Advanced Studies in Pure Mathematics 66, 2015
Singularities in Geometry and Topology 2011
pp. 123–142

On $(4,3)$ line degenerated torus curves and torus decompositions

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Abstract.

Let $C=\{f=0\}$ be an affine plane curve. In this paper, we study $(4,3)$ line degenerations of torus curves. Line degenerations of torus curves are divided into two types which are called visible or invisible degenerations. We will show that there does not exist a $(4,3)$ line degenerated torus curve which has two types decompositions.

§1. Introduction

Let $\mathbb{P}^2$ be a complex projective plane with homogeneous coordinates $[X,Y,Z]$ and let $\mathbb{C}^2=\mathbb{P}^2\setminus\{Z=0\}$ be the affine space with affine coordinates $(x,y)=(X/Z,Y/Z)$. We study plane curves in $\mathbb{P}^2$ and $\mathbb{C}^2$. Let $\mathcal{M}(d)$ (resp. $\mathcal{M}^a(d)$) be the set of projective (resp. affine) plane curves of degree d.

For a given curve $C\in\mathcal{M}(d)$ or $\mathcal{M}^a(d)$, we are interested in the topological invariant which is called the Alexander polynomial of C and *torus decompositions.* To explain this, we recall several curves which are called *torus curves*, *quasi torus curves* and *line degeneration of torus curves.* Let p and q be positive integers such that $p>q\geq 2$.

Definition 1.1. We say that $C=\{f=0\}\in\mathcal{M}^a(d)$ *torus curve of type* (p,q) if f is written as $f=f_a^p+f_b^q$ where f_j is a polynomial in $\mathbb{C}[x,y]$ of degree j. Put $\mathcal{T}(p,q;d)$ as the set of (p,q) torus curves of degree d.

Definition 1.2. We say that $C=\{f=0\}\in\mathcal{M}^a(d)$ *quasi torus curve of type* (p,q) if there exist three polynomials f_a, f_b and f_c such that they do not have a common component and they satisfy the following

Received June 19, 2012.
Revised November 30, 2012.
2010 *Mathematics Subject Classification.* 14H20, 14H30, 14H45.
Key words and phrases. Torus curve, torus decomposition.

relation:

$$f_c^{pq} f = f_a^p + f_b^q \quad \text{in} \quad \mathbb{C}[x, y], \qquad \deg f_j = j.$$

Put $\mathcal{QT}(p,q;d)$ as the set of (p,q) quasi torus curves of degree d.

For a given curve $C \in \mathcal{M}^a(d)$, we say that C has *a torus decomposition* (resp. *quasi torus decomposition*) if C is in $\mathcal{T}(p,q;d)$ (resp. $\mathcal{QT}(p,q;d)$) for some (p,q).

Example 1.1. There is an interesting phenomenon. Let $Q = \{f = 0\} \in \mathcal{M}^a(4)$ be a three cuspidal quartic. Then Q has two torus and one quasi torus decompositions ([3]):

$$f = f_1^3 + f_2^2, \quad f = g_2^3 + g_3^2, \quad h_1^6 f = h_3^3 + h_5^2 \tag{1.1}$$

where $\deg f_i = i$, $\deg g_i = i$ and $\deg h_i = i$. Furthermore its *tangential Alexander polynomial* is $(t^2-t+1)^2$ ([5]). For other quartics, if there exist a torus decomposition, then it is unique and its tangential Alexander polynomial is $t^2 - t + 1$.

We will want to consider the relation between the number of torus decompositions and the degree of its Alexander polynomial. To study this, we consider whether there exist a plane curve which has several torus decompositions.

To construct two torus decompositions in Example 1.1, we used *line degenerations of torus curves*. Now we recall line degenerations of torus curves which are defined by M. Oka in [5].

Let $C = \{F = F_q^p + F_p^q = 0\} \in \mathcal{M}(pq)$ be a projective (p,q) torus curve. Suppose that F has the following form:

$$F(X,Y,Z) = Z^j\, G(X,Y,Z) \tag{1.2}$$

where $G(X,Y,Z)$ is a reduced homogeneous polynomial of degree $pq-j$. We call a curve $D = \{G = 0\}$ *a line degenerated torus curve of type* (p,q) *of order* j and the line $L_\infty = \{Z = 0\}$ *the limit line of the degeneration.* Put $\mathcal{LT}_j(p,q;d)$ as the set of line degenerated torus curves of type (p,q) of order j and $\mathcal{LT}(p,q)$ is the union of $\mathcal{LT}_j(p,q;d)$ with respect to j.

To state our theorem, we divide the situation (1.2) into two cases which are called *visible degenerations* and *invisible degenerations.* Put the integer $r_k := \max\{r \in \mathbb{Z} \mid Z^r \text{ divides } F_k\}$ for $k = p, q$.

Visible case. Suppose that $r_p \cdot r_q \neq 0$ and $qr_p \neq pr_q$. Then F_q and F_p are written as $F_q(X,Y,Z) = F'_{q-r_q}(X,Y,Z)Z^{r_q}$ and $F_p(X,Y,Z) = F'_{p-r_p}(X,Y,Z)Z^{r_p}$. Putting $j := \min\{qr_p, pr_q\}$, we can factor F as

$F(X,Y,Z) = Z^j G(X,Y,Z)$. Then G is written using F'_{p-r_p} and F'_{q-r_q} as

(1.3)

$$G(X,Y,Z) = \begin{cases} F'_{q-r_q}(X,Y,Z)^p + F'_{p-r_p}(X,Y,Z)^q Z^{qr_p - pr_q} & \text{if } j = pr_q, \\ F'_{q-r_q}(X,Y,Z)^p Z^{pr_q - qr_p} + F'_{p-r_p}(X,Y,Z)^q & \text{if } j = qr_p. \end{cases}$$

We call such a factorization *visible factorization* and D is called *a visible degeneration of* (p,q) *torus curve.* We denote the set of visible degenerations of order j by $\mathcal{LT}_j^V(p,q;pq-j)$ and the union $\cup_j \mathcal{LT}_j^V(p,q;pq-j)$ by $\mathcal{LT}^V(p,q)$.

Example 1.2. We give an example of a visible degeneration. We take $(p,q) = (4,3)$, $(r_4, r_3) = 1$, $F_3(X,Y,Z) = (X^2+Y^2+Z^2)Z$ and $F_4(X,Y,Z) = (X^3+Y^3+Z^3)Z$. Then the order is 3 and $F(X,Y,Z) = Z^3 G(X,Y,Z)$ where

$$G(X,Y,Z) = (X^2+Y^2+Z^2)^4 Z + (X^3+Y^3+Z^3)^3.$$

Thus we have $D = \{G = 0\} \in \mathcal{LT}_3^V(4,3;9)$ and $\operatorname{Sing} D = \{6E_6\}$.

Invisible case. Either $r_p = 0$ or $r_q = 0$ but F can be written as (1.2). Then D is called *an invisible degeneration of* (p,q) *torus curve.* In this case, write $F_p^q + F_q^p = \sum_{i=0}^{pq} C_i(X,Y)Z^i$. Then $C_i(X,Y) = 0$ for i is less than $j-1$ and therefore Z^j divides F. We denote the set of invisible degenerations of order j by $\mathcal{LT}_j^I(p,q;pq-j)$ and the union $\cup_j \mathcal{LT}_j^I(p,q;pq-j)$ by $\mathcal{LT}^I(p,q)$.

Example 1.3. We give an example of an invisible degeneration. We take $(p,q) = (4,3)$, $F_3(X,Y,Z) = 3YZ^2 + L^3$ and $F_4(X,Y,Z) = z^4 + 4YZ^2L + L^4$ where $L = x+y$. Then the order is 3 and $F(X,Y,Z) = F_3(X,Y,Z)^4 - F_4(X,Y,Z)^3 = Z^3 G(X,Y,Z)$ where

$$\begin{aligned} G(X,Y,Z) &= -Z^8 - 12YLZ^6 + \varphi_1(X,Y)Z^4 + \varphi_2(X,Y)z^2 + \varphi_3(X,Y), \\ \varphi_1(X,Y) &= 81y^4 - 3L^4 - 48y^2L^2, \\ \varphi_2(X,Y) &= 4yL^3(11y^2 - 6L^2), \\ \varphi_3(X,Y) &= 3L^6(2y^2 - L^2). \end{aligned}$$

Thus we have $D = \{G = 0\} \in \mathcal{LT}_4^I(4,3;8)$ and $\operatorname{Sing} D = \{4E_6, B_{4,6}\}$. where $B_{4,6}$ is defined as $u^4 + v^6 = 0$.

Using these terminologies, two torus decompositions of 3-cuspidal quartic in Example 1.1 are written as

$$f_1^3 + f_2^2 \in \mathcal{LT}_2^V(3,2;4), \quad g_2^3 + g_3^2 \in \mathcal{LT}_2^I(3,2;4).$$

This shows that Q is in the both space $\mathcal{LT}_2^V(3,2;4) \cap \mathcal{LT}_2^I(3,2;4)$.

In this paper, we consider whether such a phenomenon occur for the case $(p,q) = (4,3)$. By the definitions and simple calculations, $\mathcal{LT}^V(4,3)$ and $\mathcal{LT}^I(4,3)$ have following decompositions:

$$\mathcal{LT}^V(4,3) = \mathcal{LT}_3^V(4,3;9) \cup \mathcal{LT}_4^V(4,3;8) \cup \mathcal{LT}_6^V(4,3;6) \cup \mathcal{LT}_8^V(4,3;4).$$

$$\mathcal{LT}^I(4,3) = \bigcup_{j=1}^{6} \mathcal{LT}_i^I(4,3;12-j).$$

Hence we consider only the cases order 3, 4 and 6.

Theorem 1. Suppose that $D \in \mathcal{LT}(4,3)$ does not consist of lines.

(1) There exist $C \in \mathcal{LT}_3^V(4,3;9)$ and $D \in \mathcal{LT}_3^I(4,3;9)$ such that

$$\operatorname{Sing} C = \operatorname{Sing} D = \{6B_{4,3}, B_{6,3}\}.$$

(2) However it is not possible to find such C, D with $C = D$ that is to say

$$\mathcal{LT}^V(4,3) \cap \mathcal{LT}^I(4,3) = \emptyset.$$

To express singularities, we use the same notations as in [7], [2]. In particular, we use important class of singularities which is called Brieskorn–Pham singularities $B_{n,m}$ which is defied by $u^n + t^m = 0$ where $n, m \geq 2$. We also use the notations:

$$B_{n,m} \circ B_{r,s} \quad : \quad (u^n + t^m)(u^r + t^s) = 0, \qquad m/n < s/r$$

and $*^\infty$ which express singularities on the limit line L_∞.

§2. Preliminaries

2.1. Line degenerations

Let U be an open neighborhood of 0 in $\mathbb{C}$ and let $\{C_s \mid s \in U\}$ be an analytic family of irreducible curves of degree d which degenerates into $C_0 := D + j\,L_\infty$ $(1 \leq j < d)$ where D is an irreducible curve of degree $d-j$ and L_∞ is a line. We assume that there is a point $B \in L_\infty \setminus L_\infty \cap D$ such that $B \in C_s$ and the multiplicity of C_s at P is j for any non-zero $s \in U$. We call such a degeneration *a line degeneration of order* j and we call L_∞ *the limit line* of the degeneration and B is called *the base point* of the degeneration. In [5], M. Oka showed that there exists a canonical surjection:

$$\varphi : \pi_1(\mathbb{C}^2 \setminus D) \to \pi_1(\mathbb{C}^2 \setminus C_s), \quad s : \text{sufficiently small},$$

where $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ and as a corollary he showed the divisibility among the Alexander polynomials of a line degeneration family:

$$\Delta_{C_s}(t) \mid \Delta_{D_0}(t).$$

He also showed that a visible type of torus curve of type (p,q) can be expressed as a line degeneration of irreducible torus curves of degree pq. Hence the Alexander polynomial of visible degenerations are not trivial.

2.2. Local singularities of visible degenerations

Let D be a visible degeneration of type (p,q) which is defined as (1.3). We put two polynomials $F_a := F'_{q-r_q}$ and $F_b := F'_{p-r_p}$ and two plane curves $C_a := \{F_a = 0\}$ and $C_b := \{F_b = 0\}$. Using these notations, equations (1.3) is written as

$$(1.3') \quad G(X,Y,Z) = \begin{cases} F_a(X,Y,Z)^p + F_b(X,Y,Z)^q Z^{ap-bq} & \text{if } j = pr_q, \\ F_a(X,Y,Z)^p Z^{bq-ap} + F_b(X,Y,Z)^q & \text{if } j = qr_p. \end{cases}$$

A singular point $P \in D \cap \mathbb{C}^2$ is called *inner* if $P \in C_a \cap C_b$. Otherwise $P \in D$ is called *outer.*

It is known that the topological type of the non-degenerate germ (C,P) is determined by its Newton principal part and does not depend on the terms with higher degree ([4], [1]). Moreover inner singularities depend on the intersection multiplicity of C_a and C_b at P and topological types C_a and C_b at P.

In this section, we consider possibilities of local singularities of D. If $P \in D$ is an inner singularity, we denote the intersection multiplicity of C_a and C_b at P by ι. If $P \in C_i \cap L_\infty$, then we denote the intersection multiplicity of C_i and L_∞ by ι_i for $i = a, b$. By the same argument with Lemma 1 in [1] and Lemma 3 in [2], we have following.

Lemma 2.1. *Let* $D = \{G = 0\}$ *be a* (p,q) *visible degenerations which is defined as*

$$D: \quad G = F_a^p + F_b^q Z^k = 0, \quad k := ap - bq > 0.$$

Let P *be a singular point of* D. *Assume that both curves* C_a *and* C_b *are smooth at* P *if* P *is on* C_j *for* $j = a, b$. *Then a singularity of* D *at* P *is as the following:*

(1) *If* P *is an inner singularity, then* $(D,P) \sim B_{p\iota,q}$.
(2) *If* $P \in C_a \cap L_\infty \setminus C_b$, *then* $(D,P) \sim B_{\iota_a k,p}$ *if* $p \le k$ *and* $B_{\iota_a p,k}$ *if* $p > k$.

(3) *If $P \in C_a \cap C_b \cap L_\infty$ and $p < q + k$, then*

$$(D, P) \sim \begin{cases} B_{p, p\iota_a} & (k-p)\iota_a + q\iota_b = 0, \\ B_{\iota q + \iota_a k,\, p} & (k-p)\iota_a + q\iota_b > 0, \\ B_{\iota_a p - \iota_b q, k} \circ B_{\iota_b q, p-k} & (k-p)\iota_a + q\iota_b < 0. \end{cases}$$

Proof. We prove only for the case (3). If $(k-p)\iota_a + q\iota_b = 0$, then the assertion is clear. We assume $(k-p)\iota_a + q\iota_b > 0$. The Newton boundary of $G(x, z)$ is given as the left side of Fig. 1. Then we can take a suitable local coordinates (u, v) so that G is defined as $v^p + c\,(v - u^\iota)^q (v - u^{\iota_a})^k = 0$ where c is a non-zero constant. Then the Newton boundary of $G(u, v)$ has non-degenerate one face and hence $(D, P) \sim B_{\iota q + \iota_a k,\, p}$. If $(k-p)\iota_a + \iota_b q < 0$, then we have two non-degenerate faces in the Newton boundary (see the right side of Fig. 1). Hence we have the assertion.

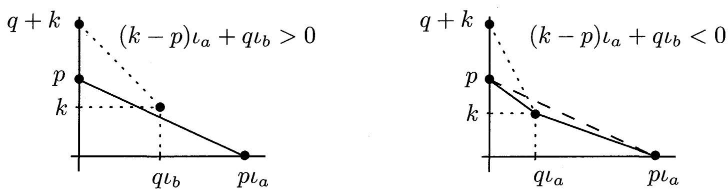

Fig. 1

Q.E.D.

2.3. Invisible degenerations

In this section, we consider invisible degenerations of order 1, 2 and 3 under assumptions p and q are relatively prime such that $2q > p$. Let $C_{p,q}$ be a (p, q) torus curve which is defined as $F(X, Y, Z) = F_q(X, Y, Z)^p - F_p(X, Y, Z)^q$ where

$$F_q(X, Y, Z) = \sum_{i=0}^{q} j_i(X, Y) Z^i, \quad F_p(X, Y, Z) = \sum_{j=0}^{p} k_j(X, Y) Z^j.$$

Here $j_i(X, Y)$ and $k_j(X, Y)$ are homogeneous polynomials in $\mathbb{C}[X, Y]$ of degree $q - i$ and $p - j$ respectively. Let $K(Z) = \sum_{i=0}^{m} a_i Z^i$ be an one variable polynomial. Using binomial theorem, we have

$$\text{Coeff}\,(K^n, 1) = a_0^n, \quad \text{Coeff}\,(K^n, Z) = n a_0^{n-1} a_1,$$
$$\text{Coeff}\,(K^n, Z^2) = n a_0^{n-2} \left(\frac{n-1}{2} a_1^2 + a_0 a_2 \right)$$

where Coeff $(K^n, *)$ is the coefficient of $*$ in the polynomial K^n. We regard F as a Z-variable polynomial and then we have

$$\text{Coeff}\,(F,1) = j_0^p - k_0^q, \quad \text{Coeff}\,(F,Z) = pj_0^{p-1}j_1 - qk_0^{q-1}k_1,$$
$$\text{Coeff}\,(F,Z^2) = pj_0^{p-2}\left(\frac{p-1}{2}j_1^2 + j_0j_2\right) - qk_0^{q-2}\left(\frac{q-1}{2}k_1^2 + k_0k_2\right).$$

First we consider the case that the order is 1, that is Z divide $F_q^p - F_p^q$, then we have $j_0^p = k_0^q$. As we assumed that p and q are relatively prime, there exist a linear form $\ell \in \mathbb{C}[X,Y]$ such that $j_0 = \ell^q$ and $k_0 = \ell^p$. Put $R := \{\ell = 0\} \cap L_\infty$. If coefficients of j_i and k_j are generic for i, $j \geq 1$, then, by simple calculations, C_p and C_q are smooth at R, the intersection multiplicity of C_p and C_q is q and the singularity of D at R is given by

$$(D,R) \sim B_{p(q-1),q-1}.$$

Next we consider the case that the order is 2, that is $Z^2 \mid F_q^p - F_p^q$, then we have $\ell^{p(q-1)}(qk_1 - pj_1\ell^{p-q}) = 0$. As $\ell \neq 0$, k_1 is written as $k_1 = \dfrac{p}{q}j_1\ell^{p-q}$. If coefficients of j_i and k_j are generic for i, $j \geq 1$, then C_q is smooth at R and C_q has $B_{p,2}$ singularity at R. Their intersection multiplicities at R is p and we have

$$(D,R) \sim B_{q(p-2),p-2}.$$

Finally assume that Z^3 divides $F_q^p - F_p^q$. Then Coeff (F_q^p, Z^2) must be equal to Coeff (F_p^q, Z^2) where

$$\text{Coeff}\,(F_q^p, Z^2) = p\ell(X,Y)^{q(p-2)}\left(\frac{p-1}{2}j_1(X,Y)^2 + \ell(X,Y)^q j_2(X,Y)\right)$$
$$\text{Coeff}\,(F_p^q, Z^2) = q\ell(X,Y)^{q(p-2)}\Big(\frac{p^2(q-1)}{2q^2}j_1(X,Y)^2 +$$
$$\ell(X,Y)^{2q-p}k_2(X,Y)\Big).$$

We solve the equation Coeff $(F_q^p, Z^2) =$ Coeff (F_p^q, Z^2). Then we have

$$\frac{p(p-q)}{2q}j_1(X,Y)^2 + p\ell(X,Y)^q j_2(X,Y) - q\ell(X,Y)^{2q-p}k_2(X,Y) = 0.$$

This implies j_1 must be divided by ℓ^s where $s = q - \left[\dfrac{p}{2}\right]$ and we put $j_1(X,Y) = \ell^s\tilde{j_1}(X,Y)$. Hence we have

$$k_{p-2}(X,Y) = \frac{p(p-q)}{2q^2}\ell(X,Y)^{\varepsilon}\tilde{j_1}(X,Y)^2 + \frac{p}{q}\ell(X,Y)^{p-q}j_2(X,Y)$$

where ε is 0 if p is even and 1 if p is odd. If p is even, then ε is 0 and $(C_q, R) \sim B_{q-1,1} \circ B_{1,1}$ and $(C_p, R) \sim B_{p,2}$. The intersection multiplicity is $p+2$ and

$$(D, R) \sim B_{p(q-\frac{3}{2}),q}$$

§3. Proof of Theorem 1

To prove Theorem 1, we take following steps:

- Classify possibilities of singularities of invisible degenerations of order 3, 4 and 6.
- Classify possibilities of singularities of visible degenerations of order 3, 4 and 6.
- Compare with singularities which are classified by the above 2 steps.
- If there is a pair such that they have the same configurations of singularities, then we consider that whether these curves are the same or not.

3.1. Singularities of invisible degenerations of order 3, 4 and 6

Let $D = \{G = 0\}$ be a $(4, 3)$ invisible degenerations of order j. The defining polynomial G satisfies the relation $Z^j G = F_4^3 - F_3^4$. In this section, we study singularities of D. By the argument in §2.3, there is a linear form ℓ such that C_4 and C_3 intersect with L_∞ at $\{\ell = 0\} \cap L_\infty$. We denote the intersection point by R and the intersection multiplicity $I(C_3, C_4; R)$ by ι.

3.1.1. Order is 3 Suppose that $D = \{G = 0\}$ is in $\mathcal{LT}_3^I(4, 3; 9)$. First we consider possibilities of singularity of D on L_∞. By the argument in §2.3, F_3 and F_4 are written as

$$\begin{aligned} F_3(X,Y,Z) &= \ell(X,Y)^3 + \ell(X,Y)\ell_1(X,Y)Z + \ell_2(X,Y)Z^2 + b\,Z^3 \\ F_4(X,Y,Z) &= \ell(X,Y)^4 + \frac{4}{3}\ell(X,Y)^2\ell_1(X,Y)Z \\ &\quad + \left(\frac{2}{9}\ell_1(X,Y)^2 + \frac{4}{3}\ell(X,Y)\ell_2(X,Y)\right) Z^2 \\ &\qquad + \ell_3(X,Y)\,Z^3 + a\,Z^4 \end{aligned}$$

where ℓ_1, ℓ_2 and ℓ_3 are suitable linear forms. We showed in §2.3 the following:

- $C_3 \cap L_\infty = C_4 \cap L_\infty = \{R\}$.
- $(C_3, R) \sim A_1$ and $(C_4, R) \sim A_3$.

- If coefficients of F_3 and F_4 are generic, then $\iota = 6$ and $(D, R) \sim B_{6,3}$.

Now we consider degenerations of the singularity of D at R.

Assume that the Newton boundary of D at R is degenerate. Then doing similar arguments in §2.3, we can show that $(C_3, R) \sim A_2$, $(C_4, R) \sim B_{3,2} \circ B_{1,1}$ and

$$(D, R) \sim B_{2,1} \circ B_{6,4}, \quad \iota = 8.$$

Moreover we assume that its second face which corresponds to $B_{6,4}$ is degenerate, then C_3 consists of three lines and $(C_4, R) \sim B_{4,3}$ and

$$(D, R) \sim B_{8,6}, \quad \iota = 9.$$

Finally assume that the face of its Newton boundary is degenerate. Then C_4 also consists of four lines and D consists of nine lines. Hence we have

$$(D, R) \sim B_{9,9}, \quad \iota = 12.$$

Lemma 3.1. *Under the above notations, configurations of singularities of D is one of the following.*

(1) *If $\iota = 6$, then we have*

$$\{B_{4n,3}, (6-n)B_{4,3}, B_{6,3}^{\infty}\} \ (n = 1, 2, 3, 4), \quad \{B_{6,4}, 4B_{4,3}, B_{6,3}^{\infty}\}.$$

(2) *If $\iota = 8$, then we have*

$$\{B_{4n,3}, (4-n)B_{4,3}, (B_{2,1} \circ B_{6,4})^{\infty}\} \ (n = 1, 2, 3).$$

(3) *If $\iota = 9$, then we have*

$$\{B_{4n,3}, (3-n)B_{4,3}, B_{8,6}^{\infty}\} \ (n = 1, 2, 3), \ \{B_{6,4}, 2B_{4,3}, B_{8,6}^{\infty}\}.$$

(4) *If $\iota = 12$, then we have $\{B_{9,9}^{\infty}\}$.*

Proof. First we note that if P is an inner singularity of D, then either C_3 or C_4 is smooth at P. Indeed, if both curves are singular at P and we may assume that $P = O$, then, by the form of the defining polynomials, we have $a = b = \ell_2 = \ell_3 = 0$ and hence ℓ_1^3 divides G. Thus G is a non-reducible curve. As we consider only reducible curves, this is a contradiction. Therefore we consider only the cases either C_3 or C_4 is smooth in the affine space $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$.

We assume that both curves C_3 and C_4 are smooth at P with the intersection multiplicity 1. Then D has $B_{4,3}$ singularity at P. If $\iota = k$, then C_3 generically intersects with C_4 at distinct $12 - k$ points in $\mathbb{C}^2$

since $C_3 \cap C_4 \cap L_\infty = \{R\}$. Hence generic configurations of singularities are as follows:

$$\operatorname{Sing} D = \begin{cases} \{6B_{4,3}, B_{6,3}^{\infty}\} & \text{if } \iota = 6, \\ \{4B_{4,3}, (B_{2,1} \circ B_{6,4})^{\infty}\} & \text{if } \iota = 8, \\ \{3B_{4,3}, B_{8,6}^{\infty}\} & \text{if } \iota = 9, \\ \{B_{9,9}^{\infty}\} & \text{if } \iota = 12. \end{cases}$$

Existence of other configurations of singularities is shown by simple computations. Suppose $\iota = 6$. Singularity $B_{4n,3}$ appear as an inner singularity with intersection multiplicity n and both curves are smooth at the intersection point. If $n \geq 5$, then we can show that ℓ divide F_4 and F_3. Then G is non-reduced and hence the cases $n \geq 5$ do not occur. Singularity $B_{6,4}$ also appear as an inner singularity under the assumptions C_3 is smooth, C_4 has A_1 or A_2 singularity at the intersection point with the intersection multiplicity is 2. We can also show that the case its intersection multiplicity is greater than 2 do not occur. Doing the same arguments for other cases, we have the assertions. Q.E.D.

3.1.2. Order is 4 Suppose that $D = \{G = 0\}$ is in $\mathcal{LT}_4^I(4,3;8)$. By the same argument in §2.4, F_3 and F_4 are written as

$$F_3(X,Y,Z) = \ell(X,Y)^3 + \frac{1}{2}\ell(X,Y)^2 Z + \ell_1(X,Y)Z^2 + b\,Z^3$$
$$F_4(X,Y,Z) = \ell(X,Y)^4 + \frac{2}{3}\ell(X,Y)^3 Z + \left(\frac{1}{18}\ell(X,Y) + \frac{4}{3}\ell_1(X,Y)\right)\ell Z^2$$
$$+ \left(\left(\frac{4}{3}b - \frac{1}{162}\right)\ell(X,Y) + \frac{2}{9}\ell_1(X,Y)\right) Z^3 + a\,Z^4$$

where ℓ_1 is a linear form. Under this situation, they satisfy the following:

- $C_3 \cap L_\infty = C_4 \cap L_\infty = \{R\}$.
- $(C_3, R) \sim A_2$ and $(C_4, R) \sim B_{3,2} \circ B_{1,1}$.
- If coefficients of F_3 and F_4 are generic, then $\iota = 8$ and $(D, R) \sim B_{6,4}$.

By the same argument of the case order 3, we have two configurations global singularities of D:

$$\{4B_{4,3}, B_{6,4}^{\infty}\}, \quad \{B_{8,8}^{\infty}\}.$$

3.1.3. Order is 6 Suppose that $D = \{G = 0\}$ is in $\mathcal{LT}_6^I(4,3;6)$. By the same argument in §2.3, F_3 and F_4 are written as

$$F_3(X,Y,Z) = \ell(X,Y)^3 + \frac{1}{2}\ell(X,Y)^2 Z + \frac{1}{12}\ell(X,Y)Z^2 + b\,Z^3$$

$$F_4(X,Y,Z) = \ell(X,Y)^4 + \frac{2}{3}\ell(X,Y)^3 Z + \frac{1}{6}\ell(X,Y)^2 Z^2 + \left(\frac{4}{3}b + \frac{1}{81}\right)\ell(X,Y)\,Z^3 + a\,Z^4$$

where $a = -\frac{1}{3888} + \frac{2}{9}b$. Thus C_3 consists of three lines and C_4 consists of four lines. They intersect at R with intersection multiplicity 12. Hence D has one singularity at R and $(D,R) \sim B_{6,6}$.

3.2. Singularities of visible degenerations of order 3, 4 and 6

Let $D = \{G = 0\}$ be a $(4,3)$ visible degeneration of order j for $j = 3,4,6$. Then the defining polynomial G has one of the following form:

(1) If the order is 3:

$$① \; G = F_3^3 + G_2^4 Z, \qquad ② \; G = F_3^3 + G_1^4 Z^5.$$

(2) If the order is 4:

$$③ \; G = F_2^3 Z^2 + G_2^4, \qquad ④ \; G = F_1^3 Z^5 + G_2^4.$$

(3) If the order is 6:

$$⑤ \; G = F_2^3 + G_1^4 Z^2.$$

We will classify local singularities for above 5 cases. To classify, we refer to the method of Pho and Oka in [7], [6]. We omit the proof of Lemma $4,\dots,9$ as our proof are mainly computational and they are done by a computer program "Maple". Let P be a singularity of D and put $C_i := \{F_i = 0\}$ $D_j := \{G_j = 0\}$.

3.2.1. Local singularities of the case ①. We divide our considerations as follows:

(i) C_3 is smooth at P.
(ii) C_3 has A_1 singularity at P.
(iii) C_3 has A_2 singularity at P.
(iv) C_3 consists of a smooth conic and a line such that the line is tangent to the conic at P. That is C_3 has A_3 singularity at P.
(v) C_3 has multiplicity 3 at P.

Moreover each case has 3 subcases:

(1) C_2 is smooth at P.
(2) C_2 consists of distinct two lines and they intersect at P. That is C_2 has A_1 singularity at P.
(3) C_2 is a line with multiplicity 2.

First assume that P is in affine space $\mathbb{C}^2$. Put $\iota^a := I(D_2, C_3; P)$.

Lemma 3.2. *Under the above notations, we have the following.*

(i) *If C_3 is smooth at P, then $(D,P) \sim B_{4\iota^a,3}$ for $\iota^a = 1, \dots, 6$.*
(ii) *Assume that C_3 has A_1 singularity at P.*
 (ii-1) *If D_2 is smooth at P, then*

$$(D,P) \sim \begin{cases} B_{3\iota^a,4} & \iota^a = 2,3,4 \\ B_{3,1} \circ B_{4\iota^a-7,3} & \iota^a = 5,6. \end{cases}$$

 (ii-2) *If D_2 has A_1 singularity at P, then $(D,P) \sim B_{4\iota^a-11,3}$ $\circ B_{3,5}$ for $\iota^a = 4, 5$.*
 (ii-3) *If D_2 is a line with multiplicity 2, then $(D,P) \sim B_{4\iota^a-11,3}$ $\circ B_{3,5}$ for $\iota^a = 4, 6$.*
(iii) *Assume that C_3 has A_2 singularity at P.*
 (iii-1) *If D_2 is smooth at P, then $B_{3\iota^a,4}$ for $\iota^a = 2, 3$.*
 (iii-2) *If D_2 has A_1 singularity at P, then*

$$(D,P) \sim \begin{cases} B_{8,6} & \iota^a = 4 \\ (B_{3,2}^3)^{B_{3,2}} & \iota^a = 5. \end{cases}$$

 (iii-3) *If D_2 is a line with multiplicity 2, then*
 - *If $\iota^a = 4$, then $(D,P) \sim B_{8,6}$.*
 - *If $\iota^a = 6$, then $(D,P) \sim (B_{3,2}^3)^{B_{3,6}}$.*

(iv) *Assume that C_3 has A_3 singularity at P.*
 (iv-1) *If D_2 is smooth at P, then $(D,P) \sim B_{3\iota^a,4}$ for $\iota^a = 2, \dots, 6$.*
 (iv-2) *If D_2 has A_1 singularity at P, then $(D,P) \sim B_{8,6}$ for $\iota^a = 4$.*
 (iv-3) *If D_2 is a line with multiplicity 2, then $(D,P) \sim B_{8,6}$ for $\iota^a = 4$.*
(v) *Assume that C_3 has multiplicity 3 at P.*
 (v-1) *If D_2 is smooth at P, then $(D,P) \sim B_{3\iota^a,4}$ for $\iota^a = 3, \dots, 6$.*
 (v-2) *If D_2 has A_1 singularity at P, then $(D,P) \sim B_{5,4} \circ B_{4,5}$ for $\iota^a = 6$.*

(v-3) *If D_2 is a line with multiplicity 2, then $(D, P) \sim B_{9,8}$ for $\iota^a = 6$.*

Next we assume that P is in $C_3 \cap L_\infty \setminus D_2$. By the form of the defining polynomial of D, D is smooth at P and intersects L_∞ with intersection multiplicity $3\iota_3$ where $\iota_3 = I(C_3, L_\infty; P)$. Hence P is a flex point of D.

Next we assume that P is in $D_2 \cap C_3 \cap L_\infty$. We may assume that P is $[0 : 1 : 0]$. We consider combinations of the intersection multiplicities $(\iota_2, \iota_3, \iota)$ where $\iota_2 = I(D_2, L_\infty; P)$ and $\iota = I(D_2, C_3; P)$. For example, we consider the case that C_3 has A_1 singularity at P, D_2 is smooth at P and $(\iota_2, \iota_3, \iota) = (1, 2, 2)$. Let (x, z) be local coordinates at P which are obtained as $(x, z) = (X/Y, Z/Y)$ and let $g_2(x, z)$ and $f_3(x, z)$ be defining polynomials of D_2 and C_3:

$$g_2(x, z) = a_{10}x + a_{01}z + a_{20}x^2 + a_{11}xz + a_{02}z^2,$$
$$f_3(x, z) = (z - ax)(z - bx) + b_{30}x^3 + b_{21}x^2z + b_{12}xz^2 + b_{03}z^3$$

where $a \neq b$ as $(C_3, P) \sim A_1$. In this coordinate, the limit line L_∞ is defined as $\{z = 0\}$. The condition $(\iota_2, \iota_3, \iota) = (1, 2, 2)$ is equivalent to $a_{10} \neq 0$, $ab \neq 0$ and $(a_{10}+aa_{01})(a_{10}+ba_{01}) \neq 0$. Under these conditions, the Newton boundary of $g(x, z)$ consists of two faces Δ_1 and Δ_2. Each face function is defined as

$$g_{\Delta_1}(x, z) = (a_{10}x + a_{01}z)^4 z, \quad g_{\Delta_2}(x, z) = (a_{10}^4 z + (ab)^3 x^2)x^4.$$

As the first face is degenerate, we take new local coordinates $(x, z_1) = (x, a_{10}x + a_{01}z)$. Then the Newton principal part of $g(x, z_1)$ is given as

$$a_{01}^4 z_1^5 - a_{01}^3 a_{10} x z_1^4 + \frac{(a_{10} + aa_{01})^3 (a_{10} + ba_{01})^3}{a_{01}^6} x^6.$$

Hence we have a non-degenerate singularity of type $B_{5,4} \circ B_{1,1}$.

Thus to obtain singularities, we consider local geometries of D_2 and C_3 at P and all combinations of intersection multiplicities $(\iota_2, \iota_3, \iota)$. There are 36 combinations but all combinations do not exist. For example, if both curves are smooth at P and ι_3, $\iota > 1$, then the case $(1, \iota_3, \iota)$ does not exist. Assumption $\iota_3 > 1$ means that L_∞ is the tangent line of C_3 at P and assumption $\iota > 1$ means that C_3 is tangent to C_2 at P. This is a contradiction to $\iota_2 = 1$.

Lemma 3.3. *Under the above notations, we have the following.*

(1) *Suppose that D_2 is smooth at P and intersects transversely. That is $\iota_2 = 1$.*

(a) *Assume that* C_3 *is smooth at* P.
- *If* $(\iota_3, \iota) = (1, \iota)$, *then* $(D, P) \sim B_{4\iota+1,3}$ *for* $\iota = 2, \dots, 6$.
- *If* $(\iota_3, \iota) = (2, 1)$, *then* $(D, P) \sim B_{6,3}$.
- *If* $(\iota_3, \iota) = (3, 1)$, *then* $(D, P) \sim B_{5,1} \circ B_{4,2}$.

(b) *Assume that* C_3 *has* A_1 *singularity at* P. *Then* $\iota_3 = 2$ *or* 3.
- *For* $\iota = 2, 3$, $(D, P) \sim B_{3\iota-1,4} \circ B_{1,1}$.
- *For* $\iota = 4, 5, 6$, $(D, P) \sim B_{4\iota-6,3} \circ B_{2,1}$.

(c) *Assume that* C_3 *has* A_2 *singularity at* P.
- *If* $(\iota_3, \iota) = (2, \iota)$, *then* $\iota = 2, 3$ *and* $(D, P) \sim B_{3\iota-1,4} \circ B_{1,1}$.
- *If* $(\iota_3, \iota) = (1, \iota)$, *then* $\iota = 2$ *and* $(D, P) \sim B_{5,4} \circ B_{1,1}$.

(d) *If* C_3 *has* A_3 *singularity at* P, *then* $(D, P) \sim B_{3\iota-1,4} \circ B_{1,1}$ *for* $\iota = 2, 4, 5, 6$ *and* $\iota_3 = 2$. *The case* $\iota_3 = 3$ *does not occur.*

(e) *If* C_3 *is three lines, then* $\iota_3 = 3$ *and* $(D, P) \sim B_{3\iota-1,4} \circ B_{1,1}$.

(2) *Suppose that* D_2 *is smooth at* P *and tangent to* L_∞. *In this case,* $\iota_2 = 2$.

(a) *Assume that* C_3 *is smooth at* P.
- *If* $\iota_3 = 1$, *then* $\iota = 1$ *and* $(D, P) \sim B_{5,3}$.
- *If* $\iota_3 = 2$, *then* $(D, P) \sim B_{4\iota+2,3}$ *for* $\iota = 2, \dots, 6$.
- *If* $\iota_3 = 3$, *then* $\iota = 1$ *and* $(D, P) \sim B_{7,3}$.

(b) *Assume that* C_3 *has* A_1 *singularity at* P.
- *If* $\iota_3 = 2$, *then* $\iota = 2$ *and* $(D, P) \sim B_{6,5}$.
- *If* $\iota_3 = 3$, *then* $(D, P) \sim B_{4\iota-5,3} \circ B_{3,2}$ *for* $\iota = 3, \dots, 6$.

(c) *Assume that* C_3 *has* A_2 *singularity at* P, *then*
- *If* $\iota_3 = 2$, *then* $\iota = 2$ *and* $(D, P) \sim B_{6,5}$.
- *If* $\iota_3 = 3$, *then* $\iota = 3$ *and* $(D, P) \sim B_{9,5}$.

(d) *If* C_3 *has* A_3 *singularity at* R, *then* $\iota_3 = \iota = 2$ *and* $(D, P) \sim B_{6,5}$.

(e) *If* C_3 *consists of three lines, then* $(D, P) \sim B_{9,5}$.

(3) *Suppose that* D_2 *is distinct two lines at* P. *In this case,* ι_2 *is also* 2.

(a) *Assume that* C_3 *is smooth at* P.
- *If* $(\iota_3, \iota) = (1, \iota)$, *then* $(D, P) \sim B_{4\iota+1,3}$ *for* $\iota = 2, \dots, 4$.
- *If* $\iota_3 = 2$, *then* $\iota = 2$ *and* $(D, P) \sim B_{10,3}$.
- *If* $\iota_3 = 2$, *then* $\iota = 2$, *then* $(D, P) \sim B_{11,3}$.

(b) *Assume that C_3 has A_1 singularity at P.*
- *If $\iota_3 = 2$, then $\iota = 4, 5, 6$ and*

$$(D,P) \sim \begin{cases} (B^3_{2,2})^{2B_{3,3}} & (\iota = 4) \\ (B^3_{2,2})^{B_{7,3}+B_{3,3}} & (\iota = 5) \\ (B^3_{2,2})^{2B_{7,3}} & (\iota = 6). \end{cases}$$

- *If $\iota_3 = 3$, then $(D,P) \sim B_{7,3} \circ B_{3,4\iota-10}$ for $\iota = 4, 5$.*

(c) *Assume that C_3 has A_2 singularity at P.*
- *If $\iota_3 = 2$, then $\iota = 4$ or 5 and $(D,P) \sim B_{9,6}$ or $(B^3_{2,3})^{B_{4,3}}$ respectively.*
- *If $\iota_3 = 3$, then $\iota = 4$ and $(B^3_{3,2})^{B_{3,1}}$.*

(d) *If C_3 has A_3 singularity at R, then $(D,P) \sim B_{9,6}$ for $\iota_3 = 2$ and $\iota = 4$.*

(e) *If C_3 consists of three lines, then $(D,P) \sim B_{9,9}$.*

(4) *Suppose that D_2 consists of a line with multiplicity 2 ($\iota_2 = 2$).*

(a) *Assume that C_3 is smooth at P.*
- *If $\iota_3 = 1$, then $(D,R) \sim B_{4\iota+1,3}$ for $\iota = 2, 4, 6$.*
- *If $\iota_3 = 2, 3$, then $\iota = 2$ and $(D,R) \sim B_{10,3}$ ($\iota_3 = 2$) or $B_{11,3}$ ($\iota_3 = 3$) respectively.*

(b) *Assume that C_3 has A_1 singularity at P.*
- *If $\iota_3 = 2$, then $\iota = 4$ or 6 and $(D,P) \sim B_{6,3} \circ B_{3,6}$ or $B_{14,3} \circ B_{3,6}$ respectively.*
- *If $\iota_3 = 3$, then $\iota = 4$ or 6 and $(D,P) \sim B_{7,3} \circ B_{3,6}$ or $B_{7,3} \circ B_{3,14}$ respectively.*

(c) *Assume that C_3 has A_2 singularity at P.*
- *If $\iota_3 = 2$, then $\iota = 4$ or 6 and $(D,P) \sim B_{9,6}$ or $(B^3_{2,3})^{B_{8,3}}$ respectively.*
- *If $\iota_3 = 3$, then $\iota = 4$ and $(D,P) \sim (B^3_{3,2})^{B_{3,1}}$.*

(d) *If C_3 is a line and conic and has A_3 singularity at R, then $(D,P) \sim B_{9,6}$ for $\iota_3 = 2$ and $\iota = 4$.*

(e) *If C_3 consists of distinct three lines, then $\iota = 6$ and $(D,P) \sim B_{9,9}$.*

3.2.2. Local singularities of the case ② First assume that P is in affine space $\mathbb{C}^2$. By the same argument in the case ①, we have the following local singularities.

	C_3 is smooth	A_1	A_2	A_3	3 lines
$\iota = 1$	$B_{4,3}$	—	—	—	—
$\iota = 2$	$B_{8,3}$	$B_{6,4}$	$B_{6,4}$	$B_{6,4}$	—
$\iota = 3$	$B_{12,3}$	$B_{9,4}$	$B_{9,4}$	—	$B_{9,4}$

Next we consider the case $P \in L_\infty$. We divided this situation into 2 cases:

$$P \in C_3 \cap L_\infty \setminus D_1 \quad \text{or} \quad C_3 \cap D_1 \cap L_\infty$$

For the later case, ι_1 is always 1 and we consider geometry of C_3 at P and all combinations of intersection multiplicities $(\iota_1, \iota_3, \iota) = (1, \iota_3, \iota)$.

Lemma 3.4. *Under the above notations, we have the following singularities.*

(1) *Suppose that $P \in C_3 \cap L_\infty \setminus D_1$.*
  (a) *If C_3 is smooth at P, then $(D, P) \sim B_{5\iota,3}$ for $\iota_3 = 1, 2, 3$.*
  (b) *Assume that C_3 has A_1 singularity at P.*
    (i) *If $\iota_3 = 2$, then $(D, P) \sim B_{6,5}$.*
    (ii) *If $\iota_3 = 3$, then $(D, P) \sim B_{7,3} \circ B_{3,2}$.*
  (c) *Assume that C_3 has A_2 singularity at P.*
    (i) *If $\iota_3 = 2$, then $(D, P) \sim B_{6,5}$.*
    (ii) *If $\iota_3 = 3$, then $(D, P) \sim B_{9,5}$.*
  (d) *If C_3 has A_3 singularity at P, then $(D, P) \sim B_{6,5}$.*
  (e) *If C_3 consists of three lines, then $(D, P) \sim B_{9,9}$.*

(2) *Suppose that $P \in C_3 \cap D_1 \cap L_\infty$.*
  (a) *Assume that C_3 is smooth at P.*
    (i) *If $\iota = 1$, then $(D, P) \sim B_{3,5\iota_3+4}$.*
    (ii) *If $\iota_3 = 1$, then $(D, P) \sim B_{3,4\iota+5}$.*
  (b) *Assume that C_3 has A_1 singularity at P.*
    (i) *If $\iota = \iota_3 = 2$, then $(D, P) \sim (B^3_{2,2})^{2B_{3,3}}$.*
    (ii) *If $\iota = 2$ and $\iota_\infty = 3$, then $(D, P) \sim B_{11,2} \circ B_{3,6}$.*
    (iii) *If $\iota = 3$ and $\iota_\infty = 2$, then $(D, P) \sim B_{6,3} \circ B_{3,10}$.*
  (c) *Assume that C_3 has A_2 singularity at P.*
    (i) *If $\iota = \iota_3 = 2$, then $(D, P) \sim B_{9,6}$.*
    (ii) *If $\iota = 2$ and $\iota_\infty = 3$, then $(D, P) \sim (B^3_{3,2})^{B_{5,3}}$.*
    (iii) *If $\iota = 3$ and $\iota_\infty = 2$, then $(D, P) \sim (B^3_{3,2})^{B_{4,3}}$.*
  (d) *If C_3 has A_3 singularity at P, then $(D, P) \sim B_{9,6}$.*
  (e) *If C_3 consists of three lines, then $(D, P) \sim B_{9,9}$.*

3.2.3. Local singularities of the case ③ For the case that C_2 and D_2 are smooth is already classified by Lemma 1 and 2. Hence we assume that C_2 or D_2 consists of two lines at P.

Lemma 3.5. *Under the above assumptions, we have the following.*

(1) *Suppose that $P \in C_2 \cap D_2 \setminus L_\infty$.*
  (a) *Assume that D_2 is smooth at P and C_2 consists of two lines ℓ_1 and ℓ_2 such that $P \in \ell_1 \cap \ell_2$.*
    (i) *If $\ell_1 \neq \ell_2$, then $(D, P) \sim B_{3\iota 4}$ for $\iota = 2, 3$.*
    (ii) *If $\ell_1 = \ell_2$, then $(D, P) \sim B_{3\iota 4}$ for $\iota = 2, 4$.*

(b) *Assume that C_2 is smooth at P and D_2 consists of two lines L_1 and L_2 such that $P \in L_1 \cap L_2$.*
(i) *If $L_1 \neq L_2$, then $(D,P) \sim B_{4\iota 3}$ for $\iota = 2,3$.*
(ii) *If $L_1 = L_2$, then $(D,P) \sim B_{4\iota 3}$ for $\iota = 2,4$.*
(c) *If C_2, D_2 consist of two lines, then $(D,P) \sim (B^3_{2,2})^{2B_{3,2}}$.*

(2) *Suppose that $P \in D_2 \cap L_\infty \setminus C_2$ and D_2 consists of two lines L_1 and L_2 such that $P \in L_1 \cap L_2$. Then $(D,P) \sim B_{8,2}$.*

(3) *Suppose that $P \in C_2 \cap D_2 \cap L_\infty$.*
(a) *Assume that D_2 is smooth at P and C_2 consists of two lines ℓ_1 and ℓ_2 such that $P \in \ell_1 \cap \ell_2$.*
(i) *If $\ell_1 \neq \ell_2$, then $(D,P) \sim B_{3\iota+2,4}$ for $\iota = 2,3$.*
(ii) *If $\ell_1 = \ell_2$, then $(D,P) \sim B_{3\iota+2,4}$ for $\iota = 2,4$.*
(b) *Assume that C_2 is smooth at P and D_2 consists of two lines L_1 and L_2 such that $P \in L_1 \cap L_2$.*
(i) *If $L_1 \neq L_2$, then $(D,P) \sim B_{8,4} \circ (B^3_{1,1})^{B_{4\iota-1,3}}$ for $\iota = 2,3$.*
(ii) *If $L_1 = L_2$, then $(D,P) \sim B_{8,4} \circ (B^3_{1,1})^{B_{4\iota-1,3}}$ for $\iota = 2,4$.*
(c) *If C_2 and D_2 consist of two lines, then $(D,P) \sim B_{8,8}$.*

3.2.4. Local singularities of the case ④ For the case that D_2 are smooth is already classified by Lemma 1 and 2. Hence we assume that D_2 consists of two lines.

Lemma 3.6. *Under the above assumptions, we have the following.*

(1) *If $P \in C_1 \cap D_2 \setminus L_\infty$, then $(D,P) \sim B_{8,3}$.*
(2) *If $P \in D_2 \cap L_\infty \setminus C_1$, then $(D,P) \sim B_{8,5}$.*
(3) *If $P \in D_2 \cap C_1 \cap L_\infty$, then $(D,P) \sim B_{8,8}$.*

3.2.5. Local singularities of the case ⑤ For the case that C_2 are smooth is already classified by Lemma 1 and 2. Hence we assume that C_2 consists of two lines.

Lemma 3.7. *Under the above assumptions,*

(1) *If $P \in C_2 \cap D_1 \setminus L_\infty$, then $(D,P) \sim B_{6,4}$.*
(2) *If $P \in C_2 \cap L_\infty \setminus D_1$, then $(D,P) \sim B_{6,2}$.*
(3) *If $P \in C_2 \cap D_1 \cap L_\infty$, then $(D,P) \sim B_{6,6}$.*

3.3. Compare with classified singularities

By the argument in §3.1, the singularity of $(4,3)$ invisible degenerations on L_∞ is one of the following:

(1) Order is 3: $B_{6,3}$, $B_{2,1} \circ B_{6,4}$, $B_{8,6}$, $B_{9,9}$.
(2) Order is 4: $B_{6,4}$, $B_{8,8}$.
(3) Order is 6: $B_{6,6}$.

For $j = 3, 4$ and 6, we consider that whether there is a pair $(C, D) \in \mathcal{LT}_j^V(4,3;d) \times \mathcal{LT}_j^I(4,3;d)$ such that $\operatorname{Sing} C = \operatorname{Sing} D$ for some d. As we assumed that D does not consist of lines in Theorem 1, we exclude singularities $B_{9,9}$, $B_{8,8}$ and $B_{6,6}$.

By local classifications of visible degenerations of order 3, there is a visible degeneration $C \in \mathcal{LT}_3^V(4,3;9)$ such that $\operatorname{Sing} C$ contains either $B_{6,3}$ or $B_{8,6}$ singularity.

Let $C = \{F_3^3 + G_2^4 Z = 0\}$ be a $(4,3)$ visible degeneration of order 3 which has $B_{6,3}$ singularity at P. Then the corresponding curves $C_3 = \{F_3 = 0\}$ and $D_2 = \{G_2 = 0\}$ satisfy the following conditions:

- P is in L_∞.
- D_2 is smooth at P and intersects transversely with L_∞.
- C_3 is smooth at P and tangent to L_∞.

Under these conditions, the intersection multiplicity $I(D_2, C_3; P)$ is 1. Hence $\operatorname{Sing} C$ is $\{5B_{4,3}, B_{6,3}^\infty\}$ generically. On the other hand, there is a $(4,3)$ invisible degeneration D such that its configurations of singularities is $\{6B_{4,3}, B_{6,3}^\infty\}$ by the argument in §3.1. Comparing with above 2 singularities, one $B_{4,3}$ singularity is shortage. To cover this shortage, we consider outer singularities of C.

Lemma 3.8. *Under the above notations, we assume that C has $B_{6,3}$ singularity at P. Then outer singularities of C of multiplicity* 3 *is one of the following*:

$$B_{3,3}, \quad B_{4,3}, \quad B_{2,1} \circ B_{3,2}, \quad B_{5,3}.$$

We omit the proof as it is parallel to that of the proof of Proposition 1 in [6].

Using Lemma 3.8, we have a pair $(C, D) \in \mathcal{LT}_3^V(4,3;9) \times \mathcal{LT}_3^I(4,3;9)$ such that $\operatorname{Sing} C = \operatorname{Sing} D = \{6B_{4,3}, B_{6,3}^\infty\}$.

Let $C = \{F_3^3 + G_2^4 Z = 0\}$ be a $(4,3)$ visible degeneration of order 3 which has $B_{8,6}$ singularity at P. Then the corresponding curves C_3 and D_2 satisfy the following conditions:

- P is an inner singularity.
- D_2 consists of two lines such that they intersect at P.
- C_3 has A_2 singularity at P.
- The intersection multiplicity $I(C_3, D_2; P)$ is 4.

By the above conditions, $\operatorname{Sing} C$ is $\{2B_{4,3}, B_{8,6}^\infty\}$ generically. On the other hand, there is a $(4,3)$ invisible degenerations D such that its configurations of singularities is $\{3B_{4,3}, B_{8,6}^\infty\}$ by the argument in §3.1. In this case, C cannot have outer singularities of order 3 by simple calculations. Therefore there is not a pair which have $B_{8,6}$ singularity.

For order 4 and 6, there is not such a pair by local classifications.

3.4. Proof of Theorem

Let $C \in \mathcal{LT}_3^V(4,3;9)$ and $D \in \mathcal{LT}_3^I(4,3;9)$ be line degenerations such that $\operatorname{Sing} C = \operatorname{Sing} D = \{6B_{4,3}, B_{6,3}^{\infty}\}$. Now we will show that such curves C and D never coincide. Let $F = F_3^3 + F_2^4 Z$ and $G = G_3^4 - G_4^3$ be the defining polynomials of C and D respectively.

Suppose that $C = D$ and we put the singular locus $\Sigma(C) = \Sigma(D) = \{P_1, \dots, P_5, Q, R\}$ such that

$$(C, P_i) \sim (C, Q) \sim B_{4,3}, \quad (C, R) \sim B_{6,3}$$

where P_i are inner singularities of C, Q is an outer singular point of C and $R \in L_\infty$. By previous arguments, $C_2 = \{F_2 = 0\}$ and $D_3 = \{G_3 = 0\}$ satisfy as the following:

(1) C_2 intersects transversely with L_∞.
(2) D_3 has A_1 singularity at R and $I(D_3, L_\infty; R) = 3$.
(3) C_2 and D_3 pass through $P_1, \dots, P_5$ and R.
(4) $Q \in D_3 \setminus C_2$.

By these conditions, we have $I(C_2, D_3) \geq 5 \cdot 1 + 2 = 7$. This is a contraction to Bézout Theorem if D_3 is irreducible. Hence D_3 is a union C_2 and L where L is the line pass through R and Q. Such a decomposition of D_3 is impossible since $I(D_3, L_\infty; R) = 3$. Q.E.D.

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