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Stability analysis of asymptotic profiles for fast diffusion equations

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Abstract.

This note is devoted to reviewing the authors' recent work [1] on the stability analysis of asymptotic profiles of solutions to the Cauchy–Dirichlet problem for the fast diffusion equation.

§1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We are concerned with the Cauchy–Dirichlet problem for fast diffusion equations of the form

$$\partial_t \left(|u|^{m-2}u\right) = \Delta u \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

$$u(\cdot,0) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $\partial_t = \partial/\partial t$, $2 < m < 2^* := 2N/(N-2)_+$ and $u_0 \in H_0^1(\Omega)$. Fast diffusion equations appear in the study of plasma physics (see [2]). It is well known that every solution $u = u(x,t)$ of (1)–(3) for $u_0 \neq 0$ vanishes at a finite time $t_* > 0$ with the explicit rate of $(t_* - t)^{1/(m-2)}$. Hence one can define the *asymptotic profile* $\phi = \phi(x)$ of each solution $u = u(x,t)$:

$$\phi(x) := \lim_{t \nearrow t_*} (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{in } H_0^1(\Omega).$$

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Berryman and Holland [3] first studied asymptotic profiles for classical positive solutions and characterized them as nontrivial solutions of the Dirichlet problem for the Emden–Fowler equation,

$$-\Delta\phi = \lambda_m|\phi|^{m-2}\phi \ \text{ in } \Omega, \quad \phi = 0 \ \text{ on } \partial\Omega$$

with the constant $\lambda_m := (m-1)/(m-2) > 0$. Kwong [11] extended results of [3] to nonnegative weak solutions, and furthermore, Savaré and Vespri treated sign-changing solutions. Recently, further detailed analysis has been done for nonnegative weak solutions by Bonforte et al [4]. The stability of positive profiles has been also discussed within the frame of positive solutions in [3], [11] and [4].

In this paper, we address ourselves to the stability and instability of asymptotic profiles for (possibly) sign-changing solutions of (1)–(3). To this end, we first set up the notions of stability and instability of asymptotic profiles under a wider class of perturbations to initial data, which might be sign-changing. We next present criteria of stability and instability for some sorts of sign-definite and sign-changing profiles. Our stability analysis is based on an infinite dimensional dynamical system on a surface $\mathcal{X}$ in $H_0^1(\Omega)$, and one of novelties of this work lies in such a view point. In the final section, we also reveal a role of this surface in the dynamical system.

§2. Asymptotic profiles of vanishing solutions

Let us first give an explicit definition of solutions for (1)–(3).

Definition 2.1 (Solution of (1)–(3)). *A function $u : \Omega\times(0,\infty) \to \mathbb{R}$ is said to be a (weak) solution of* (1)–(3)*, if the following conditions are all satisfied:*

- $u \in C([0,\infty); H_0^1(\Omega))$ *and* $|u|^{m-2}u \in C^1([0,\infty); H^{-1}(\Omega))$.
- *For all* $t \in (0,\infty)$ *and* $\psi \in C_0^\infty(\Omega)$,

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(|u|^{m-2}u\right)(t), \psi \right\rangle_{H_0^1} + \int_\Omega \nabla u(x,t)\cdot\nabla\psi(x)\mathrm{d}x = 0.$$

- $u(\cdot,t) \to u_0$ *strongly in* $H_0^1(\Omega)$ *as* $t \to +0$.

By using standard methods, one can prove the existence and uniqueness of solutions of (1)–(3) for any $u_0 \in H_0^1(\Omega)$ (see, e.g., [5] and [16], [17]). Moreover, every solution $u = u(x,t)$ of (1)–(3) for $u_0 \neq 0$ vanishes at a finite time $t_* > 0$ with the rate of $(t_* - t)^{1/(m-2)}$ (see [3], [11], [14]).

Proposion 1 (Extinction rate of solutions). *Assume that* $2 < m \leq 2^*$. *Then for any* $u_0 \in H^1_0(\Omega) \setminus \{0\}$, *the unique solution* $u = u(x,t)$ *of* (1)–(3) *vanishes at a finite time* $t_* = t_*(u_0) > 0$. *Moreover, it holds that*

$$(t_* - t)^{1/(m-2)} \leq C_1 \|u(t)\|_{L^m(\Omega)} \leq C_2 \|u(t)\|_{H^1_0(\Omega)} \leq C_3 (t_* - t)^{1/(m-2)}$$

with some constants C_i $(i = 1, 2, 3)$. *Hence* $\|u(t)\|_{H^1_0(\Omega)} := \|\nabla u(t)\|_{L^2(\Omega)}$ *and* $\|u(t)\|_{L^m(\Omega)}$ *vanish at the rate of* $(t_* - t)^{1/(m-2)}$.

The finite time $t_* = t_*(u_0)$ is called *extinction time* (of the unique solution u) for a data u_0. Here t_* can be regarded as a functional,

$$t_* : H^1_0(\Omega) \to [0, \infty),$$

which maps u_0 to the extinction time $t_*(u_0)$ corresponding to the initial data u_0.

Now, by virtue of the common explicit rate of the extinction of solutions, one can define *asymptotic profiles* $\phi = \phi(x)$ for all solutions of (1)–(3) in the following way.

Definition 2.2 (Asymptotic profiles). *Let* $u_0 \in H^1_0(\Omega) \setminus \{0\}$ *and let* $u = u(x,t)$ *be a solution for* (1)–(3) *vanishing at a finite time* $t_* > 0$. *A function* $\phi \in H^1_0(\Omega) \setminus \{0\}$ *is called an asymptotic profile of* u *if there exists an increasing sequence* $t_n \to t_*$ *such that* $(t_* - t_n)^{-1/(m-2)} u(t_n) \to \phi$ *strongly in* $H^1_0(\Omega)$ *as* $n \to \infty$.

As in the previous studies, let us apply the following transformation,

$$(4) \quad v(x,s) := (t_* - t)^{-1/(m-2)} u(x,t) \quad \text{and} \quad s := \log(t_*/(t_* - t)) \geq 0.$$

Then s tends to infinity as $t \nearrow t_*$, and moreover, the asymptotic profile $\phi = \phi(x)$ of $u = u(x,t)$ is rewritten as $\phi(x) := \lim_{s_n \nearrow \infty} v(x, s_n)$ in $H^1_0(\Omega)$. Furthermore, we derive the following Cauchy–Dirichlet problem for $v = v(x,s)$ from (1)–(3):

$$(5) \qquad \partial_s \left(|v|^{m-2} v\right) = \Delta v + \lambda_m |v|^{m-2} v \quad \text{in } \Omega \times (0, \infty),$$

$$(6) \qquad v = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

$$(7) \qquad v(\cdot, 0) = v_0 \quad \text{in } \Omega,$$

where the initial data v_0 and the constant λ_m are given by

$$(8) \quad v_0 = t_*(u_0)^{-1/(m-2)} u_0 \quad \text{and} \quad \lambda_m = (m-1)/(m-2) > 0.$$

The existence of asymptotic profiles and related dynamics have been studied in a couple of papers (see [3], [11], [14] and also [4]). The following result is a slight modification of previous results and it covers sign-changing solutions (cf. in [14], the asymptotic profiles for sign-changing solutions were studied with a weaker sense of convergence).

Theorem 1 (Existence of asymptotic profiles and their characterization [1]). *For any sequence $s_n \to \infty$, there exist a subsequence (n') of (n) and $\phi \in H^1_0(\Omega) \setminus \{0\}$ such that $v(s_{n'}) \to \phi$ strongly in $H^1_0(\Omega)$. Moreover, ϕ is a nontrivial stationary solution of* (5)–(7), *that is, ϕ solves the Dirichlet problem,*

$$(9) \qquad -\Delta\phi = \lambda_m |\phi|^{m-2}\phi \quad in \quad \Omega, \qquad \phi = 0 \quad on \quad \partial\Omega.$$

Remark 1.

(i) If ϕ is a nontrivial solution of (9), then the function $U(x,t) = (1-t)_+^{1/(m-2)}\phi(x)$ solves (1)–(3) with $U(0) = \phi(x)$. Hence $t_*(\phi) = 1$ and the profile of $U(x,t)$ coincides with $\phi(x)$.

(ii) Hence, by Theorem 1, the set of all asymptotic profiles of solutions for (1)–(3) coincides with the set of all nontrivial solutions of (9). We shall denote these sets by $\mathcal{S}$.

(iii) Due to [15], the asymptotic profile is uniquely determined for each nonnegative data $u_0 \geq 0$.

§3. Stability and instability of asymptotic profiles

To start our stability analysis, let us first recall the transformation (4) and the rescaled problem (5)–(7). We particularly focus on the relation $v_0 = t_*(u_0)^{-1/(m-2)}u_0$, and newly introduce the set

$$\mathcal{X} := \left\{ t_*(u_0)^{-1/(m-2)}u_0 \colon u_0 \in H^1_0(\Omega) \setminus \{0\} \right\}.$$

Now, the (asymptotic) stability and instability of asymptotic profiles are defined as follows:

Definition 3.1 (Stability and instability of profiles [1]). *Let $\phi \in H^1_0(\Omega)$ be an asymptotic profile of vanishing solutions for* (1)–(3).

(i) *ϕ is said to be stable, if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that any solution v of* (5)–(7) *satisfies*

$$v(0) \in \mathcal{X} \cap B(\phi;\delta) \quad \Rightarrow \quad \sup_{s\in[0,\infty)} \|v(s) - \phi\|_{H^1_0(\Omega)} < \varepsilon,$$

where $B(\phi;\delta) := \{w \in H^1_0(\Omega) \colon \|\phi - w\|_{H^1_0(\Omega)} < \delta\}$.

(ii) *ϕ is said to be unstable, if ϕ is not stable.*

(iii) *ϕ is said to be asymptotically stable, if ϕ is stable, and moreover, there exists $\delta_0 > 0$ such that any solution v of* (5)–(7) *satisfies*

$$v(0) \in \mathcal{X} \cap B(\phi;\delta_0) \quad \Rightarrow \quad \lim_{s \nearrow \infty} \|v(s) - \phi\|_{H^1_0(\Omega)} = 0.$$

We can observe the following properties of the set $\mathcal{X}$.

(i) If $v_0 \in \mathcal{X}$, then $v(s) \in \mathcal{X}$ for all $s \geq 0$.
(ii) $\mathcal{X} = \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) = 1\}$, which is homeomorphic to a unit sphere in $H_0^1(\Omega)$.
(iii) $\mathcal{S} := \{\text{nontrivial solutions of (9)}\} \subset \mathcal{X}$ (because, $t_*(\phi) = 1$ for $\phi \in \mathcal{S}$ by (ii) of Remark 1).
(iv) If $v_0 \in \mathcal{X}$, then $v(s_n) \to \phi$ strongly in $H_0^1(\Omega)$ with some $\phi \in \mathcal{S}$ along some sequence $s_n \to \infty$ (by Theorem 1).

Hence (5)–(7) generates a dynamical system on the phase surface $\mathcal{X}$. Then solutions of (9) can be regarded as stationary points of the dynamical system. Therefore the notions of stability and instability of asymptotic profiles defined above are regarded as those in Lyapunov's sense for the stationary points. Moreover, (5)–(7) can be formulated as a (generalized) gradient system of the form,

$$\frac{\mathrm{d}}{\mathrm{d}s}|v|^{m-2}v(s) = -J'(v(s)), \quad s > 0, \qquad v(0) = v_0 \in \mathcal{X},$$

where J' stands for the Fréchet derivative of the energy functional

$$J(w) = \frac{1}{2}\int_\Omega |\nabla w(x)|^2 \mathrm{d}x - \frac{\lambda_m}{m}\int_\Omega |w(x)|^m \mathrm{d}x \quad \text{for } w \in H_0^1(\Omega).$$

Then multiplying (5) by $\partial_s v(x,s)$ and integrating this over Ω, we observe that $s \mapsto J(v(s))$ is non-increasing, and hence, $J(\cdot)$ is a Lyapunov functional for the dynamical system. Here let us recall that ϕ is an asymptotic profile if and only if ϕ is a nontrivial solution of (9) (equivalently, $J'(\phi) = 0$ and $\phi \neq 0$). Therefore one can reveal the stability/instability of profiles by investigating variational properties of the functional J over $\mathcal{X}$. However, some difficulties may arise due to the lack of explicit representation of the functional $t_*(\cdot)$ (cf. we can obtain upper and lower estimates for $t_*(\cdot)$ in terms of initial data).

Remark 2. Since $m > 2$, J forms a mountain pass structure over the whole of $H_0^1(\Omega)$. Hence 0 is the unique local minimizer of J and all nontrivial critical points are saddle points of J. However, our stability analysis will be carried out on the surface $\mathcal{X}$ in $H_0^1(\Omega)$. Hence our conclusion on the stability of profiles will differ from this observation, and moreover, it would be troublesome to show the instability of profiles due to the restriction of $\mathcal{X}$.

§4. Stability criteria

Let d_1 be the *least energy* of J over nontrivial solutions, i.e.,

$$d_1 := \inf_{v \in \mathcal{S}} J(v) \quad \text{with} \quad \mathcal{S} = \{\text{nontrivial solutions of (9)}\}.$$

A *least energy solution* ϕ of (9) means $\phi \in \mathcal{S}$ satisfying $J(\phi) = d_1$. By the strong maximum principle, every least energy solution of (9) is sign-definite.

In [1], the authors obtained the following criteria for the stability and instability of asymptotic profiles:

Theorem 2 (Stability of profiles [1]). *Let ϕ be a least energy solution of* (9)*. Then it follows that*

(i) *ϕ is a stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other least energy solutions.*
(ii) *ϕ is an asymptotically stable profile, if ϕ is isolated in $H_0^1(\Omega)$ from the other sign-definite solutions.*

Theorem 3 (Instability of profiles [1]). *Let ϕ be a sign-changing solution of* (9)*. Then it follows that*

(i) *ϕ is not an asymptotically stable profile.*
(ii) *ϕ is an unstable profile, if ϕ is isolated in $H_0^1(\Omega)$ from any $\psi \in \mathcal{S}$ satisfying $J(\psi) < J(\phi)$.*

The isolation of profiles assumed above actually occurs in several cases of Ω and m. We first note by the strong maximum principle that sign-definite solutions are isolated in $H_0^1(\Omega)$ from all sign-changing solutions. In the following cases, each least energy solution is also isolated from all sign-definite ones.

Corollary 1 (Asymptotically stable profiles [1]). *Least energy solutions of* (9) *are asymptotically stable profiles in the following cases*:

- *Ω is a ball and $2 < m < 2^*$ (see Gidas–Ni–Nirenberg* [9]*).*
- *$\Omega \subset \mathbb{R}^2$ is bounded and convex and $2 < m < 2^*$ (see Lin* [12] *and also Dancer* [6]*, Pacella* [13]*).*
- *$\Omega \subset \mathbb{R}^N$ is bounded and $2 < m < 2+\delta$ with a sufficiently small $\delta > 0$ (see Dancer* [7]*).*
- *$\Omega \subset \mathbb{R}^N$ is symmetric with respect to the planes $[x_i = 0]$ and convex in the axes x_i for all $i = 1, 2, \ldots, N$ and $2^* - \delta < m < 2^*$ with a sufficiently small $\delta > 0$ (see Grossi* [10]*).*

Now, let us move on to examples of unstable profiles.

Corollary 2 (Instability of sign-changing least energy profiles [1]). *Least energy solutions among sign-changing solutions (sign-changing least energy solutions, for short) of* (9) *are unstable profiles.*

Since $m < 2^*$ and Ω is bounded, one can always assure the existence of sign-changing least energy solutions of (9). Moreover, sign-changing least energy solutions are distinct from all nontrivial solutions of (9) with lower energies.

As for the one-dimensional case, the Emden–Fowler equation,

$$-\phi'' = \lambda_m |\phi|^{m-2}\phi \ \text{ in } (0,1), \quad \phi(0) = \phi(1) = 0, \tag{10}$$

can be explicitly solved. Moreover, the set $\mathcal{S}$ of all nontrivial solutions for (10) consists of the sign-definite ones $\pm\phi_1$ and the sign-changing ones $\pm\phi_n$ with $(n-1)$ zeros in $(0,1)$ for $n = 2, 3, \ldots$ satisfying

$$J(\pm\phi_1) < J(\pm\phi_2) < \cdots < J(\pm\phi_n) \to \infty \quad \text{as } n \to \infty,$$

which means the isolation of each nontrivial solution. Hence we can completely classify all the asymptotic profiles in terms of their stability.

Corollary 3 (1D case [1]). *Sign-definite profiles are asymptotically stable. All the other profiles are unstable.*

§5. Global dynamics for the rescaled problem

The final section is devoted to further discussion of the surface $\mathcal{X}$, which was a phase space in our stability analysis, and then, we shall reveal the global dynamics of solutions to the rescaled problem (5)–(7) for any data $v_0 \in H_0^1(\Omega)$.

The following proposition classifies the whole of the energy space $H_0^1(\Omega)$ in terms of large-time behaviors of solutions for (5)–(7) (cf. see [8] for the semilinear heat equation), and in particular, $\mathcal{X}$ is a separatrix between the stable and unstable sets.

Proposion 2 (Characterization of $\mathcal{X}$). *Let $v(s)$ be a solution of* (5)–(7) *with $v(0) = v_0$. Then it follows that*

(i) *If $v_0 \in \mathcal{X}$, then $v(s_n)$ converges to some nontrivial solution ϕ of* (9) *strongly in $H_0^1(\Omega)$ along some sequence $s_n \to \infty$.*
(ii) *If $v_0 \in \mathcal{X}^+ := \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) > 1\}$, then $v(s)$ blows up in infinite time. Hence $\mathcal{X}^+$ is an unstable set.*
(iii) *If $v_0 \in \mathcal{X}^- := \{v_0 \in H_0^1(\Omega) \colon t_*(v_0) < 1\}$, then $v(s)$ vanishes in finite time. Hence $\mathcal{X}^-$ is a stable set.*

Moreover, $\mathcal{X}$ does not coincide with the Nehari manifold of J,

$$\mathcal{N} := \left\{ w \in H_0^1(\Omega) \setminus \{0\} \colon \langle J'(w), w \rangle = 0 \right\}.$$

Furthermore, $\mathcal{X}$ is surrounded by $\mathcal{N}$ (i.e., $\mathcal{N} \subset \mathcal{X} \cup \mathcal{X}^+$) and $\mathcal{N} \cap \mathcal{X} = \mathcal{S}$.

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