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Large-time asymptotics for Hamilton–Jacobi equations with noncoercive Hamiltonians appearing in crystal growth

Yoshikazu Giga, Qing Liu and Hiroyoshi Mitake

Abstract.

We investigate the large-time behavior of viscosity solutions of Hamilton–Jacobi equations with noncoercive Hamiltonian in a multi-dimensional Euclidean space. Our motivation comes from a model describing growing faceted crystals recently discussed by E. Yokoyama, Y. Giga and P. Rybka (Phys. D, **237** (2008), no. 22, 2845–2855). We prove that the average growth rate of a solution is constant only in a subset, which will be called *effective domain*, of the whole domain and give the asymptotic profile in the subset. This means that the large-time behavior for noncoercive problems may depend on the space variable in general, which is different from the usual results under the coercivity condition. Moreover, on the boundary of the effective domain, the gradient with respect to the x-variable of solutions blows up as time goes to infinity. Therefore, we are naturally led to study *singular Neumann problems* for stationary Hamilton–Jacobi equations. We establish the existence and comparison results for singular Neumann problems and apply the results for a large-time asymptotic profile on the effective domain.

§1. Introduction

In this paper we consider the Cauchy problem for Hamilton–Jacobi (HJ) equations

$$\text{(C)} \qquad \begin{cases} u_t + H(x, Du) = f(x) & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

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with *noncoercive Hamiltonian* $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ of the form

$$H(x,p) = \sigma(x)m(|p|). \tag{1}$$

Here $\sigma, f : \mathbb{R}^N \to [0,\infty)$ and $m : [0,\infty) \to [0,1)$ are given continuous functions. Moreover $m(r)$ is assumed to be Lipschitz continuous and satisfy

$$\text{strictly increasing and } m(r) \to 1 \text{ as } r \to \infty. \tag{2}$$

The function $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$ is an unknown function while $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a given initial value which is assumed to be Lipschitz continuous. To be consistent with the theory of crystal growth [1, 7] we call σ, f and m a *surface supersaturation*, *external force* at point x and a *kinetic coefficient*, respectively. Throughout the paper, we denote $u_t := \partial u/\partial t$ and $Du := (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$.

A very primitive example which we have in mind is

$$\begin{cases} u_t + \dfrac{2}{\pi}\arctan(u_x^2) = |x| & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = 0 & \text{for all } x \in \mathbb{R}. \end{cases} \tag{3}$$

If u is a solution of (3), then the large-time asymptotic behavior of u can be described by

$$u(\cdot,t) \to u_\infty \text{ uniformly on } [-1,1],$$

where

$$u_\infty(x) = \int_0^{|x|} (\tan(\frac{\pi}{2}y))^{1/2}\, dy \text{ for all } x \in [-1,1],$$

and

$$u(\cdot,t) \to +\infty \text{ uniformly on each compact subset of } (-\infty,-1) \cup (1,\infty)$$

as $t \to \infty$. This large time behavior is easily obtained by the method of characteristics as in [4, Section 2]. From this example we learn that the growth rate of u may depend on the x-variable explicitly. We emphasize that this phenomenon seems to be new at least from the viewpoint of study for the large-time behavior of solutions of HJ equations. The typical result of study for this asymptotic problem for HJ equations with (*coercive*) Hamiltonian shows that solutions converge (locally) uniformly with the constant growth rate in the whole domain which is considered as time goes to infinity. (See [5] and references therein for instance.)

Roughly speaking, on the one hand, the viscosity solution of (C) has the constant growth rate asymptotically in a subset in the whole domain

$\mathbb{R}^N$, which we will call the *effective domain* for (C), and on the other hand, outside of the effective domain, the viscosity solution of (C) has an unstable growth rate. In [7], an effective domain is called a maximal stable region of a growing facet. The other feature to be noted is that *gradient grow-up* (or *infinite time gradient blow-up*) of solutions happens. More precisely, let u be the solution of (C), $\Omega_e \subset \mathbb{R}^N$ be the effective domain and c be the growth rate on Ω_e and then the normal derivative with respect to the x-variable of $u - ct$ blows up on the boundary $\partial\Omega_e$ of Ω_e as time goes to infinity, i.e.,

$$D\big(u(x,t) - ct\big) \cdot n(x) \to +\infty \text{ for all } x \in \partial\Omega_e \text{ as } t \to \infty,$$

whereas $u - ct$ remains bounded on $\overline{\Omega_e} = \Omega_e \cup \partial\Omega_e$.

One of the aims of this paper is to investigate the large-time behavior of viscosity solutions of (C). More precisely, we give the formulas of the effective domain and the growth rate and prove that viscosity solutions of (C) converge uniformly on the effective domain and that outside of the effective domain they have growth rates which are higher than that on the effective domain.

It turns out that the asymptotic profile on the effective domain is reduced to stationary problems. As we state above, we encounter a difficulty related to boundary-value problems for stationary HJ equations. More precisely, we are led to consider the *singular Neumann problem* for stationary HJ equations

$$\text{(S)} \quad \begin{cases} |Dv| = m^{-1}\Big(\dfrac{f(x)+c}{\sigma(x)}\Big) & \text{in } \Omega_e, \quad (4) \\ \dfrac{\partial v}{\partial n} = +\infty & \text{on } \partial\Omega_e, \\ \sup\limits_{\Omega_e} |v(x)| < +\infty, & \quad (5) \end{cases}$$

where c and Ω_e will be decided by σ, m, f. See Section 3 below.

We use the following definition of solutions of (S) which has been introduced in [6, Section V.1].

Definition 1 (Definition of solutions of (S)). *Let v be a function on $\overline{\Omega_e}$ with values in $\mathbb{R}$. We call v a subsolution of* (S) *if $v \in \mathrm{USC}\,(\overline{\Omega_e})$ is a viscosity subsolution of* (4). *We call v a supersolution of* (S) *if $v \in \mathrm{LSC}\,(\overline{\Omega_e})$ is a viscosity supersolution of* (4) *and satisfies*

$$v - \phi \textit{ never has a local minimum on } \overline{\Omega_e}$$
$$\textit{at the boundary } \partial\Omega_e \textit{ for any } \phi \in C^1(\overline{\Omega_e}).$$

We call v a solution of (S) *if $v \in C(\overline{\Omega}_e)$ is a subsolution and a supersolution of* (S).

In order to distinguish a viscosity solution of (4) which blows up at some points on boundary, i.e., $u(x) \to +\infty$ as $x \to x_0 \in \partial\Omega$, we impose (5). In fact, notice that in the example above, we have

$$(u_\infty)_x(x)\,|_{x=\pm 1} \cdot \pm 1 = +\infty,$$

while u is bounded on $[-1,1]$.

We present the explanation for the derivation of (C). We establish the existence and comparison results for (S) in Section 3.1 and we use these results for (S) for the study of the large-time behavior of viscosity solutions of (C) in Section 3.2.

§2. Explanation for the derivation of (C)

Before stating main results on (S) and (C), we describe how the noncoercive Hamiltonian (1) is derived in this section. We derive problem (C) from the evolution of hypersurface $\{\Gamma_t^\varepsilon\}_{t\geq 0} \subset \mathbb{R}^{N+1}$ moving according to the law of propagation

$$V_\varepsilon(x, x_{N+1}, t) = \tilde{\sigma}(x, x_{N+1}) m\left(\frac{|p|}{\varepsilon}\right) - \tilde{f}(x, x_{N+1}) \quad \text{on } \Gamma_t^\varepsilon, \tag{6}$$

where $\varepsilon > 0$, V_ε is the normal growth rate at the surface, p is the step density and $\tilde{\sigma}, \tilde{f} : \mathbb{R}^{N+1} \to \mathbb{R}$ are given functions which satisfy

$$\tilde{\sigma}(x, 0) = \sigma(x), \;\; \tilde{f}(x, 0) = f(x) \text{ for all } x \in \mathbb{R}^N.$$

Due to the physical requirement it is important to consider the function m which satisfies (2) and we refer to the literature [1] for the background of the physical model in crystal growth. Let us consider the graph representation of the above evolution and therefore we introduce the function v^ε which satisfies $\Gamma_t^\varepsilon = \{(x, -v^\varepsilon(x,t)) \mid x \in \mathbb{R}^N\}$. Then the step density and the growth rate perpendicular to x-axis are expressed by the gradient of v^ε, i.e., $p = Dv^\varepsilon(x)$ and v_t^ε, respectively. Thus, the above surface evolution equation can be written by

$$v_t^\varepsilon + H_\varepsilon(x, v^\varepsilon, Dv^\varepsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{7}$$

where

$$H_\varepsilon(x,r,p) := \left(\tilde{\sigma}(x,-r) m\left(\frac{|p|}{\varepsilon}\right) - \tilde{f}(x,-r)\right)\sqrt{|p|^2+1}.$$

We approximate v^ε by using the *microscopic time variable*, i.e., $\tau = t/\varepsilon$ so that

$$v^\varepsilon(x, \varepsilon\tau) = \varepsilon u(x, \tau) + o(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

where $o : [0, \infty) \to [0, \infty)$ is a function which satisfies $o(r)/r \to 0$ as $r \to 0$.

Proposition 2.1 ([4, Proposition 5.1]). *Assume that* u_0 *and* $\tilde{\sigma}, \tilde{f}$ *are bounded in* $\mathbb{R}^N$ *and* $\mathbb{R}^{N+1}$*, respectively. Let* v^ε *be the viscosity solutions of* (7) *with the initial value* u_0*. Then,*

$$u^\varepsilon(x, t) := \frac{1}{\varepsilon} v^\varepsilon(x, \varepsilon t)$$

converges to the viscosity solution of (C) *uniformly on every compact set of* $\mathbb{R}^N \times [0, \infty)$ *as* $\varepsilon \to 0$.

The main theorem below, Theorem 3.3, gives a clear view of the solution v^ε on the effective domain Ω_e of (C). We have

$$\begin{aligned} v^\varepsilon(x, t) &= \varepsilon u(x, \frac{t}{\varepsilon}) + o(\varepsilon) \\ &= \varepsilon(\phi_\infty(x) + ct/\varepsilon) + o(\varepsilon) \\ &= \varepsilon\phi_\infty(x) + ct + o(\varepsilon) \end{aligned}$$

for all $x \in \Omega_e$ and $t = O(\varepsilon)$ and c, ϕ_∞ are the constant and the function given by Theorem 3.3. Therefore, we see that roughly speaking, the growing facet moving according to (6) is flat up to order ε with speed c on the effective domain Ω_e.

§3. Main results

3.1. Singular Neumann problems

We use the following assumptions on σ, m, f.

(A1) The function $m : [0, \infty) \to [0, 1)$ is a Lipschitz function with $m(0) =: m_0 \in [0, 1)$, and satisfies (2).

(A2) The function f satisfies $f \geq 0$ in $\mathbb{R}^N$ and

$$\mathcal{A} := \{x \in \mathbb{R}^N \mid f(x) = 0, \sigma(x) = \overline{\sigma}\} \neq \emptyset, \tag{8}$$

where we set $\overline{\sigma} := \sup\{\sigma(x) \mid x \in \mathbb{R}^N\}$.

(A3) The set $\Omega_e := \{x \in \mathbb{R}^N \mid \sigma(x) - f(x) > c\}$ is a bounded domain, where $c := \overline{\sigma} m(0)$.

(A4) $\sigma \in C^1(\mathbb{R}^N)$ and $D\sigma(x) \neq 0$ on $\partial\Omega_e$.

We call the set Ω_e the effective domain for (C). For any $x \in \Omega_e$, $p \in \mathbb{R}^N$ we have

$$\frac{f(x)+c}{\sigma(x)} < 1.$$

Thus we can define the function $h : \Omega_e \to \mathbb{R}$ by

$$h(x) := m^{-1}\Big(\frac{f(x)+c}{\sigma(x)}\Big) \quad \text{for all } x \in \Omega_e.$$

In view of (A4), there exists a modulus ω such that

$$\int_0^1 h(sx+(1-s)y)|x-y|\,ds \le \omega(|x-y|)$$

for all $x, y \in \Omega$ such that $[x,y] \subset \Omega$, where $[x,y] := \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$. This observation implies that any subsolution $u \in \mathrm{USC}\,(\overline{\Omega}_e)$ of (4) is uniformly continuous in Ω_e. (See [4, Lemma 3.3].)

Moreover we have

Proposition 3.1 (Existence of solutions of (S), [4, Thorem 3.2]). *Assume that* (A1)–(A4) *hold. There exists a solution* $v \in C(\overline{\Omega}_e)$ *of* (S).

We add the following assumption.

(A5) $\Omega_e^\alpha := \{\sigma(\cdot) - f(\cdot) > c - \alpha\}$ are convex for all $\alpha \in (0, \alpha_0]$ for a small $\alpha_0 > 0$.

Proposition 3.2 (Comparison principle for (S), [4, Thorem 3.8]). *Assume that* (A1)–(A5) *hold. Let* $v \in C(\overline{\Omega}_e)$ *and* $w \in \mathrm{LSC}\,(\overline{\Omega}_e)$ *be a subsolution and a supersolution of* (S), *respectively. If* $v \le w$ *on* $\mathcal{A}$, *then* $v \le w$ *on* $\overline{\Omega}_e$, *where* $\mathcal{A}$ *is defined by* (8).

Noting that

$$\mathcal{A} = \{x \in \mathbb{R}^N \mid h(x) = 0\},$$

we see that the set $\mathcal{A}$ coincides with the Aubry–Mather set for (4), i.e., an attractor set for the geodesics associated with the Lax–Oleinik formula. (See [2] for instance.)

3.2. Large-time asymptotics

In this section we state the main result of the large-time behavior of solutions of (C).

Theorem 3.3 (Main result, [4, Thorem 4.1]). *Assume that* (A1)–(A5) *holds. Let* u *be a solution of* (C). *We have the result of large-time asymptotics given by*

$$u(\cdot,t) + ct \to \phi_\infty \quad \textit{uniformly on each compact subset of } \Omega_e$$

and

$$u(\cdot,t)+ct \to +\infty \quad \textit{uniformly on each compact subset of } \mathbb{R}^n \setminus \overline{\Omega}_e$$

as $t \to +\infty$, *where* ϕ_∞ *is a solution of* (S). *Moreover* ϕ_∞ *can be represented by*

$$\phi_\infty(x) := \min\{d(x,y)+\phi_-(y) \mid y \in \mathcal{A}\}, \tag{9}$$

where

$$\begin{aligned} \phi_-(x) &:= \inf_{t\geq 0}(u(x,t)+ct), \\ d(x,y) &:= \sup\{v(x)-v(y) \mid v \textit{ is a viscosity subsolution of } (4)\} \quad (10) \\ &= \inf\{\int_0^t h(\gamma(s))\,ds \mid t>0, \gamma \in \mathcal{C}(x,t;y,0), |\dot\gamma(s)| \leq 1\}, \quad (11) \end{aligned}$$

where

$$\mathcal{C}(x,t;y,0) := \{\gamma \in \mathrm{AC}([0,t];\Omega_e) \mid \gamma(t)=x, \gamma(0)=y\}$$

and we denote the set of absolutely continuous functions on $[0,t]$ *with values in* Ω_e *by* $\mathrm{AC}([0,t];\Omega_e)$.

Finally, we give an example which we can calculate the function ϕ_∞ given by (9) concretely. Let $N=1$ and let us set $u_0 \equiv 0$, $\sigma(x)=\overline{\sigma}(1-x^2)_+$ and $f \equiv 0$, where $r_+ := \max\{r,0\}$ for $r \in \mathbb{R}$. Let m be a function which satisfies (A1) with $m_0 = m(0) > 0$. Then we have $c=\overline{\sigma}m_0$, $\Omega_e = (-\sqrt{1-m_0}, \sqrt{1-m_0})$, $\mathcal{A}=\{0\}$ and $h(x)=m^{-1}(m_0/(1-x^2)_+)$. Note that 0 is a subsolution of (C) at this case and therefore we have

$$0 \leq u \leq u+ct \text{ on } \mathbb{R}\times[0,\infty),$$

which implies $\phi_-(x) = \inf_{t\geq 0}(u(x,t)+ct) = 0$. Therefore we have $\phi_\infty(x) = \min_{y\in\mathcal{A}}\{d(x,y)+\phi_-(y)\} = d(x,0)$. By (11) we have for any $x \in \Omega_e$

$$\begin{aligned} d(x,0) &= \inf\{\int_0^t h(\gamma(s))\,ds \mid t>0, \gamma\in\mathcal{C}(x,t;0,0), |\dot\gamma(s)|\leq 1\} \\ &= \int_0^x h(s)\,ds = \int_0^x m^{-1}(\frac{m_0}{(1-s^2)_+})\,ds. \end{aligned}$$

Thus we obtain

$$\phi_\infty(x) = \int_0^x m^{-1}(\frac{m_0}{(1-s^2)_+})\,ds \text{ for all } x \in \overline{\Omega}_e,$$

which is same as the result in [7].

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Yoshikazu Giga
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914
Japan
and
Department of Mathematics, Faculty of Sciences
King Adbulaziz University
P.O. Box 80203, Jeddah 21589
Saudi Arabia

Qing Liu
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260
U.S.A.

Hiroyoshi Mitake
Department of Applied Mathematics, Faculty of Science
Fukuoka University
Fukuoka 814-0180
Japan

E-mail address: `labgiga@ms.u-tokyo.ac.jp`
`qingliu@pitt.edu`
`mitake@math.sci.fukuoka-u.ac.jp`