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Two kinds of conditionings for stable Lévy processes

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Abstract.

Two kinds of conditionings for one-dimensional stable Lévy processes are discussed via h-transforms of excursion measures: One is to stay positive, and the other is to avoid the origin.

§1. Introduction

It is well-known that a one-dimensional Brownian motion conditioned to stay positive is a three-dimensional Bessel process. As an easy consequence, it follows that the former conditioned to avoid the origin is a symmetrized one of the latter.

The aim of the present article is to give a brief survey with some new results on these two different kinds of conditionings for one-dimensional stable Lévy processes via h-transforms of Itô's excursion measures.

The organization of this article is as follows. In Section 2, we recall the conditionings for Brownian motions. In Section 3, we give a review on the conditioning for stable Lévy processes to stay positive. In Section 4, we present results on the conditioning for symmetric stable Lévy processes to avoid the origin.

§2. Conditionings for Brownian motions

We recall the conditionings for Brownian motions. For the details, see, e.g., [8, §III.4.3] and [10, §VI.3 and Chap.XII].

Let (X_t) denote the coordinate process on the space of càdlàg functions and $(\mathcal{F}_t)$ its natural filtration. Set $\mathcal{F}_\infty = \sigma(\cup_{t>0}\mathcal{F}_t)$. For $t \geq 0$, we write θ_t for the shift operator: $X_s \circ \theta_t = X_{t+s}$. For $0 < t < \infty$, a functional Z_t is called *$\mathcal{F}_t$-nice* if Z_t is of the form $Z_t = f(X_{t_1}, \dots, X_{t_n})$ for

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some $0 < t_1 < \ldots < t_n < t$ and some continuous function $f : \mathbb{R}^n \to \mathbb{R}$ which vanishes at infinity.

Let W_x denote the law of the one-dimensional Brownian motion starting from $x \in \mathbb{R}$.

2.1. Brownian motions conditioned to stay positive

For any fixed $t > 0$, we define a probability law $W^{\uparrow,(t)}$ on $\mathcal{F}_t$ as

$$W^{\uparrow,(t)}(\cdot) = \frac{\boldsymbol{n}^+(\cdot;\ \zeta > t)}{\boldsymbol{n}^+(\zeta > t)} \tag{2.1}$$

where $\boldsymbol{n}^+$ stands for the *excursion measure of the reflecting Brownian motion* (see, e.g., [8, §III.4.3] and [10, §XII.4]) and ζ for the *lifetime*. The process $(X_s : s \leq t)$ under $W^{\uparrow,(t)}$ is called the *Brownian meander*. Durrett–Iglehart–Miller [7, Thm.2.1] have proved that

$$W^{\uparrow,(t)}[Z_t] = \lim_{\varepsilon \to 0+} W_0 \left[Z_t \,\middle|\, \forall u \leq t,\ X_u \geq -\varepsilon \right] \tag{2.2}$$

for any bounded continuous $\mathcal{F}_t$-measurable functional Z_t; in particular, for any $\mathcal{F}_t$-nice functional Z_t. We may represent (2.2) symbolically as

$$W^{\uparrow,(t)}(\cdot) = W_0 \left(\cdot \,\middle|\, \forall u \leq t,\ X_u \geq 0 \right); \tag{2.3}$$

that is, the Brownian meander is the Brownian motion *conditioned to stay positive until time t*. As another interpretation of (2.3), we have

$$W^{\uparrow,(t)}[Z_t] = \lim_{\varepsilon \to 0+} W_0 \left[Z_t \circ \theta_\varepsilon \,\middle|\, \forall u \leq t,\ X_u \circ \theta_\varepsilon \geq 0 \right] \tag{2.4}$$

for any $\mathcal{F}_t$-nice functional Z_t. The proof of (2.4) will be given in Theorem 3.5 in the settings of stable Lévy processes.

We write $W_0^{\uparrow}$ for the law of a *three-dimensional Bessel process*, that is, the law of the radius $\sqrt{(B_t^{(1)})^2 + (B_t^{(2)})^2 + (B_t^{(3)})^2}$ of a three-dimensional Brownian motion $(B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$. We remark that $W_0^{\uparrow}$ is locally equivalent to $\boldsymbol{n}^+$:

$$\mathrm{d}W_0^{\uparrow}|_{\mathcal{F}_t} = \frac{1}{C^{\uparrow}} X_t \mathrm{d}\boldsymbol{n}^+|_{\mathcal{F}_t} \tag{2.5}$$

where $C^{\uparrow} = \boldsymbol{n}^+[X_t]$ is a constant independent of $t > 0$. We say that $W_0^{\uparrow}$ is the *h-transform of the excursion measure $\boldsymbol{n}^+$ with respect to the function $h(x) = x$*. Then it holds (see, e.g., Theorem 3.6) that

$$W_0^{\uparrow}[Z] = \lim_{t \to \infty} W^{\uparrow,(t)}[Z] \tag{2.6}$$

for any $\mathcal{F}_t$-nice functional Z with $0 < t < \infty$. We may represent (2.6) symbolically as

$$W_0^{\uparrow}(\cdot) = W_0 \left(\cdot \,\middle|\, \forall u,\ X_u \geq 0 \right); \tag{2.7}$$

that is, $W_0^{\uparrow}$ is the Brownian motion *conditioned to stay positive during the whole time.*

2.2. Brownian motions conditioned to avoid the origin

For any fixed $t > 0$, we define a probability law $W^{\times,(t)}$ on $\mathcal{F}_t$ as

$$W^{\times,(t)} = \frac{W^{\uparrow,(t)} + W^{\downarrow,(t)}}{2} \tag{2.8}$$

where $W^{\downarrow,(t)}$ stands for the law of the process $(-X_s : s \leq t)$ under $W^{\uparrow,(t)}$. The law $W^{\times,(t)}$ may be represented as

$$W^{\times,(t)}(\cdot) = \frac{\boldsymbol{n}(\cdot;\ \zeta > t)}{\boldsymbol{n}(\zeta > t)} \tag{2.9}$$

where $\boldsymbol{n}$ stands for the *excursion measure of the Brownian motion*; in fact, $\boldsymbol{n} = \frac{\boldsymbol{n}^+ + \boldsymbol{n}^-}{2}$ where $\boldsymbol{n}^-$ is the image measure of the process $(-X_t)$ under $\boldsymbol{n}^+$. Immediately from (2.4) and continuity of paths, we have

$$W^{\times,(t)}[Z_t] = \lim_{\varepsilon \to 0+} W_0 \left[Z_t \circ \theta_\varepsilon \,\middle|\, \forall u \leq t,\ X_u \circ \theta_\varepsilon \neq 0 \right] \tag{2.10}$$

for any $\mathcal{F}_t$-nice functional Z_t. We may represent (2.10) symbolically as

$$W^{\times,(t)}(\cdot) = W_0 \left(\cdot \,\middle|\, \forall u \leq t,\ X_u \neq 0 \right); \tag{2.11}$$

that is, $W^{\times,(t)}$ is the Brownian motion *conditioned to avoid the origin until time t.*

We define

$$W_0^{\times} = \frac{W_0^{\uparrow} + W_0^{\downarrow}}{2} \tag{2.12}$$

where $W_0^{\downarrow}$ stands for the law of the process $(-X_t)$ under $W_0^{\uparrow}$. In other words, the law $W_0^{\times}$ is the symmetrization of the three-dimensional Bessel process. We also remark that $W_0^{\times}$ is locally equivalent to $\boldsymbol{n}$:

$$\mathrm{d}W_0^{\times}|_{\mathcal{F}_t} = \frac{1}{C^{\times}} |X_t| \mathrm{d}\boldsymbol{n}|_{\mathcal{F}_t} \tag{2.13}$$

where $C^{\times} = \boldsymbol{n}[|X_t|]$ is a constant independent of $t > 0$. We say that $W_0^{\times}$ is the *h-transform with respect to the function* $h(x) = |x|$. Then it is immediate from (2.6) that

$$W_0^{\times}[Z] = \lim_{t \to \infty} W^{\times,(t)}[Z] \tag{2.14}$$

for any $\mathcal{F}_t$-nice functional Z with $0 < t < \infty$. We may represent (2.14) symbolically as

$$W_0^{\times}(\cdot) = W_0\left(\cdot \,\middle|\, \forall u,\ X_u \neq 0\right); \tag{2.15}$$

that is, $W_0^{\times}$ is the Brownian motion *conditioned to avoid the origin during the whole time.*

§3. Stable Lévy processes conditioned to stay positive

Let us review the theory of strictly stable Lévy processes conditioned to stay positive. For references, see, e.g., [1] and [6]. We refer to these textbooks also about the theory of conditioning to stay positive for *spectrally negative Lévy processes*, where we do not go into the details.

For a Borel set F, we denote the first hitting time of F by

$$T_F = \inf\{t > 0 : X_t \in F\}. \tag{3.1}$$

Define

$$\underline{X}_t = \inf_{s \le t} X_s, \quad \underline{R}_t = X_t - \underline{X}_t \tag{3.2}$$

and call the process $(\underline{R}_t)$ the *reflected process.*

Let (P_x) denote the law of a *strictly stable Lévy process of index* $0 < \alpha \le 2$, that is, a process with càdlàg paths and with stationary independent increments satisfying the following scaling property:

$$(k^{-\frac{1}{\alpha}} X_{kt} : t \ge 0) \overset{\text{law}}{=} (X_t : t \ge 0) \quad \text{under } P_0 \tag{3.3}$$

for any $k > 0$. Note that the Brownian case corresponds to $\alpha = 2$. From the scaling property (3.3), it is immediate that the quantity

$$\rho := P_0(X_t \ge 0) \tag{3.4}$$

does not depend on $t > 0$, which is called the *positivity parameter.* The possible values of ρ range over $[0,1]$ if $0 < \alpha < 1$, $(0,1)$ if $\alpha = 1$, and

$[1-\frac{1}{\alpha}, \frac{1}{\alpha}]$ if $1 < \alpha \le 2$. Let $(Q_x : x > 0)$ denote the law of the process killed at $T_{(-\infty,0)}$:

$$(3.5) \qquad Q_x(\Lambda_t;\ t < \zeta) = P_x\left(\Lambda_t;\ t < T_{(-\infty,0)}\right), \quad x > 0,\ \Lambda_t \in \mathcal{F}_t.$$

Note that the function

$$(3.6) \qquad (x,t) \mapsto Q_x(t < \zeta) = P_x\left(t < T_{(-\infty,0)}\right)$$

is jointly continuous in $x > 0$ and $t > 0$.

Let us **exclude** the case where $|X|$ is a subordinator, i.e.,

$$(3.7) \qquad 0 < \alpha < 1 \quad \text{and} \quad \rho = 0, 1.$$

Then the reflected process $(\underline{R}_t)$ under (P_x) is a Feller process where the origin is regular for itself, and hence there exists the continuous local time process $(\underline{L}_t)$ at level 0 of the reflected process $(\underline{R}_t)$ such that

$$(3.8) \qquad P_0\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}\underline{L}_t\right] = q^{\rho-1}, \quad q > 0.$$

Let $\boldsymbol{n}^\uparrow$ denote the corresponding excursion measure away from 0 of the reflected process $(\underline{R}_t)$. The Markov property of $\boldsymbol{n}^\uparrow$ may be expressed as

$$(3.9) \qquad \boldsymbol{n}^\uparrow\left(1_\Lambda \circ \theta_t;\ \Lambda_t,\ t < \zeta\right) = \boldsymbol{n}^\uparrow\left[Q_{X_t}(\Lambda);\ \Lambda_t,\ t < \zeta\right]$$

for any $\Lambda \in \mathcal{F}_\infty$ and $\Lambda_t \in \mathcal{F}_t$. We introduce the following function:

$$(3.10) \qquad h^\uparrow(x) = P_0\left[\int_0^\infty 1_{\{\underline{X}_t \ge -x\}}\mathrm{d}\underline{L}_t\right], \quad x \ge 0.$$

Since the *ladder height process* $H := \underline{X} \circ \underline{L}^{-1}$ is a stable Lévy process of index $\alpha(1-\rho)$, we see that

$$(3.11) \qquad h^\uparrow(x) = P_0\left[\int_0^\infty 1_{\{H_u \ge -x\}}\mathrm{d}u\right] = C_1^\uparrow x^{\alpha(1-\rho)}$$

for some constant $C_1^\uparrow > 0$ independent of $x \ge 0$. The following theorem is due to Silverstein [14, Thm.2]; another proof can be found in Chaumont–Doney [5, Lem.1] (see also Doney [6, Lem.10 in §8.3]).

Theorem 3.1 ([14]). *It holds that*

$$(3.12) \qquad Q_x[(X_t)^{\alpha(1-\rho)};\ t < \zeta] = x^{\alpha(1-\rho)}, \quad x > 0,\ t > 0,$$

$$(3.13) \qquad \boldsymbol{n}^\uparrow[(X_t)^{\alpha(1-\rho)};\ t < \zeta] = C_2^\uparrow, \quad t > 0$$

for some constant $C_2^\uparrow$ independent of $t > 0$.

By virtue of this theorem, we may define the h-transform by

$$\mathrm{d}P_x^{\uparrow}|_{\mathcal{F}_t} = \begin{cases} \left(\frac{X_t}{x}\right)^{\alpha(1-\rho)} \mathrm{d}Q_x|_{\mathcal{F}_t} & \text{if } x > 0, \\ \frac{1}{C_2^{\uparrow}}(X_t)^{\alpha(1-\rho)} \mathrm{d}\boldsymbol{n}^{\uparrow}|_{\mathcal{F}_t} & \text{if } x = 0; \end{cases} \tag{3.14}$$

indeed, the family $(P_x^{\uparrow}|_{\mathcal{F}_t} : t \geq 0)$ is proved to be consistent by the Markov property of $\boldsymbol{n}^{\uparrow}$.

Theorem 3.2 ([3]). *The process $((X_t), (P_x^{\uparrow}))$ is a Feller process.*

This theorem is due to Chaumont [3, Thm.6], where he proved weak convergence $P_x^{\uparrow} \to P_0^{\uparrow}$ as $x \to 0+$ in the càdlàg space equipped with Skorokhod topology. Bertoin–Yor [2, Thm.1] proved the weak convergence for general positive self-similar Markov processes. Tanaka [15, Thm.4] proved the Feller property for quite a general class of Lévy processes.

Now let us discuss the conditionings (2.3) and (2.7) for stable Lévy processes. The following theorem is an immediate consequence of Chaumont [4, Lem.1] and of the continuity of (3.6).

Theorem 3.3 ([4]). *Let $t > 0$ be fixed. Then the function*

$$[0, \infty) \ni x \mapsto P_x^{\uparrow}\left[(X_t)^{-\alpha(1-\rho)}\right] \tag{3.15}$$

is continuous and vanishes at infinity.

Define a probability law $M^{\uparrow,(t)}$ on $\mathcal{F}_t$ as

$$M^{\uparrow,(t)}(\Lambda_t) = \frac{\boldsymbol{n}^{\uparrow}(\Lambda_t;\ \zeta > t)}{\boldsymbol{n}^{\uparrow}(\zeta > t)} = \frac{P_0^{\uparrow}[(X_t)^{-\alpha(1-\rho)};\ \Lambda_t]}{P_0^{\uparrow}[(X_t)^{-\alpha(1-\rho)}]} \tag{3.16}$$

for $\Lambda_t \in \mathcal{F}_t$. The following theorem, which generalizes (2.2), can be found in Bertoin [1, Thm.VIII.18].

Theorem 3.4 ([1]). *For any $t > 0$, it holds that*

$$M^{\uparrow,(t)}[Z_t] = \lim_{\varepsilon \to 0+} P_0\left[Z_t \,\middle|\, \forall u \leq t,\ X_u \geq -\varepsilon\right] \tag{3.17}$$

for any $\mathcal{F}_t$-nice functional Z_t.

Now we give the following version of (2.4) for stable Lévy processes.

Theorem 3.5. *For any $t > 0$, it holds that*

$$M^{\uparrow,(t)}[Z_t] = \lim_{\varepsilon \to 0+} P_0\left[Z_t \circ \theta_\varepsilon \,\middle|\, \forall u \leq t,\ X_u \circ \theta_\varepsilon \geq 0\right] \tag{3.18}$$

for any $\mathcal{F}_t$-nice functional Z_t.

Proof. By the Markov property, the expectation of the right hand side of (3.18) is equal to

$$(3.19)\quad \frac{P_0\left[P_{X_\varepsilon}\left[Z_t;\ \forall u\le t,\ X_u\ge 0\right]\right]}{P_0\left[P_{X_\varepsilon}\left(\forall u\le t,\ X_u\ge 0\right)\right]} = \frac{P_0\left[(X_\varepsilon)^{-\gamma}P^{\uparrow}_{X_\varepsilon}\left[Z_t(X_t)^{-\gamma}\right]\right]}{P_0\left[(X_\varepsilon)^{-\gamma}P^{\uparrow}_{X_\varepsilon}\left[(X_t)^{-\gamma}\right]\right]},$$

where we put $\gamma=\alpha(1-\rho)$. By the scaling property, this is equal to

$$(3.20)\quad \frac{P_0\left[(X_1)^{-\gamma}P^{\uparrow}_{\varepsilon^{1/\alpha}X_1}\left[Z_t(X_t)^{-\gamma}\right]\right]}{P_0\left[(X_1)^{-\gamma}P^{\uparrow}_{\varepsilon^{1/\alpha}X_1}\left[(X_t)^{-\gamma}\right]\right]}.$$

By Theorems 3.2 and 3.3 and by the dominated convergence theorem, we see that this quantity converges as $\varepsilon\to 0+$ to

$$(3.21)\quad \frac{P^{\uparrow}_0\left[Z_t(X_t)^{-\gamma}\right]}{P^{\uparrow}_0\left[(X_t)^{-\gamma}\right]},$$

which coincides with $M^{\uparrow,(t)}[Z_t]$ by the definition (3.16). Q.E.D.

The following theorem generalizes (2.6).

Theorem 3.6. *It holds that*

$$(3.22)\quad P^{\uparrow}_0[Z]=\lim_{t\to\infty}M^{\uparrow,(t)}[Z]$$

for any $\mathcal{F}_t$-nice functional Z with $0<t<\infty$.

The proof can be done in the same way as Theorem 3.5.

§4. Symmetric stable Lévy processes conditioned to avoid the origin

Let us discuss the conditioning for symmetric stable Lévy processes to avoid the origin. This has been introduced by Yano–Yano–Yor [18] in order to extend some of the *penalisation problems* for Brownian motions by Roynette–Vallois–Yor [11], [12] and Najnudel–Roynette–Yor [9], to symmetric stable Lévy processes.

We assume that $\rho=\frac{1}{2}$, i.e., the process $((X_t),(P_x))$ is *symmetric*, and that the index α satisfies $1<\alpha\le 2$. For simplicity, we assume that $P_0[\mathrm{e}^{i\lambda X_1}]=\mathrm{e}^{-|\lambda|^\alpha}$. Then the origin is regular for itself, and there exists the continuous resolvent density:

$$(4.1)\quad u_q(x)=\frac{1}{\pi}\int_0^\infty\frac{\cos x\lambda}{q+\lambda^\alpha}\mathrm{d}\lambda,\quad q>0,\ x\in\mathbb{R}.$$

Moreover, there exists the local time process at level 0 of the process (X_t), which we denote by (L_t), such that

$$P_0\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t\right] = u_q(0) = \frac{q^{\frac{1}{\alpha}-1}}{\alpha\sin\frac{\pi}{\alpha}}, \quad q>0. \tag{4.2}$$

The corresponding excursion measure away from 0 will be denoted by $\boldsymbol{n}^\times$. We introduce the following function:

$$h^\times(x) = \lim_{q\to 0+}\{u_q(0)-u_q(x)\} = \frac{1}{C^\times}|x|^{\alpha-1}, \quad x\in\mathbb{R}; \tag{4.3}$$

see, e.g., [17, Appendix] for the exact value of the constant $C^\times$. Note that the function $h^\times(x)$ may also be represented as

$$h^\times(x) = P_0[L_{T_{\{x\}}}], \quad x\in\mathbb{R}; \tag{4.4}$$

see, e.g., [1, Lem.V.11]. Let (P_x^0) denote the law of the process $((X_t),(P_x))$ killed at $T_{\{0\}}$:

$$P_x^0(\Lambda_t;\ t<\zeta) = P_x\left(\Lambda_t;\ t<T_{\{0\}}\right), \quad x\neq 0,\ \Lambda_t\in\mathcal{F}_t. \tag{4.5}$$

By [18, Thm.3.5], we see that the function

$$(x,t)\mapsto P_x^0(t<\zeta) = P_x\left(t<T_{\{0\}}\right) \tag{4.6}$$

is jointly continuous in $x\in\mathbb{R}\setminus\{0\}$ and $t>0$. The following theorem is due to Salminen–Yor [13, eq.(3)] and Yano–Yano–Yor [18, Thm.4.7].

Theorem 4.1 ([13], [18]). *It holds that*

$$P_x^0[|X_t|^{\alpha-1}] = |x|^{\alpha-1}, \quad x\neq 0,\ t>0, \tag{4.7}$$

$$\boldsymbol{n}^\times[|X_t|^{\alpha-1};\ t<\zeta] = C^\times, \quad t>0. \tag{4.8}$$

By virtue of this theorem, we may define the h-transform by

$$\mathrm{d}P_x^\times|_{\mathcal{F}_t} = \begin{cases} \left|\frac{X_t}{x}\right|^{\alpha-1}\mathrm{d}P_x^0|_{\mathcal{F}_t} & \text{if } x\neq 0,\\ \frac{1}{C^\times}|X_t|^{\alpha-1}\mathrm{d}\boldsymbol{n}^\times|_{\mathcal{F}_t} & \text{if } x=0;\end{cases} \tag{4.9}$$

indeed, the family $(P_x^\times|_{\mathcal{F}_t} : t\geq 0)$ is proved to be consistent by the Markov property of $\boldsymbol{n}^\times$. The following theorem is due to [16, Thm.1.5].

Theorem 4.2 ([16]). *Suppose that* $1<\alpha<2$. *Then the process* $((X_t),(P_x^\times))$ *is a Feller process.*

Remark 4.3. Yano [16, Thm.1.4 and Cor.1.9] obtained the following long-time behavior of paths: If $1 < \alpha < 2$, then

$$P_0^{\times}\left(\limsup_{t\to\infty} X_t = \limsup_{t\to\infty}(-X_t) = \lim_{t\to\infty} |X_t| = \infty\right) = 1. \tag{4.10}$$

Remark 4.4. In the Brownian case ($\alpha = 2$), the process $((X_t), (P_x^{\times}))$ is *not* a Feller process. Indeed, $P_0^{\times}$ is not irreducible (see (2.12)). Contrary to (4.10), the long-time behavior in this case is as follows:

$$P_0^{\times}\left(\lim_{t\to\infty} X_t = \infty\right) = P_0^{\times}\left(\lim_{t\to\infty} X_t = -\infty\right) = \frac{1}{2}. \tag{4.11}$$

Now let us discuss the conditionings (2.11) and (2.15) for symmetric stable Lévy processes. The following theorem is an immediate consequence of Yano–Yano–Yor [18, Lem.4.10] and of the continuity of (4.6).

Theorem 4.5 ([18]). *Let $t > 0$ be fixed. Then the function*

$$\mathbb{R} \ni x \mapsto P_x^{\times}\left[|X_t|^{-(\alpha-1)}\right] \tag{4.12}$$

is continuous and vanishes at infinity.

Define a probability law $M^{\times,(t)}$ on $\mathcal{F}_t$ as

$$M^{\times,(t)}(\Lambda_t) = \frac{\boldsymbol{n}^{\times}(\Lambda_t;\ \zeta > t)}{\boldsymbol{n}^{\times}(\zeta > t)} = \frac{P_0^{\times}[|X_t|^{-(\alpha-1)};\ \Lambda_t]}{P_0^{\uparrow}[|X_t|^{-(\alpha-1)}]} \tag{4.13}$$

for $\Lambda_t \in \mathcal{F}_t$. The following theorem generalizes (2.10).

Theorem 4.6. *For any $t > 0$, it holds that*

$$M^{\times,(t)}[Z_t] = \lim_{\varepsilon\to 0+} P_0\left[Z_t \circ \theta_\varepsilon \,\middle|\, \forall u \le t,\ X_u \circ \theta_\varepsilon \neq 0\right] \tag{4.14}$$

for any $\mathcal{F}_t$-nice functional Z_t.

The proof can be done in the same way as Theorem 3.5 by virtue of Theorems 4.2 and 4.5. The following theorem generalizes (2.14).

Theorem 4.7 ([18]). *It holds that*

$$P_0^{\times}[Z] = \lim_{t\to\infty} M^{\times,(t)}[Z] \tag{4.15}$$

for any $\mathcal{F}_t$-nice functional Z with $0 < t < \infty$.

This is a special case of [18, Thm.4.9].

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