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Computation of principal $\mathcal{A}$-determinants through dimer dynamics

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Abstract.

$\mathcal{A}$ is a set of N vectors in $\mathbb{Z}^{N-2}$ situated in a hyperplane not through 0 and spanning $\mathbb{Z}^{N-2}$ over $\mathbb{Z}$. Gulotta's algorithm [4] constructs from $\mathcal{A}$ a dimer model. A theorem in [6] states that the principal $\mathcal{A}$-determinant equals the determinant of (a suitable form of) the Kasteleyn matrix of that dimer model. In the present note we translate Gulotta's pictorial description of the algorithm into matrix operations. As a result one obtains an algorithm for computing the principal $\mathcal{A}$-determinant, which is much faster than the algorithm in [5].

§1. Introduction

$\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ is a set of vectors in $\mathbb{Z}^{N-2}$ situated in a hyperplane not through 0 and spanning $\mathbb{Z}^{N-2}$ over $\mathbb{Z}$. The principal $\mathcal{A}$-determinant, defined in [2], describes the singularities of Gelfand–Kapranov–Zelevinsky's $\mathcal{A}$-hypergeometric system of partial differential equations [3]. It also describes for which N-tuples of coefficients $u_1, \ldots, u_N \in \mathbb{C}$ the Laurent polynomial $\sum_{j=1}^{N} u_j \mathbf{x}^{\mathbf{a}_j}$ in $N-2$ variables is singular (see [2] for details). It is a polynomial with integer coefficients in the variables $u_1, \ldots, u_N$. The restriction $\operatorname{rank} \mathcal{A} = \sharp\mathcal{A} - 2$ means that the corresponding hypergeometric functions are essentially functions in two variables and that in the Laurent polynomial the number of terms exceeds the number of variables by 2. Even in this case the definition of the principal $\mathcal{A}$-determinant is fairly complicated and only a few of its coefficients could explicitly be calculated in [2]. In [1] Dickenstein and Sturmfels re-examined the definition of the principal $\mathcal{A}$-determinant and related

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it to Chow forms which is another important concept from [2]. In [6] it was shown how these Chow forms and the principal $\mathcal{A}$-determinant can be easily computed as the determinant of a suitable version of the Kasteleyn matrix of a dimer model associated with $\mathcal{A}$. When writing [6] I had only the algorithm in [5] to construct that dimer model. Later Gulotta gave another algorithm [4] for constructing the appropriate dimer model. He describes the algorithm as a process that transforms certain doubly periodic configurations of curves in the plane. In Sections 2, 3, 4 we give a faithfull reproduction of those configurations of curves by matrices and of Gulotta's algorithm by row and column operations on these matrices. In that form Gulotta's algorithm, which is a fast converging iterative process, is much more efficient than the algorithm in [5], which is a search with many trial-and-errors. Moreover, unlike for the algorithm in [5] for Gulotta's algorithm it can be guaranteed that it finds a desired dimer model. On the other hand there are cases in which [5] yields two different models and [4] gives only one.

In this note we do not need formal definitions of 'dimer model' and 'principal $\mathcal{A}$-determinant'. Dimer models are implicitly present through their patterns of zigzags. This is briefly explained in Remark 2.4. Principal $\mathcal{A}$-determinants appear only in Section 6 in a quotation from [6].

In Section 5 we recall from [5] how a pattern of zigzags (alias dimer model) is faithfully represented by a matrix $\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$, which is in fact a suitable *generalization of the Kasteleyn matrix of the dimer model.* In Section 6 we recall from [6] the theorem that expresses the principal $\mathcal{A}$-determinant as the determinant of $\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$. Sections 5 and 6 can be read immediately after Definitions 2.2 and 2.3.

§2. Patterns of zigzags on the torus

2.1. This note is about patterns of zigzags on the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$. Here *zigzag* means the image (modulo $\mathbb{Z}^2$) of an oriented connected curve C in the plane (i.e. the image of a continuous map from $\mathbb{R}$ to $\mathbb{R}^2$) such that the two coordinate functions restrict to monotone functions on C and such that $\mathbf{t} + C = C$ for some non-zero vector $\mathbf{t} \in \mathbb{Z}^2$.

We denote by ε_1 (resp. ε_2) the zigzags in $\mathbb{T}$ coming from the first (resp. second) coordinate axis of $\mathbb{R}^2$. The homology classes of ε_1 and ε_2 form the standard basis for the homology group $H_1(\mathbb{T}, \mathbb{Z})$. We denote the intersection number of two elements α, $\beta \in H_1(\mathbb{T}, \mathbb{Z})$ by $\alpha \wedge \beta$. For two zigzags Z, Z' on $\mathbb{T}$ we write $Z \wedge Z'$ for the intersection number of their homology classes. For two zigzags Z, Z' which intersect in only a finite number of points and for which the intersections are transverse, each intersection point contributes $+1$ or -1 to $Z \wedge Z'$ according to the orientation; e.g. $\varepsilon_1 \wedge \varepsilon_1 = \varepsilon_2 \wedge \varepsilon_2 = 0$, $\varepsilon_1 \wedge \varepsilon_2 = -\varepsilon_2 \wedge \varepsilon_1 = 1$.

Writing $[Z]$ for the homology class of a zigzag Z we have

$$[Z] = (Z \wedge \varepsilon_2)[\varepsilon_1] - (Z \wedge \varepsilon_1)[\varepsilon_2]. \tag{1}$$

2.2. Definition In this note *pattern of zigzags* means a finite sequence $\mathcal{Z} = (Z_1, \dots, Z_p)$ of zigzags on $\mathbb{T}$ which satisfies the following conditions:

1. The homology classes $[Z_1], \dots, [Z_p]$ span $H_1(\mathbb{T}, \mathbb{Q})$. Their sum is 0.
2. For every j the greatest common divisor of $(Z_j \wedge \varepsilon_1, Z_j \wedge \varepsilon_2)$ is 1.
3. Every point of $\mathbb{T}$ lies on at most two zigzags in $\mathcal{Z}$.
4. Every pair of zigzags Z_i, Z_j intersects in only a finite number of points and the intersections are transverse.
5. The 2-cells (i.e. connected components) of $\mathbb{T} \setminus \bigcup_{i=1}^{p} Z_i$ are divided into three types: those (called $+$-cells) of which the boundary is positively oriented, those (called $-$-cells) of which the boundary is negatively oriented and those of which the boundary is not oriented. It is required that for every intersection point x of zigzags in $\mathcal{Z}$ one of the four 2-cells having x in their boundary is a $+$-cell, one is a $-$-cell and two have an unoriented boundary; see Figures 1–10.
6. The number of $+$-cells equals the number of $-$-cells.

2.3. Definition We say that a pattern of zigzags $\mathcal{Z} = (Z_1, \dots, Z_p)$ is *good* if in addition to the above conditions, it also satisfies:

7. Write ζ_i for the column vector $[Z_i \wedge \varepsilon_1,\ Z_i \wedge \varepsilon_2]^t$. Then the determinants $\det(\zeta_i, \zeta_{i+1})$ for $i = 1, \dots, p-1$ and $\det(\zeta_p, \zeta_1)$ are not negative. Moreover, $\zeta_i \neq -\zeta_{i+1}$ for $i = 1, \dots, p-1$ and $\zeta_p \neq \pm\zeta_1$.
8. Every ordered pair (Z_i, Z_j) of zigzags has the same orientation at all its intersection points. This is equivalent with:

$$\forall i, j : \qquad |Z_i \wedge Z_j| = \sharp(Z_i \cap Z_j).$$

Condition 7 means that the homology classes are ordered counterclockwise with increasing indices and that for every homology class the indices of the zigzags in that class form a connected interval in the index set $\{1, \dots, p\}$. In 2.12 we formulate a condition which also restrains the ordering of zigzags within their homology class.

2.4. Remark The dimer model for a pattern of zigzags is the graph with a node $\bullet$ for every $+$-cell and a node $\circ$ for every $-$-cell. Two nodes are connected by an edge if the cells have a vertex in common. Figure 1 shows an example.

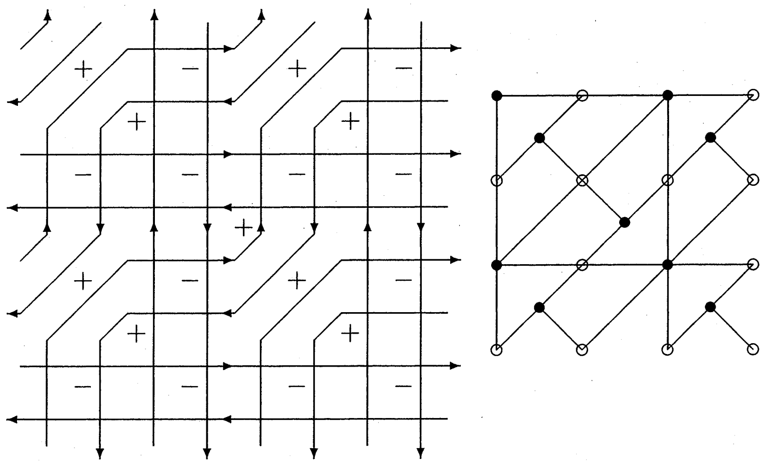


Fig. 1. *Zigzag pattern and corresponding dimer model*

2.5. To a pattern of zigzags $\mathcal{Z} = (Z_1, \ldots, Z_p)$ we assign the $2\times p$-matrix

$$B_{\mathcal{Z}} = \begin{bmatrix} Z_1 \wedge \varepsilon_1 & ,\ldots, & Z_p \wedge \varepsilon_1 \\ Z_1 \wedge \varepsilon_2 & ,\ldots, & Z_p \wedge \varepsilon_2 \end{bmatrix},$$

displaying the intersection numbers of the zigzags in the pattern with the two curves ε_1 and ε_2. We assume that all intersections of zigzags with ε_1 and ε_2 are transverse. The intersection numbers are visible in the pictures as follows. Represent $\mathbb{T}$ by the unit square with opposite sides identified. Then $Z_j \wedge \varepsilon_2$ is the number of times the zigzag Z_j crosses the right-hand vertical edge from left to right minus the number of times it crosses from right to left. Similarly, $Z_j \wedge \varepsilon_1$ is the number of times Z_j crosses the top horizontal edge downwards minus the number of times it crosses upwards.
Condition 2.2.1. is equivalent with

the rank of $B_{\mathcal{Z}}$ *is* 2 *and the sum of its columns is* 0.

2.6. From (1) we see:

$$Z_i \wedge Z_j = (Z_i \wedge \varepsilon_1)(Z_j \wedge \varepsilon_2) - (Z_i \wedge \varepsilon_2)(Z_j \wedge \varepsilon_1) = \det \begin{bmatrix} (Z_i \wedge \varepsilon_1) & (Z_j \wedge \varepsilon_1) \\ (Z_i \wedge \varepsilon_2) & (Z_j \wedge \varepsilon_2) \end{bmatrix}.$$

This together with Condition 2.2.2. implies

$$Z_i \wedge Z_j = 0 \qquad \Longleftrightarrow \qquad [Z_i] = \pm[Z_j]\,.$$

And thus, $\operatorname{rank} B_{\mathcal{Z}} = 2$ if and only if $[Z_i] \neq \pm[Z_j]$ for some i, j.

2.7. Let $\mathcal{Z} = \{Z_1, \ldots, Z_p\}$ be a pattern of zigzags on $\mathbb{T}$. Pick a point $\star$ in one of the $-$-cells. To every 2-cell c we associate a row vector in $\mathbb{Z}^p$ as follows. Take any path γ on $\mathbb{T}$ starting at $\star$ and ending in (the interior of) c, such that γ intersects zigzags transversely. Each point in $Z_j \cap \gamma$ contributes, depending on the orientation, $+1$ or -1 to the intersection number $Z_j \wedge \gamma$. Then to c we associate the *vector of intersection numbers*, or briefly *intersection vector*, $[Z_1 \wedge \gamma, \ldots, Z_p \wedge \gamma]$. Choosing another path γ' from $\star$ to c changes this vector by a $\mathbb{Z}$-linear combination of the rows of the matrix $B_{\mathcal{Z}}$.

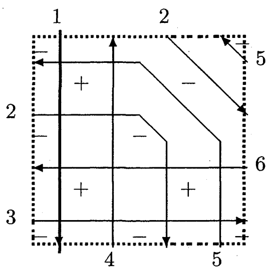
It follows from Condition 2.1.5 that at an intersection point of zigzags Z_i and Z_j the vectors for the two cells with unoriented boundary and the cell with negatively oriented boundary can be obtained from the vector for the $+$-cell by subtracting 1 from the i-th coordinate, respectively 1 from the j-th coordinate, respectively 1 from both i-th and j-th coordinate. We can thus capture all relevant information of the pattern of zigzags $\mathcal{Z}$ in the matrix $B_{\mathcal{Z}}$ and two additional matrices $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$, defined as follows.

2.8. Definition The columns of $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ correspond with the zigzags $Z_1, \ldots, Z_p$. The rows of $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ correspond with the intersection points of pairs of zigzags in $\mathcal{Z}$. The row of matrix $I_{\mathcal{Z}}$ for a point $x \in Z_i \cap Z_j$ has 1 in positions i and j and 0 elsewhere. Matrix $P_{\mathcal{Z}}$ has in the row for intersection point x an intersection vector of the $+$-cell which has x in its boundary; see Figures 2 and 10 for examples.

It is also convenient to have the short notation $Q_{\mathcal{Z}} = P_{\mathcal{Z}} - I_{\mathcal{Z}}$. Then matrix $Q_{\mathcal{Z}}$ has in the row for an intersection point x an intersection vector of the $-$-cell which has x in its boundary.

2.9. Remark Due to the $\mathbb{Z}^2 B_{\mathcal{Z}}$-ambiguity in the choice of the intersection vectors there is also a $\mathbb{Z}^2 B_{\mathcal{Z}}$-ambiguity in the rows of the matrices $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$ in Definition 2.8. In the algorithm we start with well-defined matrices $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$. In the course of the algorithm we only delete rows and perform the same operations on the columns of the matrices $B_{\mathcal{Z}}$, $P_{\mathcal{Z}}$, $Q_{\mathcal{Z}}$ and $I_{\mathcal{Z}}$ simultaneously. So the algorithm is also unambiguous. It does however happen that rows of $P_{\mathcal{Z}}$ (resp. $Q_{\mathcal{Z}}$) which correspond to the same $+$-cell (resp. $-$-cell) are not the same, but differ by a vector in $\mathbb{Z}^2 B_{\mathcal{Z}}$.

In Section 5 we pass to $\mathbb{Z}^p / \mathbb{Z}^2 B_{\mathcal{Z}}$.



$$
\begin{array}{c}
\begin{array}{r} B_{\mathcal{Z}} \\ I_{\mathcal{Z}} \;\; P_{\mathcal{Z}} \end{array}
\begin{bmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix} \\
\begin{bmatrix} 110000 \\ 100010 \\ 010100 \\ 000110 \\ 101000 \\ 100001 \\ 001100 \\ 000101 \\ 011000 \\ 010001 \\ 001010 \\ 000011 \end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & -1 & 1 & 1 \\
1 & 0 & 0 & -1 & 1 & 1 \\
1 & 0 & 0 & -1 & 1 & 1 \\
1 & 0 & 0 & -1 & 1 & 1
\end{bmatrix}
\end{array}
$$

Fig. 2. *Pattern of zigzags and the corresponding matrices*

2.10. Notation For a matrix M we denote j-th column as $M(:,j)$, the i-th row as $M(i,:)$ and the (i,j)-entry as $M(i,j)$.

2.11. Definition Let Z_j and Z_k be two zigzags such that $[Z_j] = -[Z_k]$. We say that $(Z_j,\, Z_k)$ is a *+-opposite pair* (resp. *−-opposite pair*) if

$$Q_{\mathcal{Z}}(:,j) = -Q_{\mathcal{Z}}(:,k) \quad (\text{resp. } P_{\mathcal{Z}}(:,j) = -P_{\mathcal{Z}}(:,k)).$$

Suppose $Z_j \cap Z_k = \emptyset$. Then $(Z_j,\, Z_k)$ is a +-opposite pair (resp. −-opposite pair) if and only if there are between Z_j and Z_k no −-cells (resp. no +-cells). Most pictures in this note contain examples of opposite pairs. The term 'opposite pair' without $\pm$ was introduced in [4] §5.3.

2.12. Definition A good pattern of zigzags $\mathcal{Z} = (Z_1, \ldots, Z_p)$ is said to be *very good* if it satisfies:

9. For every homology class $[Z]$ the sequence of all zigzags $(Z_i, \ldots, Z_{i+r})$ in homology class $[Z]$ and the sequence of all zigzags $(Z_j, \ldots, Z_{j+s})$ in homology class $-[Z]$ satisfy:

$$
\begin{array}{ll}
 & \textit{for } 0 \le t \le \min(r,s): \quad (Z_{i+t}, Z_{j+t}) \textit{ is a +-opposite pair} \\
 & \text{and either:} \\
(2) & \qquad \textit{for } 0 \le t < \min(r,s): (Z_{i+t+1}, Z_{j+t}) \textit{ is a −-opposite pair} \\
 & \text{or: } \textit{for } 0 \le t < \min(r,s): (Z_{i+t}, Z_{j+t+1}) \textit{ is a −-opposite pair}
\end{array}
$$

§3. The moves in the algorithm

Gulotta's algorithm transforms in an iterative way a very good pattern of zigzags $\mathcal{Z}$ into another very good one $\mathcal{Z}'$. In [4] the algorithm is mainly described by transforming a drawing of $\mathcal{Z}$ into a drawing of $\mathcal{Z}'$. We will present the same algorithm by row and column operations on the matrices $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$.

3.1. Merging move *The basic move in the algorithm merges two zigzags Z_i and Z_j which intersect in exactly one point, as shown in Figure 3.* It is evident that the merging moves preserve Conditions 1–6 in 2.2.

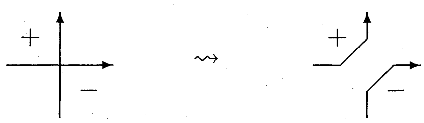


Fig. 3. *Merging move*

It was pointed out in [4], that when two zigzags Z_i and Z_j of a pattern $\mathcal{Z}$ are merged and become one zigzag Z in $\mathcal{Z}'$ then

$$\begin{bmatrix} Z \wedge \varepsilon_1 \\ Z \wedge \varepsilon_2 \end{bmatrix} = \begin{bmatrix} Z_i \wedge \varepsilon_1 \\ Z_i \wedge \varepsilon_2 \end{bmatrix} + \begin{bmatrix} Z_j \wedge \varepsilon_1 \\ Z_j \wedge \varepsilon_2 \end{bmatrix}. \tag{3}$$

Actually this means $[Z] = [Z_i] + [Z_j]$ and, hence, also the column of the matrix $P_{\mathcal{Z}'}$ which corresponds with the zigzag Z is the sum of the i-th and the j-th columns of $P_{\mathcal{Z}}$. Moreover, as the picture indicates, the point of intersection $Z_i \cap Z_j$ disappears. The following statement also specifies where we put the new zigzag Z in the list of zigzags for $\mathcal{Z}'$.
Conclusion: *The merging of Z_i and Z_j for $Z_i \wedge Z_j = 1$ is given by the same column operation on $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$, namely: add the j-th column to the i-th and subsequently delete the j-th column. It also deletes from $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ the row for $Z_i \cap Z_j$.*

Merging moves performed on a *very good* pattern of zigzags need not preserve Conditions 2.3.7–8 and 2.12.9. Some repairing may be needed in order to turn the pattern of zigzags produced by the merging moves into a very good one again.

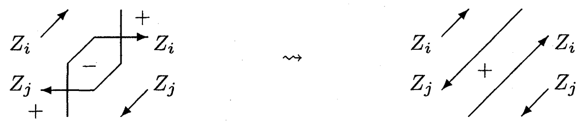


Fig. 4. *Repairing move 1*

3.2. Repairing move 1 *The first type of repairing move is shown in Figure 4.* This is used when $[Z_i] = -[Z_j]$ and $\sharp(Z_i \cap Z_j) = 2$, while the area between the zigzags is just one $-$-cell (as suggested in the picture) or one $+$-cell (interchange $+$ and $-$ in the picture). From this picture one immediately comes to the conclusion:

Conclusion: *Repairing move 1 just deletes the rows for the two points of $Z_i \cap Z_j$ from $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$.*

3.3. Repairing move 2 *The second type of repairing move is shown in Figure 5.* This is used when $[Z_i] = [Z_j]$ and $\sharp(Z_i \cap Z_j) = 2$. In this case

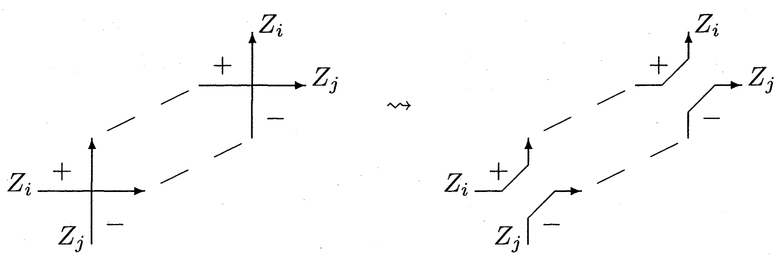


Fig. 5. *Repairing move 2*

one may distinguish three kinds of rows in the matrix $P_{\mathcal{Z}}$, according to whether the i-th entry minus the j-th entry is equal to $m-2$, $m-1$ or m, for some integer m (depending on i and j). The rows of the latter kind correspond in the picture with $+$-cells in the area between the two zigzags. When the intervals of Z_i and Z_j between the points of $Z_i \cap Z_j$ are swopped (as suggested by the right-hand picture) one must add 1 to the j-th coordinate and subtract 1 from the i-th coordinate in all rows of $P_{\mathcal{Z}}$ corresponding with a $+$-cell in the area between Z_i and Z_j. One must also interchange the i-th and j-th entries in the rows of $I_{\mathcal{Z}}$ which correspond with intersections with the intervals of Z_i and Z_j between the two points of $Z_i \cap Z_j$. And one must delete from $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ the two rows corresponding to the two points of $Z_i \cap Z_j$.

Conclusion: *Repairing move 2 operates on the columns of $P_{\mathcal{Z}}$ as follows: Write $H(r) = P_{\mathcal{Z}}(r,i) - P_{\mathcal{Z}}(r,j)$ and $m = \max_r(H(r))$. Then*

$$\begin{array}{llll} P_{\mathcal{Z}}(r,i) \rightsquigarrow P_{\mathcal{Z}}(r,i) - 1\,, & P_{\mathcal{Z}}(r,j) \rightsquigarrow P_{\mathcal{Z}}(r,j) + 1 & \text{if} & H(r) = m\,, \\ P_{\mathcal{Z}}(r,i) \rightsquigarrow P_{\mathcal{Z}}(r,i)\,, & P_{\mathcal{Z}}(r,j) \rightsquigarrow P_{\mathcal{Z}}(r,j) & \text{if} & H(r) \neq m\,. \end{array}$$

It operates on the columns of $I_{\mathcal{Z}}$ by:

$$\begin{array}{l} I_{\mathcal{Z}}(r,i) \rightsquigarrow I_{\mathcal{Z}}(r,j)\,, \quad I_{\mathcal{Z}}(r,j) \rightsquigarrow I_{\mathcal{Z}}(r,i) \\ \quad \text{if} \quad H(r) = m \text{ and } I_{\mathcal{Z}}(r,i) = 1, \text{ or } \; H(r) = m-1 \text{ and } I_{\mathcal{Z}}(r,j) = 1, \\ I_{\mathcal{Z}}(r,i) \rightsquigarrow I_{\mathcal{Z}}(r,i)\,, \quad I_{\mathcal{Z}}(r,j) \rightsquigarrow I_{\mathcal{Z}}(r,j) \quad \text{otherwise}. \end{array}$$

Finally, it deletes from $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ the rows for the points of $Z_i \cap Z_j$.

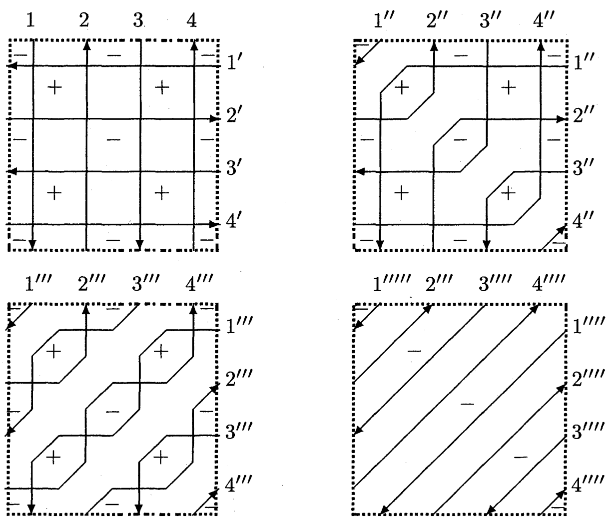


Fig. 6. *Transforming an alternating sequence of opposite pairs*

3.4. Example Let $(Z_1, \ldots, Z_{2s})$ and $(Z'_1, \ldots, Z'_{2s})$ be two sequences of zigzags in a pattern $\mathcal{Z}^{(1)}$ such that $\sharp(Z_i \cap Z'_j) = 1$ for all i, j and $Z_i \cap Z_j = Z'_i \cap Z'_j = \emptyset$ for all $i \neq j$ and such that (Z_{j-1}, Z_j) and (Z'_{j-1}, Z'_j) are $(-1)^j$-opposite pairs for $j = 2, \ldots, 2s$. There are two well controllable cases in which merging of these two sequences followed by repairing moves 2 and 1 yields a sequence of zigzags $(Z''''_1, \ldots, Z''''_{2s})$

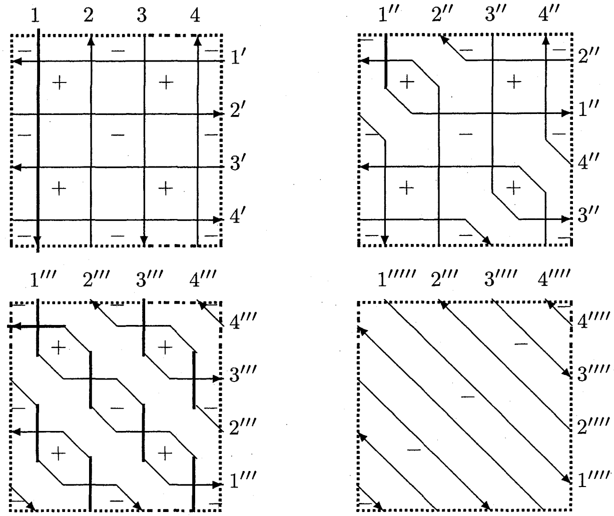


Fig. 7. *Transforming an alternating sequence of opposite pairs*

such that $Z_i'''' \cap Z_j'''' = \emptyset$ for all $i \neq j$ and such that (Z_{j-1}'''', Z_j'''') is a $(-1)^j$-opposite pair for $j = 2, \ldots, 2s$.

Case 1: $\quad$ *Z_j and Z'_j merge for $j = 1, \ldots, 2s$.*

Case 2: $\quad$ *Z_j and $Z'_{j-(-1)^j}$ merge for $j = 1, \ldots, 2s$.*

Figures 6 and 7 show this for $s = 2$ and clearly generalize to arbitrary s.

In either case let $(Z_1'', \ldots, Z_{2s}'')$ be the sequence of zigzags in the pattern $\mathcal{Z}^{(2)}$ which results from the merging. Now transform $\mathcal{Z}^{(2)}$ by applying repairing moves of type 2 at the points of $Z_i'' \cap Z_j''$ for all $i \neq j$ for which $[Z_i''] = [Z_j'']$. The result is the sequence of zigzags $(Z_1''', \ldots, Z_{2s}''')$ in the pattern $\mathcal{Z}^{(3)}$. Then $Z_i''' \cap Z_j''' \neq \emptyset$ only if $[Z_i'''] = -[Z_j''']$. Next transform $\mathcal{Z}^{(3)}$ by applying repairing moves of type 1 at the points of $Z_i''' \cap Z_j'''$ for all i, j. Call the resulting pattern of zigzags $\mathcal{Z}^{(4)}$ and the relevant sequence of zigzags $(Z_1'''', \ldots, Z_{2s}'''')$. In this last sequence $Z_i'''' \cap Z_j'''' = \emptyset$ for all $i \neq j$ and (Z_{j-1}'''', Z_j'''') is a $(-1)^j$-opposite pair for $j = 2, \ldots, 2s$.

It is instructive to perform the moves in Figures 6 and 7 also for the matrices $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$. For the top-left picture $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$ are given in Figure 10. For the other pictures one may follow the description of the merging and repairing moves.

3.5. Remark In 3.4 we have chosen the labels for the zigzags while drawing the pictures. If one uses the matrix operations instead, the algorithm determines the labels and it may be necessary to reorder the columns of the matrices $P_{\mathcal{Z}}$, $Q_{\mathcal{Z}}$, $I_{\mathcal{Z}}$ to meet the requirements of Equation (2). Example 3.4 shows that such a reordering is always possible.

3.6. Repairing move 3 *The third type of repairing move is shown in Figure 8.* It is used for a zigzag Z_0 and a sequence of zigzags $(Z_1, \ldots, Z_{2s})$ which satisfy the following conditions. Firstly, $Z_i \cap Z_j = \emptyset$ for all $i > j \geq 1$ and (Z_{j-1}, Z_j) is a $(-1)^j$-opposite pair for $j = 2, \ldots, 2s$. Secondly $[Z_0] = \pm[Z_1]$ and $\sharp(Z_0 \cap Z_j) = 2$ for $j = 1, \ldots, 2s$. Reversing if necessary the labeling in the sequence $(Z_1, \ldots, Z_{2s})$ we may without loss of generality assume $[Z_0] = (-1)^j[Z_j]$ for $j = 1, \ldots, 2s$.

Gulotta's instructions (cf. [4] §5.3) in this situation are to remove $(Z_1, \ldots, Z_{2s})$ and to insert a sequence of zigzags $(Z'_1, \ldots, Z'_{2s})$ such that $Z'_i \cap Z'_j = \emptyset$ for all $j > i \geq 0$ and such that (Z'_{j-1}, Z'_j) is a $(-1)^j$-opposite pair for $j = 1, \ldots, 2s$. For notational convenience we write here and below $Z'_0 = Z_0$.

In terms of the matrices $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ this means that we first delete from $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ all rows which correspond with an intersection point on one of the zigzags $Z_1, \ldots, Z_{2s}$ and subsequently replace, for $j = 1, \ldots, 2s$, the column of $P_{\mathcal{Z}}$ which corresponds with the zigzag Z_j by $(-1)^j$ times the column of $P_{\mathcal{Z}}$ which corresponds with the zigzag Z_0.

Next we expand every row of $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ which corresponds with an intersection point of Z_0 and a zigzag $Z_\infty \neq Z_0, Z_1, \ldots, Z_{2s}$ to $1+2s$ rows which correspond with the intersection points of Z_∞ with $Z'_0, Z'_1, \ldots, Z'_{2s}$ (see Figure 9). The columns of $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ are labeled in such a way that the column which originally corresponded to Z_j now corresponds to Z'_j for $j = 0, \ldots, 2s$.

Thus a row of $I_{\mathcal{Z}}$ for an intersection point x of Z_∞ and Z_0 is replaced by $1 + 2s$ rows of which the j-th one (for $j = 0, \ldots, 2s$) has entry 1 in the columns corresponding with the zigzags Z_∞ and Z'_j. Of the $1 + 2s$ new rows of $P_{\mathcal{Z}}$ the 0-th one is equal to the row of $P_{\mathcal{Z}}$ which corresponds to x. Figure 9 shows that for odd j the j-th row is obtained from the $(j-1)$-st one by adding -1 in the column for Z'_{j-1} and $+1$ in the column for Z'_j; for even $j \geq 2$ the j-th row is equal to the $(j-1)$-st one.

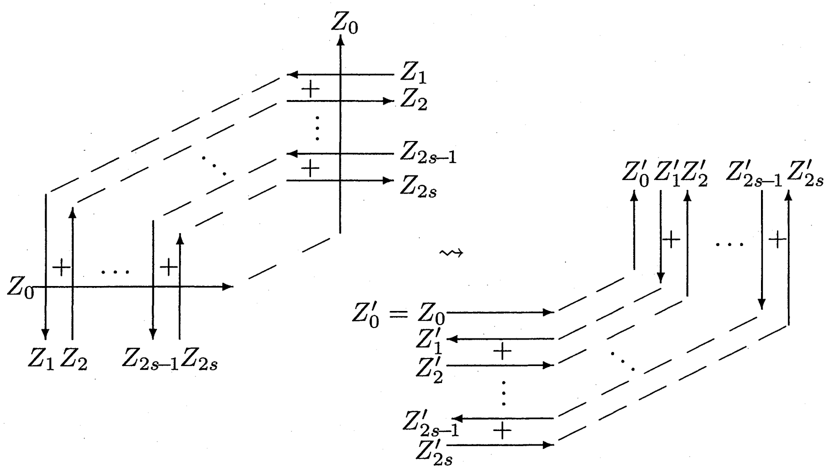


Fig. 8. *Repairing move 3*

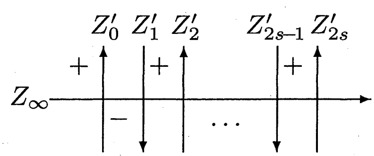


Fig. 9. *Intersections with alternating sequence of opposite pairs*

§4. Running the algorithm

In this Section we translate the algorithm described by Gulotta in terms of pictures, into an iterative process operating on matrices.

4.1. In order to prepare the input for the algorithm from the set $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ (see §1) we take a $2 \times N$-matrix $B_\mathcal{A}$ such that its rows are a $\mathbb{Z}$-basis for the lattice $\{(\ell_1, \dots, \ell_N) \in \mathbb{Z}^N \mid \ell_1\mathbf{a}_1 + \dots + \ell_N\mathbf{a}_N = 0\}$.

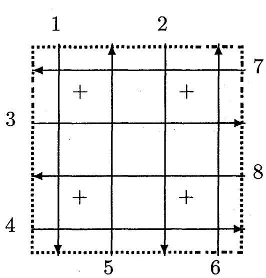
The aim of the algorithm is to create a very good pattern of zigzags $\mathcal{Z}$ *such that* $B_\mathcal{Z}$ *results by permuting and splitting up the columns of* $B_\mathcal{A}$ *as follows*

$$\text{(4)}\qquad \text{column } \begin{bmatrix} r \\ s \end{bmatrix} \text{ of } B_\mathcal{A} \rightsquigarrow d = \text{g.c.d.}(r,s) \text{ columns } \frac{1}{d}\begin{bmatrix} r \\ s \end{bmatrix} \text{ in } B_\mathcal{Z}.$$

4.2. The algorithm starts with a very good pattern of zigzags $\mathcal{Z}$ for which $B_{\mathcal{Z}}$ is the $2 \times (2n_1 + 2n_2)$-matrix

$$\begin{bmatrix} \overbrace{1\ldots1}^{n_1} & \overbrace{0\ldots0}^{n_2} & \overbrace{-1\cdots-1}^{n_1} & \overbrace{0\ldots\ \ 0}^{n_2} \\ 0\ldots0 & 1\ldots1 & 0\ldots\ \ 0 & -1\cdots-1 \end{bmatrix},$$

where n_1 (resp. n_2) is the sum of the positive entries in the first (resp. second) row of $B_{\mathcal{A}}$. This matrix is realized by a pattern with straight lines, n_1 vertically down, n_1 vertically up, n_2 horizontally left-to-right and n_2 horizontally right-to-left. To get a very good pattern one takes the vertical lines alternatingly down and up, and the horizontal lines alternatingly left-to-right and right-to-left. The matrices $B_{\mathcal{Z}}$, $P_{\mathcal{Z}}$ and $I_{\mathcal{Z}}$ for the initial pattern have $2n_1 + 2n_2$ columns. There are $4n_1n_2$ intersection points and, hence, $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ have $4n_1n_2$ rows. We build $I_{\mathcal{Z}}$ and $P_{\mathcal{Z}}$ as follows: the $+$-cells are given by pairs (a, b) in $\{1, \ldots, n_1\} \times \{1, \ldots, n_2\}$. The rows of $I_{\mathcal{Z}}$ which correspond to the four vertices of the $+$-cell (a, b) are have 1 in positions a, $b + n_1$, resp. a, $b + 2n_1 + n_2$, resp. $a + n_1 + n_2$, $b + n_1$, resp. $a + n_1 + n_2$, $b + 2n_1 + n_2$. All other entries in these rows of $I_{\mathcal{Z}}$ are 0. The non-zero entries of the four rows of $P_{\mathcal{Z}}$ which correspond to the vertices of the $+$-cell (a, b) are 1 in position j if $1 \leq j \leq a$ or $2n_1 + n_2 + 1 \leq j \leq 2n_1 + n_2 + b$ and -1 in position j if $n_1 + n_2 + 1 \leq j \leq n_1 + n_2 + a - 1$ or $n_1 + 1 \leq j \leq n_1 + b - 1$. Figure 10 shows an example with $n_1 = n_2 = 2$.



$$\begin{array}{r} B_{\mathcal{Z}} \\ I_{\mathcal{Z}} \quad P_{\mathcal{Z}} \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 10100000 \\ 10000010 \\ 00101000 \\ 00001010 \\ 10010000 \\ 10000001 \\ 00011000 \\ 00001001 \\ 01100000 \\ 01000010 \\ 00100100 \\ 00000110 \\ 01010000 \\ 01000001 \\ 00010100 \\ 00000101 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 & 1 & 1 \end{bmatrix}$$

Fig. 10. *Pattern to start from*

4.3. At the beginning of an iteration step we have a very good pattern of zigzags $\mathcal{Z} = \{Z_1, \dots, Z_p\}$. The merging moves are determined from the positions of the columns of the matrix $B_{\mathcal{A}}$ with respect to the columns of the matrix $B_{\mathcal{Z}}$. The columns of $B_{\mathcal{Z}}$ are vectors in the plane $\mathbb{R}^2$ and the ordering by increasing index coincides with the counter-clockwise cyclic ordering. In agreement with this cyclic structure we treat the first column of $B_{\mathcal{Z}}$ as consecutive to the last column.

A column v of $B_{\mathcal{A}}$ is either a positive integer multiple of a column of $B_{\mathcal{Z}}$ or there is a unique pair of consecutive columns v_1 and v_2 of $B_{\mathcal{Z}}$ such that $v = c_1v_1 + c_2v_2$ with $c_1,\ c_2 \in \mathbb{Q}_{>0}$.

4.4. The **algorithm terminates** automatically when all columns of $B_{\mathcal{A}}$ are multiples of columns of $B_{\mathcal{Z}}$. Since the merging moves decrease the number of zigzags the algorithm will surely terminate.

4.5. Cramer's rule explicates the relation $v = c_1v_1 + c_2v_2$:

$$\det(v_1, v_2)\, v \;=\; \det(v, v_2)\, v_1 \;-\; \det(v, v_1)\, v_2\,. \tag{5}$$

In the pattern we start with the determinants of consecutive pairs of non-equal columns of $B_{\mathcal{Z}}$ are 1. The merging of two zigzags in a very good pattern $\mathcal{Z}$ with exactly one intersection point replaces the corresponding columns of $B_{\mathcal{Z}}$ by their sum. Thus in the next iteration step in the algorithm Equation (5) becomes

$$\det(v_1, v_2)\, v \;=\; (\det(v, v_2) - m)\, v_1 \;+\; m(v_1 + v_2) \;-\; (\det(v, v_1) + m)\, v_2$$

with $m \;=\; \min(\det(v_1, v), \det(v, v_2))$. This then gives v either as a multiple of $v_1 + v_2$ or as a positive linear combination of v_1 and $v_1 + v_2$ or of $v_1 + v_2$ and v_2. Note that $\det(v_1, v_1 + v_2) = \det(v_1 + v_2, v_2) = \det(v_1, v_2)$.

Conclusion: *In all cases in which Equation (5) is used in the algorithm* $\det(v_1, v_2) = 1$ *and the equation actually reads*

$$v \;=\; \det(v, v_2)\, v_1 \;-\; \det(v, v_1)\, v_2\,. \tag{6}$$

Column v of $B_{\mathcal{A}}$ thus leads to the command that m zigzags of the pattern $\mathcal{Z}$ in the homology class corresponding with the column v_1 of $B_{\mathcal{Z}}$ must merge with m zigzags in the homology class corresponding with the column v_2; here $m = \min(\det(v_1, v), \det(v, v_2))$.

4.6. As Equation (6) indicates we need the determinants of the 2×2-matrices with first column from $B_{\mathcal{A}}$ and second column from $B_{\mathcal{Z}}$. These are simultaneously given as the entries of the $N \times p$-matrix

$$S = B_{\mathcal{A}}^t \, J \, B_{\mathcal{Z}} \qquad \text{with} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The columns of $B_{\mathcal{A}}$ correspond with the rows of S. One can implement the discussion in 4.3 and 4.5 for all columns of $B_{\mathcal{A}}$ simultaneously as follows. Let $S^c = S(:, [2:p, 1])$ be the $N \times p$-matrix obtained from S by cyclically permuting the columns so that the first column comes in the last position. Let R be the $N \times p$-matrix with (i,j)-entry

$$\begin{array}{rcll} R_{ij} & = & \frac{1}{2}(|S_{ij}| + |S^c_{ij}| - |S_{ij} + S^c_{ij}|) & \text{if } S_{ij} < 0\,, \\ R_{ij} & = & 0 & \text{if } S_{ij} \geq 0\,. \end{array}$$

We define functions ρ and λ on $\{1, \ldots, p\}$ by

$$\rho(j) = \sum_{i=1}^{N} R_{ij}\,, \qquad \lambda(j) = \rho(j-1) \text{ if } j > 1\,, \quad \lambda(1) = \rho(p)\,.$$

Next we define for a homology class of zigzags $[Z]$ in the pattern $\mathcal{Z} = (Z_1, \ldots, Z_p)$:

$$\begin{array}{rcl} \widetilde{\rho}([Z]) & = & \max\{\rho(j) \mid Z_j \in [Z]\}\,, \\ \widetilde{\lambda}([Z]) & = & \max\{\lambda(j) \mid Z_j \in [Z]\}\,, \\ \widetilde{\mu}([Z]) & = & \sharp([Z]) - \widetilde{\rho}([Z]) - \widetilde{\lambda}([Z])\,. \end{array}$$

Then $\widetilde{\rho}([Z])$ (resp. $\widetilde{\lambda}([Z])$) is the number of zigzags in $[Z]$ which must merge with a zigzag in the homology class immediately after (resp. before) $[Z]$ and $\widetilde{\mu}([Z])$ is the number of zigzags in $[Z]$ which must not merge with another zigzag. The merging step defined in 4.8 decreases the number of zigzags by

$$q = \sum_{h=1}^{p} \lambda(h)$$

and uses the map $\varphi : \{1, \ldots, p\} \longrightarrow \{1, \ldots, p-q\}$,

$$\varphi(j) = \begin{cases} j - \sum_{h=1}^{j} \lambda(h) & \text{if} \quad j > \lambda(1) \\ p - q + j - \sum_{h=1}^{j} \lambda(h) & \text{if} \quad j \leq \lambda(1)\,. \end{cases} \tag{7}$$

4.7. In order to eventually satisfy Requirement 2.12.9 and to benefit from Example 3.4 we permute the zigzags in each homology class as follows. First we define for each homology class of zigzags $[Z]$ for which the opposite class $-[Z]$ also occurs in the pattern $\mathcal{Z}$:

$$\widehat{\rho}([Z]) = \min\{\widetilde{\rho}([Z]), \widetilde{\rho}(-[Z])\}, \qquad \widehat{\lambda}([Z]) = \min\{\widetilde{\lambda}([Z]), \widetilde{\lambda}(-[Z])\},$$
$$\widehat{\mu}([Z]) = \min\{\widetilde{\mu}([Z]), \widetilde{\mu}(-[Z])\}.$$

If $-[Z]$ does not occur in $\mathcal{Z}$ we put $\widehat{\rho}([Z]) = \widehat{\lambda}([Z]) = \widehat{\mu}([Z]) = 0$.
Next we write for every homology class $[Z]$ in $\mathcal{Z}$:

$$\overline{\rho}([Z]) = \widetilde{\rho}([Z]) - \widehat{\rho}([Z]),\ \overline{\lambda}([Z]) = \widetilde{\lambda}([Z]) - \widehat{\lambda}([Z]),\ \overline{\mu}([Z]) = \widetilde{\mu}([Z]) - \widehat{\mu}([Z]).$$

The permutation we apply to the zigzags in homology class $[Z]$ is a so-called shuffle. This means that $[Z]$ is split into disjoint intervals which are permuted, while inside each interval the ordering is unchanged. The shuffle we apply to the zigzags in $[Z]$ is depicted in Figure 11; the numbers $\widehat{\lambda}([Z])$ etc. indicate the length of the interval.

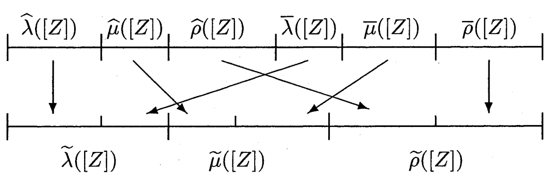


Fig. 11. *Shuffle within one homology class*

Such shuffles must be applied to each homology class in the pattern $\mathcal{Z} = (Z_1, \ldots, Z_p)$. The composite result is the shuffle permutation

$$(8) \qquad \sigma : \{1, \ldots, p\} \longrightarrow \{1, \ldots, p\}.$$

4.8. Definition We define the *merging matrix* $M_{\mathcal{AZ}}$ for the set $\mathcal{A}$ and the pattern of zigzags $\mathcal{Z}$ to be the $p \times (p-q)$-matrix with (i,j)-entry

$$(9) \qquad (M_{\mathcal{AZ}})_{ij} = 1 \text{ if } j = \varphi(\sigma(i)), \quad (M_{\mathcal{AZ}})_{ij} = 0 \text{ if } j \neq \varphi(\sigma(i)),$$

with φ and σ as in (7) and (8).

The *merging step* in the algorithm multiplies the matrices $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$ from the right with the matrix $M_{\mathcal{AZ}}$ and subsequently deletes the rows which correspond to intersection points in the pattern $\mathcal{Z}$ which disappear in the merging process. These are recognized as the rows of $I_{\mathcal{Z}}\, M_{\mathcal{AZ}}$ with only one non-zero entry (namely 2).

Thus the merging $\mathcal{Z} \rightsquigarrow \mathcal{Z}'$ is realized by

$$(10) \qquad \begin{cases} B_{\mathcal{Z}'} = B_{\mathcal{Z}}\, M_{\mathcal{A}\mathcal{Z}} \\ I_{\mathcal{Z}'} = \text{delete rows from } I_{\mathcal{Z}}\, M_{\mathcal{A}\mathcal{Z}} \\ P_{\mathcal{Z}'} = \text{delete rows from } P_{\mathcal{Z}}\, M_{\mathcal{A}\mathcal{Z}} \\ Q_{\mathcal{Z}'} = \text{delete rows from } Q_{\mathcal{Z}}\, M_{\mathcal{A}\mathcal{Z}}. \end{cases}$$

4.9. The transformation of patterns of zigzags $\mathcal{Z} \rightsquigarrow \mathcal{Z}'$ in (10) has been organized so that if $\mathcal{Z}$ is a very good pattern, then $\mathcal{Z}'$ satisfies the Conditions 2.2.1–6, 2.3.7 and the property formulated on the first line of Equation (2). However, $\mathcal{Z}'$ need not satisfy 2.3.8, i.e. the equality

$$|Z_i' \wedge Z_j'| = \sharp(Z_i' \cap Z_j')$$

need not hold for all pairs of zigzags (Z_i', Z_j') in $\mathcal{Z}'$. This equality can only be violated if Z_i' results from merging zigzags Z_{i_1} and Z_{i_2} from $\mathcal{Z}$ such that $Z_{i_1} \wedge Z_{i_2} = 1$ and $(Z_{i_1} \wedge Z_j')(Z_{i_2} \wedge Z_j') < 0$. Since $[Z_j']$ can only be the homology class of a zigzag in $\mathcal{Z}$ or the sum of two such, this can only happen if $[Z_j'] = \pm([Z_{i_1}] + [Z_{i_2}]) = \pm[Z_i']$.

The merging process in (10) is such that if Z_i' results from merging zigzags Z_{i_1} and Z_{i_2} from $\mathcal{Z}$, then every zigzag Z_j' in the homology class $[Z_i']$ (resp. in $-[Z_i']$) is the result of merging a zigzag Z_{j_1} from $[Z_{i_1}]$ (resp. $-[Z_{i_1}]$) with a zigzag Z_{j_2} from $[Z_{i_2}]$ (resp. $-[Z_{i_2}]$). Since $\mathcal{Z}$ satisfies Condition 2.3.8. and $Z_{i_1} \wedge Z_{i_2} = 1$ we have in that case

$$Z_{i_1} \cap Z_{j_1} = Z_{i_2} \cap Z_{j_2} = \emptyset\,, \qquad \sharp(Z_{i_1} \cap Z_{j_2}) = \sharp(Z_{i_2} \cap Z_{j_1}) = 1\,.$$

Whence if $Z_j' \in \pm[Z_i']$ and $Z_i' \neq Z_j'$, then $\sharp(Z_i' \cap Z_j') = 2$.
Conclusion: *For a pair of zigzags (Z_i', Z_j') in $\mathcal{Z}'$ we have:*

$$|Z_i' \wedge Z_j'| \neq \sharp(Z_i' \cap Z_j') \iff Z_i' \wedge Z_j' = 0 \text{ and } \sharp(Z_i' \cap Z_j') = 2\,.$$

4.10. We now transform the pattern of zigzags $\mathcal{Z}'$ created in (10) into a very good one. First we apply repairing moves 2 (see 3.3) to those pairs of zigzags (Z_i', Z_j') which satisfy $[Z_i'] = [Z_j']$ and $|Z_i' \wedge Z_j'| \neq \sharp(Z_i' \cap Z_j')$ and which either both do or both do not belong to a $+$-opposite pair (cf. Definition 2.11). Next we apply repairing moves 1 (see 3.2) wherever possible. See also Example 3.4 and Remark 3.5 for the effect of repairing moves 2 and 1 on alternating sequences of opposite pairs. Thus with repairing moves 2 and 1 and possibly a permutation of columns in $I_{\mathcal{Z}'}$, $P_{\mathcal{Z}'}$, $Q_{\mathcal{Z}'}$ we now have a pattern of zigzags which satisfies also the second half of Equation (2) and in which two zigzags Z_i, Z_j with $[Z_i] = \pm[Z_j]$

do not intersect unless one is member of an opposite pair and the other is not.

Finally we apply repairing moves 3 (see 3.6) as follows for all homology classes $[Z]$ and $-[Z]$ which have resulted from merging and which satisfy $\sharp([Z]) > \sharp(-[Z]) > 0$. In these circumstances the zigzags in $-[Z]$ and the first $\sharp(-[Z])$ zigzags in $[Z]$ form a sequence $(Z_1, \ldots, Z_{2s})$ as in the beginning of 3.6. For the zigzag Z_0 in 3.6 we take the zigzag in $[Z]$ with the highest index. Note that while performing repairing move 3 we have first deleted from the matrices $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$ all rows which corresponded with intersection points on one of the zigzags in the sequence $(Z_1, \ldots, Z_{2s})$. Subsequently we have inserted rows for intersection points of a zigzag in the new sequence $(Z_1', \ldots, Z_{2s}')$ with a zigzag which also intersects $Z_0' = Z_0$. The previously applied repairing moves 2 had already removed all intersection points of Z_0 with the zigzags in $[Z]$ which were not in $(Z_1, \ldots, Z_{2s})$. So after applying repairing moves 3 two zigzags in $[Z] \cup (-[Z])$ do not intersect.

4.11. After the above merging and repairing moves we have produced a very good pattern of zigzags and now **return to 4.6 for the next iteration.**

§5. From $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$, $Q_{\mathcal{Z}}$ to $\mathbb{K}_{\mathcal{Z}}$ and back

5.1. The conversion works for every good pattern $\mathcal{Z}$ of, say p, zigzags. Condition 2.12.9 is not needed here. The rows of $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$ must be taken modulo the row space of $B_{\mathcal{Z}}$. This is achieved by multiplying $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$ from the right by a $p \times (p-2)$-matrix $A_{\mathcal{Z}}$ with entries in $\mathbb{Z}$, such that $\mathrm{rank}(A_{\mathcal{Z}}) = p - 2$ and $B_{\mathcal{Z}}\, A_{\mathcal{Z}} = \mathbf{0}$.

The rows of $P_{\mathcal{Z}}\, A_{\mathcal{Z}}$ represent points in $\mathbb{Z}^p/\mathbb{Z}^2 B_{\mathcal{Z}}$. We denote the set of these points by $\mathfrak{B}$, because these are in fact the black nodes in the dimer model. In the zigzag pattern these are the $+$-cells. Similarly, we denote the set of rows of $Q_{\mathcal{Z}}\, A_{\mathcal{Z}}$ by $\mathfrak{W}$. These are the white nodes in the dimer model and the $-$-cells in the zigzag pattern.

5.2. Definition(cf. [5] Definition 8.2, [6] Definition 1) The *generalized Kasteleyn matrix* $\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$ of a good pattern of zigzags $\mathcal{Z} = (Z_1, \ldots, Z_p)$ is defined as follows. The rows of $\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$ correspond $1:1$ with the elements of $\mathfrak{B}$ and the columns correspond $1:1$ with the elements of $\mathfrak{W}$. The entries of $\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$ are polynomials in two sets of variables $\mathbf{z}$ and $\mathbf{u}$. The variables in $\mathbf{u} = \{u_1, \ldots, u_p\}$ correspond $1:1$ with the zigzags in $\mathcal{Z}$, and, hence, with the columns of $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$. The variables in $\mathbf{z} = \{z_e\}$ correspond $1:1$ with the intersection points e of $\mathcal{Z}$ and, hence, with the rows of $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$ and $Q_{\mathcal{Z}}$.

For an intersection point e we denote by $\mathbf{b}(e)$ the element of $\mathfrak{B}$ which "is" the e-th row of $P_{\mathcal{Z}}\,A_{\mathcal{Z}}$. Similarly, $\mathbf{w}(e) \in \mathfrak{W}$ "is" the e-th row of $Q_{\mathcal{Z}}\,A_{\mathcal{Z}}$. Finally, $i(e)$ and $j(e)$ are such that $I_{\mathcal{Z}}(e, i(e)) = I_{\mathcal{Z}}(e, j(e)) = 1$.

Finally, for $\mathbf{b} \in \mathfrak{B}$ and $\mathbf{w} \in \mathfrak{W}$ we define:

(11) *the* $(\mathbf{b}, \mathbf{w})$*-entry of* $\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$ *is* $\displaystyle\sum_{e:\,\mathbf{b}(e)=\mathbf{b},\,\mathbf{w}(e)=\mathbf{w}} z_e u_{i(e)} u_{j(e)}$.

5.3. Example For the pattern of zigzags in Figure 2 the generalized Kasteleyn matrix is:

$$\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u}) = \begin{bmatrix} z_1u_1u_5 & z_2u_1u_2 + z_3u_4u_5 & z_4u_2u_4 \\ z_5u_1u_3 & z_6u_1u_6 + z_7u_3u_4 & z_8u_4u_6 \\ z_9u_3u_5 & z_{10}u_5u_6 + z_{11}u_2u_3 & z_{12}u_2u_6 \end{bmatrix}.$$

5.4. Definition (cf. [6] Definition 3) The *complementary generalized Kasteleyn matrix* $\mathbb{K}^c_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$ of a good pattern $\mathcal{Z}$ of p zigzags is

$$\mathbb{K}^c_{\mathcal{Z}}(\mathbf{z}, \mathbf{u}) = u_1 \cdot \ldots \cdot u_p \mathbb{K}_{\mathcal{Z}}(\mathbf{z}, u_1^{-1}, \ldots, u_p^{-1}).$$

5.5. Remark The information in $\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$ is in fact equivalent with that in $B_{\mathcal{Z}}$, $I_{\mathcal{Z}}$, $P_{\mathcal{Z}}$. By Theorem 9.3 and §3.7 in [5] the columns of $B_{\mathcal{Z}}$ are the primitive vectors along the sides of the Newton polygon of $\det \mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})$ w.r.t. $u_1, \ldots, u_p$. One recovers $P_{\mathcal{Z}}$ and $I_{\mathcal{Z}}$ as follows. Let $\mathcal{P}$ and $\mathcal{Q}$, respectively, be the sets of exponent vectors of the monomials in $u_1, \ldots, u_p$ which appear in the first columns of the matrices

$$\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})\ \left(\mathbb{K}^t_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})\,\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})\right)^n \quad \text{resp.} \quad \left(\mathbb{K}^t_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})\,\mathbb{K}_{\mathcal{Z}}(\mathbf{z}, \mathbf{u})\right)^{n+1}$$

for $n \in \mathbb{Z}_{\geq 0}$. Translations on $\mathbb{Z}^p$ by vectors in $\mathbb{Z}^2 B_{\mathcal{Z}}$ preserve $\mathcal{P}$ and $\mathcal{Q}$. Now take a 'fundamental domain' $\mathcal{P}^* \subset \mathcal{P}$ for the $\mathbb{Z}^2 B_{\mathcal{Z}}$-action on $\mathcal{P}$. For each vector α in $\mathcal{P}^*$ let $\mathcal{Q}_\alpha$ be the set of vectors β in $\mathcal{Q}$ such that $\alpha - \beta$ has precisely two non-zero entries and these are both 1. The rows of $P_{\mathcal{Z}}$ resp. $I_{\mathcal{Z}}$ are α resp. $\alpha - \beta$ with $\alpha \in \mathcal{P}^*$ and $\beta \in \mathcal{Q}_\alpha$.

§6. The principal $\mathcal{A}$-determinant

6.1. The principal $\mathcal{A}_{\mathcal{Z}}$-determinant for a good pattern of zigzags Let $\mathcal{Z} = (Z_1, \ldots, Z_p)$ be a good pattern of zigzags and let $\mathcal{A}_{\mathcal{Z}}$ denote the set of rows of the matrix $A_{\mathcal{Z}}$ (see 5.1). Let ι be the homomorphism

$$\iota : \mathbb{Z}[z_e \,|\, e \text{ intersection point in } \mathcal{Z}] \longrightarrow \mathbb{Z}, \qquad \iota(z_e) = |Z_{i(e)} \wedge Z_{j(e)}|\,.$$

The *principal $\mathcal{A}_{\mathcal{Z}}$-determinant* is defined in [2] and Theorem 3 in [6] states that it is equal to

$$\iota\left(\det \mathbb{K}^{c}_{\mathcal{Z}}(\mathbf{z},\mathbf{u})\right) . \tag{12}$$

The relation

$$I_{\mathcal{Z}} = P_{\mathcal{Z}} - Q_{\mathcal{Z}}$$

is precisely the one required in Condition 2 of [6].

6.2. Concluding remark about the principal $\mathcal{A}$-determinant

For the principal $\mathcal{A}$-determinant of the set $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ in the Introduction we may have to make some slight adaptations to formula (12), which reverse in a sense the transformation from $B_{\mathcal{A}}$ to $B_{\mathcal{Z}}$ in 4.1.

In (12) the variables in $\mathbf{u} = (u_1, \dots, u_p)$ correspond $1:1$ with the columns of $B_{\mathcal{Z}}$. Take a new set of variables $\mathbf{v} = (v_1, \dots, v_N)$ which correspond $1:1$ with the columns of $B_{\mathcal{A}}$. Recall that $B_{\mathcal{Z}}$ is obtained from $B_{\mathcal{A}}$ by permuting and splitting up columns as in (4). Then, to reverse the transformation from $B_{\mathcal{A}}$ to $B_{\mathcal{Z}}$ one must set $u_i = d_k\, v_k$ if the i-th column of $B_{\mathcal{Z}}$ comes from the k-th column of $B_{\mathcal{A}}$; here d_k is the g.c.d. of the two entries in the k-th column of $B_{\mathcal{A}}$.

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