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On derived categories of K3 surfaces, symplectic automorphisms and the Conway group

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Dedicated to Professor Shigeru Mukai on the occasion of his 60th birthday.

Abstract.

In this note we interpret a recent result of Gaberdiel et al [7] in terms of derived equivalences of K3 surfaces. We prove that there is a natural bijection between subgroups of the Conway group Co_0 with invariant lattice of rank at least four and groups of symplectic derived equivalences of $\mathrm{D}^{\mathrm{b}}(X)$ of projective K3 surfaces fixing a stability condition.

As an application we prove that every such subgroup $G \subset Co_0$ satisfying an additional condition can be realized as a group of symplectic automorphisms of an irreducible symplectic variety deformation equivalent to $\mathrm{Hilb}^n(X)$ of some K3 surface.

In his celebrated paper [17] Mukai established a bijection between finite groups of symplectic automorphisms of K3 surfaces $G \subset \mathrm{Aut}_s(X)$ and finite subgroups $G \subset M_{23}$ of the Mathieu group M_{23} with at least five orbits. An alternative approach relying on Niemeier lattices was given by Kondō in [15].

More recently, physicists observed that groups of supersymmetry preserving automorphisms of non-linear σ-models on K3 surfaces are linked to subgroups of the larger Mathieu group M_{24} and the even larger Conway group Co_1, both sporadic finite simple groups. The lattice theory used in [7], ultimately going back to Kondō, can be reinterpreted in purely mathematical terms to prove the following result about derived autoequivalences of K3 surfaces which should be seen as a derived version of Mukai's classical result.

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Theorem 0.1. *For a group G the following conditions are equivalent:*
i) G is isomorphic to a subgroup of the group $\mathrm{Aut}_s(\mathrm{D^b}(X),\sigma)$ *of some complex projective K3 surface X endowed with a stability condition* $\sigma\in\mathrm{Stab}^{\mathrm{o}}(X)$.
ii) G is isomorphic to a subgroup of the Conway group Co_0 with invariant lattice of rank at least four.

Here, $\mathrm{D^b}(X)=\mathrm{D^b}(\mathrm{Coh}(X))$ is the bounded derived category of coherent sheaves on X and

$$\mathrm{Stab}^{\mathrm{o}}(X)\subset\mathrm{Stab}(X)$$

is Bridgeland's distinguished connected component of the space of stability conditions on $\mathrm{D^b}(X)$, see [3]. By $\mathrm{Aut}(\mathrm{D^b}(X))$ we denote the group of isomorphism classes of $\mathbb{C}$-linear exact autoequivalences of the triangulated category $\mathrm{D^b}(X)$ and by $\mathrm{Aut}_s(\mathrm{D^b}(X))\subset\mathrm{Aut}(\mathrm{D^b}(X))$ the finite index subgroup of symplectic autoequivalences (see below for details of these definitions). Finally, we write $\mathrm{Aut}(\mathrm{D^b}(X),\sigma)\subset\mathrm{Aut}(\mathrm{D^b}(X))$ for the subgroup of autoequivalences Φ with $\Phi^*\sigma=\sigma$ and let

$$\mathrm{Aut}_s(\mathrm{D^b}(X),\sigma):=\mathrm{Aut}(\mathrm{D^b}(X),\sigma)\cap\mathrm{Aut}_s(\mathrm{D^b}(X)).$$

To explain the condition in ii), recall that the Conway group Co_0 is by definition the orthogonal group of the Leech lattice N, i.e.

$$Co_0:=\mathrm{O}(N).$$

So, for a subgroup $G\subset Co_0$ we can consider the invariant lattice N^G and ii) means $\mathrm{rk}\,N^G\geq 4$. Note that whenever N^G is non-trivial, then G does not contain $-\mathrm{id}$ and can therefore be realized as a subgroup of the Conway group $Co_1:=Co_0/\{\pm\mathrm{id}\}$. We think of the condition $\mathrm{rk}\,N^G\geq 4$ as G acting with at least four orbits, analogously to Mukai's condition on subgroups $G\subset M_{23}$ acting with at least five orbits on $\{1,\ldots,24\}$.

Note that a finite group $G\subset\mathrm{Aut}(X)$ always leaves invariant one ample class $\alpha\in H^{1,1}(X,\mathbb{Z})$ and, therefore, can be lifted to a subgroup of $\mathrm{Aut}(\mathrm{D^b}(X),\sigma_\alpha)$, where σ_α is the canonical stability condition with stability function $Z=\langle\exp(i\alpha),~\rangle$ constructed in [3, Sec. 6 & 7]. Also, as we shall see, the group $\mathrm{Aut}(\mathrm{D^b}(X),\sigma)$ is automatically finite for any $\sigma\in\mathrm{Stab}^{\mathrm{o}}(X)$, so that all groups in i) (and of course also in ii)) are finite. Thus, Theorem 0.1 can indeed be seen as a true generalization of Mukai's result on finite groups of symplectic automorphisms [17].

Furthermore, Theorem 0.1 can be used to prove that most of the above groups can be realized as symplectic automorphisms on higher dimensional analogues of K3 surfaces.

Theorem 0.2. *Assume $G \subset Co_0$ is a subgroup with invariant lattice of rank at least four satisfying condition $(*)$ (see Section 4). Then there exists a projective irreducible symplectic variety Y deformation equivalent to the Hilbert scheme $\mathrm{Hilb}^n(X)$ of subschemes of length n on a K3 surface X such that G is isomorphic to a subgroup of $\mathrm{Aut}_s(Y)$.*

A more complete result concerning groups of symplectic automorphisms of deformations of $\mathrm{Hilb}^n(X)$ has recently been announced by Giovanni Mongardi, see Remark 4.1.

Outline. Following [1], a mathematician should think of a non-linear σ-model on a K3 surface as a pair of orthogonal positive planes

$$P_1 \perp P_2 \subset \Lambda \otimes \mathbb{R}$$

without any integral (-2)-classes in $(P_1 \oplus P_2)^\perp$. Here, $\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$ is the unique even, unimodular lattice of signature $(4, 20)$. In fact, Λ is isomorphic to the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$ (which is nothing but $H^*(X, \mathbb{Z})$ with the sign reversed in the pairing of H^0 and H^4) of any K3 surface X.

Geometrically this comes about as follows. To any K3 surface X with a Kähler class $\alpha \in H^{1,1}(X, \mathbb{R})$ one can naturally associate two positive planes

$$P_X := (H^{2,0} \oplus H^{0,2})(X) \cap H^2(X, \mathbb{R}) \text{ and } P_\alpha := \mathbb{R} \cdot \alpha \oplus \mathbb{R} \cdot (1 - (\alpha)^2/2)$$

in $\widetilde{H}(X, \mathbb{R})$. A (-2)-class $\delta \in \widetilde{H}(X, \mathbb{Z})$ is orthogonal to P_X if and only if δ is algebraic, i.e. $\delta \in \widetilde{H}^{1,1}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^{1,1}(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$. If $\delta \in H^{1,1}(X, \mathbb{Z})$, then $(\alpha.\delta) \neq 0$ and, therefore, $\delta \notin P_\alpha^\perp$. In fact it can be shown that for $\alpha \in H^{1,1}(X, \mathbb{Z}) \otimes \mathbb{R}$, e.g. α an ample class, with $(\alpha)^2 > 2$ none of the algebraic (-2)-classes is orthogonal to P_α.

Note that the positive four space $P_1 \oplus P_2$ associated with a pair of orthogonal positive planes $P_1, P_2 \subset H^*(M, \mathbb{R})$ can always be written as $\exp(B) \cdot (P_X \oplus P_\alpha)$ for some K3 surface structure X on the underlying differentiable manifold M endowed with a class $\alpha \in H^{1,1}(X, \mathbb{R})$ with $(\alpha)^2 > 0$ and a class $B \in H^2(X, \mathbb{R})$. Here, $\exp(B) = 1 + B + (B)^2/2 \in H^*(X, \mathbb{R})$ acts by multiplication. See [12, Prop. 3.6].

The main result in [7] describes all finite subgroups $G \subset \mathrm{O}(\Lambda)$ acting trivially on $P_1 \oplus P_2$ for some non-linear σ-model $P_1 \perp P_2$ as above. So the main task of this note is to pass from this lattice theoretic condition on finite groups of isometries to one that can be phrased in terms of derived categories $\mathrm{D^b}(X)$ of complex projective K3 surfaces. In fact [7] also contains a more precise description of the occurring groups which of course includes all groups of Mukai's list (as any finite group of automorphisms preserves one ample class). But it also contains groups of the form $(\mathbb{Z}/3\mathbb{Z})^{\oplus 4}.A_6$, which in particular is not contained in M_{24} as its order does not divide $|M_{24}|$.

In Section 1 we establish a bijection between groups of autoequivalences fixing a stability condition $\sigma \in \mathrm{Stab}^{\mathrm{o}}(X)$ and groups of Hodge isometries of $\widetilde{H}(X,\mathbb{Z})$ acting trivially on the positive four space that comes with σ. In Section 2 we give a brief sketch of the proof of [7]. The final Section 3 contains the proof of Theorem 0.1.

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§1. Lifting Hodge isometries

We shall link Hodge isometries of the Mukai lattice $\widetilde{H}(X,\mathbb{Z})$ of a complex projective K3 surface X fixing an additional positive plane in $\widetilde{H}^{1,1}(X,\mathbb{Z})\otimes\mathbb{R}$ to autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$ fixing a stability condition.

1.1.

Let Λ be the lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$ (or any lattice of signature $(4,n)$) and consider a K3 Hodge structure on Λ, i.e. a Hodge structure of weight two given by an orthogonal decomposition

$$\Lambda \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$$

such that $\Lambda^{2,0}$ is isotropic and $(\Lambda^{2,0} \oplus \Lambda^{0,2}) \cap (\Lambda \otimes \mathbb{R}) \subset \Lambda \otimes \mathbb{R}$ is a positive plane.

A *Hodge isometry* $\varphi : \Lambda \xrightarrow{\sim} \Lambda$ is an orthogonal transformation $\varphi \in \mathrm{O}(\Lambda)$ such that its $\mathbb{C}$-linear extension satisfies $\varphi(\Lambda^{2,0}) = \Lambda^{2,0}$. We say that φ is *symplectic* if $\varphi|_{\Lambda^{2,0}} = \mathrm{id}$ and *positive* if there exists a positive plane $P \subset \Lambda^{1,1} \cap (\Lambda \otimes \mathbb{R})$ with $\varphi|_P = \mathrm{id}$. If P is given and $\varphi|_P = \mathrm{id}$, then φ is called *P-positive.*

1.2.

Let now $\widetilde{H}(X,\mathbb{Z})$ be the Mukai lattice of a complex K3 surface X with its natural Hodge structure given by $\widetilde{H}^{2,0}(X) = H^{2,0}(X)$ and $\widetilde{H}^{1,1}(X) = (H^0 \oplus H^{1,1} \oplus H^4)(X)$. The group of all Hodge isometries resp. of all symplectic Hodge isometries shall be denoted

$$\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z})) \text{ resp. } \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z})).$$

For a positive plane $P \subset \widetilde{H}^{1,1}(X,\mathbb{R})$ we write

$$\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}), P) \subset \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$$

for the subgroup of all P-positive Hodge isometries and $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}),P)$ for its intersection with $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}))$. If $P = P_Z$ is a positive plane spanned by $\mathrm{Re}(Z)$ and $\mathrm{Im}(Z)$ of some $Z \in \widetilde{H}^{1,1}(X,\mathbb{Z}) \otimes \mathbb{C}$, then let

$$\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}), Z) := \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}), P_Z) \subset \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z})).$$

Example 1.1. Any Kähler class $\alpha \in H^{1,1}(X)$ gives rise to a positive plane $P_\alpha \subset \widetilde{H}^{1,1}(X,\mathbb{R})$ spanned by $\alpha \in H^{1,1}(X)$ and $1 - (\alpha)^2/2 \in (H^0 \oplus H^4)(X,\mathbb{R})$. We write

$$\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}), \alpha) := \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}), P_\alpha).$$

If $Z = \exp(i\alpha) = 1 + i\alpha - (\alpha)^2/2$, then $P_\alpha = P_Z$ and $\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\alpha) = \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}), Z)$. The setting can be generalized to the positive plane generated by real and imaginary part of $\exp(B + i\alpha) \in \widetilde{H}^{1,1}(X)$, where $\alpha \in H^{1,1}(X,\mathbb{R})$ with $(\alpha)^2 > 0$ and $B \in H^{1,1}(X,\mathbb{R})$.

Let X be a complex K3 surface and $f : X \xrightarrow{\sim} X$ an automorphism. If f acts as id on $H^{2,0}(X)$, then $f^* : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,\mathbb{Z})$ is a symplectic Hodge isometry, i.e. $f^* \in \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}))$. If $f^*\alpha = \alpha$ for a Kähler class α or, more generally, for some $\alpha \in H^{1,1}(X,\mathbb{R})$ with $(\alpha)^2 > 0$, then f^* is P_α-positive, i.e. $f^* \in \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),\alpha)$.

If an automorphism $f : X \xrightarrow{\sim} X$ is of finite order, then f^* is always positive. Indeed, take any Kähler class $\alpha \in H^{1,1}(X,\mathbb{R})$ and consider $\widetilde{\alpha} := \sum_i f^{i*}\alpha$. Then $\widetilde{\alpha}$ is a Kähler class that satisfies $f^*(\widetilde{\alpha}) = \widetilde{\alpha}$ and hence $f^* = \mathrm{id}$ on $P_{\widetilde{\alpha}}$. So any symplectic automorphism of finite order $f : X \xrightarrow{\sim} X$ of a K3 surface X gives rise to a positive symplectic Hodge isometry of $\widetilde{H}(X,\mathbb{Z})$.

Remark 1.2. i) As a sort of converse of the above, one observes that $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}),P) = \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z})) \cap \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),P)$ is a finite group: Indeed, it is a discrete subgroup of the compact group

$$\mathrm{O}(((H^{2,0} \oplus H^{0,2})(X,\mathbb{R}) \oplus P)^{\perp}) \simeq \mathrm{O}(20,\mathbb{R}).$$

In particular, any Hodge isometry which is symplectic and positive is automatic of finite order. In this sense, Mukai's classification of finite groups of symplectic automorphisms of K3 surfaces is part of the broader classification of all groups of symplectic P-positive Hodge isometries of $\widetilde{H}(X,\mathbb{Z})$ for some K3 surface X endowed with a positive plane $P \subset \widetilde{H}^{1,1}(X,\mathbb{R})$.

ii) In fact, if X is projective, then already $\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),P)$ is finite. Indeed, in this case the transcendental lattice $T(X)$ is non-degenerate and irreducible and hence $\mathrm{Aut}(T(X))$ is a discrete subgroup of a compact group and hence finite. But the kernel of $\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),P) \longrightarrow \mathrm{Aut}(T(X))$ is contained in the finite group $\mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}),P)$. The finiteness of $\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}),P)$ is the analogue of the finiteness of the group of all automorphisms $f : X \xrightarrow{\sim} X$ fixing an ample line bundle L or a Kähler class α.

1.3.

Let from now on the complex K3 surface X also be projective. We denote its bounded derived category of coherent sheaves by $\mathrm{D^b}(X) := \mathrm{D^b}(\mathrm{Coh}(X))$. Furthermore, let $\mathrm{Aut}(\mathrm{D^b}(X))$ be the group of isomorphism classes of exact $\mathbb{C}$-linear autoequivalences $\Phi : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$. To any $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X))$ one associates the Hodge isometry $\varphi := \Phi^H : \widetilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(X,\mathbb{Z})$ which defines a homomorphism

$$\rho : \mathrm{Aut}(\mathrm{D^b}(X)) \longrightarrow \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z})).$$

This goes back to Mukai's article [18], see [11, Ch. 10] for further details, references, and notations. The image of ρ is the index two subgroup

$$\mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))^{+} \subset \mathrm{Aut}(\widetilde{H}(X,\mathbb{Z}))$$

of Hodge isometries preserving the (natural) orientation of positive planes $P \subset \widetilde{H}^{1,1}(X,\mathbb{R})$, see [9]. We say that Φ is symplectic if $\varphi \in \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}))$ and we let

$$\mathrm{Aut}_s(\mathrm{D^b}(X)) \subset \mathrm{Aut}(\mathrm{D^b}(X))$$

denote the subgroup of all symplectic autoequivalences.

In the following we shall denote by $\mathrm{Stab}(X)$ the space of all stability conditions $\sigma = (\mathcal{P}, Z)$ on $\mathrm{D^b}(X)$ and by $\mathrm{Stab^o}(X) \subset \mathrm{Stab}(X)$ the distinguished connected component introduced and studied in [3]. Tacitly, all stability conditions are required to be locally finite. For a brief survey of the main features see also [8].

The group $\mathrm{Aut}(\mathrm{D^b}(X))$ acts on $\mathrm{Stab}(X)$ and we let

$$\mathrm{Aut^o}(\mathrm{D^b}(X)) \subset \mathrm{Aut}(\mathrm{D^b}(X))$$

be the subgroup of all autoequivalences fixing $\mathrm{Stab^o}(X)$. Note that it is known that the restriction $\rho : \mathrm{Aut^o}(\mathrm{D^b}(X)) \twoheadrightarrow \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))^+$ is still surjective. Indeed, in the original argument, due to Mukai and Orlov see [11, Ch. 10.2], one only has to check that universal families of μ-stable bundles preserve the distinguished component $\mathrm{Stab^o} \subset \mathrm{Stab}$ and for this see e.g. [10, Prop. 5.2].

Conjecturally, $\mathrm{Stab^o}(X) = \mathrm{Stab}(X)$ or, at least, $\mathrm{Aut^o}(\mathrm{D^b}(X)) = \mathrm{Aut}(\mathrm{D^b}(X))$. In fact, a proof of the conjecture for the case $\rho(X) = 1$ has recently been given by Bayer and Bridgeland [4]. In any case, as shown in [3], the group $\mathrm{Aut^o}(\mathrm{D^b}(X)) \cap \mathrm{Ker}(\rho)$ can be identified with the group of deck transformations of the covering

$$\pi : \mathrm{Stab^o}(X) \twoheadrightarrow \mathcal{P}_0^+(X), \ \sigma = (\mathcal{P}, Z) \longmapsto Z.$$

Here,

$$\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta_X} \delta^\perp \subset \widetilde{H}^{1,1}(X, \mathbb{Z}) \otimes \mathbb{C}$$

with $\mathcal{P}^+(X)$ the connected component containing $1 + i\alpha - (\alpha)^2/2$ with α ample of the open set $\mathcal{P}(X)$ of all $Z \in \widetilde{H}^{1,1}(X, \mathbb{Z}) \otimes \mathbb{C}$ with real and imaginary part spanning a positive plane

$$P_Z := \mathbb{R} \cdot \mathrm{Re}(Z) \oplus \mathbb{R} \cdot \mathrm{Im}(Z) \subset \widetilde{H}^{1,1}(X, \mathbb{R}).$$

By $\Delta_X \subset \widetilde{H}^{1,1}(X, \mathbb{Z})$ we denote the set of all (-2)-classes $\delta \in \widetilde{H}^{1,1}(X, \mathbb{Z})$. Here and in the sequel, the stability function $Z : \widetilde{H}^{1,1}(X, \mathbb{Z}) \longrightarrow \mathbb{C}$ is, via Poincaré duality, identified with an element $Z \in \widetilde{H}^{1,1}(X, \mathbb{Z}) \otimes \mathbb{C}$.

Definition 1.3. An exact $\mathbb{C}$-linear autoequivalence $\Phi : \mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X)$ is *positive* if there exists a stability condition $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab^o}(X)$ with $\Phi^* \sigma = \sigma$ (and then Φ is called *σ-positive*). The group of all σ-positive autoequivalences is denoted $\mathrm{Aut}(\mathrm{D^b}(X), \sigma)$.

Note that in particular $\mathrm{Aut}(\mathrm{D^b}(X), \sigma) \subset \mathrm{Aut^o}(\mathrm{D^b}(X))$.

1.4.

The next proposition is a derived version of the Global Torelli theorem for automorphisms of polarized K3 surfaces (X, L) which can be stated as

$$\mathrm{Aut}(X, L) \xrightarrow{\sim} \mathrm{Aut}(H^2(X, \mathbb{Z}), \ell)$$

for $\mathrm{Aut}(X, L)$ the group of automorphisms $f : X \xrightarrow{\sim} X$ with $f^*L \simeq L$ and $\mathrm{Aut}(H^2(X, \mathbb{Z}), \ell)$ the group of Hodge isometries of $H^2(X, \mathbb{Z})$ fixing the ample class $\ell := \mathrm{c}_1(L)$.

Proposition 1.4. *For* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{o}}(X)$*, the homomorphism* ρ *induces isomorphisms*

$$\mathrm{Aut}(\mathrm{D^b}(X), \sigma) \xrightarrow{\sim} \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z)$$

and

$$\mathrm{Aut}_s(\mathrm{D^b}(X), \sigma) \xrightarrow{\sim} \mathrm{Aut}_s(\widetilde{H}(X, \mathbb{Z}), Z).$$

Proof. Clearly, for $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X), \sigma)$ the induced Hodge isometry $\varphi := \rho(\Phi)$ satisfies $\varphi(Z) = Z$ and hence $\varphi \in \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z)$.

To prove injectivity, let $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X), \sigma)$ with $\varphi = \mathrm{id}$. Then by [3], Φ acts as a deck transformation of the covering map $\pi : \mathrm{Stab}^{\mathrm{o}}(X) \twoheadrightarrow \mathcal{P}_0^+(X)$. But a deck transformation that fixes a point $\sigma \in \mathrm{Stab}^{\mathrm{o}}(X)$ has to be the identity. Hence, $\Phi \simeq \mathrm{id}$.

For the surjectivity, let $\varphi \in \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z)$. As $\mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z) \subset \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))^+$, there exists an autoequivalence $\Phi_0 \in \mathrm{Aut}^{\mathrm{o}}(\mathrm{D^b}(X))$ with $\rho(\Phi_0) = \varphi$. Now $\sigma, \sigma_0 := \Phi_0^*\sigma \in \mathrm{Stab}^{\mathrm{o}}(X)$ both map to $Z = \pi(\sigma) = \pi(\sigma_0) \in \mathcal{P}_0^+(X)$ and, therefore, differ by a unique $\Psi \in \mathrm{Aut}^{\mathrm{o}}(\mathrm{D^b}(X))$ with $\rho(\Psi) = \mathrm{id}$. But then $\Phi := \Phi_0 \circ \Psi \in \mathrm{Aut}^{\mathrm{o}}(\mathrm{D^b}(X))$ satisfies $\rho(\Phi) = \rho(\Phi_0) = \varphi$ and $\Phi^*\sigma = \sigma$, i.e. $\Phi \in \mathrm{Aut}(\mathrm{D^b}(X), \sigma)$.

The second isomorphism follows from the first. $\square$

By Remark 1.2 the proposition immediately yields the following. (Note that whenever $\mathrm{D^b}(X)$ is used the surface X is assumed to be projective.)

Corollary 1.5. *The groups* $\mathrm{Aut}(\mathrm{D^b}(X), \sigma)$ *and* $\mathrm{Aut}_s(\mathrm{D^b}(X), \sigma)$ *are finite.* $\square$

Remark 1.6. As was explained to me by Tom Bridgeland, the finiteness of the stabilizer $\mathrm{Aut}(\mathrm{D^b}(X), \sigma)$ of a stability condition σ is a general phenomenon. Roughly, for any triangulated category $\mathcal{D}$ the quotient $\mathrm{Aut}(\mathcal{D}, \sigma)/\mathrm{Aut}(\mathcal{D}, \mathrm{Stab}^{\mathrm{o}}(\mathcal{D}))$ is finite. Here, $\mathrm{Aut}(\mathcal{D}, \mathrm{Stab}^{\mathrm{o}}(\mathcal{D}))$ is the subgroup of autoequivalences acting trivially on the connected component $\mathrm{Stab}^{\mathrm{o}}(\mathcal{D})$ containing the stability condition σ.

As another consequence we find

Corollary 1.7. *Let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{o}}(X)$. Then for a group G the following conditions are equivalent:*
i) G is isomorphic to a subgroup of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X), \sigma)$ *resp.* $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X), \sigma)$.
ii) G is isomorphic to a subgroup of $\mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}), Z)$ *resp.* $\mathrm{Aut}_s(\widetilde{H}(X, \mathbb{Z}), Z)$. □

Remark 1.8. i) The arguments show more generally that for any subgroup $G \subset \mathrm{Aut}^{\mathrm{o}}(\mathrm{D}^{\mathrm{b}}(X))$ of positive autoequivalences the restriction $\rho : G \longrightarrow \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))$ is injective. Indeed, $H := \mathrm{Ker}(\rho : G \longrightarrow \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z})))$ acts trivially on $\widetilde{H}(X, \mathbb{Z})$ and, therefore, consists of deck transformations of $\pi : \mathrm{Stab}^{\mathrm{o}}(X) \twoheadrightarrow \mathcal{P}_0^+(X)$. However, by assumption on G, there exists for any $\Phi \in H \subset G$ a stability condition $\sigma \in \mathrm{Stab}^{\mathrm{o}}(X)$ with $\Phi^*\sigma = \sigma$. Therefore, $\Phi = \mathrm{id}$, since non-trivial deck transformations act without fixed points.

ii) In a different direction, one can generalize to groups of autoequivalences that fix a simply connected open set of stability functions. Assume $G \subset \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))^+$ is a subgroup of autoequivalences such that there exists a contractible open set $U \subset \mathcal{P}_0^+(X)$ with $\varphi(U) = U$ for all $\varphi \in G$. Then there exists a non-canonical group homomorphism lifting the inclusion

$$\begin{tikzcd} & G \arrow[dl, dashed] \arrow[d, hook] \\ \mathrm{Aut}^{\mathrm{o}}(\mathrm{D}^{\mathrm{b}}(X)) \arrow[r, two heads] & \mathrm{Aut}(\widetilde{H}(X, \mathbb{Z}))^+. \end{tikzcd}$$

Indeed, since U is simply connected, any connected component $U' \subset \pi^{-1}(U) \subset \mathrm{Stab}^{\mathrm{o}}(X)$ of $\pi^{-1}(U)$ maps homeomorphically onto U. Pick one $U' \subset \pi^{-1}(U)$ and argue as above: For any $\varphi \in G$, there exists $\Phi_0 \in \mathrm{Aut}^{\mathrm{o}}(\mathrm{D}^{\mathrm{b}}(X))$ with $\rho(\Phi_0) = \varphi$. Pick $\sigma = (\mathcal{P}, Z) \in U'$. Then $\pi(\Phi_0^*\sigma) \in U$ and thus there exists a unique $\Psi \in \mathrm{Aut}^{\mathrm{o}}(\mathrm{D}^{\mathrm{b}}(X))$ with $\rho(\Psi) = \mathrm{id}$ and $\Psi^*\Phi_0^*\sigma \in U'$. The new $\Phi := \Phi_0 \circ \Psi \in \mathrm{Aut}^{\mathrm{o}}(\mathrm{D}^{\mathrm{b}}(X))$ satisfies $\rho(\Phi) = \rho(\Phi_0) = \varphi$ and $\Phi^*(U') = (U')$. Moreover, Φ is unique with these properties and we define the lift $G \longrightarrow \mathrm{Aut}^{\mathrm{o}}(\mathrm{D}^{\mathrm{b}}(X))$ by $\varphi \longmapsto \Phi$. To see that this defines a group homomorphism consider $\varphi_1, \varphi_2 \in G$ and let $\varphi_3 := \varphi_1 \circ \varphi_2$. For the (unique) lifts Φ_i of φ_i with $\Phi_i^*(U') = U'$ the composition $\Psi := \Phi_3^{-1} \circ (\Phi_1 \circ \Phi_2)$ satisfies $\rho(\Psi) = \mathrm{id}$ and $\Psi^*(U') = U'$. Hence, $\Psi = \mathrm{id}$ and, therefore, $\Phi_3 = \Phi_1 \circ \Phi_2$.

Remark 1.9. In [12] the notion of generalized K3 structures on the differentiable manifold M underlying a K3 surface was introduced as an orthogonal pair of generalized Calabi–Yau structures $\varphi, \varphi' \in \mathcal{A}^{2*}_{\mathbb{C}}(M)$. The period of a generalized K3 structure was defined as the pair of positive planes $P_\varphi, P_{\varphi'} \subset \widetilde{H}(M, \mathbb{R})$ spanned by real and imaginary parts of the cohomology classes $[\varphi]$ resp. $[\varphi']$. Isomorphisms between generalized K3 structures include $\mathrm{Diff}(M)$ and B-field twists $\exp(B)$ by closed forms $B \in \mathcal{A}^2(M)$ with integral cohomology class $\beta := [B] \in H^2(M, \mathbb{Z})$. Note that $\mathrm{Diff}(M)$ surjects onto the index two subgroup $\mathrm{O}(H^2(M, \mathbb{Z}))^+$ and that $\mathrm{O}(\widetilde{H}(M, \mathbb{Z}))$ is generated by $\mathrm{O}(H^2(M, \mathbb{Z}))$, $\exp(\beta)$ with $\beta \in H^2(M, \mathbb{Z})$, and $\mathrm{O}((H^0 \oplus H^4)(M, \mathbb{Z}))$. (Only the latter has no interpretation on the level of forms.) In this sense $\mathrm{Aut}_s(\widetilde{H}(X, \mathbb{Z}), Z)$ may be seen as the group of automorphisms of the generalized K3 structure given by $\varphi_1 = \omega$, a holomorphic two-form on X, and $\varphi_2 = \exp(B + i\alpha) \in \mathcal{A}^{2*}_{\mathbb{C}}(M)$ representing Z. So Proposition 1.4 seems to suggests that automorphisms of (φ_1, φ_2) can be interpreted as automorphisms of a stability condition σ on $\mathrm{D^b}(X)$. It would be interesting to find a more direct approach to this not relying on the period description of both sides.

§2. Groups of Hodge isometries as subgroups of Co_1

For the convenience of the reader, we recall the lattice theoretic arguments in [7] which relate groups of positive symplectic Hodge isometries with subgroups of the Conway group Co_1. Section 2.2 is later used in Section 3 to prove that groups of symplectic σ-positive autoequivalences can be realized as subgroups of Co_1, whereas the converse is based on Section 2.3.

2.1.

We go back to the abstract setting of Section 1.1 and let $\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}$. We also fix a positive subspace of dimension four $\Pi \subset \Lambda \otimes \mathbb{R}$ such that no (-2)-class $\delta \in \Lambda$ is contained in $\Pi^\perp$.

Then consider the subgroup

$$\mathrm{Aut}(\Lambda, \Pi) \subset \mathrm{O}(\Lambda)$$

of all isometries $\varphi : \Lambda \xrightarrow{\sim} \Lambda$ such that its $\mathbb{R}$-linear extension satisfies $\varphi = \mathrm{id}$ on Π. Thus, $\mathrm{Aut}(\Lambda, \Pi) \subset \mathrm{O}(\Pi^\perp)$ and since $\Pi^\perp$ is negative definite and hence $\mathrm{O}(\Pi^\perp)$ compact, $\mathrm{Aut}(\Lambda, \Pi)$ is finite. For a subgroup $G \subset \mathrm{Aut}(\Lambda, \Pi)$ we denote by

$$\Lambda^G \text{ and } \Lambda_G := (\Lambda^G)^\perp$$

the invariant part resp. its orthogonal complement. The group G can be chosen arbitrary and one can even take $G = \mathrm{Aut}(\Lambda, \Pi)$.

Following the classical line of arguments, one first proves

Lemma 2.1. *The lattice Λ_G is negative definite of rank* $\mathrm{rk}\,\Lambda_G \leq 20$ *and does not contain any (-2)-classes. The induced action of G on its discriminant $A_{\Lambda_G} = \Lambda_G^*/\Lambda_G$ is trivial and the minimal number of generators of A_{Λ_G} is bounded by $\ell(A_{\Lambda_G}) \leq 24 - \mathrm{rk}\,\Lambda_G$.*

Proof. As $\Lambda_G \subset \Pi^\perp$, the assumption on Π implies that the lattice Λ_G does not contain any (-2)-class. Moreover, since $\Pi^\perp \subset \Lambda \otimes \mathbb{R}$ is a negative definite subspace of dimension ≤ 20, also Λ_G is negative definite and of rank $\mathrm{rk}\,\Lambda_G \leq 20$. As Λ is unimodular and, as is easy to check, also Λ^G is non-degenerate, there exists an isomorphism $A_{\Lambda_G} \simeq A_{\Lambda^G}$ which is compatible with the induced action of G. Since the action on the latter is trivial, it is so on A_{Λ_G}. The isomorphism also yields $\ell(A_{\Lambda_G}) = \ell(A_{\Lambda^G}) \leq \mathrm{rk}\,\Lambda^G = \mathrm{rk}\,\Lambda - \mathrm{rk}\,\Lambda_G$. The arguments are quite standard, for more details see [14] and references therein. $\square$

The key idea in [15] is to embed Λ_G (or rather $\Lambda_G \oplus A_1(-1)$) into some Niemeier lattice, i.e. into one of the 24 negative definite, even, unimodular lattices of rank 24. In our situation, Kondō's approach is easy to adapt if the stronger inequality

$$\ell(A_{\Lambda_G}) < 24 - \mathrm{rk}\,\Lambda_G, \tag{2.1}$$

is assumed. Indeed, then by [19, Thm. 1.12.2] there exists a primitive embedding

$$\Lambda_G \hookrightarrow N_i$$

into one of the Niemeier lattices N_i. Moreover, as G acts trivially on A_{Λ_G}, its action on Λ_G can be extended to an action of G on N_i which is trivial on $\Lambda_G^\perp \subset N_i$, see [19, Thm. 1.6.1, Cor. 1.5.2]. In Kondō's approach $\Lambda_G \oplus A_1(-1)$ is embedded into a Niemeier lattice N_i. This excludes N_i from being the Leech lattice N which does not contain any (-2)-class. So, under the additional assumption (2.1), the group G can be realized as a subgroup of $\mathrm{O}(N_i)$ of a certain Niemeier lattice different from the Leech lattice. If N_i is the Niemeier lattice with root lattice $A_1(-1)^{\oplus 24}$, this eventually leads to an embedding $G \hookrightarrow M_{24}$. Recall that in this case $\mathrm{O}(N_i) \simeq M_{24} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\oplus 24}$. The $(\mathbb{Z}/2\mathbb{Z})^{\oplus 24}$ is avoided by G, as Λ_G does not contain any (-2)-classes. Indeed, if $g(e_i) = -e_i$ for some $g \in G$ and a root e_i, then e_i would be orthogonal to Λ^G and hence contained in Λ_G. However, if N_i is the Leech lattice N, then one only gets an embedding into the much larger Conway group Co_0

$$G \hookrightarrow \mathrm{O}(N) =: Co_0.$$

Note that in both cases, the invariant lattice N_i^G satisfies $\operatorname{rk} N_i^G \geq 4$.

By a clever twist of the argument, the authors of [7] manage to prove the existence of an embedding into the Leech lattice only assuming the weak inequality in (2.1) which holds due to Lemma 2.1.

Proposition 2.2. (Gaberdiel, Hohenegger, Volpato) *For a group G the following conditions are equivalent:*
i) G is isomorphic to a subgroup of $\operatorname{Aut}(\Lambda, \Pi)$ for some positive four space $\Pi \subset \Lambda \otimes \mathbb{R}$ without (-2)-class contained in $\Pi^\perp$.
ii) G is isomorphic to a subgroup of the Conway group Co_0 with invariant lattice of rank at least four.

For completeness sake, we sketch the argument. Section 2.2 shows that i) implies ii) and Section 2.3 deals with the converse. We follow the original [7] closely.

2.2.

We shall show that there exists a primitive embedding

$$\Lambda_G \hookrightarrow N$$

into the Leech lattice N inducing an inclusion

$$G \hookrightarrow Co_0$$

with $\operatorname{rk} N^G \geq 4$.

If $\ell(A_{\Lambda_G}) = 24 - \operatorname{rk} \Lambda_G$, then an embedding into a Niemeier lattice can be found if for odd prime p the p-Sylow group $(A_{\Lambda_G})_p$ of A_{Λ_G} satisfies the stronger inequality $\ell((A_{\Lambda_G})_p) < 24 - \operatorname{rk} \Lambda_G$ and for $p = 2$ the discriminant form (A_{Λ_G}, q) splits off (A_{A_1}, q), see [19, Thm 1.12.2]. Of course, if Λ_G splits off a summand $A_1(-1)$ both conditions are satisfied, but in general it seems difficult to check.

Instead, in [7] Nikulin's criterion is applied to $\Lambda'_G := \Lambda_G \oplus A_1(-1)$. This is of course inspired by Kondō's original argument, but unfortunately in the present situation one cannot hope for embeddings of Λ'_G into a Niemeier lattice. Instead, one obtains primitive embeddings

$$\Lambda_G \hookrightarrow \Lambda'_G \hookrightarrow \Gamma := E_8(-1)^{\oplus 3} \oplus U.$$

Note that Γ is the unique even unimodular lattice of signature $(1, 25)$ which is often denoted $\mathrm{II}_{1,25}$. As G acts trivially on A_{Λ_G}, the action on Λ_G can be extended to Γ by id on $\Lambda_G^\perp \subset \Gamma$. Note that then $\Gamma^G = \Lambda_G^\perp$ is a non-degenerate lattice of signature $(1, 25 - \operatorname{rk} \Lambda_G)$. In particular, $\Gamma^G \otimes \mathbb{R}$ intersects the positive cone $\mathcal{C} \subset \Gamma \otimes \mathbb{R}$. More precisely, $\Gamma^G \otimes \mathbb{R}$ intersects one of the chambers of $\mathcal{C}$, defined as usual by means of the set of all (-2)-classes $\Delta_\Gamma \subset \Gamma$. Indeed, otherwise there exists one (-2)-class $\delta \in \Gamma$ with $\Gamma^G \subset \delta^\perp$ which would yield the contradiction $\delta \in \Lambda_G$.

Next choose an isomorphism $\Gamma \simeq N \oplus U$, where N is the Leech lattice, and consider a standard generator of U as an isotropic vector $w \in \Gamma$ (the *Weyl vector*). Then the (-2)-classes δ with $(\delta.w) = 1$ are called *Leech roots*. The Weyl group $W \subset \mathrm{O}(\Gamma)$ is in fact generated by the reflections s_δ associated with Leech roots (see [6, Ch. 27]) or, equivalently, there exists one chamber $\mathcal{C}_0 \subset \mathcal{C}$ that is described by the condition $(\delta.\mathcal{C}_0) > 0$ for all Leech roots δ.

Thus, after applying elements of the Weyl group W to the embedding $\Lambda_G \hookrightarrow \Gamma$ if necessary, one can assume that the distinguished chamber $\mathcal{C}_0$ is fixed by G. Then G is contained in the subgroup $Co_\infty \subset \mathrm{O}(\Gamma)$ of all isometries fixing $\mathcal{C}_0$. The group Co_∞ is also known to fix the isotropic vector $w \in \Gamma$ (see [5]) and hence $w \in \Gamma^G$. One obtains a primitive embedding of $\Lambda_G \hookrightarrow N$ as the composition

$$\Lambda_G \hookrightarrow w^\perp \twoheadrightarrow w^\perp/\mathbb{Z}\cdot w \simeq N.$$

Finally, by using $G \subset Co_\infty \longrightarrow Co_0 = \mathrm{O}(N)$ (or by applying Nikulin's general result once more) one extends the action of G from Λ_G to N by the identity on $\Lambda_G^\perp \subset N$. Then $\Lambda_G^\perp \subset N^G$ (in fact, equality holds) and, by Lemma 2.1, we ensure $\operatorname{rk} N^G \geq 4$:

So we proved that i) implies ii).

2.3.

For the converse of Proposition 2.2, let $G \subset Co_0$ be a subgroup with $\operatorname{rk} N^G \geq 4$. One needs to show that then $G \subset \mathrm{Aut}(\Lambda, \Pi)$ for some positive four space $\Pi \subset \Lambda \otimes \mathbb{R}$ without (-2)-classes contained in $\Pi^\perp$. This is proved in [7] as follows:

Firstly, one shows the existence of a primitive embedding

$$N_G = (N^G)^\perp \hookrightarrow \Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}. \tag{2.2}$$

Such an embedding exists if there exists an orthogonal lattice, i.e. an even lattice M with signature $(4, 20 - \operatorname{rk} N_G)$ and discriminant form $(A_M, q_M) \simeq (A_{N_G}, -q_{N_G})$, see [19, Prop. 1.5.1]. But [19, Thm. 1.12.4] implies the existence of a primitive embedding $N^G \hookrightarrow E_8(-1) \oplus U^{\oplus \operatorname{rk} N_G - 4}$ and its orthogonal complement M has the required properties.

Secondly, as G acts trivially on $A_{N_G} \simeq A_{N^G}$, its action on N_G can be extended by id to Λ. The orthogonal $N_G^\perp \subset \Lambda$ has signature $(4, 20 - \operatorname{rk} N_G)$ and, therefore, $N_G^\perp \otimes \mathbb{R}$ contains a positive four space Π on which G acts trivially

Thirdly, N_G as a sublattice of the Leech lattice N does not contain any (-2)-classes and for generic choice of $\Pi \subset N_G^\perp \otimes \mathbb{R}$ neither does $\Pi^\perp$.

This concludes the proof of Proposition 2.2. $\square$

§3. Proof of Theorem 0.1

Combining the previous two sections one can now complete the proof of Theorem 0.1.

3.1.

The proof of one direction of Theorem 0.1 is easy. Indeed, if $G \subset \mathrm{Aut}_s(\mathrm{D^b}(X), \sigma)$ for some $\sigma \in \mathrm{Stab}^{\mathrm{o}}(X)$, then $G \subset \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}), Z)$ by Proposition 1.4. If we define $\Pi_{X,Z}$ as the positive four space

$$\Pi_{X,Z} := P_X \oplus P_Z = (H^{2,0} \oplus H^{0,2})(X,\mathbb{R}) \oplus \mathbb{R} \cdot \mathrm{Re}(Z) \oplus \mathbb{R} \cdot \mathrm{Im}(Z)$$

and view it as a subspace of $\Lambda \otimes \mathbb{R} \simeq \widetilde{H}(X,\mathbb{R})$, then

$$G \subset \mathrm{Aut}_s(\widetilde{H}(X,\mathbb{Z}), Z) \simeq \mathrm{Aut}(\Lambda, \Pi_{X,Z}).$$

Thus, the discussion in Section 2.2 and Proposition 2.2 apply and show that there exists an injection $G \hookrightarrow Co_1$ with invariant lattice of rank at least four.

Remark 3.1. Whenever Λ_G can be embedded into a Niemeier lattice N_i that is not the Leech lattice, then one can argue as in [15] and deduce the existence of an embedding $G \hookrightarrow M_{24}$. But unfortunately, the Leech lattice cannot be excluded and one really has to deal with Co_1. Concrete examples have been given in [7].

3.2.

For the proof of the converse, an additional problem has to be addressed that was not present in [7]: One needs to ensure that Π in Section 2.3 can be chosen of the form $\Pi_{X,Z}$ with X projective and $Z \in H^{1,1}(X,\mathbb{Z}) \otimes \mathbb{C}$.

To achieve this, fix an isomorphism $\Lambda \simeq \Lambda_0 \oplus U_0$, where we think of $\Lambda_0 = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ as the K3 lattice $H^2(Y,\mathbb{Z})$ and of $U_0 \simeq U$ as $(H^0 \oplus H^4)(Y,\mathbb{Z})$. For a subgroup $G \subset Co_0$ with $\mathrm{rk}\, N^G \geq 4$ choose a primitive embedding $N_G \hookrightarrow \Lambda$ as in (2.2).

Now, for an arbitrary positive definite primitive sublattice $L \subset N_G^\perp \subset \Lambda$ of rank four, the lattice $L \cap \Lambda_0$, which is the kernel of the projection $L \longrightarrow U_0$, is of rank at least two. Hence, there exists a positive sublattice $L_1 \subset L \cap \Lambda_0$ of rank two. We let $P_1 := L_1 \otimes \mathbb{R}$ be the associated positive plane. Due to the surjectivity of the period map, there exists a K3 surface X with a marking $H^2(X,\mathbb{Z}) \simeq \Lambda_0$ inducing $(H^{2,0} \oplus H^{0,2})(X,\mathbb{R}) \simeq P_1$. As the lattice $L_1^\perp \cap \Lambda_0$, which has signature $(1,19)$, is contained in $P_1^\perp \subset \Lambda_0 \otimes \mathbb{R}$, there exists a class $\alpha \in H^{1,1}(X,\mathbb{Z})$ with $(\alpha)^2 > 0$ and, therefore, X is projective. (In other words, any K3 surface of maximal Picard number 20 is projective.)

It remains to find a second positive plane $P_2 \subset P_1^\perp \cap (N_G^\perp \otimes \mathbb{R})$ such that $(P_1 \oplus P_2)^\perp$ does not contain any (-2)-class. In fact, any such P_2 contains elements with non-trivial H^0 component and is, therefore, of the form P_Z for some $Z = \exp(B + i\alpha)$ with $B, \alpha \in H^{1,1}(X,\mathbb{Z}) \otimes \mathbb{R}$. Thus, $\Pi := P_1 \oplus P_2$ is of the form $\Pi_{X,Z}$ as required.

In order to show the existence of P_2, observe that if the intersection of the usual period domain $Q \subset \mathbb{P}(L_1^\perp \otimes \mathbb{C})$ with $\mathbb{P}((L_1^\perp \cap N_G^\perp) \otimes \mathbb{C})$ is contained in the union of all hyperplanes $\delta^\perp$ orthogonal to some (-2)-class $\delta \in L_1^\perp$, then there exists in fact one $\delta \in L_1^\perp$ orthogonal to $N_G^\perp$. But this would imply that N_G contains a (-2)-class which is absurd for a sublattice of the Leech lattice.

This concludes the proof of Theorem 0.1. □

It might be worth pointing out, that the K3 surfaces constructed in the proof above have all maximal Picard number $\rho(X) = 20$. However, that any group that can be realized at all can also be realized on one of this type, can also be proved directly.

§4. Symplectic automorphisms of deformations of Hilbert schemes

Let $G \subset Co_0 = \mathrm{O}(N)$ be a subgroup with $\operatorname{rk} N^G \geq 4$ and choose as before a primitive embedding

$$N_G := (N^G)^\perp \hookrightarrow \Lambda$$

into the extended K3 lattice

$$\Lambda := E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}.$$

We also fix a decomposition $\Lambda = \Lambda_0 \oplus U_0$, into the K3 lattice $\Lambda_0 := E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ and a copy U_0 of U. By $N_G^\perp \subset \Lambda$ we denote the orthogonal complement of N_G in Λ. Then $N_G^\perp$ has $\operatorname{rk}(N_G^\perp) \geq 4$, more precisely it is a lattice of signature $(4,m)$, and the action of G on N_G can be extended by id on $N_G^\perp$ to an action of Λ.

In order that a given G can act on a deformation of $\operatorname{Hilb}^n(X)$ it needs to satisfy an additional condition:

($*$) The lattice $N_G^\perp$ contains a primitive positive definite lattice $L \subset N_G^\perp$ with

$$\ell(L) < \mathrm{rk}(L) = 3.$$

We can now state the theorem from the introduction in the following precise form.

Theorem 0.2. *If $G \subset Co_0$ with $\mathrm{rk}(N^G) \geq 4$ satisfies ($*$), then there exist an $n > 0$ and a projective irreducible symplectic variety Y deformation equivalent to $\mathrm{Hilb}^n(X)$ of a K3 surface X such that G is isomorphic to a subgroup of the group of symplectic automorphisms $\mathrm{Aut_s}(Y)$ of Y.*

Proof. Consider a primitive, positive definite lattice of rank three $L \subset N_G^\perp$. Choose bases of L and of its dual L^* such that the natural inclusion $i\colon L \hookrightarrow L^*$ is given by a diagonal matrix $\mathrm{diag}(a_1, a_2, a_3)$ with $a_i | a_{i+1}$. Then $\ell(L) < 3$ if and only of $a_1 = \pm 1$. So, by assumption we can assume that $L \subset N_G^\perp$ contains a vector $0 \neq v \in L$ which is primitive in the overlattice $L \subset L^*$.

Next, we shall apply [19, Thm. 1.14.4] twice to $L_1 := v^\perp \cap L$. The first time, to ensure that there exists a primitive embedding $L_1 \hookrightarrow \Lambda_0$, as $\ell(L_1) + 2 \leq 4 \leq \mathrm{rk}(\Lambda_0) - \mathrm{rk}(L_1) = 20$, and a second time to conclude that the induced embedding $L_1 \hookrightarrow \Lambda_0 \hookrightarrow \Lambda$ is unique up to $\mathrm{O}(\Lambda)$. Hence, the given embedding $L_1 \subset L \subset N_G^\perp \subset \Lambda$ can be modified by some $\varphi \in \mathrm{O}(\Lambda)$ such that $\varphi(L_1) \subset \Lambda_0$. By modifying the original embedding of N_G by φ, we may therefore assume that in fact $L_1 \subset \Lambda_0$.

Due to the surjectivity of the period map, there exists a K3 surface X and a marking $H^2(X, \mathbb{Z}) \simeq \Lambda_0$ such that $(H^{2,0} \oplus H^{0,2})(X) \simeq L_1 \otimes \mathbb{C}$. Note that X is automatically projective. We denote by $P_1 := L_1 \otimes \mathbb{R}$ the real positive plane associated to L_1. Now choose a generic real positive plane in $P_2 \subset P_1^\perp \cap (N_G^\perp \otimes \mathbb{R})$ with $v \in P_2$ and such that $(P_1 \oplus P_2)^\perp$ does not contain any (-2)-class. The latter is possible, because otherwise there would be a (-2)-class δ with $(\delta.v') = 0$ for any class $v' \in N_G^\perp \otimes \mathbb{R}$ in an open subset and, therefore, one in N_G, which is absurd. Any such P_2 contains elements with non-trivial H^0 component and is, therefore, spanned by the real and the imaginary part of some $Z = \exp(B + i\alpha)$ with $B, \alpha \in H^{1,1}(X, \mathbb{R})$.

As by construction there are no (-2)-classes in $\widetilde{H}^{1,1}(X, \mathbb{Z})$ orthogonal to Z, there exists a stability condition of the form $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{o}}(X)$. We may furthermore assume $Z(v) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Via the marking G can be seen a subgroup of the group $\mathrm{Aut}_s(\widetilde{H}(X, \mathbb{Z}), Z)$ of symplectic Hodge isometries fixing Z, for the period L_1 and the stability function Z are by construction both G-invariant.

Due to Theorem 0.1, G can thus be realized as a subgroup of $\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X),\sigma)$.

Claim: The stability condition σ is v-generic.

Suppose E is semistable with $v(E)=v$ and phase $\phi(E)\in(0,\pi]$ and suppose there exists a semistable subobject $F\hookrightarrow E$ in the heart of the stability condition with $\phi(F)=\phi(E)$. Decompose $v(F)$ as $v(F)=v_1+v_2$ according to the finite index inclusion

$$L\oplus L^{\perp}\subset\widetilde{H}(X,\mathbb{Z}),$$

i.e. $v_1\in L\otimes\mathbb{Q}$ and $v_2\in L^{\perp}\otimes\mathbb{Q}$. From $v(F)\in P_1^{\perp}$ and $L^{\perp}\subset P_1^{\perp}$, one concludes that v_1 is contained in $(L\otimes\mathbb{Q})\cap P_2$ which is spanned by v. Hence, $v_1=\lambda\cdot v$ for some $\lambda\in\mathbb{Q}$. Decomposing v_2 further as $v_2=v_2'+v_2''$ with $v_2'\in P_2\cap v^{\perp}$ and $v_2''\in(P_1\oplus P_2)^{\perp}$ allows one to write $Z(F)=\lambda\cdot Z(v)+Z(v_2')$. Now, $Z(F)\in\mathbb{R}_{>0}\cdot Z(v)$, as $\phi(F)=\phi(E)$, and hence $Z(v_2')\in\mathbb{R}\cdot Z(v)$. However, using the injectivity of $Z\colon P_2\hookrightarrow\mathbb{C}$ one finds $v_2'=0$ and, therefore, $Z(F)=\lambda\cdot Z(v)$ with $0<\lambda\leq 1$. On the other hand, the projection of $v(F)$ under $\Lambda\simeq\Lambda^*\longrightarrow L^*\subset L\otimes\mathbb{Q}$ is $v_1=\lambda\cdot v$. As by construction v is primitive in L^*, this implies $\lambda=1$. But then the semistable quotient E/F would have Mukai vector $-v_2=-v_2''$ which is annihilated by Z. This yields a contradiction unless $v_2=0$, in which case $F=E$.

Consider the moduli space $M_\sigma(v)$ of σ-stable objects with Mukai vector v and phase $\phi\in(0,\pi]$. Then we know by [2] that $M_\sigma(v)$ is a smooth projective variety birational to a moduli space of stable sheaves on X and therefore, due to [13], deformation equivalent to it and, eventually, also deformation equivalent to $\mathrm{Hilb}^n(X)$.

Any $\Phi\in G\subset\mathrm{Aut}_s(\mathrm{D}^{\mathrm{b}}(X),\sigma)$ fixes σ and v and, therefore, acts as an automorphism

$$\Phi_v\colon M_\sigma(v)\xrightarrow{\sim}M_\sigma(v).$$

This yields a homomorphism

$$G\longrightarrow\mathrm{Aut}(M_\sigma(v)),\,\Phi\longmapsto\Phi_v.$$

First one observes that all Φ_v are symplectic. For this it is enough to check that Φ_v preserves the natural symplectic structure on $M_\sigma(v)$. This can in fact be verified in one point, say $[E]\in M_\sigma(v)$. So one has to argue that if $\Phi\colon\mathrm{D}^{\mathrm{b}}(X)\xrightarrow{\sim}\mathrm{D}^{\mathrm{b}}(X)$ acts as id on $H^{2,0}(X)$, then the isomorphism $\mathrm{Ext}^1(E,E)\simeq\mathrm{Ext}^1(\Phi(E),\Phi(E))$ respects the natural pairing given by Serre duality, which is obvious. Second, the map $\Phi\longmapsto\Phi_v$ is injective, as it is compatible with the isomorphism $v^{\perp}\simeq H^2(M_\sigma(v),\mathbb{Z})$. (For simplicity, we assume here that $M_\sigma(v)$ is fine. Otherwise $M_\sigma(v)$

has to be twisted and the action of G on the twisted cohomology and hence on the moduli space itself is faithful.) $\square$

Remark 4.1. In [16], based on Markman's monodromy operators, Mongardi also finds a sufficient condition for a group $G \subset Co_0$ to act as a group of symplectic automorphisms on a variety deformation equivalent to a Hilbert scheme. Moreover, he shows that his condition is in fact equivalent to $(*)$. The methods in [16] should be powerful enough to eventually give a complete characterization of such groups, whereas it seems unlikely that one can obtain a necessary condition by our methods.

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