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From GW invariants of symmetric product stacks to relative invariants of threefolds

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Abstract.

In this note, we relate the equivariant GW invariants of the symmetric product stacks of any nonsingular toric surface X in genus zero to the equivariant relative GW invariants of the threefold $X \times \mathbb{P}^1$ in all genera. We give an example for which an equivalence between these two theories exists.

§1. Introduction

1.1. Overview

Throughout the note, we let X be a nonsingular toric surface, $\mathbb{T} = (\mathbb{C}^*)^2$ a torus, and n a positive integer.

The symmetric group $\mathfrak{S}_n$ acts on X^n by permuting coordinates. This gives rise to a quotient variety $\mathrm{Sym}^n(X) := X^n/\mathfrak{S}_n$, the n-fold symmetric product of X, and a quotient stack $[\mathrm{Sym}^n(X)]$, the n-fold symmetric product *stack* of X. The stack $[\mathrm{Sym}^n(X)]$ is an orbifold, i.e., a nonsingular Deligne–Mumford stack, and its coarse moduli space is $\mathrm{Sym}^n(X)$, which is singular when $n \geq 2$.

Our main purpose is to study the enumerative geometry of the symmetric product stack $[\mathrm{Sym}^n(X)]$ and the threefold $X \times \mathbb{P}^1$. We are particularly interested in the following conjecture.

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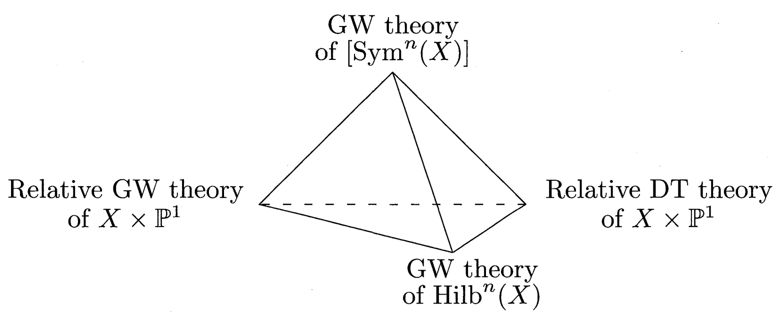
Conjecture A. *For any cohomology-weighted partitions* $\lambda_1(\overrightarrow{\eta_1})$, $\lambda_2(\overrightarrow{\eta_2})$, $\lambda_3(\overrightarrow{\eta_3})$ *of* n, *and* $\beta \in H_2(X;\mathbb{Z})$, *we have the following equality:*

$$F_\beta^{X\times\mathbb{P}^1}(\lambda_1(\overrightarrow{\eta_1}),\lambda_2(\overrightarrow{\eta_2}),\lambda_3(\overrightarrow{\eta_3})) \\ = u^{n-\sum_{i=1}^3 \ell(\lambda_i)} F_\beta^{[\mathrm{Sym}^n(X)]}(\lambda_1(\overrightarrow{\eta_1}),\lambda_2(\overrightarrow{\eta_2}),\lambda_3(\overrightarrow{\eta_3})).$$

The term $F_\beta^{X\times\mathbb{P}^1}(\lambda_1(\overrightarrow{\eta_1}),\lambda_2(\overrightarrow{\eta_2}),\lambda_3(\overrightarrow{\eta_3}))$ is the generating series encoding $\mathbb{T}$-equivariant relative Gromov–Witten (GW) invariants of the threefold $X \times \mathbb{P}^1$ in all genera while $F_\beta^{[\mathrm{Sym}^n(X)]}(\lambda_1(\overrightarrow{\eta_1}),\lambda_2(\overrightarrow{\eta_2}),\lambda_3(\overrightarrow{\eta_3}))$ denotes the generating series encoding $\mathbb{T}$-equivariant extended GW invariants of the symmetric product stack $[\mathrm{Sym}^n(X)]$ in genus zero. We will discuss later that the cohomology-weighted partitions do generate a basis for the $\mathbb{T}$-equivariant Chen–Ruan cohomology of the symmetric product stack.

1.2. Motivation

There are two other theories, the $\mathbb{T}$-equivariant GW theory of the Hilbert scheme $\mathrm{Hilb}^n(X)$ of n points on X and the $\mathbb{T}$-equivariant relative Donaldson–Thomas (DT) theory of the threefold $X \times \mathbb{P}^1$, relating to the theories discussed in Conjecture A. In fact, this note is motivated by a conjectural tetrahedron of equivalences:



The base triangle includes the conjectural GW/DT correspondence for relative invariants formulated by Maulik, Nekrasov, Okounkov and Pandharipande [20], which relates the cohomology-weighted partitions of n to the Nakajima basis for the cohomology of $\mathrm{Hilb}^n(X)$, and its conjectural relationship to the GW theory of $\mathrm{Hilb}^n(X)$. The reader may want to consult [19] and [20] for the formulation of the GW/DT correspondence for absolute invariants. Remarkably, it is shown in [23] that the absolute version holds for all nonsingular toric threefolds.

On the other hand, the SYM/HILB correspondence between the GW theories of $[\mathrm{Sym}^n(X)]$ and $\mathrm{Hilb}^n(X)$ in genus zero is predicted by the crepant resolution conjectures (CRC) of Ruan [28] and Bryan–Graber [3]; see also [9] and [10] for other formulations. A strengthened version of Ruan's CRC, i.e., the SYM/HILB correspondence between the (extended) invariants of $[\mathrm{Sym}^n(X)]$ in degree zero and the invariants of $\mathrm{Hilb}^n(X)$ in extremal classes, is confirmed to be valid in [7]. Furthermore, using the results of Graber [12] on the GW theory of $\mathrm{Hilb}^2(\mathbb{P}^2)$, Wise [29] verifies Bryan–Graber's CRC for $[\mathrm{Sym}^2(\mathbb{P}^2)]$ and $\mathrm{Hilb}^2(\mathbb{P}^2)$. However, except for this and other special examples (see the next paragraph), it is not known if Bryan–Graber's CRC holds for a general toric surface X.

It is worth mentioning that the tetrahedron of equivalences holds when X is the affine plane (cf. [3], [4], [25], [26]) and is also valid for divisor operators when X is the minimal resolution $\mathcal{A}_r$ of the quotient $\mathbb{C}^2/\boldsymbol{\mu}_{r+1}$ where $\boldsymbol{\mu}_{r+1}$ is the group of $(r+1)$-th roots of unity (cf. [8], [18], [21], [22]). Despite striking similarities, there is no direct geometric relationship between the underlying moduli spaces of these four theories. All of the equivalences are achieved by matching the formulas on each side possibly after an appropriate change of variables.

1.3. Outline

In Section 2, we recall some basic notions concerning orbifold and relative GW theories. In Section 3, we provide evidence for Conjecture A. In particular, the validity for $\beta = 0$ is explained. The rest of Section 3 is an introduction to the works of Cheong–Gholampour [8] and Maulik [18]. To keep things simple, we focus our attention to the calculations of the orbifold invariants of the symmetric square $\mathrm{Sym}^2(\mathcal{A}_1)$ and the relative invariants of the threefold $\mathcal{A}_1 \times \mathbb{P}^1$. The solutions to these two theories verify Conjecture A for any β.

§2. Preliminaries

2.1. Twisted stable maps and evaluations

For any curve class $\beta \in H_2(\mathrm{Sym}^n(X); \mathbb{Z})$, we denote by

$$\overline{M}_{0,k}([\mathrm{Sym}^n(X)], \beta)$$

the moduli of genus zero, k-pointed, twisted stable map to the symmetric product stack $[\mathrm{Sym}^n(X)]$ of class β (cf. Chen–Ruan [6] and Abramovich–Graber–Vistoli [2]). Note here that each marking of the domain curve is *twisted*, i.e., it is the classifying stack $\mathcal{B}\boldsymbol{\mu}_s = [\mathrm{Spec}\,\mathbb{C}/\boldsymbol{\mu}_s]$ for some s, and each node is *balanced*, i.e., locally near each node, the domain curve looks like the stack $[\mathrm{Spec}\ \mathbb{C}[u,v]/(uv)/\boldsymbol{\mu}_s]$ for some s, where the cyclic group $\boldsymbol{\mu}_s$ acts on $\mathrm{Spec}\ \mathbb{C}[u,v]$ by $\xi\cdot(u,v)=(\xi u,\xi^{-1}v)$. We refer to [1] and [2] for a more detailed discussion.

Let $\overline{\mathcal{I}}[\mathrm{Sym}^n(X)]$ be the stack of cyclotomic gerbes in $[\mathrm{Sym}^n(X)]$. That is, it is the disjoint union

$$\coprod_{s\in\mathbb{N}} \mathcal{I}_{\boldsymbol{\mu}_s}([\mathrm{Sym}^n(X)])\,/\!\!/\boldsymbol{\mu}_s \tag{2.1}$$

where $\mathcal{I}_{\boldsymbol{\mu}_s}([\mathrm{Sym}^n(X)]) := \mathrm{HomRep}(\mathcal{B}\boldsymbol{\mu}_s, [\mathrm{Sym}^n(X)])$ is the stack of representable morphisms from $\mathcal{B}\boldsymbol{\mu}_s$ to $[\mathrm{Sym}^n(X)]$, and $\mathcal{I}_{\boldsymbol{\mu}_s}([\mathrm{Sym}^n(X)])/\!\!/\boldsymbol{\mu}_s$ is the *rigidification of* $\mathcal{I}_{\boldsymbol{\mu}_s}([\mathrm{Sym}^n(X)])$ *along* $\boldsymbol{\mu}_s$; roughly speaking, it is obtained by removing the cyclic group $\boldsymbol{\mu}_s$ from the automorphisms of $\mathcal{I}_{\boldsymbol{\mu}_s}([\mathrm{Sym}^n(X)])$. We refer the reader to Section 5 of [1] for a technical discussion of the rigidification; see also [2] and [27]. (Note that the stack $\mathcal{I}_{\boldsymbol{\mu}_s}([\mathrm{Sym}^n(X)])/\!\!/\boldsymbol{\mu}_s$ is $\mathcal{I}_{\boldsymbol{\mu}_s}([\mathrm{Sym}^n(X)])^{\boldsymbol{\mu}_s}$ in [1]. The notation "$/\!\!/$" was introduced by Romagny in [27] and has become widely used.)

Evaluating twisted stable maps at their markings does not necessarily give points in $[\mathrm{Sym}^n(X)]$. In fact, it gives points in $\overline{\mathcal{I}}[\mathrm{Sym}^n(X)]$ in general because the markings are twisted points. More precisely, we have the i-th evaluation morphism

$$\mathrm{ev}_i\colon\ \overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)\to\overline{\mathcal{I}}[\mathrm{Sym}^n(X)] \tag{2.2}$$

given by sending the twisted stable map $f\colon\ (\mathcal{C},\mathcal{P}_1,\dots,\mathcal{P}_k)\to[\mathrm{Sym}^n(X)]$ to $f|_{\mathcal{P}_i}\colon\ \mathcal{P}_i\to[\mathrm{Sym}^n(X)]$ where $\mathcal{P}_i$ denotes the i-th marking of $\mathcal{C}$.

2.2. Chen–Ruan cohomology

The stack $\overline{\mathcal{I}}[\mathrm{Sym}^n(X)]$ is isomorphic to a disjoint union of orbifolds, which correspond bijectively to partitions of n. The component $\overline{\mathcal{I}}_\lambda$ corresponding to the partition λ is the quotient stack $[X^n_\sigma/\overline{C(\sigma)}]$ where $\sigma\in\mathfrak{S}_n$ is of cycle type λ, X^n_σ is the σ-fixed locus of X^n, and $\overline{C(\sigma)}$ denotes the quotient group of the centralizer $C(\sigma)$ of σ by $\langle\sigma\rangle$. Note that the coarse moduli space of $[X^n_\sigma/\overline{C(\sigma)}]$ is simply $X^n_\sigma/C(\sigma)$, which we denote by I_λ.

The Chen–Ruan cohomology $H^*_{\mathrm{CR}}([\mathrm{Sym}^n(X)])$ of the n-fold symmetric product stack of X is the cohomology $H^*(\overline{\mathcal{I}}[\mathrm{Sym}^n(X)];\mathbb{Q})$; see

Chen–Ruan [5]. Thus, it is simply $\bigoplus_{|\lambda|=n} H^*(I_\lambda;\mathbb{Q})$. As X is toric, there are induced $\mathbb{T}$-actions on the stack of cyclotomic gerbes in the orbifold $[\mathrm{Sym}^n(X)]$. So we may put the Chen–Ruan cohomology into a $\mathbb{T}$-equivariant context. We denote by

$$H^*_{\mathrm{CR},\mathbb{T}}([\mathrm{Sym}^n(X)])$$

the $\mathbb{T}$-equivariant Chen–Ruan cohomology of $[\mathrm{Sym}^n(X)]$. Now we sketch a basis for $H^*_{\mathbb{T}}(I_\lambda;\mathbb{Q})$ for any partition λ (cf. Cheong [7]).

Let $\lambda = (\lambda_1,\ldots,\lambda_{\ell(\lambda)})$ be a partition of n and $\eta_1,\ldots,\eta_{\ell(\lambda)}$ $\mathbb{T}$-equivariant classes on X. Suppose $\sigma \in \mathfrak{S}_n$ is of cycle type λ. It admits a cycle decomposition $\sigma = \sigma_1 \ldots \sigma_{\ell(\lambda)}$ where σ_i is a λ_i-cycle. We consider the class

$$\sum_{\tau\in C(\sigma)} \bigotimes_{i=1}^{\ell(\lambda)} \tau^{-1}\sigma_i\tau(\eta_i) \in H^*_{\mathbb{T}}(X^n_\sigma;\mathbb{Q})^{C(\sigma)} = H^*_{\mathbb{T}}(I_\lambda;\mathbb{Q}). \tag{2.3}$$

Here $\tau^{-1}\sigma_i\tau(\eta_i)$ is the pullback of η_i by the isomorphism $X^{\lambda_i}_{\tau^{-1}\sigma_i\tau} \cong X$. The expression (2.3) does not depend on the decomposition of σ. We use the cohomology-weighted partition

$$\lambda(\overrightarrow{\eta}) := \lambda_1(\eta_1)\ldots\lambda_{\ell(\lambda)}(\eta_{\ell(\lambda)})$$

to denote the class in (2.3) divided by the factor $|\mathrm{Aut}(\lambda(\overrightarrow{\eta}))|\prod_{i=1}^{\ell(\lambda)}\lambda_i$ where $\mathrm{Aut}(\lambda(\overrightarrow{\eta}))$ is the group of permutations on $\{1,\ldots,\ell(\lambda)\}$ which fixes $\big(\lambda_1(\eta_1),\ldots,\lambda_{\ell(\lambda)}(\eta_{\ell(\lambda)})\big)$.

Let $\mathfrak{B}$ be a basis for $H^*_{\mathbb{T}}(X;\mathbb{Q})$. The classes $\lambda(\overrightarrow{\eta})$'s, with all $\eta_i \in \mathfrak{B}$, serve as a basis for $H^*_{\mathbb{T}}(I_\lambda;\mathbb{Q})$.

2.3. Extended orbifold GW invariants

We let t_1 and t_2 be the generators of the $\mathbb{T}$-equivariant cohomology of a point. For $\mathbb{T}$-equivariant Chen–Ruan cohomology classes $\alpha_i \in H^*_{\mathrm{CR},\mathbb{T}}([\mathrm{Sym}^n(X)])$, $i = 1,\ldots,k$, and curve class $\beta \in H_2(\mathrm{Sym}^n(X);\mathbb{Z})$, the $\mathbb{T}$-equivariant, k-point, orbifold GW invariant of $[\mathrm{Sym}^n(X)]$ is defined by

$$\langle\alpha_1,\ldots,\alpha_k\rangle_\beta := \int_{[\overline{M}_{0,k}([\mathrm{Sym}^n(X)],\beta)]^{\mathrm{vir}}_{\mathbb{T}}} \mathrm{ev}_1^*(\alpha_1)\cdots\mathrm{ev}_k^*(\alpha_k), \tag{2.4}$$

where $[\ ]^{\mathrm{vir}}_{\mathbb{T}}$ indicates the $\mathbb{T}$-equivariant virtual fundamental class. The expression (2.4) is not really well-defined as the underlying moduli space is not necessarily compact, but we can interpret the right side of (2.4) as a sum of residue integrals via virtual localization (cf. [13]). In general, $\mathbb{T}$-equivariant orbifold GW invariants take values in $\mathbb{Q}(t_1, t_2)$.

From now on, we denote by

$$(2)$$

the divisor class $2(1)1(1)^{n-2}$.

In this note, we are more interested in the extended version of orbifold invariants. In fact, for any nonnegative integer a, curve class $\beta \in H_2(\mathrm{Sym}^n(X); \mathbb{Z})$, and Chen–Ruan cohomology classes $\alpha_1, \ldots, \alpha_k \in H^*_{\mathrm{CR},\mathbb{T}}([\mathrm{Sym}^n(X)])$, we call the following term

$$\frac{1}{a!}\langle \alpha_1, ..., \alpha_k, (2)^a \rangle_\beta \tag{2.5}$$

a k-point extended GW invariant of $[\mathrm{Sym}^n(X)]$ and denote it by

$$\langle \alpha_1, ..., \alpha_k \rangle_{(a,\beta)}.$$

We encode the extended invariants in a generating series. That is, we define

$$F_\beta^{[\mathrm{Sym}^n(X)]}(\alpha_1, ..., \alpha_k) = \sum_{a=0}^{\infty} \langle \alpha_1, ..., \alpha_k \rangle_{(a,\beta)} u^a.$$

2.4. Relative GW invariants

Let us fix some notation on the moduli space of stable relative maps. We refer to Graber–Vakil [14], Li [15], and Li [16] for a detailed discussion of the geometry of the space.

Let $(\mathbb{P}^1, p_1, \ldots, p_k)$ be the projective line with k distinct marked points. Given a positive integer n, partitions $\lambda_1, \ldots, \lambda_k$ of n, and curve class $\beta \in H_2(X; \mathbb{Z})$. We denote by

$$\overline{M}^\circ_g(X \times \mathbb{P}^1, (\beta, n); \lambda_1, \ldots, \lambda_k)$$
$$\left(\text{resp. } \overline{M}^\bullet_g(X \times \mathbb{P}^1, (\beta, n); \lambda_1, \ldots, \lambda_k)\right)$$

the moduli space parametrizing stable relative maps from connected (resp. possibly disconnected) genus g curves to $X \times \mathbb{P}^1$ of homology class $(\beta, n) \in H_2(X \times \mathbb{P}^1; \mathbb{Z}) = H_2(X; \mathbb{Z}) \oplus \mathbb{Z}$, with ramification profile λ_i over the relative divisor $X \times p_i$ for $i = 1, \ldots, k$. The ramification points are taken to be marked and ordered. For the disconnected case, the connected components of the source curve are required to be non-contracted.

Let us denote $\overline{M}^{\circ}_g(X \times \mathbb{P}^1, (\beta, n); \lambda_1, \ldots, \lambda_k)$ by $\overline{M}^{\circ}$ temporarily. There is an evaluation map

$$\mathrm{ev}_{ij} \colon \overline{M}^{\circ} \to X \times p_i \cong X$$

determined by the ramification points of type λ_{ij} over the divisor $X \times p_i$ for $j = 1, \ldots, \ell(\lambda_i)$ and $i = 1, \ldots, k$. (Here $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{i\ell(\lambda_i)})$).

The connected relative GW invariant is defined as follows: Given cohomology-weighted partitions $\lambda_1(\overrightarrow{\eta_1}), \ldots, \lambda_k(\overrightarrow{\eta_k})$ of n with entries of $\overrightarrow{\eta_i}$ being classes on X, define $\langle \lambda_1(\overrightarrow{\eta_1}), \ldots, \lambda_k(\overrightarrow{\eta_k}) \rangle^{\circ}_{g,\beta}$ by

$$\frac{1}{\prod_{i=1}^{k} |\mathrm{Aut}(\lambda_i(\overrightarrow{\eta_i}))|} \int_{[\overline{M}^{\circ}]^{\mathrm{vir}}_{\mathbb{T}}} \prod_{i=1}^{k} \prod_{j=1}^{l(\lambda_i)} \mathrm{ev}_{ij}^{*}(\eta_{ij}). \tag{2.6}$$

Again, we apply virtual localization in case this is not truly well-defined. Note that the virtual class of $\overline{M}^{\circ}$ is of dimension

$$-K_X \cdot \beta + 2n - \sum_{i=1}^{k} (n - \ell(\lambda_i)).$$

The corresponding disconnected relative invariants over the moduli space $\overline{M}^{\bullet}_g(X \times \mathbb{P}^1, (\beta, n); \lambda_1, \ldots, \lambda_k)$, denoted by

$$\langle \lambda_1(\overrightarrow{\eta_1}), \ldots, \lambda_k(\overrightarrow{\eta_k}) \rangle^{\bullet}_{g,\beta},$$

is assumed to obey the product rule: The source curve is a union of connected components, the disconnected invariant is set to be the product of the connected invariants corresponding to these components.

Just like the GW theory of symmetric product stacks, the disconnected relative invariants are encoded in a generating series: Define

$$F_{\beta}^{[\mathrm{Sym}^n(X)]}(\lambda_1(\overrightarrow{\eta_1}), \ldots, \lambda_k(\overrightarrow{\eta_k})) = \sum_{g} \langle \lambda_1(\overrightarrow{\eta_1}), \ldots, \lambda_k(\overrightarrow{\eta_k}) \rangle^{\bullet}_{g,\beta} \, u^{2g-2}.$$

§3. Comparison of two theories

3.1. Evidence

Let us explain why Conjecture A is true for $\beta = 0$. The toric surface X has finitely many $\mathbb{T}$-fixed points, which we denote by $x_1, \ldots, x_m$. For each i, x_i is contained in an affine open subset U_i, which is isomorphic to the affine plane $\mathbb{C}^2$.

In this subsection, we work on the basis given by cohomology-weighted partitions whose parts are labeled by $\mathbb{T}$-fixed point classes on X. For partitions $\rho_i = (\rho_{i1}, \dots, \rho_{i\ell(\rho_i)})$, $i = 1, \dots, m$, we denote the class

$$\rho_{11}([x_1]) \cdots \rho_{1\ell(\rho_1)}([x_1]) \cdots \rho_{m1}([x_m]) \cdots \rho_{m\ell(\rho_m)}([x_m])$$

by

$$\widetilde{\rho} := (\rho_1, \dots, \rho_m).$$

The classes $\widetilde{\rho}$'s form a basis for $H^*_{\mathrm{CR},\mathbb{T}}([\mathrm{Sym}^n(X)]) \otimes_{\mathbb{Q}[t_1,t_2]} \mathbb{Q}(t_1, t_2)$.

Assume that for each $i = 1, \dots, m$, ρ_i, μ_i, and ν_i are arbitrary partitions of n_i, and $\sum_{i=1}^m n_i = n$. According to [3] and [4] for the $\mathbb{C}^2$-case, we have

$$F_0^{U_i \times \mathbb{P}^1}(\widetilde{\rho_i}, \widetilde{\mu_i}, \widetilde{\nu_i}) = u^{n_i - \ell(\rho_i) - \ell(\mu_i) - \ell(\nu_i)} F_0^{\mathrm{Sym}^{n_i}(U_i)}(\widetilde{\rho_i}, \widetilde{\mu_i}, \widetilde{\nu_i}).$$

Since $\mathbb{T}$-fixed stable relative maps collapse to $x_1, \dots, x_m$ by the projection $X \times \mathbb{P}^1 \to X$, the generating series $F_0^{X \times \mathbb{P}^1}(\widetilde{\rho}, \widetilde{\mu}, \widetilde{\nu})$ is simply the product $\prod_{k=1}^m F_0^{X \times \mathbb{P}^1}(\widetilde{\rho_i}, \widetilde{\mu_i}, \widetilde{\nu_i})$. By results of [7], we have a similar phenomenon on the symmetric product side. That is,

$$F_0^{[\mathrm{Sym}^n(X)]}(\widetilde{\rho}, \widetilde{\mu}, \widetilde{\nu}) = \prod_{k=1}^m F_0^{[\mathrm{Sym}^{n_i}(U_i)]}(\widetilde{\rho_i}, \widetilde{\mu_i}, \widetilde{\nu_i}).$$

Moreover, $F_0^{X \times \mathbb{P}^1}(\widetilde{\rho_i}, \widetilde{\mu_i}, \widetilde{\nu_i})$ and $F_0^{U_i \times \mathbb{P}^1}(\widetilde{\rho_i}, \widetilde{\mu_i}, \widetilde{\nu_i})$ obviously coincide. Thus,

$$F_0^{X \times \mathbb{P}^1}(\widetilde{\rho}, \widetilde{\mu}, \widetilde{\nu}) = u^{n - (\ell(\widetilde{\rho}) + \ell(\widetilde{\mu}) + \ell(\widetilde{\nu}))} F_0^{[\mathrm{Sym}^n(X)]}(\widetilde{\rho}, \widetilde{\mu}, \widetilde{\nu}).$$

Here, for example, $\ell(\widetilde{\rho}) = \sum_{i=1}^m \ell(\rho_i)$. The above equality is clear if the condition $|\rho_i| = |\mu_i| = |\nu_i|$ is violated for some i, and so we deduce the following.

Proposition 3.1. *For any cohomology-weighted partitions* $\lambda_1(\overrightarrow{\eta_1})$, $\lambda_2(\overrightarrow{\eta_2})$, $\lambda_3(\overrightarrow{\eta_3})$ *of* n,

$$\begin{aligned} F_0^{X \times \mathbb{P}^1}&(\lambda_1(\overrightarrow{\eta_1}), \lambda_2(\overrightarrow{\eta_2}), \lambda_3(\overrightarrow{\eta_3})) \\ &= u^{n - \sum_{i=1}^3 \ell(\rho_i)} F_0^{[\mathrm{Sym}^n(X)]}(\lambda_1(\overrightarrow{\eta_1}), \lambda_2(\overrightarrow{\eta_2}), \lambda_3(\overrightarrow{\eta_3})). \end{aligned}$$

3.2. Connected version

To proceed, we must introduce connected GW invariants of symmetric product stacks.

Suppose $f\colon (\mathcal{C}, \mathcal{P}_1, \ldots, \mathcal{P}_k) \to [\mathrm{Sym}^n(X)]$ is a k-pointed twisted stable map. Let $P_{\mathcal{C}} = \mathcal{C} \times_{[\mathrm{Sym}^n(X)]} X^n$, which is a scheme due to the representability of f. The map f induces an $\mathfrak{S}_n$-equivariant morphism $g\colon P_{\mathcal{C}} \to X^n$. By taking g mod $\mathfrak{S}_{n-1}$ and composing with the n-th projection, we obtain a morphism $\widetilde{f}\colon \widetilde{C} \to X$ where $\widetilde{C}$ is a degree n admissible (possibly disconnected) cover of the coarse moduli space of $\mathcal{C}$ branched over the markings. We call $\widetilde{C}$ the cover associated to f or simply the cover associated to $\mathcal{C}$.

Let

$$\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(X)], \beta)$$

denote the locus in $\overline{M}_{0,k}([\mathrm{Sym}^n(X)], \beta)$ which parametrizes twisted stable maps with associated covers being connected. The k-point connected invariant

$$\langle \alpha_1, \ldots, \alpha_k \rangle^{\mathrm{conn}}_{\beta}$$

is defined by replacing the underlying moduli $\overline{M}_{0,k}([\mathrm{Sym}^n(X)], \beta)$ of the invariant (2.4) with $\overline{M}^{\circ}_{0,k}([\mathrm{Sym}^n(X)], \beta)$. Note that the connected invariants in nonzero degrees are polynomials in t_1, t_2 as the underlying moduli space is compact. The k-point extended connected invariant $\langle \alpha_1, \ldots, \alpha_k \rangle^{\mathrm{conn}}_{(a,\beta)}$ may be defined in a similar way.

3.3. The resolved surface $\mathcal{A}_1$

From here on, we will focus on the case when $X = \mathcal{A}_1$, which is also the cotangent line bundle of $\mathbb{P}^1$. The surface $\mathcal{A}_1$ comes equipped with a $\mathbb{T}$-action induced by the natural $\mathbb{T}$-action on $\mathbb{C}^2/\{\pm 1\}$. It has two $\mathbb{T}$-fixed points, denoted by q_0 and q_∞, and a unique compact rational curve E, which is $\mathbb{T}$-invariant.

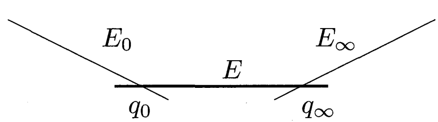


Fig. 1

In Fig. 1, E_0 and E_∞ are two noncompact curves attached to the $\mathbb{T}$-fixed points q_0 and q_∞ respectively. The $\mathbb{T}$-weights of the tangent spaces $T_{q_0}\mathcal{A}_1$ and $T_{q_\infty}\mathcal{A}_1$ are given by

$$(2t_1, t_2 - t_1),\ (t_1 - t_2, 2t_2)$$

respectively. The homology $H_2(\mathcal{A}_1;\mathbb{Z}) \cong \mathbb{Z}$ has a basis given by $[E]$. We denote by ω the class Poincaré dual to $[E]$, that is,

$$\omega = -\frac{1}{2}[E].$$

We will determine 3-point extended invariants of $[\mathrm{Sym}^2(\mathcal{A}_1)]$ and compare them to the relative invariants of $\mathcal{A}_1 \times \mathbb{P}^1$. Our determination is based on [8] and [18].

3.4. Evaluation of three-point extended invariants

We identify $H_2(\mathrm{Sym}^2(\mathcal{A}_1);\mathbb{Z})$ with $H_2(\mathcal{A}_1;\mathbb{Z})$, and so it is generated by the rational curve class $[E]$. Fix integers $a \geq 0$, $d > 0$, and partitions λ_1, λ_2 of 2 throughout the rest of this subsection.

First of all, we wish to calculate 2-point extended invariants

$$\langle \lambda_1(\overrightarrow{\eta_1}), \lambda_2(\overrightarrow{\eta_2})\rangle_{(a,\, d[E])}$$

where each entry of $\overrightarrow{\eta_i}$'s is 1 or $[E]$. This will determine all 2-point extended invariants as the insertions generate the equivariant Chen–Ruan cohomology. We will also show that 3-point extended invariants may be expressed in terms of 2-point extended invariants.

Our calculation involves virtual localization. For each $\mathbb{T}$-fixed connected component Γ of the underlying moduli of twisted stable maps, we denote the virtual normal bundle to Γ by

$$N_\Gamma^{\mathrm{vir}}.$$

Moreover, we say that a $\mathbb{T}$-fixed connected component Γ is *distinguished* if it has such a configuration: Suppose $f\colon \mathcal{C} \to [\mathrm{Sym}^2(\mathcal{A}_1)]$ represents a point in Γ. The source curve $\mathcal{C}$ has a unique noncontracted irreducible component, whose associated cover has a unique irreducible component not contracted by the morphism to $\mathcal{A}_1$.

Let us see how a fixed component Γ contributes to the inverse Euler class $1/e_{\mathbb{T}}(N_\Gamma^{\mathrm{vir}})$. We will see in a moment that the contribution has positive (t_1+t_2)-valuation, which may be further classified according to whether Γ is distinguished or not.

Let $f\colon \mathcal{C} \to [\mathrm{Sym}^2(\mathcal{A}_1)]$ be a twisted stable map in Γ. It induces a stable map $f_c\colon C \to \mathrm{Sym}^2(\mathcal{A}_1)$ of coarse moduli spaces. As discussed above, we also have a map $\widetilde{f}\colon \widetilde{C} \to \mathcal{A}_1$ where $\widetilde{C}$ is a degree 2 cover of C. There are three situations to consider.

(1) *Infinitesimal deformations and obstructions of f with $\mathcal{C}$ held fixed:*

- Contracted components: Let $\mathcal{C}'$ be a contracted component. Consider a connected component Σ of the cover associated to $\mathcal{C}'$. As Σ maps to q_0 or q_∞ under the morphism to $\mathcal{A}_1$, its contribution, being $e_{\mathbb{T}}(H^1(\Sigma, \widetilde{f}^*T\mathcal{A}_1))e_{\mathbb{T}}(H^0(\Sigma, \widetilde{f}^*T\mathcal{A}_1))^{-1}$, is congruent modulo $t_1 + t_2$ to
$$\frac{\Lambda^\vee(2t_1)\Lambda^\vee(t_2 - t_1)}{-(2t_1)^2} \equiv (-1)^{g-1}(2t_1)^{2g-2}$$
by Mumford's relation. Here g is the genus of Σ and $\Lambda^\vee(t) = \sum_{i=0}^{g} c_i(H^0(\Sigma, \omega_\Sigma)^\vee)t^{g-i}$. This also means that the contribution from $\mathcal{C}'$ is not divisible by $t_1 + t_2$ as it is the product of the contributions from (at most two) such Σ's.
- The nodes joining contracted curves to noncontracted curves have zero $(t_1 + t_2)$-valuation because they give a product of tangent weights, which is a positive power of $-(2t_1)^2$ modulo $(t_1 + t_2)$.
- Noncontracted components: A noncontracted component Σ contributes
$$e_{\mathbb{T}}(H^1(\widetilde{\Sigma}, \widetilde{f}^*T\mathcal{A}_1))^m(e_{\mathbb{T}}(H^0(\widetilde{\Sigma}, \widetilde{f}^*T\mathcal{A}_1))^m)^{-1}$$
where $\widetilde{\Sigma}$ is the (possibly disconnected) cover associated to Σ, and $(\ )^m$ indicates the moving part. It is clear from the above discussion that each $\widetilde{f}$-contracted component of $\widetilde{\Sigma}$ contributes no factor of $t_1 + t_2$.

  Let us analyze how an $\widetilde{f}$-noncontracted, irreducible component Σ of $\widetilde{\Sigma}$ contributes. Assume that $\widetilde{f}$ restricted to this component is of degree k. Since the curve E is a (-2)-curve, the invertible sheaf $\omega_\Sigma \otimes \widetilde{f}^* N_{E/\mathcal{A}_1}^\vee$ has degree $2k-2$. The moving part of $e_{\mathbb{T}}(H^1(\Sigma, \widetilde{f}^*T\mathcal{A}_1))$ comes from the term $H^1(\Sigma, \widetilde{f}^*N_{E/\mathcal{A}_1}) = H^0(\Sigma, \omega_\Sigma \otimes \widetilde{f}^*N_{E/\mathcal{A}_1}^\vee)^\vee$, and so it is congruent modulo $(t_1+t_2)^2$ to
$$(t_1 + t_2) \prod_{i=0;\, i\neq k-1}^{2(k-1)} \frac{i(\frac{k-1}{k}(2t_1)) + (2k-2-i)(\frac{1-k}{k}(2t_1))}{2k-2}.$$

On the other hand, $e_{\mathbb{T}}(H^0(\Sigma, \widetilde{f}^*T\mathcal{A}_1))^m$, which agrees with $e_{\mathbb{T}}(H^0(\Sigma, \widetilde{f}^*TE))^m$, is given by

$$\prod_{i=0;\, i\neq k}^{2k} \frac{i(-2t_1)+(2k-i)(2t_1)}{2k} \quad \text{mod } (t_1+t_2).$$

As a consequence, the component Σ contributes

$$\frac{-(t_1+t_2)}{(2t_1)^2} \quad \text{mod } (t_1+t_2)^2.$$

(2) *Infinitesimal automorphisms of $\mathcal{C}$*: Only those nonspecial points p that lie on noncontracted curves Σ and map to $\mathbb{T}$-fixed points contribute. The contributions are the tangent weights of Σ at p, which have zero (t_1+t_2)-valuation.

(3) *Infinitesimal deformations of $\mathcal{C}$*: Given any node $\mathcal{P}$ joining two curves $\mathcal{C}_1$ and $\mathcal{C}_2$. Let P, C_1, C_2 be coarse moduli spaces of $\mathcal{P}, \mathcal{C}_1, \mathcal{C}_2$ respectively and $o_{\mathcal{P}}$ the order of the stabilizer of $\mathcal{P}$. The node-smoothing contribution may be divided into two cases.

- $\mathcal{C}_1$ and $\mathcal{C}_2$ are noncontracted: If f_c restricted to C_i is a d_i-sheeted covering of the rational curve $f_c(C_i)$. Then the node-smoothing contribution with respect to $\mathcal{P}$ is

$$o_{\mathcal{P}}\ (w_1/d_1 + w_2/d_2)^{-1}$$

  where w_i is the tangent weight of $f_c(C_i)$ at $f_c(P)$. Thus, it is proportional to $(t_1+t_2)^{-1}$ only if $d_1 = d_2$ and w_1+w_2 is a multiple of t_1+t_2.
- $\mathcal{C}_1$ is noncontracted but $\mathcal{C}_2$ is contracted: The node smoothing contributes

$$\frac{o_{\mathcal{P}}}{w-\psi}$$

  where w is the tangent weight of C_1 at the node P and ψ is the first Chern class of the tautological line bundle formed by the cotangent space $T^*_{\mathcal{P}}\mathcal{C}_2$.

The above analysis shows that Γ gives positive (t_1+t_2)-valuation because the number of noncontracted curves is more than the number of nodes connecting them.

If Γ is distinguished, then it contributes (t_1+t_2)-valuation 1 as there is a unique noncontracted rational components for the cover associated to $\mathcal{C}$. Suppose Γ is not distinguished, then the associated cover has at least two noncontracted rational curves. As one noncontracted component contributes a factor of $(t_1 + t_2)$, the fixed component contributes $(t_1 + t_2)$-valuation of at least 2. Summing up, we have the following.

Proposition 3.2. *If Γ is distinguished, then it contributes $(t_1 + t_2)$-valuation 1 to $1/e_{\mathbb{T}}(N_\Gamma^{\mathrm{vir}})$. Otherwise, it contributes $(t_1 + t_2)$-valuation of at least 2.*

We may use this proposition to calculate the 2-point extended connected invariants.

Proposition 3.3. *The connected invariant*

$$\langle \lambda_1(\overrightarrow{\eta_1}), \lambda_2(\overrightarrow{\eta_2})\rangle^{\mathrm{conn}}_{(a,\, d[E])} \tag{3.1}$$

vanishes if at least one entry of $\overrightarrow{\eta_1}$ or $\overrightarrow{\eta_2}$ is the class 1.

Proof. The virtual dimension of

$$\overline{M}_{0,a+2}([\mathrm{Sym}^2(\mathcal{A}_1)], d[E]) \cap \mathrm{ev}_1^{-1}(\overline{\mathcal{I}}_{\lambda_1}) \cap \mathrm{ev}_2^{-1}(\overline{\mathcal{I}}_{\lambda_2}) \cap \bigcap_{i=1}^{a} \mathrm{ev}_{i+2}^{-1}(\overline{\mathcal{I}}_{(2)})$$

is $\ell(\lambda_1) + \ell(\lambda_2) - 1$, which is equal to the the maximal sum of the degrees of the insertions. However, the invariant (3.1) is a polynomial in t_1, t_2 and is divisible by $t_1 + t_2$ by Proposition 3.2. This forces (3.1) to vanish. Q.E.D.

It remains to determine connected invariants when each entry of $\overrightarrow{\eta_1}$ and $\overrightarrow{\eta_2}$ is the class $[E]$. For simplicity, if each entry of $\overrightarrow{\eta_i}$ is ξ, we use the following notation

$$\overrightarrow{\eta_i} = \widehat{\xi}.$$

Let us recall the notion of the double Hurwitz number as it will be useful for evaluating our connected invariants. For partitions μ, ν of n, the double Hurwitz number $H^g_{\mu,\nu}$ is the weighted number

$$\sum_\pi \frac{1}{|\mathrm{Aut}(\pi)|}$$

where the sum is taken over all genus g (possibly disconnected) covers $\pi\colon X \to \mathbb{P}^1$ which are branched over 0 and ∞ with ramifications μ and ν respectively and are simply branched elsewhere. Here $|\mathrm{Aut}(\pi)|$ is the size of the automorphism group of π. The number of simple branched points is $2g - 2 + \ell(\mu) + \ell(\nu)$ by the Riemann–Hurwitz formula.

Theorem 3.4. *The connected invariant*

$$\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]})\rangle^{\mathrm{conn}}_{(a,\, d[E])} \tag{3.2}$$

is given by

$$(t_1+t_2)(-1)^{g+a}(-2)^{\ell(\lambda_1)+\ell(\lambda_2)-a+2}d^{a-1}\sum_{a_1+a_2=a}\frac{H^{g_{a_1}}_{\lambda_1,(2)}H^{g_{a_2}}_{\lambda_2,(2)}}{a_1!a_2!}.$$

Here $g=\frac{1}{2}(a-\ell(\lambda_1)-\ell(\lambda_2)+2)$ *and* $g_{a_i}=\frac{1}{2}(a_i-\ell(\lambda_i)+1)$ *are integers,* $i=1,2$.

Proof. As the invariant (3.2) is a multiple of t_1+t_2 by dimension constraints, it suffices to prove the equality modulo $(t_1+t_2)^2$. Furthermore, any $\mathbb{T}$-fixed component that contributes (t_1+t_2)-valuation of at least 2 may be ruled out.

Let E_0 and E_∞ be noncompact curves in $\mathcal{A}_1$ at q_0 and q_∞ respectively (see Fig. 1). We have

$$[E_0]=-\frac{1}{2}[E]+t_2\cdot 1,\ \ [E_\infty]=-\frac{1}{2}[E]+t_1\cdot 1.$$

By Proposition 3.3, the invariant

$$\langle \lambda_1(\widehat{[E_0]}), \lambda_2(\widehat{[E_\infty]})\rangle^{\mathrm{conn}}_{(a,\, d[E])} \tag{3.3}$$

is $(-1/2)^{\ell(\lambda_1)+\ell(\lambda_2)}$ times the original invariant (3.2).

It remains to evaluate (3.3) modulo $(t_1+t_2)^2$. Because of the constraints on the (t_1+t_2)-valuation and insertions, the source curve $\mathcal{C}$ decomposes into three pieces $\mathcal{C}_{a_1}\cup\Sigma\cup\mathcal{C}_{a_2}$: the intermediate curve Σ is the unique noncontracted component, and its associated cover must be totally ramified at nodes; for $i=1,2$, $\mathcal{C}_{a_i}$ is a contracted component and carries the marking corresponding to λ_i and a_i unordered markings corresponding to (2), $\mathcal{C}_{a_1}$ and $\mathcal{C}_{a_2}$ are disjoint but are connected by Σ. ($\mathcal{C}_{a_i}$'s are twisted points whenever $a_i=0$). We denote this fixed component by Γ_{a_1,a_2}.

Let $\overline{M}_i=\overline{M}(\mathcal{B}\mathfrak{S}_2,\lambda_i,(2);a_i)$ for $i=1,2$. There is a morphism

$$\Gamma_{a_1,a_2}\to\overline{M}_1\times\overline{M}_2$$

obtained by sending $\mathcal{C}$ to $(\mathcal{C}_{a_1},\mathcal{C}_{a_2})$. It is of degree $(d\,a_1!a_2!)^{-1}$.

Let $\epsilon_i\colon \overline{M}_i\to\overline{M}_{0,a_i+2}$ be the natural morphism mapping $\mathcal{C}_{a_i}$ to its coarse moduli space C_{a_i} (the node $\mathcal{C}_{a_i}\cap\Sigma$ is mapped to the marking

Q_i), and let ψ_i be the first Chern class of tautological line bundle formed by the cotangent space $T^*_{Q_i}C_{a_i}$ for $i = 1, 2$. The morphism ϵ_i has degree given by the double Hurwitz number $H^{g_{a_i}}_{\lambda_i,(2)}$.

The contribution of the component Γ_{a_1,a_2} to the invariant (3.3) is congruent modulo $(t_1+t_2)^2$ to

$$(-2t_1)^{\ell(\lambda_1)}(2t_1)^{\ell(\lambda_2)}\frac{1}{d\,a_1!a_2!}(-4t_1^2)(t_1+t_2)$$
$$\times\quad (-1)^{a_1}\prod_{i=1}^{2}\frac{(-1)^{g_{a_i}-1}(2t_1)^{2g_{a_i}-2}}{2^{a_i-1}}(\frac{d}{2t_1})^{a_i}\int_{\overline{M}_i}\epsilon_i^*\psi_i^{a_i-1}.$$

The above expression can be simplified to

$$\frac{(t_1+t_2)(-1)^g d^{a-1}}{2^{a-2}a_1!a_2!}\prod_{i=1}^{2}\deg(\epsilon_i)\int_{\overline{M}_{0,a_i+2}}\psi_i^{a_i-1}.$$

(We leave the case where $a_1 = 0$ or $a_2 = 0$ to the reader). Summing over all possible pairs (a_1, a_2) with $a_1 + a_2 = a$, the invariant (3.3) is thus given by

$$\frac{(t_1+t_2)(-1)^g d^{a-1}}{2^{a-2}}\sum_{a_1+a_2=a}\frac{H^{g_{a_1}}_{\lambda_1,(2)}H^{g_{a_2}}_{\lambda_2,(2)}}{a_1!a_2!}\mod (t_1+t_2)^2,$$

and we are done. Q.E.D.

Remark 3.5. The double Hurwitz numbers $H^{g_{a_i}}_{\lambda_i,(2)}$ in Theorem 3.4 can be explicitly calculated. Indeed, for any partition $\mu = (\mu_1, \ldots, \mu_{\ell(\mu)})$ of n, Goulden, Jackson and Vakil [11] obtain the following formula:

$$\sum_{g=0}^{\infty}\frac{|\mathrm{Aut}(\mu)|\,H^g_{\mu,(n)}}{(2g+\ell(\mu)-1)!\,n^{2g+\ell(\mu)-2}}t^{2g}$$
$$=\left(\frac{\sinh(t/2)}{t/2}\right)^{-1}\prod_{i=1}^{\ell(\mu)}\frac{\sinh(\mu_i t/2)}{\mu_i t/2}.$$

Only very few 2-point extended invariants of the symmetric square are not connected invariants. These invariants are of the form

$$\langle 1(\eta_{11})1(\eta_{12}), 1(\eta_{21})1(\eta_{22})\rangle_{(0,\,d[E])}$$

where η_{ij} are 1 or $[E]$. To calculate them, notice that the source curves have exactly two connected components, exactly one of which is non-contracted due to (t_1+t_2)-valuation. With this description, we deduce that

$$(3.4)\qquad \langle 1(\eta_{11})1(\eta_{12}), 1(\eta_{21})1(\eta_{22})\rangle_{0,\, d[E]} = \sum \langle \xi_{11} \,|\, \xi_{11}\rangle \langle \xi_{21}, \xi_{22}\rangle_{d[E]}.$$

Here $\langle \xi_{11} \,|\, \xi_{11}\rangle$ and $\langle \xi_{21}, \xi_{22}\rangle_{d[E]}$ are simply a Poincaré pairing and a 2-point GW invariant of $\mathcal{A}_1$ respectively, and the sum is taken over all possible ξ_{ij}'s such that $1(\eta_{i1})1(\eta_{i2}) = 1(\xi_{i1})1(\xi_{i2})$ for $i = 1, 2$.

The pairing on $\mathcal{A}_1$ is determined by

$$\langle 1 \,|\, 1\rangle = \frac{1}{2t_1t_2},\ \langle 1 \,|\, [E]\rangle = 0,\ \langle [E] \,|\, [E]\rangle = -2.$$

We may evaluate 2-point GW invariants of $\mathcal{A}_1$ by virtual localization just as Theorem 3.4. In fact, $\langle \xi_{21}, \xi_{22}\rangle_{d[E]}$ vanishes if one of ξ_{21} and ξ_{22} is 1 and

$$\langle [E], [E]\rangle_{d[E]} = \frac{4(t_1 + t_2)}{d}.$$

As a consequence, the right side of (3.4) can be counted explicitly.

Proposition 3.6. *Two-point extended invariants determine three-point extended invariants.*

Proof. We may give an (extended) quantum multiplication $*$ on the equivariant Chen–Ruan cohomology $H^*_{\mathrm{CR},\mathbb{T}}([\mathrm{Sym}^2(\mathcal{A}_1)])$. Indeed, we define

$$\alpha_1 * \alpha_2 = \sum_{\beta\in\mathfrak{B}} \beta^\vee \sum_{d=0}^{\infty} F_{d[E]}^{[\mathrm{Sym}^2(\mathcal{A}_1)]}(\alpha_1, \alpha_2, \beta)\, v^d$$

where $\mathfrak{B} = \{1([E])1([E]),\ 2([E]),\ 1(1)1([E]),\ 2(1),\ 1(1)1(1)\}$ is an ordered basis for $H^*_{\mathrm{CR},\mathbb{T}}([\mathrm{Sym}^2(\mathcal{A}_1)])$, $\beta^\vee$ is a class Poincaré dual to β, and v is a formal parameter. Note that the associativity follows from the WDVV equation, and $1(1)1(1)$ is the multiplicative identity.

By divisor equation,

$$\langle \lambda_1(\vec{\eta_1}), \lambda_2(\vec{\eta_2}), 1(1)1(\omega)\rangle_{(a,\, d[E])} = d\langle \lambda_1(\vec{\eta_1}), \lambda_2(\vec{\eta_2})\rangle_{(a,\, d[E])}.$$

Use the results on 2-point extended invariants, we obtain the matrix representation of the operator $1(1)1(\omega) * -$ with respect to $\mathfrak{B}$:

$$\begin{pmatrix} \frac{2\theta(v^2-1)}{f(u,v)} & \frac{-2\theta v \sin u}{f(u,v)} & -1 & 0 & 0 \\ \frac{4\theta v \sin u}{f(u,v)} & 2\theta(\frac{v^2 - v\cos u}{f(u,v)} + \frac{1}{v-1}) & 0 & -1 & 0 \\ 2t_1t_2 & 0 & \frac{\theta(v+1)}{v-1} & 0 & -\frac{1}{2} \\ 0 & 4t_1t_2 & 0 & 0 & 0 \\ 0 & 0 & 4t_1t_2 & 0 & 0 \end{pmatrix}$$

(here $f(u,v) = v^2 - 2v\cos u + 1$, $\theta = t_1 + t_2$). This matrix has distinct eigenvalues, and so a Vandermonde argument (see [4] and [8]) shows that the quantum ring structure is determined by the quantum multiplication by $1(1)1(\omega)$. As a result, 3-point extended orbifold invariants are also determined. Q.E.D.

3.5. Review of Maulik's work

Unless otherwise stated, the results in this subsection are due to Maulik. Again, we fix integers $g \geq 0$ and $d > 0$ throughout the subsection. We will sketch Maulik's equivariant calculation [18] for relative GW invariants of $\mathcal{A}_1 \times \mathbb{P}^1$ of class $(d[E], 2)$.

From now on, we also fix cohomology-weighted partitions $\lambda_1(\overrightarrow{\eta_1})$ and $\lambda_2(\overrightarrow{\eta_2})$ of 2 where each entry of $\overrightarrow{\eta_1}$ and $\overrightarrow{\eta_2}$ is 1 or $[E]$.

Proposition 3.7. *The connected invariant*

$$\langle \lambda_1(\overrightarrow{\eta_1}), \lambda_2(\overrightarrow{\eta_2}), 1(1)1(1)\rangle^{\circ}_{g,\, d[E]}$$

vanishes. Moreover, if some entries of η_i's is 1, the invariant

$$\langle \lambda_1(\overrightarrow{\eta_1}), \lambda_2(\overrightarrow{\eta_2}), 1(1)1(\omega)\rangle^{\circ}_{g,\, d[E]}$$

vanishes as well.

Proof. Performing virtual localization as in the proof of Proposition 3.2, we can show that these connected invariants are divisible by $t_1 + t_2$. Both statements then follow from dimension constraints. Q.E.D.

We would like to determine the connected invariant

$$\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}), 1(1)1(\omega)\rangle^{\circ}_{g,\, d[E]}.$$

We assume that the first two marked points of $\mathbb{P}^1$ are 0 and ∞ respectively. However, as the $\mathbb{T}$-fixed loci of the underlying moduli spaces involve stable relative maps to *nonrigid* targets, we must study the nonrigid invariants

$$\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}) \rangle^{\sim}_{g,\, d[E]},$$

which are defined similarly to $\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}) \rangle^{\circ}_{g,\, d[E]}$ (cf. (2.6)), but the underlying moduli space $\overline{M}^{\circ}_g(\mathcal{A}_1 \times \mathbb{P}^1, (d[E], 2);\ \lambda_1, \lambda_2)$ is replaced with the moduli space

$$\overline{M}^{\sim}_g(\mathcal{A}_1 \times \mathbb{P}^1, (d[E], 2);\ \lambda_1, \lambda_2)$$

of genus g stable maps to a nonrigid target $\mathcal{A}_1 \times \mathbb{P}^1$ of homology class $(d[E], 2)$, relative to divisors $\mathcal{A}_1 \times 0$ and $\mathcal{A}_1 \times \infty$ with ramifications λ_1 and λ_2 respectively, and up to an equivalence given by $\mathbb{C}^*$-scaling on the $\mathbb{P}^1$-factor; see [18] and [24]. The above two moduli spaces are very close. The only difference is that two nonrigid relative maps are declared isomorphic if they are isomorphic after applying an automorphism of $\mathbb{P}^1$ fixing 0 and ∞.

Theorem 3.8. *The nonrigid invariant* $\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}) \rangle^{\sim}_{g,\, d[E]}$ *is*

$$(t_1 + t_2)(-1)^{g+\ell(\lambda_1)+\ell(\lambda_2)} 2^{4-2g} d^{2g-3+\ell(\lambda_1)+\ell(\lambda_2)} \cdot \sum_{g_1+g_2=g} \frac{H^{g_1}_{\lambda_1,(2)} H^{g_2}_{\lambda_2,(2)}}{(2g_1 - 1 + \ell(\lambda_1))!(2g_2 - 1 + \ell(\lambda_2))!}.$$

Sketch of Proof. Let us calculate the case where $d = 1$. Just as in the proof of Theorem 3.4, it is enough to show that $\langle \lambda_1(\widehat{[E_0]}), \lambda_2(\widehat{[E_\infty]}) \rangle_{g,\, [E]}$ is congruent modulo $(t_1 + t_2)^2$ to

$$\text{(3.5)} \quad \frac{(t_1 + t_2)(-1)^g}{2^{2g-4+\ell(\lambda_1)+\ell(\lambda_2)}} \sum_{g_1+g_2=g} \frac{H^{g_1}_{\lambda_1,(2)} H^{g_2}_{\lambda_2,(2)}}{(2g_1 - 1 + \ell(\lambda_1))!(2g_2 - 1 + \ell(\lambda_2))!}.$$

As $d = 1$, for any genera g_1 and g_2 that add up to g, there is a unique fixed locus allowed: The source curve breaks into three pieces $C_{g_1} \cup \Sigma \cup C_{g_2}$: The intermediate Σ is the unique component not contracted by the projection to $\mathcal{A}_1$ and connects C_{g_1} and C_{g_2}. The curve C_{g_1} is a genus g_1 curve and maps to $\{q_0\} \times \mathbb{P}^1$ with ramifications λ_1 over $(q_0, 0)$ and (2) over (q_0, ∞), and C_{g_2} is a genus g_2 curve and maps to $\{q_\infty\} \times \mathbb{P}^1$ with ramifications λ_2 over (q_∞, ∞) and (2) over $(q_\infty, 0)$. So this fixed locus may be described as $\overline{M}^{\sim}_{g_1}(\mathbb{P}^1;\, \lambda_1,\, (2)) \times \overline{M}^{\sim}_{g_2}(\mathbb{P}^1;\, \lambda_2, (2))$ and its contribution is congruent modulo $(t_1+t_2)^2$ to

$$\begin{aligned} &4(-2t_1)^{\ell(\lambda_1)}(2t_1)^{\ell(\lambda_2)}(-4t_1^2)(t_1+t_2) \\ &\quad \cdot (-1)^{g_1-1}(2t_1)^{2g_1-2}\int_{[\overline{M}^{\sim}_{g_1}(\mathbb{P}^1;\,\lambda_1,\,(2))]^{\mathrm{vir}}} \frac{1}{2(t_2-t_1)-c_1(\mathbb{L}_0)} \\ &\qquad \cdot (-1)^{g_2-1}(2t_1)^{2g_2-2}\int_{[\overline{M}^{\sim}_{g_2}(\mathbb{P}^1;\,\lambda_2,\,(2))]^{\mathrm{vir}}} \frac{1}{2(t_1-t_2)-c_1(\mathbb{L}_\infty)} \end{aligned}$$

where $\mathbb{L}_0$ (resp. $\mathbb{L}_\infty$) is the tautological line bundle at the relative divisor 0 (resp. ∞). As the virtual dimension of the moduli $\overline{M}^{\sim}_{g_i}(\mathbb{P}^1;\, \lambda_i,\, (2))$ is $2g_i - 2 + \ell(\lambda_i)$, which is 1 less than the virtual dimension of the moduli $\overline{M}^{\circ}_{g_i}(\mathbb{P}^1;\, \lambda_i,\, (2))$ for $i = 1, 2$, we simplify the expression to obtain

$$C \cdot \int_{[\overline{M}^{\sim}_{g_1}(\mathbb{P}^1;\,\lambda_1,\,(2))]^{\mathrm{vir}}} \psi_0^{2g_1-2+\ell(\lambda_1)} \cdot \int_{[\overline{M}^{\sim}_{g_2}(\mathbb{P}^1;\,\lambda_2,\,(2))]^{\mathrm{vir}}} \psi_\infty^{2g_2-2+\ell(\lambda_2)}$$

where $C = \dfrac{(-1)^g(t_1+t_2)}{2^{2g-4+\ell(\lambda_1)+\ell(\lambda_2)}}$. It is then given by

$$C \cdot \frac{H^{g_1}_{\lambda_1,(2)}}{(2g_1-1+\ell(\lambda_1))!} \cdot \frac{H^{g_2}_{\lambda_2,(2)}}{(2g_2-1+\ell(\lambda_2))!}$$

due to the result of Liu–Liu–Zhou [17] on double Hurwitz numbers. Summing over all g_1, g_2 such that $g_1 + g_2 = g$, we obtain (3.5).

The method presented in Maulik–Pandharipande [24] may be applied to deal with the case $d > 1$. In fact, it allows us to show that $\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]})\rangle^{\sim}_{g,\, d[E]}$ is

$$d^{2g-3+\ell(\lambda_1)+\ell(\lambda_2)}\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]})\rangle^{\sim}_{g,\, [E]}$$

(here we refer the details to [18]), and the theorem follows. Q.E.D.

Theorem 3.9.

$$\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}), 1(1)1(\omega)\rangle^{\circ}_{g,\, d[E]} = d\,\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]})\rangle^{\sim}_{g,\, d[E]}. \tag{3.6}$$

Proof. By the divisor equation [24],

$$\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}), \omega\rangle^{\sim}_{g,\, d[E]} = d\,\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]})\rangle^{\sim}_{g,\, d[E]}.$$

The marking corresponding to the class ω fixes the $\mathbb{C}^*$-scaling on the $\mathbb{P}^1$ factor, and so the invariant on the left-hand side becomes

$$\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}) \mid \iota_*\omega\rangle^{\circ}_{g,\, d[E]}. \tag{3.7}$$

This invariant is defined by cupping the integrand of the connected invariant $\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]})\rangle^{\circ}_{g,\, d[E]}$ with an extra $\mathrm{ev}^*(\iota_*\omega)$ where $\iota\colon \mathcal{A}_1 \to \mathcal{A}_1 \times \mathbb{P}^1$ is the natural inclusion and ev is the evaluation at the extra non-relative marking. By degenerating $\mathbb{P}^1$, we may arrange to have two components Σ_1 and Σ_2 so that the relative markings lie over Σ_1 and the non-relative marking lies over Σ_2. The degeneration formula expresses the invariant (3.7) in the following form:

$$\text{(3.8)}\qquad \sum \langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}), \theta(\overrightarrow{\xi})\rangle^{\bullet}_{g_1,\, d_1[E]}\, \langle \theta(\overrightarrow{\xi})^{\vee} \,|\, \iota_*\omega\rangle^{\bullet}_{g_2,\, d_2[E]}$$

where the sum is taken over all cohomology-weighted partitions $\theta(\overrightarrow{\xi})$ with each entry of $\overrightarrow{\xi}$ being either 1 or ω, all configurations of connected domain components whose gluing is connected, and all splittings of $g = g_1 + g_2$ and $d = d_1 + d_2$. (The class $\theta(\overrightarrow{\xi})^{\vee}$ denotes the dual to $\theta(\overrightarrow{\xi})$ with respect to the orbifold pairing, and so the gluing term has already been added to the formula (3.8)).

In the expression (3.8), d_2 must be 0. Otherwise, there would be a connected component C not contracted by the projection $\mathcal{A}_1 \times \mathbb{P}^1 \to \mathcal{A}_1$. The dimension constraint shows that in order to make nonvanishing contribution, C must be collapsed by the projection to $\mathbb{P}^1$. But the total configuration would then be disconnected.

Moreover, $\langle \theta(\overrightarrow{\xi})^{\vee} \,|\, \iota_*\omega\rangle^{\bullet}_{g_2,0}$ vanishes unless $\theta(\overrightarrow{\xi}) = 1(1)1(\omega)$ and $g_2 = 0$ (cf. Bryan–Pandharipande [4]). This forces the other configuration to be given by connected curves of genus g. Thus, (3.7) is indeed

$$\langle \lambda_1(\widehat{[E]}), \lambda_2(\widehat{[E]}), 1(1)1(\omega)\rangle^{\circ}_{g,\, d[E]},$$

and the theorem follows. Q.E.D.

We have evaluated all 3-point invariants with one insertion $1(1)1(\omega)$ as long as they are connected. It remains to calculate invariants of the form

$$\text{(3.9)}\qquad \langle 1(\eta_{11})1(\eta_{12}), 1(\eta_{21})1(\eta_{22}), 1(1)1(\omega)\rangle^{\bullet}_{0,\, d[E]}.$$

Each of the domain curves involved has exactly two components. By Proposition 3.7, the projection $\mathcal{A}_1 \times \mathbb{P}^1 \to \mathcal{A}_1$ collapses the component with the third marking being labeled with 1. Thus, the invariants are given by

$$\sum \langle \xi_{11} \,|\, \xi_{11}\rangle \langle \xi_{21}, \xi_{22}, \omega\rangle_{d[E]}.$$

Here $\langle \xi_{11} \,|\, \xi_{11} \rangle$ and $\langle \xi_{21}, \xi_{22}, \omega \rangle_{d[E]}$ are simply a Poincaré pairing and a 3-point GW invariant of $\mathcal{A}_1$ respectively, and the sum is taken over all possible ξ_{ij}'s such that $1(\eta_{i1})1(\eta_{i2}) = 1(\xi_{i1})1(\xi_{i2})$ for $i = 1, 2$. Thus, the relative invariant (3.9) coincides with the symmetric square invariant $\langle 1(\eta_{11})1(\eta_{12}), 1(\eta_{21})1(\eta_{22}), 1(1)1(\omega) \rangle_{(0,\, d[E])}$.

3.6. An equivalence

The calculations of the GW invariants of the symmetric product stack $[\mathrm{Sym}^n(\mathcal{A}_1)]$ and relative invariants of the threefold $\mathcal{A}_1 \times \mathbb{P}^1$ for $n = 2$ are very similar. In fact, we have the equivalence:

Theorem 3.10. *Suppose $X = \mathcal{A}_1$. Then Conjecture A is valid for $n = 2$ and any class β.*

Proof. The case $\beta = 0$ is included in Proposition 3.1. Suppose β is a positive multiple of $[E]$. By a similar argument to the proof of Proposition 3.6, 3-point relative invariants of $\mathcal{A}_1 \times \mathbb{P}^1$ may be expressed in terms of those with an insertion $1(1)1(\omega)$. As a result, we need only show that the conjecture is true for $\lambda_3(\overrightarrow{\eta_3}) = 1(1)1(\omega)$. But this follows by a direct comparison of the formulas in Sections 3.4 and 3.5. Q.E.D.

We can apply the arguments in Sections 3.4 and 3.5 to compute connected GW invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ and connected relative GW invariants of $\mathcal{A}_r \times \mathbb{P}^1$ when one of the insertions is $1(1)1(\omega)$ or (2) for any r and n. The disconnected relative invariants of $\mathcal{A}_r \times \mathbb{P}^1$ are not difficult to compute due to the product rule. But the corresponding invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ are far from easy to evaluate; see [8, Lemma 3.2], which plays a key role in the evaluation.

We have the following result. The proof is omitted, and the reader is referred to [8] and [18] for a detailed treatment.

Theorem 3.11. *Suppose $X = \mathcal{A}_r$ where r is an arbitrary positive integer. Then Conjecture A is valid for any positive integer n and any class β when $\lambda_3(\overrightarrow{\eta_3}) = 1(1)1(\omega)$ or (2).*

It is not known if the 2-point invariants of $[\mathrm{Sym}^n(\mathcal{A}_r)]$ determine the 3-point invariants for $(n, r) \neq (2, 1)$. But once we assume the Generation Conjecture (cf. [18, Section 4.5]), this will be true and Theorem 3.11 will be established even if $\lambda_3(\overrightarrow{\eta_3})$ is an arbitrary cohomology-weighted partition. The details can be found in Section 5.2 of [8].

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