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Topology of curves on a surface and lattice-theoretic invariants of coverings of the surface

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Abstract.

Let S be a smooth simply-connected complex projective surface, and let A be a finite abelian group. We define invariants T_A, F_A and σ_A for curves B on S by means of étale Galois coverings of the complement of B with the Galois group A, and show that they are useful in finding examples of Zariski pairs of curves on S. We also investigate the relation between these invariants and the fundamental group of the complement of B.

§1. Introduction

We work over the complex number field $\mathbb{C}$. Let S be a smooth projective surface. Throughout this paper, *we assume that S is simply-connected.* By a *curve* on S, we mean a reduced (possibly reducible) curve on S.

Let B and B' be curves on S.

Definition 1.1. We say that a homeomorphism $f : B \xrightarrow{\sim} B'$ *preserves the classes of irreducible components* if we have $[B_i] = [f(B_i)]$ in $H^2(S, \mathbb{Z})$ for any irreducible component B_i of B.

Note that, since S is simply-connected, the equality $[B_i] = [f(B_i)]$ in $H^2(S, \mathbb{Z})$ is equivalent to the equality $[B_i] = [f(B_i)]$ in the Picard group $\mathrm{Pic}(S)$ of S.

Following [5, Definition 2], we make the following:

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Definition 1.2. We say that B and B' have *the same embedding topology* and write $B \sim_{\rm top} B'$ if there exists a homeomorphism between (S, B) and (S, B') such that the induced homeomorphism $B \xrightarrow{\sim} B'$ preserves the classes of irreducible components.

Definition 1.3. A *map of equi-configuration* is a homeomorphism $(\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$, where $\mathcal{T} \subset S$ is a tubular neighborhood of B and $\mathcal{T}' \subset S$ is a tubular neighborhood of B', such that the induced homeomorphism $B \xrightarrow{\sim} B'$ preserves the classes of irreducible components.

Definition 1.4. We say that B and B' are *of the same configuration type* and write $B \sim_{\rm cfg} B'$ if there exist a tubular neighborhood $\mathcal{T} \subset S$ of B, a tubular neighborhood $\mathcal{T}' \subset S$ of B', and a map of equi-configuration $(\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$.

It is obvious that $B \sim_{\rm top} B'$ implies $B \sim_{\rm cfg} B'$.

Definition 1.5. A pair $[B, B']$ of curves on S is said to be a *Zariski pair* if $B \sim_{\rm cfg} B'$ but $B \not\sim_{\rm top} B'$.

By a *plane curve*, we mean a curve on $\mathbb{P}^2$. Since the work of Artal-Bartolo [2], Zariski pairs of plane curves have been studied by many authors. See the survey paper [5]. The most classical example of Zariski pairs is the following (Zariski [28], see also Oka [13] and Shimada [15]):

Example 1.6. There exist irreducible plane curves B and B' of degree 6 with six ordinary cusps as their only singularities such that $\pi_1(\mathbb{P}^2 \setminus B) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, while $\pi_1(\mathbb{P}^2 \setminus B') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

As in this example, the fundamental group $\pi_1(\mathbb{P}^2 \setminus B)$ has been a main tool in finding the examples of Zariski pairs of plane curves.

In this paper, we fix a finite abelian group A and define three invariants $T_A(S, B, \gamma)$, $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$ of curves B on S by means of étale Galois coverings $W_\gamma \to S \setminus B$ with the Galois group A, where γ is a homomorphism $H^2(B, \mathbb{Z}) \to A$ describing the Galois covering. The invariants $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$ are defined in terms of the algebraic cycles on a smooth projective completion X of W_γ, while the invariant $T_A(S, B, \gamma)$ involves the transcendental cycles of X. Using these invariants, we can distinguish topological types of curves on S in the same configuration type, and find many Zariski pairs.

The idea of the invariant $T_A(S, B, \gamma)$ comes from Shioda's observation [20, Lemma 3.1] that the transcendental lattice of a smooth projective surface is a birational invariant.

These invariants have been defined and studied for the double coverings of the projective plane branching along plane curves of degree 6 with only simple singularities ([1], [19], [16], [18]). In particular, the invariant $F_A(S, B, \gamma)$ was intensively studied in [18] in terms of Z-splitting curves.

The plan of this paper is as follows. In §2, we describe all étale Galois coverings of $S \setminus B$ with the Galois group A, and define the invariants $F_A(S, B, \gamma)$ and $T_A(S, B, \gamma)$ in Definition 2.3. In §3, we investigate $T_A(S, B, \gamma)$, and show that, under certain conditions, $T_A(S, B, \gamma)$ is an invariant of the embedding topology of curves (Theorem 3.1). In §4, we define a new invariant $\sigma_A(S, B, \gamma)$, and show that it is an invariant of the configuration types of curves (Theorem 4.3). The invariants $F_A(S, B, \gamma)$ and $T_A(S, B, \gamma)$ are related via $\sigma_A(S, B, \gamma)$ (Proposition 4.7). We then present a method of finding examples of Zariski pairs by means of these invariants (Corollary 4.9). In §5, a relation between $T_A(S, B, \gamma)$, $F_A(S, B, \gamma)$, $\sigma_A(S, B, \gamma)$ and $\pi_1(S \setminus B)$ is presented. We then give several sufficient conditions for $\pi_1(S \setminus B)$ to be non-abelian (Corollaries 5.11 and 5.12). This result generalizes the theory of dihedral coverings, which has been studied by several authors. (See, for example, Artal et al. [3], [4], [6], Tokunaga [22], [23], [24], Degtyarev [8], [9], Degtyarev–Oka [10]). We conclude this paper by a remark on the computation of these invariants in §6.

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Conventions.

- A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a non-degenerate symmetric bilinear form $L \times L \to \mathbb{Z}$. For a subset R of a lattice L, we denote by $\langle R \rangle$ the submodule generated by R.
- Every (co)homology group is the singular (co)homology group with coefficients in $\mathbb{Z}$, unless otherwise stated.
- Let A be a finite abelian group. For a prime number p, we denote by A_p the p-part of A, and by $\mathrm{leng}_p(A)$ the minimal number of generators of A_p.
- For a smooth projective surface Y, we denote by $\mathrm{NS}(Y) \subset H^2(Y)/(\text{the torsion part})$ the *Néron–Severi lattice* of Y.

§2. Definition of the invariants $F_A(S, B, \gamma)$ and $T_A(S, B, \gamma)$

We fix a finite abelian group A once and for all.

Let S be a smooth simply-connected projective surface, and let B be a curve on S with the irreducible components $B_1, \dots, B_m$. We classify all étale Galois coverings of $S \setminus B$ with the Galois group A; that is, we describe all surjective homomorphisms $\pi_1(S \setminus B) \twoheadrightarrow A$. We have

$$H^2(B) = \bigoplus_{i=1}^{m} \mathbb{Z}[B_i].$$

Since S is smooth and projective, we have $H_1(S \setminus B) \cong H^3(S, B)$. Since S is simply-connected, we have $H^3(S) = 0$ and obtain an exact sequence

$$H^2(S) \xrightarrow{r} H^2(B) \longrightarrow H_1(S \setminus B) \longrightarrow 0,$$

where r is the restriction homomorphism. Hence all étale Galois coverings of $S \setminus B$ with the Galois group A are in one-to-one correspondence with the set

$$\mathcal{C}_A(S, B) := \left\{ \gamma \;\middle|\; \begin{array}{l} \gamma \text{ is a surjective homomorphism} \\ H^2(B) \twoheadrightarrow A \text{ such that } \operatorname{Im} r \subset \operatorname{Ker} \gamma \end{array} \right\}.$$

For an element γ of $\mathcal{C}_A(S, B)$, we denote by

$$\varphi_\gamma : W_\gamma \to S \setminus B$$

the étale Galois covering corresponding to γ.

Since S is simply-connected, $H^2(S)$ is torsion-free and we have a canonical isomorphism

$$H^2(S) \xrightarrow{\sim} \operatorname{Hom}(H^2(S), \mathbb{Z}) \tag{2.1}$$

by the cup-product. The restriction homomorphism

$$r_i : H^2(S) \to H^2(B_i) = \mathbb{Z}[B_i] \cong \mathbb{Z}$$

is given by $[B_i] \in H^2(S)$ under (2.1). If $\tau : (\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$ is a map of equi-configuration, then $[B_i] = [\tau(B_i)]$ holds in $H^2(S)$ and hence we have the following commutative diagram:

$$\begin{array}{ccc} H^2(S) & \xrightarrow{r} & H^2(B') \\ \| & & \downarrow \tau^* \\ H^2(S) & \xrightarrow{r} & H^2(B). \end{array}$$

Therefore τ induces a bijection

$$\tau_* : \mathcal{C}_A(S, B) \xrightarrow{\sim} \mathcal{C}_A(S, B').$$

Let

$$h : (S, B) \xrightarrow{\sim} (S, B')$$

be a homeomorphism. Restricting h to a tubular neighborhood $\mathcal{T}$ of B, we obtain a map of equi-configuration $h|_{\mathcal{T}}$, and hence we have a bijection

$$h^* = (h|_{\mathcal{T}}^{-1})_* : \mathcal{C}_A(S, B') \xrightarrow{\sim} \mathcal{C}_A(S, B).$$

For $\gamma \in \mathcal{C}_A(S, B')$, the étale Galois covering

$$\varphi_{h^*\gamma} : W_{h^*\gamma} \to S \setminus B$$

corresponding to $h^*\gamma \in \mathcal{C}_A(S, B)$ is obtained as the pull-back of the étale Galois covering $\varphi_\gamma : W_\gamma \to S \setminus B'$ by the homeomorphism of the complement $h : S \setminus B \cong S \setminus B'$. In particular, we see that $W_{h^*\gamma}$ is homeomorphic to W_γ.

Definition 2.1. A *smooth projective completion* of $\varphi_\gamma : W_\gamma \to S \setminus B$ is a morphism

$$\phi : X \to S$$

from a smooth projective surface X such that X contains W_γ as a Zariski open dense subset, and that ϕ extends $\varphi_\gamma : W_\gamma \to S \setminus B$.

Definition 2.2. A smooth projective completion $\phi : X \to S$ of $\varphi_\gamma : W_\gamma \to S \setminus B$ is said to be *A-equivariant* if the action of A on W_γ is extended to the action on X.

We choose a smooth projective completion $\phi : X \to S$ of φ_γ (not necessarily A-equivariant), and put

$$\mathcal{E}(X) := \left\{ \; E \subset X \; \middle| \; \begin{array}{l} E \text{ is a reduced irreducible curve on} \\ X \text{ such that } \phi(E) \text{ is a point on } S \end{array} \right\}.$$

We consider

$$H^2(X)' := H^2(X)/(\text{the torsion part})$$

as a lattice under the cup-product. In this lattice, we have two submodules

$$\begin{array}{rcl} \phi^*\mathrm{NS}(S) & = & \langle\, [\phi^*C] \mid C \text{ is a curve on } S \,\rangle, \quad \text{and} \\ \langle \mathcal{E}(X) \rangle & = & \langle\, [E] \mid E \in \mathcal{E}(X) \,\rangle, \end{array}$$

which are perpendicular to each other by the cup-product. Note that $\phi^*\mathrm{NS}(S)$ is a hyperbolic lattice by the Hodge index theorem, and that the intersection pairing on $\langle \mathcal{E}(X) \rangle$ is negative-definite by Mumford's result [11]. In particular, the cup-product is non-degenerate on

$$\Sigma(X) := \phi^* \mathrm{NS}(S) \oplus \langle \mathcal{E}(X) \rangle \subset H^2(X)',$$

that is, $\Sigma(X)$ is a sublattice of $H^2(X)'$. We denote by

$$\Lambda(X) := (\Sigma(X) \otimes \mathbb{Q}) \cap H^2(X)'$$

the primitive closure of $\Sigma(X)$ in $H^2(X)'$.

Definition 2.3. We put

$$F_A(S, B, \gamma) := \Lambda(X)/\Sigma(X),$$

which is a finite abelian group, and denote by

$$T_A(S, B, \gamma) := \Sigma(X)^\perp = \Lambda(X)^\perp \subset H^2(X)'$$

the orthogonal complement of $\Sigma(X)$, which is a primitive sublattice of $H^2(X)'$.

Proposition 2.4. *Neither the isomorphism class of the finite abelian group $F_A(S, B, \gamma)$ nor the isomorphism class of the lattice $T_A(S, B, \gamma)$ does depend on the choice of the smooth projective completion $\phi : X \to S$ of $\varphi_\gamma : W_\gamma \to S \setminus B$.*

Proof. Suppose that $\phi' : X' \to S$ is another smooth projective completion of $\varphi_\gamma : W_\gamma \to S \setminus B$. Then there is a commutative diagram

$$\begin{tikzcd} & X'' \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & X' \arrow[dl] \\ & S & \end{tikzcd},$$

where X'' is a smooth projective surface, and $X'' \to X$ and $X'' \to X'$ are birational morphisms that are isomorphisms over $S \setminus B$. Since a birational morphism between smooth surfaces are composite of blowing-ups at points, we obtain orthogonal direct-sum decompositions

$$\begin{aligned} &(2.2) \qquad \Sigma(X'') &= \; &\Sigma(X) \quad \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle \quad \text{and} \\ &(2.3) \qquad H^2(X'')' &= \; &H^2(X)' \oplus \langle e_1 \rangle \oplus \cdots \oplus \langle e_N \rangle, \end{aligned}$$

where $e_1, \ldots, e_N$ are classes with $e_i^2 = -1$. Hence we obtain

$$\Lambda(X)/\Sigma(X) \cong \Lambda(X'')/\Sigma(X'') \quad \text{and} \quad \Sigma(X)^\perp \cong \Sigma(X'')^\perp.$$

The same isomorphisms hold between X' and X''. Q.E.D.

We investigate the action of A on these invariants.

Proposition 2.5. *There always exists an A-equivariant smooth projective completion.*

For the proof, we need the following:

Lemma 2.6. *There exist a vector bundle $\eta_\gamma : V_\gamma \to S$ on S and a closed subvariety $\overline{W}_\gamma \subset V_\gamma$ finite over S such that A acts on V_γ over S, that $\overline{W}_\gamma$ is stable under this action, and that there exists an A-equivariant isomorphism $W_\gamma \cong \overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$.*

Proof. First we prove the case where A is cyclic of order d. We choose a generator g of A and fix an isomorphism $A \cong \mathbb{Z}/d\mathbb{Z}$ by $g \mapsto 1$. We also embed A into $\mathbb{C}^\times$ by $g \mapsto \exp\left(2\pi\sqrt{-1}/d\right)$. Let a_i be an integer such that

$$\gamma([B_i]) \equiv a_i \bmod d$$

in $A = \mathbb{Z}/d\mathbb{Z}$. Recall that the restriction map $r_i : H^2(S) \to H^2(B_i) \cong \mathbb{Z}$ is given by $[B_i] \in H^2(S)$ under (2.1). The condition $\operatorname{Im} r \subset \operatorname{Ker}\gamma$ for γ implies that there exists a line bundle $\eta_\gamma : V_\gamma \to S$ on S such that

$$a_1[B_1] + \cdots + a_m[B_m] = d\,[V_\gamma]$$

holds in $\operatorname{Pic}(S) \subset H^2(S)$. We have a section s of $V_\gamma^{\otimes d}$ such that $s = 0$ defines the divisor

$$a_1 B_1 + \cdots + a_m B_m.$$

We denote by $S_\gamma \subset V_\gamma^{\otimes d}$ the image of the section $s : S \to V_\gamma$. We have a morphism

$$\delta : V_\gamma \to V_\gamma^{\otimes d}$$

given by $\xi \mapsto \xi^d$, where ξ is a fiber coordinate of V_γ. Let $\overline{W}_\gamma$ be the pull-back of S_γ by δ. Then W_γ is isomorphic to $\overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$. The natural action of $\mathbb{C}^\times$ on V_γ and the embedding $A \hookrightarrow \mathbb{C}^\times$ induces an A-action on V_γ over S, under which $\overline{W}_\gamma$ is stable and the isomorphism $W_\gamma \xrightarrow{\sim} \overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$ is A-equivariant.

In the general case, we decompose A into a direct sum of cyclic groups $A \cong A_1 \times \cdots \times A_l$, and let

$$\gamma(j) : H^2(B) \to A_j$$

be the composite of γ with the projection $A \to A_j$. We put

$$V_\gamma := V_{\gamma(1)} \oplus \cdots \oplus V_{\gamma(l)},$$

on which A acts over S, and define the closed subvariety $\overline{W}_\gamma \subset V_\gamma$ by

$$\overline{W}_\gamma = \{\ (\xi_1, \dots, \xi_l) \in V_\gamma \ \mid\ \xi_j \in \overline{W}_{\gamma(j)} \subset V_{\gamma(j)} \ \text{ for } j = 1, \dots, l\ \},$$

which is stable under the action of A. Then W_γ is A-equivariantly isomorphic to $\overline{W}_\gamma \cap \eta_\gamma^{-1}(S \setminus B)$. Q.E.D.

Proof of Proposition 2.5. By means of the celebrated theorem of Villamayor [25, Corollary 7.6.3], we can make an equivariant embedded desingularization of $\overline{W}_\gamma \subset V_\gamma$. Q.E.D.

Combining Propositions 2.4 and 2.5, we obtain the following:

Corollary 2.7. *The Galois group A acts on the finite abelian group $F_A(S, B, \gamma)$ and on the lattice $T_A(S, B, \gamma)$.*

§3. The invariant $T_A(S, B, \gamma)$

The invariant $T_A(S, B, \gamma)$ is a topological invariant. Recall that a homeomorphism

$$h : (S, B) \xrightarrow{\sim} (S, B')$$

induces a bijection $h^* : \mathcal{C}_A(S, B') \xrightarrow{\sim} \mathcal{C}_A(S, B)$.

Theorem 3.1. *Suppose that the classes $[B_i]$ of the irreducible components of B span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$. If $h : (S, B) \xrightarrow{\sim} (S, B')$ is a homeomorphism, then the lattices $T_A(S, B, h^*\gamma)$ and $T_A(S, B', \gamma)$ are isomorphic.*

Proof. Remark that the classes $[h(B_i)]$ of the irreducible components of B' also span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$.

Since $W_{h^*\gamma}$ is homeomorphic to W_γ, it is enough to show that the lattice $T_A(S, B, \gamma)$ is determined by the homeomorphism type of the open surface W_γ. We consider the intersection pairing

$$\iota_W : H_2(W_\gamma) \times H_2(W_\gamma) \to \mathbb{Z},$$

which may be degenerate since W_γ is not compact. We put

$$\mathrm{Ker}(\iota_W) := \{\ x \in H_2(W_\gamma) \ \mid\ \iota_W(x, y) = 0 \ \text{ for all } \ y \in H_2(W_\gamma)\ \}.$$

Then ι_W induces a non-degenerate symmetric bilinear form

$$\bar{\iota}_W \ :\ H_2(W_\gamma)/\mathrm{Ker}(\iota_W) \ \times\ H_2(W_\gamma)/\mathrm{Ker}(\iota_W) \ \to\ \mathbb{Z}$$

on the free $\mathbb{Z}$-module $H_2(W_\gamma)/\mathrm{Ker}(\iota_W)$. Since the lattice

$$(H_2(W_\gamma)/\mathrm{Ker}(\iota_W),\ \bar{\iota}_W)$$

is determined by the homeomorphism type of W_γ, the proof is completed by Proposition 3.2 below, which was proved in a slightly different situation in [16] and [19]. Q.E.D.

Proposition 3.2. *Suppose that the classes* $[B_i]$ *span* $\mathrm{NS}(S)\otimes\mathbb{Q}$ *over* $\mathbb{Q}$. *Then the lattice* $H_2(W_\gamma)/\operatorname{Ker}(\iota_W)$ *is isomorphic to* $T_A(S,B,\gamma)$.

Proof. We put $D := X \setminus W_\gamma$, and let $D_1, \dots, D_M$ be the reduced irreducible components of D. Since the classes $[B_i]$ span $\mathrm{NS}(S)\otimes\mathbb{Q}$, the classes $[D_1], \dots, [D_M]$ span $\Sigma(X)\otimes\mathbb{Q}$ over $\mathbb{Q}$. We put

$$\widetilde{T} := \{\, x \in H_2(X) \mid (x, [D_i])_X = 0 \quad \text{for all } i = 1, \dots, M \,\},$$

where $(\ ,\ )_X$ is the intersection pairing on X. Then we have an isomorphism

$$T_A(S,B,\gamma) \cong \widetilde{T}/(\text{the torsion part})$$

of lattices. The image of the homomorphism

$$j_* : H_2(W_\gamma) \to H_2(X)$$

induced by $j : W_\gamma \hookrightarrow X$ is contained in $\widetilde{T}$. Note that, by definition, the homomorphism j_* preserves the intersection pairings. On the other hand, from the Poincaré–Lefschetz duality isomorphisms

$$H_2(W_\gamma) \cong H^2(X,D) \quad \text{and} \quad H_2(X) \cong H^2(X)$$

and the cohomology exact sequence

$$H^2(X,D) \ \to\ H^2(X) \ \to\ H^2(D) = \bigoplus \mathbb{Z}[D_i],$$

we see that every homology class in $\widetilde{T}$ is represented by a topological 2-cycle on W_γ. Thus the inclusion j induces a surjective homomorphism

$$\bar{j}_* : H_2(W_\gamma) \twoheadrightarrow T_A(S,B,\gamma),$$

which preserves the intersection pairings. Since the symmetric bilinear form on $T_A(S,B,\gamma)$ is non-degenerate, we can easily prove that $\operatorname{Ker}\bar{j}_*$ is equal to $\operatorname{Ker}(\iota_W)$. Q.E.D.

Definition 3.3. A plane curve $B \subset \mathbb{P}^2$ of degree 6 is called a *simple sextic* if B has only simple singularities. A simple sextic B is called a *maximizing sextic* if the total Milnor number $\mu(B)$ attains the possible maximum 19.

Example 3.4. Suppose that $B \subset \mathbb{P}^2$ is a maximizing sextic. We consider the double covering of $\mathbb{P}^2$ corresponding to

$$\gamma : H^2(B) \to A = \mathbb{Z}/2\mathbb{Z}$$

such that $\gamma([B_i]) \neq 0$ for any B_i. Then we have a $K3$ surface with the Picard number being the possible maximum 20 (i.e. a *singular* $K3$ surface in the sense of Shioda) as a smooth projective completion X of $\varphi_\gamma : W_\gamma \to \mathbb{P}^2 \setminus B$, and the invariant $T_A(\mathbb{P}^2, B, \gamma)$ is the *transcendental lattice* of X, which is a positive-definite even lattice of rank 2. Using the result of transcendental lattices of conjugate $K3$ surfaces with the maximal Picard number [14, 17], we have obtained many arithmetic Zariski pairs of degree 6 in [19].

In [1], we have exhibited a pair $[B_+, B_-]$ of maximizing sextics with the singularities of type $A_9 + A_{10}$ that are defied over $\mathbb{Q}(\sqrt{5})$ and are conjugate under the action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. The invariants T_A for them are calculated as follows:

$$T_A(\mathbb{P}^2, B_+, \gamma) \cong \begin{bmatrix} 2 & 1 \\ 1 & 28 \end{bmatrix}, \quad T_A(\mathbb{P}^2, B_-, \gamma) \cong \begin{bmatrix} 8 & 3 \\ 3 & 8 \end{bmatrix}.$$

§4. The invariants $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$

We investigate the algebraicity of the invariant $F_A(S, B, \gamma)$. For $\sigma \in \mathrm{Aut}(\mathbb{C})$ and $\gamma \in \mathcal{C}_A(S, B)$, we denote by $\gamma^\sigma \in \mathcal{C}_A(S^\sigma, B^\sigma)$ the element corresponding to the étale Galois covering of $S^\sigma \setminus B^\sigma$ obtained as the pull-back of the morphism $\varphi_\gamma : W_\gamma \to S \setminus B$ over $\mathrm{Spec}\, \mathbb{C}$ by $\sigma^* : \mathrm{Spec}\, \mathbb{C} \to \mathrm{Spec}\, \mathbb{C}$; that is,

$$\gamma^\sigma : H^2(B^\sigma) \to A$$

is given by $\gamma^\sigma([B_i^\sigma]) = \gamma([B_i])$, where B_i^σ is the conjugate of B_i by σ. The following is obvious from the definition:

Proposition 4.1. *For any $\sigma \in \mathrm{Aut}(\mathbb{C})$, the finite abelian groups $F_A(S, B, \gamma)$ and $F_A(S^\sigma, B^\sigma, \gamma^\sigma)$ are isomorphic.*

Next we define a new invariant $\sigma_A(S, B, \gamma)$, which is an invariant of the configuration type of B. The invariants $T_A(S, B, \gamma)$ and $F_A(S, B, \gamma)$ are related via this invariant.

We recall the definition of the *discriminant group* of a lattice. Let L be a lattice. Then we can canonically embed L into its dual lattice

$$L^\vee := \mathrm{Hom}(L, \mathbb{Z}).$$

The *discriminant group* $\operatorname{disc}(L)$ of L is defined by

$$\operatorname{disc}(L) := L^{\vee}/L.$$

Proposition 4.2. *The isomorphism class of the discriminant group* $\operatorname{disc}(\Sigma(X))$ *does not depend on the choice of the smooth projective completion* $\phi : X \to S$ *of* $\varphi_\gamma : W_\gamma \to S \setminus B$.

Proof. The discriminant group of a lattice $\langle e \rangle$ of rank 1 with $e^2 = -1$ is trivial. Proposition 4.2 then follows from (2.2) by the same argument as in the proof of Proposition 2.4. Q.E.D.

Thus the following is well-defined:

$$\sigma_A(S, B, \gamma) := \operatorname{disc}(\Sigma(X)).$$

We will show that $\sigma_A(S, B, \gamma)$ is an invariant of the configuration type.

Theorem 4.3. *Suppose that* $\tau : (\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$ *is a map of equi-configuration. Then* $\sigma_A(S, B, \gamma)$ *is isomorphic to* $\sigma_A(S, B', \tau_*\gamma)$.

For the proof, we recall the definition of *equisingularity* of plane curve singularities. See [26, Proposition 4.3.9] for details.

Let $P \in \operatorname{Sing} B$ be a singular point of B, and let $P' \in \operatorname{Sing} B'$ be a singular point of B'. Let $B^{(1)}, \dots, B^{(k)}$ be the local branches of B at P, and let $B'^{(1)}, \dots, B'^{(k')}$ be the local branches of B' at P'.

Definition 4.4. We say that the two germs (B, P) and (B', P') of the plane curve singularity are *equisingular* if $k = k'$ holds and there exists a bijection from $\{B^{(1)}, \dots, B^{(k)}\}$ to $\{B'^{(1)}, \dots, B'^{(k')}\}$, given by $B^{(\kappa)} \mapsto B'^{(\kappa)}$ after permutations of indices, such that $B^{(\kappa)}$ and $B'^{(\kappa)}$ have the same Puiseux characteristic for $\kappa = 1, \dots, k$ and that the equalities of intersection numbers $B^{(i)} \cdot B^{(j)} = B'^{(i)} \cdot B'^{(j)}$ hold for all $i \neq j$.

Proof of Theorem 4.3. By the equivalence of (i) and (iv) in [26, Theorem 5.5.9], we see that (B, P) and $(B', \tau(P))$ are equisingular for any singular point P of B. Let $\mu : (\widetilde{S}, \widetilde{B}) \to (S, B)$ be the minimal good embedded resolution of B, and let $\mu' : (\widetilde{S}', \widetilde{B}') \to (S, B')$ be the minimal good embedded resolution of B'. (See [26, §3.4] for the definition of minimal good embedded resolutions.) Note that μ induces an analytic isomorphism $\widetilde{S} \setminus \widetilde{B} \cong S \setminus B$, and hence induces a bijection

$$\mu_* : \mathcal{C}_A(\widetilde{S}, \widetilde{B}) \xrightarrow{\sim} \mathcal{C}_A(S, B)$$

via the isomorphism $\mu_* : \pi_1(\widetilde{S} \setminus \widetilde{B}) \xrightarrow{\sim} \pi_1(S \setminus B)$. By Theorem 8.1.7 or Proposition 8.3.1 of [26], we have a map of equi-configuration

$$\tilde{\tau} : (\widetilde{T}, \widetilde{B}) \to (\widetilde{T}', \widetilde{B}'),$$

which induces a commutative diagram of bijections

$$\begin{array}{ccc} \mathcal{C}_A(\widetilde{S}, \widetilde{B}) & \xrightarrow{\mu_*} & \mathcal{C}_A(S, B) \\ \tilde{\tau}_* \downarrow & & \downarrow \tau_* \\ \mathcal{C}_A(\widetilde{S}', \widetilde{B}') & \xrightarrow{\mu'_*} & \mathcal{C}_A(S, B'). \end{array}$$

A smooth projective completion $\widetilde{X} \to \widetilde{S}$ of an étale Galois covering $W_{\tilde{\gamma}} \to \widetilde{S} \setminus \widetilde{B}$ corresponding to $\tilde{\gamma} \in \mathcal{C}_A(\widetilde{S}, \widetilde{B})$ is a smooth projective completion of $W_{\mu_*\tilde{\gamma}} \to S \setminus B$. Therefore, by Proposition 4.2, it is enough to prove

$$\sigma_A(\widetilde{S}, \widetilde{B}, \tilde{\gamma}) \cong \sigma_A(\widetilde{S}', \widetilde{B}', \tilde{\tau}_*\tilde{\gamma})$$

for any $\tilde{\gamma} \in \mathcal{C}_A(\widetilde{S}, \widetilde{B})$; that is,*we can assume that B and B' are normal crossing divisors on S.*

Suppose that B and B' are normal crossing divisors. Recall the *finite* covering

$$\bar{\varphi}_\gamma : \overline{W}_\gamma \to S$$

constructed in Lemma 2.6. Let $\nu : Y_\gamma \to \overline{W}_\gamma$ be the normalization of $\overline{W}_\gamma$, and consider the finite covering

$$\bar{\varphi}_\gamma \circ \nu : Y_\gamma \to S.$$

Then $\operatorname{Sing} Y_\gamma$ is located over $\operatorname{Sing} B$. If $P \in \operatorname{Sing} B$ is an intersection point of B_i and B_j, then the number and the analytic isomorphism classes of singular points of Y_γ over P are determined by $\gamma([B_i]) \in A$ and $\gamma([B_j]) \in A$. We construct the finite covering

$$\bar{\varphi}_{\tau_*\gamma} \circ \nu' : Y_{\tau_*\gamma} \to S$$

of S by a normal surface $Y_{\tau_*\gamma}$ branching along B' in the same way. Then there exists a bijection

$$\operatorname{Sing} Y_\gamma \cong \operatorname{Sing} Y_{\tau_*\gamma}$$

that covers the bijection $\operatorname{Sing} B \cong \operatorname{Sing} B'$ by τ and preserves the analytic isomorphism classes of the surface singularities. Hence there exist desingularizations

$$X_\gamma \to Y_\gamma, \quad \text{and} \quad X_{\tau_*\gamma} \to Y_{\tau_*\gamma}$$

such that the sets $\mathcal{E}(X_\gamma)$ and $\mathcal{E}(X_{\tau_*\gamma})$ of exceptional curves have the same configuration. Therefore we have

$$\Sigma(X_\gamma) \cong \Sigma(X_{\tau_*\gamma}).$$

By Proposition 4.2, we complete the proof. Q.E.D.

The isomorphism class of the discriminant group $\operatorname{disc}(\Lambda(X))$ of the primitive closure $\Lambda(X)$ of $\Sigma(X)$ is also independent of the choice of the smooth projective completion X. More precisely, we have the following:

Proposition 4.5. *The discriminant group* $\operatorname{disc}(\Lambda(X))$ *is isomorphic to* $\operatorname{disc}(T_A(S, B, \gamma))$ *for any smooth projective completion* X*.*

The proof follows from Lemma 4.6 below and the fact that the lattice $H^2(X)'$ is unimodular.

Lemma 4.6. *Let* L *and* L' *be primitive sublattices of a unimodular lattice* M *such that* $L \perp L'$ *and that* $[M : L \oplus L'] < \infty$*. Then* $\operatorname{disc}(L)$ *and* $\operatorname{disc}(L')$ *are isomorphic.*

Lemma 4.6 is [12, Proposition 1.6.1] without the assumption that lattices be even. See also the proof of [17, Proposition 2.1.1].

The three invariants $T_A(S, B, \gamma)$, $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$ are related as follows:

Proposition 4.7. *For any* $\gamma \in \mathcal{C}_A(S, B)$*, we have*

$$|\operatorname{disc}(T_A(S, B, \gamma))| \cdot |F_A(S, B, \gamma)|^2 = |\sigma_A(S, B, \gamma)|.$$

Moreover, for any prime integer p*, we have*

$$\begin{aligned} \operatorname{leng}_p(\operatorname{disc}(T_A(S, B, \gamma))) &\leq \operatorname{leng}_p(\sigma_A(S, B, \gamma)) \\ &\leq \operatorname{leng}_p(\operatorname{disc}(T_A(S, B, \gamma))) + 2 \operatorname{leng}_p(F_A(S, B, \gamma)). \end{aligned}$$

This proposition follows from the following elementary lemma [12] and Proposition 4.5.

Lemma 4.8. *Let* L *be a lattice, and let* M *be a sublattice of* L *with finite index. Then we have*

$$M \subset L \subset L^\vee \subset M^\vee.$$

Since $M^\vee / L^\vee \cong L/M$*, we have* $|\operatorname{disc}(M)| = |\operatorname{disc}(L)| \cdot [L : M]^2$*, and*

$$\operatorname{leng}_p(\operatorname{disc}(L)) \leq \operatorname{leng}_p(\operatorname{disc}(M)) \leq \operatorname{leng}_p(\operatorname{disc}(L)) + 2 \operatorname{leng}_p(L/M).$$

As a corollary of Theorems 3.1, 4.3 and Proposition 4.7, we obtain the following generalization of [18, Theorem 8.5] and the idea of Xie and Yang in [27]. This corollary shows that the algebraic invariant $F_A(S, B, \gamma)$ can be used to distinguish the topological types of B.

Corollary 4.9. *Suppose that the classes* $[B_i]$ *span* $\mathrm{NS}(S) \otimes \mathbb{Q}$ *over* $\mathbb{Q}$. *Let* $\tau : (\mathcal{T}, B) \xrightarrow{\sim} (\mathcal{T}', B')$ *be a map of equi-configuration. If we have* $|F_A(S, B, \gamma)| \neq |F_A(S, B', \tau_* \gamma)|$, *then* τ *cannot be extended to a homeomorphism* $(S, B) \xrightarrow{\sim} (S, B')$.

Example 4.10. Let B and B' be the plane curves of degree 6 in Example 1.6. Consider the finite abelian group $A = \mathbb{Z}/2\mathbb{Z}$. Then each of $\mathcal{C}_A(\mathbb{P}^2, B)$ and $\mathcal{C}_A(\mathbb{P}^2, B')$ consists of a single element γ. We have $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}$ while $F_A(\mathbb{P}^2, B', \gamma) = 0$.

The six-cuspidal sextic B is defined by the torus-type equation

$$f^3 + g^2 = 0,$$

where $\deg f = 2$ and $\deg g = 3$, and f and g are chosen generally. The conic Q defined by $f = 0$ passes through $\operatorname{Sing} B$, and hence B is called a *conical six-cuspidal sextic*. The proper transform of Q by $\phi : X \to \mathbb{P}^2$ splits into the union two irreducible components $\tilde{Q}^+$ and $\tilde{Q}^-$. The class $[\tilde{Q}^+]$ is contained in the primitive closure $\Lambda(X)$, and $F_A(\mathbb{P}^2, B, \gamma) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by $[\tilde{Q}^+]$.

On the other hand, there exist no conics passing through the 6 cusps $\operatorname{Sing} B'$. The existence of such a non-conical six-cuspidal sextic B' was stated by Del Pezzo without proof, and was proved by B. Segre (see [21, page 407]). Zariski also proved the existence in [28]. The explicit defining equation of a non-conical six-cuspidal sextics was given by Oka [13].

Many Zariski pairs of simple sextics have been discovered in [18] and by Xie and Yang in [27] using the idea of Corollary 4.9.

We also have the following corollary, which plays an important role in the next section:

Corollary 4.11. *Let p be a prime integer. If we have*

$$\operatorname{leng}_p(\operatorname{disc}(T_A(S, B, \gamma))) < \operatorname{leng}_p(\sigma_A(S, B, \gamma)),$$

then we have $F_A(S, B, \gamma)_p \neq 0$. *In particular, if*

$$\operatorname{rank}(T_A(S, B, \gamma)) < \operatorname{leng}_p(\sigma_A(S, B, \gamma))$$

holds, then $F_A(S, B, \gamma)_p \neq 0$.

The second assertion follows from the observation that, for a lattice L, we have $\operatorname{leng}_p(\operatorname{disc}(L)) \leq \operatorname{rank}(L)$.

§5. The fundamental group $\pi_1(S \setminus B)$

In this section, we give a result on a relation between $\pi_1(S \setminus B)$ and the invariants $T_A(S, B, \gamma)$, $F_A(S, B, \gamma)$ and $\sigma_A(S, B, \gamma)$.

Definition 5.1. Let M be an abelian group, and let G be a group. Suppose that there exists an exact sequence

$$(5.1) \qquad 0 \longrightarrow M \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

Then we have an action $\sigma_\Gamma : G \to \mathrm{Aut}(M)$ of G on M defined by

$$\bar{\gamma}(a) := \gamma a \gamma^{-1},$$

where $\bar{\gamma} \in G$ is the image of $\gamma \in \Gamma$, and M is regarded as a normal subgroup of Γ. We call σ_Γ the *action associated with* (5.1).

In this section, we put

$$U := S \setminus B,$$

and fix a base point $b \in U$. Let $\varphi : W \to U$ be a finite étale Galois covering with the Galois group G, which is not necessarily abelian. Then the group G acts on W and hence on $H_1(W)$ in a natural way. Let

$$N := \mathrm{Ker}(\rho : \pi_1(U, b) \twoheadrightarrow G),$$

be the kernel of the surjective homomorphism $\rho : \pi_1(U, b) \twoheadrightarrow G$ associated with φ. Then N is (non-canonically) isomorphic to the fundamental group of W, and $H_1(W)$ is *canonically* identified with $N/[N, N]$.

The following is well-known, for example, in the study of Alexander polynomials [7].

Proposition 5.2. *The action of G on $H_1(W)$ is associated with the exact sequence*

$$(5.2) \qquad 0 \to H_1(W) \to \pi_1(U, b)/[N, N] \to G \to 1.$$

Corollary 5.3. *Suppose that there exists a finite étale Galois covering $W \to U$ with the Galois group G acting on $H_1(W)$ non-trivially. Then $\pi_1(U, b)$ is non-abelian.*

Corollary 5.4. *Let Γ be a group that fits in an exact sequence*

$$0 \longrightarrow M \longrightarrow \Gamma \xrightarrow{g} G \longrightarrow 1$$

with M being abelian. Suppose that there is a surjective homomorphism $\gamma : \pi_1(U, b) \twoheadrightarrow \Gamma$. Let $W \to U$ be the finite étale Galois covering associated with the composite $g \circ \gamma : \pi_1(U, b) \twoheadrightarrow G$. Then there exists a surjective homomorphism of G-modules $H_1(W) \twoheadrightarrow M$, where M is considered as a G-module by σ_Γ.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(W) & \to & \pi_1(U) & \to & G & \to & 1 \quad \text{(exact)} \\ & & & & \downarrow\!\!\downarrow & & \| & & \\ 0 & \to & M & \to & \Gamma & \to & G & \to & 1 \quad \text{(exact).} \end{array}$$

Hence we have a surjective homomorphism $\pi_1(W) \twoheadrightarrow M$, which factors through the homomorphism of G-modules $H_1(W) \twoheadrightarrow M$. Q.E.D.

We now return to the finite abelian Galois covering

$$\varphi_\gamma : W_\gamma \to U = S \setminus B$$

with the Galois group $G = A$ associated with an element $\gamma \in \mathcal{C}_A(S, B)$. Let $\phi : X \to S$ be a smooth projective completion. We put

$$D := X \setminus W_\gamma = \phi^{-1}(B),$$

and let $D_1, \dots, D_M$ be the reduced irreducible components of D. We consider the submodule

$$\Theta(X) := \langle [D_1], \dots, [D_M] \rangle \subset H^2(X)'$$

of $H^2(X)'$ generated by $[D_1], \dots, [D_M]$, and its primitive closure

$$\Xi(X) := (\Theta(X) \otimes \mathbb{Q}) \cap H^2(X)'.$$

We put

$$F'_A(S, B, \gamma) := \Xi(X)/\Theta(X).$$

We can prove the following by the same argument as Proposition 2.4:

Proposition 5.5. *The isomorphism class of the finite abelian group $F'_A(S, B, \gamma)$ is independent of the choice of the smooth projective completion $\phi : X \to S$.*

Therefore, by choosing an A-equivariant smooth completion, we see that A acts on $F'_A(S, B, \gamma)$.

Proposition 5.6. *There exists a natural A-equivariant embedding*

$$F'_A(S,B,\gamma)^\vee \hookrightarrow H_1(W_\gamma),$$

where $F'_A(S,B,\gamma)^\vee := \mathrm{Hom}(F'_A(S,B,\gamma), \mathbb{Q}/\mathbb{Z})$.

Proof. We have a canonical isomorphism $H_1(W_\gamma) \cong H^3(X,D)$. Hence the cokernel of the restriction homomorphism

$$r_X : H^2(X) \to H^2(D) = \bigoplus \mathbb{Z}[D_i]$$

is contained in $H_1(W_\gamma)$. Note that r_X factors through

$$s : H^2(X)' \to H^2(D)$$

and that $H^2(X)'$ is a unimodular lattice by the cup-product. Hence $H^2(X)'$ is self-dual. The submodule $\Theta(X)$ is the image of the dual homomorphism

$$s^\vee : H^2(D)^\vee \to H^2(X)'$$

of s. Thus we have a decomposition

$$H^2(D)^\vee \twoheadrightarrow \Theta(X) \hookrightarrow \Xi(X) \hookrightarrow H^2(X)'$$

of $s^\vee$, where $H^2(D)^\vee = \mathrm{Hom}(H^2(D), \mathbb{Z})$. The dual homomorphism $H^2(X)' \to \Xi(X)^\vee$ of the primitive embedding $\Xi(X) \hookrightarrow H^2(X)'$ is surjective. The dual homomorphism $\Xi(X)^\vee \to \Theta(X)^\vee$ of $\Theta(X) \hookrightarrow \Xi(X)$ is injective and its cokernel is canonically isomorphic to

$$F'_A(S,B,\gamma)^\vee = \mathrm{Hom}(\Xi(X)/\Theta(X), \mathbb{Q}/\mathbb{Z}).$$

The dual homomorphism $\Theta(X)^\vee \to H^2(D)$ of the surjective homomorphism $H^2(D)^\vee \twoheadrightarrow \Theta(X)$ is injective. Thus $\mathrm{Coker}(s) = \mathrm{Coker}(r_X)$ contains $F'_A(S,B,\gamma)^\vee$ in a natural way, and hence so does $H_1(W_\gamma)$. Q.E.D.

We investigate the relation between $F'_A(S,B,\gamma)$ and $F_A(S,B,\gamma)$.

Definition 5.7. For a reduced irreducible curve F on S, the *strict transform of* F is the total transform of F by $\phi : X \to S$ minus the components that are contracted to points by ϕ.

Remark that the class of the strict transform of any reduced irreducible curve on S is contained in $\Sigma(X)$.

Suppose that A is a cyclic group of prime order l. Then, for any reduced irreducible curve F on S, the strict transform of F is either reduced, or of the form lC with C being reduced and irreducible. The later occurs if and only if F is an irreducible component B_i of B such that $\gamma([B_i]) \neq 0$ in $A \cong \mathbb{Z}/l\mathbb{Z}$.

Assumption 5.8. We consider the following assumptions:

(a) the finite abelian group A is cyclic of prime order l,
(b) the classes $[B_1], \dots, [B_m]$ span $\mathrm{NS}(S) \otimes \mathbb{Q}$ over $\mathbb{Q}$, and
(c) $\gamma([B_i]) \neq 0$ for $i = 1, \dots, m$.

Proposition 5.9. *Suppose that Assumption 5.8 holds. Then, for any prime $p \neq l$, there exists a surjective homomorphism of A-modules from $F'_A(S, B, \gamma)_p$ to $F_A(S, B, \gamma)_p$.*

Proof. The assumption (b) implies that $\Theta(X) \otimes \mathbb{Q} = \Sigma(X) \otimes \mathbb{Q}$. Hence we have $\Xi(X) = \Lambda(X)$. Moreover $\Theta(X) \cap \Sigma(X)$ is of finite index in $\Lambda(X)$. We put

$$\widetilde{F}_A := \Lambda(X)/(\Theta(X) \cap \Sigma(X)).$$

The assumptions (a) and (c) imply that

$$\Theta(X)/(\Theta(X) \cap \Sigma(X)) = \mathrm{Ker}(\widetilde{F}_A \twoheadrightarrow F'_A(S, B, \gamma))$$

is an elementary l-group. Indeed, if $D_i \in \mathcal{E}(X)$, then $[D_i] \in \Sigma(X)$, while if $D_i \notin \mathcal{E}(X)$, then D_i is the reduced part of the strict transform of an irreducible component B_j of B, and hence $l[D_i] \in \Sigma(X)$. In particular, the natural projection $\widetilde{F}_A \twoheadrightarrow F'_A(S, B, \gamma)$ induces $(\widetilde{F}_A)_p \cong F'_A(S, B, \gamma)_p$ for $p \neq l$. Therefore the natural projection

$$\widetilde{F}_A \twoheadrightarrow F_A(S, B, \gamma)$$

induces a surjective homomorphism from $F'_A(S, B, \gamma)_p$ to $F_A(S, B, \gamma)_p$ for any $p \neq l$. Q.E.D.

On the other hand, we have the following:

Proposition 5.10. *Suppose that Assumption 5.8 holds. If the order of a non-zero element $f \in F_A(S, B, \gamma)$ is not equal to l, then A acts on f non-trivially.*

Proof. We choose an A-equivariant smooth projective completion $\phi : X \to S$. Suppose that R is a divisor on X such that

$$f = [R] \bmod \Sigma(X).$$

Let H be an ample divisor on S. Since $[\phi^* H] \in \Sigma(X)$, we can replace R by $R + n(\phi^* H)$ with sufficiently large n if necessary, and assume that R is effective. We write

$$R = R_1 + \cdots + R_N,$$

where $R_1, \dots, R_N$ are reduced and irreducible. Since $\langle \mathcal{E}(X) \rangle \subset \Sigma(X)$, we can assume that each R_i is not in $\mathcal{E}(X)$ and hence is mapped by ϕ to a curve on S. Let $\overline{R}_i$ be the reduced irreducible curve on S that is the image of R_i. Let d_i be the degree of $R_i \to \overline{R}_i$, which is either 1 or l. The divisor $\sum_{g \in A} g(R_i)$ on X is equal to d_i times the strict transform of $\overline{R}_i$, and hence its class is contained in $\Sigma(X)$. Therefore we have $\sum_{g \in A} g(f) = 0$. Since the order of $f \neq 0$ is not equal to $|A| = l$, we have $g(f) \neq f$. Q.E.D.

Combining all the results, we obtain the following:

Corollary 5.11. *Suppose that Assumption 5.8 holds. If we have $F_A(S, B, \gamma)_p \neq 0$ for some $p \neq l$, then $\pi_1(S \setminus B)$ acts on $H_1(W_\gamma)$ non-trivially and hence is non-abelian.*

By Corollary 4.11, we obtain the following:

Corollary 5.12. *Suppose that Assumption 5.8 holds. If we have*

$$\operatorname{leng}_p(\sigma_A(S, B, \gamma)) > \operatorname{leng}_p(\operatorname{disc}(T_A(S, B, \gamma)))$$

for some $p \neq l$, then $\pi_1(S \setminus B)$ is non-abelian. In particular, if

$$\operatorname{leng}_p(\sigma_A(S, B, \gamma)) > \operatorname{rank}(T_A(S, B, \gamma))$$

for some $p \neq l$, then $\pi_1(S \setminus B)$ is non-abelian.

We apply these corollaries to the double covering of $\mathbb{P}^2$ branching along a curve with only simple singularities. Let $B \subset \mathbb{P}^2$ be a plane curve of even degree d. Consider the double covering $\varphi_\gamma : W_\gamma \to \mathbb{P}^2 \setminus B$ corresponding to $\gamma : H^2(B) \to \mathbb{Z}/2\mathbb{Z}$ with $\gamma([B_i]) \neq 0$ for any irreducible component B_i of B. Suppose that B has only simple singularities, and let μ_B be the total Milnor number of $\operatorname{Sing} B$. Then the normal surface Y_γ constructed in the proof of Theorem 4.3 has only rational double points of the total Milnor number equal to μ_B. We choose the minimal resolution X of Y_γ as the smooth projective completion of $\varphi_\gamma : W_\gamma \to \mathbb{P}^2 \setminus B$. Then we have

$$\begin{aligned} &\operatorname{rank}(\Sigma(X)) = 1 + \mu_B \quad \text{and} \\ &b_2(X) = \operatorname{rank}(\Sigma(X)) + \operatorname{rank}(T_A(\mathbb{P}^2, B, \gamma)) = d^2 - 3d + 4. \end{aligned}$$

Therefore we obtain the following corollary, which has been proved in [23].

Corollary 5.13. *If $\mu_B + \operatorname{leng}_p(\sigma_A(\mathbb{P}^2, B, \gamma)) > d^2 - 3d + 3$ for some odd prime p, then $\pi_1(\mathbb{P}^2 \setminus B)$ is non-abelian.*

See [23] also for various applications of this corollary.

Note that $\mathrm{leng}_p(\sigma_A(\mathbb{P}^2, B, \gamma))$ is easily calculated from the *ADE*-type of $\mathrm{Sing}\, B$. Note also that μ_B and $\mathrm{leng}_p(\sigma_A(\mathbb{P}^2, B, \gamma))$ are both invariants of the configuration type of B.

Another corollary is about the relation between the existence of Z-splitting curves and $\pi_1(\mathbb{P}^2 \setminus B)$.

Definition 5.14. Let $B \subset \mathbb{P}^2$ be as above. A reduced irreducible curve $\Gamma \subset \mathbb{P}^2$ is said to be *Z-splitting* if the strict transform $\widetilde{\Gamma} \subset X$ of Γ splits into two irreducible components $\widetilde{\Gamma}^+$, $\widetilde{\Gamma}^-$ and their classes $[\widetilde{\Gamma}^+]$ and $[\widetilde{\Gamma}^-]$ are distinct elements of $\Lambda(X)$. The *class order* of a Z-splitting curve Γ is the order of $[\widetilde{\Gamma}^+]$ in the finite abelian group $F_A(\mathbb{P}^2, B, \gamma)$.

Corollary 5.15. *If B has a Z-splitting curve of class order not equal to a power of 2, then $\pi_1(\mathbb{P}^2 \setminus B)$ is non-abelian.*

Example 5.16. In [18], we have completely classified all Z-splitting curves of degree ≤ 3 for simple sextics by means of period mapping for complex $K3$ surfaces.

For example, we have found a maximizing sextic $B = C + Q$ of type $A_3 + A_5 + A_{11}$ (a union of a conic C and a quartic Q with A_5) with a Z-splitting line of class order 12. By Corollary 5.15, we see that $\pi_1(\mathbb{P}^2 \setminus B)$ is non-abelian.

§6. Computation of the invariants

We close this paper with a remark on the computation of the invariants T_A, F_A and σ_A. Suppose that we know the structure of $\mathrm{NS}(S)$. The lattice $\Sigma(X)$ and hence its discriminant group $\sigma_A(S, B, \gamma)$ can be calculated from the configuration type of B. In [1], we have developed a general method of Zariski–van Kampen type to calculate the lattice $T_A(S, B, \gamma)$. Hence the order of the finite abelian group $F_A(S, B, \gamma)$ can be also calculated. We also obtain some information about the structure of $F_A(S, B, \gamma)$ from the discriminant groups of $T_A(S, B, \gamma)$ and of $\Sigma(X)$ by using Lemma 4.8.

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