mathcal{O}_D$-module. We take an effective divisor $D\in\mathrm{Hilb}_Y^{\eta}$. Since a fiber of $K_{H'}^{\alpha}(0,\eta,b)\to\mathbb{P}^{(\eta^2)/2-1}$ is a fiber of $f^{-1}(D)\to X\times\widehat{X}$, we shall study the map $f^{-1}(D)\to X\times\widehat{X}$. For the connectivity of fibers, we may assume that D is a general member of Hilb_Y^{η}. Indeed since $\mathbb{P}^{(\eta^2)/2-1}$ is a normal variety over a field of characteristic 0, the connectivity of the generic fiber implies the connectivity for all fibers.

The following well-known result is due to Reider.

Proposition 6.2. *Let X be an abelian surface defined over an algebraically closed field k and D a divisor on X. If $(X, D) \not\cong (C_1 \times C_2, C_1 + kC_2)$ and $(D^2) > 4$, then $|D|$ is base point free. Moreover $|D|$ is fixed point free, if $(X, D) \not\cong (C_1 \times C_2, C_1 + kC_2)$ and $(D^2) = 4$.*

Lemma 6.3. *For a general point of* $\operatorname{Hilb}_Y^\eta$*, D is a normal crossing divisor such that each component D_i is smooth and the configuration is tree.*

Proof. We take a line bundle L on Y with $c_1(L) = \eta$. If $|L|$ is base point free, then Bertini's theorem implies that a general member of $|L|$ is a smooth divisor. Assume that $|L|$ has a base point. Then Proposition 6.2 implies that (i) $(\eta^2) = 4$ and L does not have a fixed component or (ii) there is an elliptic curve C on Y with $(\eta, C) = 1$. In the first case,

$$K(L) := \{x \in Y \mid T_x^*(L) \cong L\}$$

is a subgroup of order 4. Let $\operatorname{Bs}(L)$ be the set of base points. Then by the action of $K(L)$, $\operatorname{Bs}(L)$ is invariant. Therefore $\#\operatorname{Bs}(L) \geq \#K(L)$. Since L does not have a fixed component, $4 \geq \#\operatorname{Bs}(L)$. Therefore $\operatorname{Bs}(L)$ consists of 4 points. For two $D, D' \in |L|$, D and D' intersect transversally. Therefore D is smooth at base points. By using Bertini's theorem, D is smooth for a general member of $|L|$. For case (ii), there is an elliptic curve C' such that $(C, C') = 1$ and $\eta = C + nC'$, where $n = (\eta^2)/2$. Since nC' is linear equivalent to $\sum_{i=1}^n C_i$ with $C_i \cap C_j = \emptyset$ $(i \neq j)$, we get the claim. Q.E.D.

6.2. Moduli of twisted-stable sheaves on D

By Lemma 6.3, we shall study $f^{-1}(D)$ for a normal crossing divisor $D = \sum_{i=0}^m D_i$ such that D_i are smooth curves and the configuration of D_i is tree. We may assume that $D = D_0 + D_1 + \cdots + D_m$ and $p_1, p_2, ..., p_m$ are the singular points of D such that $p_i = D_{\varphi(i)} \cap D_i$ with $\varphi(i) < i$.

By looking at the dual graph of irreducible components, we have the following lemma.

Lemma 6.4. *For each singular point p_i, we have a unique decomposition $D = A^i + B^i$ with $A^i \cap B^i = \{p_i\}$.*

Lemma 6.5. *In the free abelian group generated by $D_0, D_1, .., D_m$, we have*

$$\mathbb{Z}D + \mathbb{Z}A^1 + \cdots + \mathbb{Z}A^m = \mathbb{Z}D_0 \oplus \mathbb{Z}D_1 \oplus \cdots \oplus \mathbb{Z}D_m.$$

Proof. Before proving this lemma, we note that $\mathbb{Z}D + \mathbb{Z}A^i = \mathbb{Z}D + \mathbb{Z}B^i$. Thus the left hand side is independent of the choice of A^i in the decomposition $D = A^i + B^i$.

For each D_i, let $p_{n_1}, ..., p_{n_t} \in D_i$ be the singular points of D. Then we have the decompositions $D = A^{n_j} + B^{n_j}$ with $A^{n_j} \cap B^{n_j} = \{p_{n_j}\}$. We may assume that $D_i \subset A^{n_j}$ for all j. Then $D = D_i + \sum_j B^{n_j}$. Hence $\sum_{i=0}^m \mathbb{Z}D_i \subset \mathbb{Z}D + \sum_{i=1}^m A^i$, which implies the claim. Q.E.D.

Definition 6.6. Let β be a $\mathbb{Q}$-Cartier divisor on D.

(1) For a subdivisor $D' \subset D$, $\beta(D') := \int_{D'} \beta \in \mathbb{Q}$ denotes the degree of $\beta_{|D'} \in H^2(D', \mathbb{Q})$.

(2) For a coherent sheaf E on D, we set $\chi(E(-\beta)) := \chi(E) - \beta(\mathrm{Div}(E))$.

Definition 6.7. We have a surjective homomorphism

$$\begin{array}{cccc} \deg^m : & \mathrm{Pic}(D) & \to & \bigoplus_{i=0}^m \mathbb{Z}\delta_i \\ & E & \to & \sum_{i=0}^m \deg(E_{|D_i})\delta_i. \end{array}$$

Indeed for a smooth point $p_i \in D_i$, we have a Cartier divisor and get a line bundle $\mathcal{O}_D(p_i)$ on D. Then $\deg^m(\mathcal{O}_D(p_i)) = \delta_i$.

Definition 6.8. Let H' be an ample divisor on D. A purely 1-dimensional sheaf E on D is β-twisted semi-stable, if

$$\frac{\chi(F(-\beta))}{(\mathrm{Div}(F), H')} \leq \frac{\chi(E(-\beta))}{(\mathrm{Div}(E), H')}$$

for all $0 \neq F \subset E$. If the inequality is strict for all proper subsheaf F, then E is β-twisted stable.

Since $H^2_{\text{ét}}(D, \mathcal{O}_D^\times) = 0$, by refining the covering of D, we have an α_D-twisted line bundle L on D which induces an equivalence

$$\begin{array}{ccc} \mathrm{Coh}^\alpha(D) & \to & \mathrm{Coh}(D) \\ E & \mapsto & E \otimes L^\vee. \end{array}$$

Let G be a locally free α-twisted sheaf defining twisted semi-stability of $f^{-1}(D)$. Then $G' := G_{|D} \otimes L^{-1}$ is a locally free sheaf on D. We set $\beta := c_1(G')/\operatorname{rk} G' \in H^2(D, \mathbb{Q})$. Thus we have an isomorphism $f^{-1}(D) \to M_D^\beta(v)$, where $M_D^\beta(v)$ is the moduli space of β-twisted sheaves on D with $v(E) = v$ and the polarization is $H'_{|D}$. We shall describe $M_D^\beta(v)$.

Let x be a smooth point of D. Then the stalk E_x is a free $\mathcal{O}_{D,x}$-module. Since $\mathrm{Div}(E) = D$, the classification of finitely generated $\mathcal{O}_{D,x}$-module implies that $E_x \cong \mathcal{O}_{D,x}$.

Proposition 6.9. *Assume that β is general, that is, $M_D^\beta(v)$ consists of β-twisted stable sheaves.*

(1) *$M_D^\beta(v)$ is non-empty and consists of line bundles on D.*
(2) *$M_D^\beta(v)$ is isomorphic to $\prod_i \operatorname{Pic}^0(D_i)$. In particular, $M_D^\beta(v)$ is an abelian variety.*

For the proof of Proposition 6.9, we first prove the following.

Lemma 6.10. *$M_D^\beta(v)$ consists of locally free $\mathcal{O}_D$-modules.*

Proof. For $E \in M_D^\beta(v)$, assume that $E_{|D_i}$ is torsion free. Then $E_{|D_i}$ is a locally free sheaf of rank 1 on D_i. By using Nakayama's lemma, we have a surjective homomorphim $\mathcal{O}_{X,x} \to E_x$ for all $x \in D_i$. Then we have a surjective homomorphism $\mathcal{O}_U \to E_{|U}$ for a neighborhood U of x. Since E is an $\mathcal{O}_D$-module, we have a surjective homomorphism $\psi : \mathcal{O}_{D\cap U} \to E_{|U}$. Since E is a locally free $\mathcal{O}_D$-module over $X \setminus \cup_{j\neq k} D_j \cap D_k$, $\operatorname{Supp}\ker\psi \subset \cup_{j\neq k} D_j \cap D_k$. Hence ψ is an isomorphism. Therefore it is sufficient to show the torsion freeness of $E_{|D_i}$. Assume that the torsion module T of $E_{|D_i}$ is not zero. Then there is a component D_j such that $T_{p_i} \neq 0$ at $p_i \in D_i \cap D_j$. We take the decomposition $D = A^i + B^i$ with $A^i \cap B^i = \{p_i\}$ in Lemma 6.4. We may assume that $D_i \subset A^i$ and $D_j \subset B^i$. Let T' be the torsion submodule of E_{A^i}. Then T is a direct summand of T' with $T_{p_i} = T'_{p_i}$. For the morphism $E \to E_{|A^i}/T'$, the kernel contains a submodule F fitting in an exact sequence

$$0 \to (E_{|B^i}/T'')(-p_i) \to F \to T' \to 0,$$

where T'' is the torsion submodule of $E_{|B^i}$. Then we have

$$\begin{aligned}\frac{\chi((E_{|B^i}/T'')(-p_i-\beta)) + \chi(T')}{(B^i, H')} = \frac{\chi(F(-\beta))}{(B^i, H')} &< \frac{\chi(E(-\beta))}{(D, H')} \\ &< \frac{\chi((E_{|B^i}/T'')(-\beta))}{(B^i, H')}\end{aligned}$$

Since $\chi((E_{|B^i}/T'')(-p_i-\beta)) + \chi(T') \geq \chi((E_{|B^i}/T'')(-\beta))$, we get a contradiction. Therefore $E_{|D_i}$ is torsion free, and we complete the proof. Q.E.D.

We next characterize the Mukai vectors of $E_{|D_i}$ for $E \in M_D^\beta(v)$. We set $v(E_{|A^i}) = (0, A^i, a_i)$ and $v(E_{|B^i}) = (0, B^i, b_i)$. Since $E_{|B^i}(-p_i)$ is a subsheaf of E and $E_{|B^i}$ is a quotient of E, we have

$$\frac{b_i - 1 - \beta(B^i)}{(B^i, H')} < \frac{\chi(E) - \beta(D)}{(D, H')} < \frac{b_i - \beta(B^i)}{(B^i, H')}. \tag{6.1}$$

Hence

$$b_i = \min\left\{ n \in \mathbb{Z} \,\middle|\, n > \frac{(\chi(E) - \beta(D))(B^i, H')}{(D, H')} + \beta(B^i) \right\}. \tag{6.2}$$

Conversely if (6.2) holds for a line bundle E on D, then we shall prove that E is β-twisted stable.

We first note that (6.1) holds and

$$\frac{a_i - 1 - \beta(A^i)}{(A^i, H')} < \frac{\chi(E) - \beta(D)}{(D, H')} < \frac{a_i - \beta(A^i)}{(A^i, H')}. \tag{6.3}$$

For an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0$$

such that E_1 and E_2 are purely 1-dimensional and E_1 is β-twisted stable, $\mathrm{Div}(E_1)$ is connected. We set

$$\mathrm{Div}(E_1) \cap \mathrm{Div}(E_2) := \{p_{n_1}, p_{n_2}, ..., p_{n_s}\}.$$

We note that $p_{n_j} = D_{a_j} \cap D_{b_j}$ with $\{a_j, b_j\} = \{n_j, \varphi(n_j)\}$. We may assume that $D_{a_j} \subset E_1$ and $D_{b_j} \subset E_2$. Replacing B^i by A^i, we may assume that $D_{a_j} \subset A^{n_j} \cap \mathrm{Div}(E_1)$ and $D_{b_j} \subset \mathrm{Div}(E_2) \cap B^{n_j}$ for $1 \le j \le s$. Since $A^{n_j} \setminus \{p_{n_j}\}$ is a connected component of $D \setminus \{p_{n_j}\}$, connectivity of $\mathrm{Div}(E_1) \setminus \{p_j\}$ implies that $\mathrm{Div}(E_1) \subset A^{n_j}$. Since D is a tree configuration, we also have $B^{n_j} \cap B^{n_k} = \emptyset$ for $j \neq k$. Hence we have a decomposition of $\mathrm{Div}(E_2) = D - \mathrm{Div}(E_1)$ into connected components B^j: $\mathrm{Div}(E_2) = \sum_j B^{n_j}$. By (6.1), we have

$$\frac{\chi(E(-\beta))}{(D, H')}(B^{n_j}, H') < \chi(E_{|B^{n_j}}(-\beta)).$$

Then we have

$$\sum_j \chi(E_{|B^{n_j}}(-\beta)) > \sum_j \frac{\chi(E(-\beta))}{(D, H')}(B^{n_j}, H') = \frac{\chi(E(-\beta))}{(D, H')}(\mathrm{Div}(E_2), H').$$

Since $\cup_j B^{n_j}$ is a disjoint union, we have a surjective homomorphism $E \to \oplus_j E_{|B^{n_j}}$. Since $E_1 \to E \to \oplus_j E_{|B^{n_j}}$ is a zero map, we have a surjective morphism $E_2 \to \oplus_j E_{|B^{n_j}}$. Since $\mathrm{Div}(E_2) = \sum_j B^{n_j}$ and E_2 is pure, it is an isomorphism. Therefore

$$\chi(E_2(-\beta)) > \frac{\chi(E(-\beta))}{(D, H')}(\mathrm{Div}(E_2), H'),$$

which implies E is β-twisted stable.

Remark 6.11. By the proof of Lemma 6.5, we also have an exact sequence

$$0 \to \mathcal{O}_{D_i}(-\sum_j p_{n_j}) \to \mathcal{O}_D \to \oplus_j \mathcal{O}_{B^{n_j}} \to 0.$$

Hence

$$\chi(E_{|D_i}) = (D - D_i, D_i) + \chi(E) - \sum_j b_{n_j}.$$

Definition 6.12. For a sequence of smooth curves $C_1, C_2, ..., C_s$ in X and a sequence of integers $d_1, d_2, ..., d_s$, $\mathrm{Pic}^{d_1,d_2,...,d_s}(\sum_{j=1}^s C_j)$ denotes the moduli spaces of line bundles E on $\sum_{j=1}^s C_j$ such that

$$\chi(E_{|C_i}) = d_i + (1 - g(C_i)).$$

By Lemma 6.5, we have a bijective correspondence

$$(\chi(E_{|D_0}), \chi(E_{|D_1}), ..., \chi(E_{|D_m})) \longleftrightarrow (\chi(E), b_1, b_2, ..., b_m).$$

Hence Proposition 6.9 follows from the following claim.

Lemma 6.13. *For $E \in M_D^\beta(v)$, we set*

$$d_i := \deg(E_{|D_i}) = \chi(E_{|D_i}) - (1 - g(D_i)).$$

Then we have an isomorphism

$$\mathrm{Pic}^{d_0,d_1,...,d_i}(\sum_{j=0}^{i} D_j) \cong \mathrm{Pic}^{d_0,d_1,...,d_{i-1}}(\sum_{j=0}^{i-1} D_j) \times \mathrm{Pic}^d_i(D_i).$$

In particular, $\mathrm{Pic}^{d_0,d_1,...,d_m}(\sum_{j=0}^m D_j) \cong \prod_j \mathrm{Pic}^{d_j}(D_j)$.

Proof. For $E \in \mathrm{Pic}^{d_0,d_1,...,d_i}(\sum_{j=0}^i D_j)$, we have

$$(E_{|\sum_{j<i} D_j}, E_{|D_i}) \in \mathrm{Pic}^{d_0,d_1,...,d_{i-1}}(\sum_{j=0}^{i-1} D_j) \times \mathrm{Pic}^d_i(D_i)$$

and E fits in an exact sequence

$$0 \to E_{|D_i}(-p_i) \to E \to E_{|\sum_{j<i} D_j} \to 0.$$

Since $\mathrm{Ext}^k(E_{|\sum_{j<i} D_j}, E_{|D_i}(-p_i)) = 0$ for $k \neq 1$ and

$$\begin{aligned}\mathrm{Ext}^1(E_{|\sum_{j<i} D_j}, E_{|D_i}(-p_i)) &\cong H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(E_{|\sum_{j<i} D_j}, E_{|D_i}(-p_i))) \\ &\cong H^0(X, \mathbb{C}_{p_i}),\end{aligned}$$

E is uniquely determined by $(E_{|\sum_{j<i} D_j}, E_{|D_i}(-p_i))$. Q.E.D.

6.3. Proof of Proposition 6.1

Lemma 6.14. *Let C be a smooth curve of $(C^2) > 0$ in an abelian surface Y. Then $H_1(C,\mathbb{Z}) \to H_1(Y,\mathbb{Z})$ is surjective.*

Proof. If it is not surjective, then $f : \mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(C)$ is not injective. For the abelian surface $f(\mathrm{Pic}^0(Y))$, we set $Y' := \mathrm{Pic}^0(f(\mathrm{Pic}^0(Y)))$. Since $C \to Y$ factors through $\mathrm{Alb}(C)$, we have the following diagram

$$\begin{array}{ccc} C & \longrightarrow & Y' \\ \| & & \big\downarrow g \\ C & \longrightarrow & Y \end{array}$$

Let y be a point of $\ker g$. Since $T_y^*(C)$ is algebraically equivalent to C, we have $(T_y^*(C), C) = (C, C) > 0$. Thus $T_y^*(C) \cap C \neq \emptyset$. For a point $s \in T_y^*(C) \cap C$, $g(s) = g(s+y)$ for the points $s, s+y \in C$. Thus $g_{|C}$ is not injective. Therefore $H_1(C,\mathbb{Z}) \to H_1(Y,\mathbb{Z})$ is surjective. Q.E.D.

For the divisor $D \in \mathrm{Hilb}_Y^{\eta}$ in Lemma 6.3, we take $E \in f^{-1}(D)$. Let $\mathfrak{d}$ be a Cartier divisor of D such that $\mathfrak{d} = \sum_i \mathfrak{d}_i$, $\mathfrak{d}_i = \sum_j n_{ij} p_{ij}$, $p_{ij} \in D_i \setminus \cup_{k \neq i} D_k$ and $\deg(\mathfrak{d}_i) = \sum_j n_{ij} = 0$ for all i. For $\mathcal{O}_D(\mathfrak{d}) \in \mathrm{Pic}(D)$, we have $\mathcal{O}_D(\mathfrak{d}) = \mathcal{O}_D + \sum_{i,j} n_{ij} \mathbb{C}_{p_{ij}}$ in $K(D)$. Hence

$$\Phi_{Y \to X}^{\mathbf{E}}(E(\mathfrak{d})) = \Phi_{Y \to X}^{\mathbf{E}}(E) + \sum_{i,j} n_{ij} \mathbf{E}_{p_{ij}}$$

in $K(X)$. Then we have

$$\mathfrak{a}(\Phi_{Y \to X}^{\mathbf{E}}(E(\mathfrak{d}))) = \mathfrak{a}(\Phi_{Y \to X}^{\mathbf{E}}(E)) + \sum_{i,j} n_{ij} \mathfrak{a}(\mathbf{E}_{p_{ij}}).$$

This morphism is the same as

$$\prod_i \mathrm{Pic}^0(D_i) \cong \prod_i \mathrm{Jac}(D_i) \xrightarrow{\mu} Y \xrightarrow{\mathfrak{a}} X \times \widehat{X}$$

sending $\mathcal{O}_D(\sum_{i,j} n_{ij} p_{ij})$ to the image of $\sum_{i,j} n_{ij} p_{ij} \in Y$ by $\mathfrak{a}$.

Lemma 6.15. *$\mathfrak{a} : Y \to X \times \widehat{X}$ sending $y \in Y$ to $\mathfrak{a}(\mathbf{E}_y) \in X \times \widehat{X}$ is injective.*

Proof. We set $u = (r, \xi, a)$. Replacing u by $-u$, we may assume that $r > 0$. We set $p = (r, \xi)$. Since v is primitive, $(p, a) = 1$. Since $ra = (\xi^2)/2 \in p^2\mathbb{Z}$, we may set $r = p^2 t$ and $\xi = pqH$, where H is primitive. Since $q^2(H^2)/2 = ta$ and $(q, pt) = 1$, we can set $a = q^2 s$.

Thus we get $v=(p^2t, pqH, q^2s)$, where $(pt,q)=1$, $(p,q^2s)=1$ and the type of H is $(1,ts)$. We take $E \in Y$. We have a morphism $f: X \to Y$ by sending $x \in X$ to $T_x^*(E)$. Then $X/K(E) \cong Y$, where $K(E) = \operatorname{im}(K(pqH) \overset{p^2t}{\to} X)$. We note that $g: X \to Y \to X \times \widehat{X}$ is the morphism sending x to $(q^2sx, \phi_{pqH}(x))$. We shall prove that $\ker g = K(E)$.

We set $K(pqH) = \frac{1}{pq}V_1/V_1 \oplus \frac{1}{pqts}V_2/V_2$. Then

$$K(E) = p^2tK(pqH) = \frac{p^2t}{pq}V_1/V_1 \oplus \frac{p^2t}{pqts}V_2/V_2 = \frac{1}{q}V_1/V_1 \oplus \frac{1}{qs}V_2/V_2,$$

where we used $(pt,q)V_1 = V_1$ and $(p,qt)V_2 = V_2$. For $(\frac{x_1}{pq}, \frac{x_2}{pqts}) \in \ker(q^2s) \cap K(pqH)$, we have $(\frac{q^2sx_1}{pq}, \frac{q^2sx_2}{pqts}) \in V_1 \oplus V_2$. Then we have $x_1 \in pV_1$ and $x_2 \in ptV_2$. Hence

$$\ker(q^2s) \cap K(pqH) = \frac{1}{q}V_1/V_1 \oplus \frac{1}{qs}V_2/V_2 = K(E).$$

Q.E.D.

By Proposition 6.9, Proposition 6.1 follows from the following claim.

Lemma 6.16. *If D is a normal crossing divisor of smooth curves D_i, then $\ker\mu$ is connected.*

Proof. Since D_i are smooth, it is sufficient to prove that

$$H_1(\prod_i \operatorname{Pic}^0(D_i), \mathbb{Z}) \to H_1(Y, \mathbb{Z})$$

is surjective. Since

$$H_1(\prod_i \operatorname{Pic}^0(D_i), \mathbb{Z}) \cong \bigoplus_i H_1(\operatorname{Pic}^0(D_i), \mathbb{Z}) \cong \bigoplus_i H_1(D_i, \mathbb{Z}),$$

Lemma 6.14 implies the claim unless all D_i are elliptic curves. If all D_i are elliptic curves, then $(D_0, D_1) = 1$ implies that the natural homomorphism $D_0 \times D_1 \to Y$ is an isomorphism. Hence

$$H_1(D_0, \mathbb{Z}) \oplus H_1(D_1, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$$

is an isomorphism. Therefore the claim holds. Q.E.D.

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Department of Mathematics, Faculty of Science, Kobe University, Kobe, 657, Japan

E-mail address: `yoshioka@math.kobe-u.ac.jp`