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Hartman sets, functions and sequences – a survey

Reinhard Winkler [1]

Abstract.

A complex valued function $f : G \to \mathbb{C}$ on a topological group is called a Hartman function if there is a compact group C, a continuous homomorphism $\iota : G \to C$ with image $\iota(G)$ dense in C and a function $F : C \to \mathbb{C}$ which is integrable in the Riemann sense and satisfies $F \circ \iota = f$. $H \subseteq G$ is called a Hartman set if the characteristic function $\mathbf{1}_H$ is a Hartman function. In the case $G = \mathbb{Z}$ such a function is a two sided infinite binary sequence, called a Hartman sequence.

The investigation of such objects is motivated by the interest in finitely additive invariant probability measures on groups as well as by questions from symbolic dynamics in the context of ergodic group rotations. Connections to number theory ($n\alpha$- and Beatty sequences), combinatorics (complexity of words), geometry (projection bodies of convex polytopes), dynamics (ergodic group rotations, Sturmian sequences), topology (compactifications, group topologies, in particular precompact ones) and harmonic analysis (almost periodicity and weak almost periodicity) have been studied recently. This note is an extended version of a survey talk on these topics.

§1. Introduction and summary

This article is an extended version of a survey talk on Hartman sets, functions and sequences, objects which are connected with very classical mathematical topics from number theory, measure theory, symbolic dynamics and ergodic theory, geometry, topology, harmonic analysis etc. For most results presented in this article I do not give proofs. Nevertheless, concerning the main ideas I try to keep the presentation self

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contained to a reasonable extent. Therefore some classical material, in particular on invariant means on groups, compactifications, dynamical systems and harmonic analysis is included as well. The more recent work presented in this article has been done since the late 90s of the 20th century. Some results are yet unpublished but all proofs are available at least in electronic form.

The activities were motivated by investigations of measure theoretic concepts and the distribution of sequences on structures like certain rings of algebraic integers (cf. [27], [31], [32] and [33]). Countable rings (admitting no interesting σ-additive probability measure) with sufficiently many ideals of finite index have been considered as well as completions with respect to induced ideal metrics or topologies. The investigated completions turn out to be compact such that the unique normalized Haar measure (with respect to the additive structure) is available. A crucial step is to focus on so-called continuity (or Jordan measurable) sets in the completion, i.e. sets with a topological boundary of zero Haar measure. This approach induces a nontrivial Boolean set algebra of measurable sets on the original ring. The multiplicative structure, though very useful for defining interesting sets or sequences of algebraic origin, does not have any influence on the purely additively defined Haar measure. Thus one can as well just look for compactifications of the additive structure. Section 2 is a very brief presentation of the problem of finding invariant probability measures on non-compact groups G (as $G = \mathbb{Z}$). As a standard textbook on this topic we recommend [42].

Motivated by the above consideration, the initial point in [15] (the first paper of the series I am mainly reporting about) was to replace completions of certain metric spaces by compactifications of topological groups. By allowing arbitrary group compactifications (not necessarily compatible with the multiplicative structure) one gets much bigger classes of measurable sets on the original structure. Such sets are called Hartman sets. A complete exposition of the concept of Hartman sets is the main content of Section 3.

Hartman functions are related to Hartman sets in a similar way as Riemann integrable functions are related to Jordan measurable sets in classical real analysis. This is discussed in more detail in Section 4. The special case of Hartman sequences (0-1-valued Hartman functions on the group $G = \mathbb{Z}$) is particularly emphasized.

Section 5 presents some remarks on the historic background, in particular on the connection between Hartman sets and functions on one side and Hartman's concept of uniform distribution of sequences on non-compact groups on the other side.

Section 6 is mainly based on [36] and presents examples of Hartman sets, functions and sequences, mainly in the context of the group $G = \mathbb{Z}$. For instance, arithmetic progressions and the set of primes form Hartman sets. Such examples appear already if one restricts to compactifications which are compatible with the ring structure of $\mathbb{Z}$ as well. More typical for Hartman sequences are Sturmian sequences. They come from proper group compactifications and play a fundamental role for us since, in a certain sense, they generate all Hartman sequences. On one hand they are very well understood objects in number theory (by means of continued fractions), dynamics (special coding sequences of rotations of the circle) and combinatorics (they have minimal complexity among all sequences which are not periodic). On the other hand they form, in a certain sense, a generating system for the set algebra of Hartman sets. A rigorous statement of this fact is Theorem 5. Section 6 concludes with a result indicating that Hartman sets and Hartman uniformly distributed sequences are complementary in some sense (Theorem 6), and the fact that lacunary sequences form Hartman sets (Theorem 7).

In Section 7 I discuss combinatorial properties of Hartman sequences, mainly in terms of the complexity function from symbolic dynamics which is closely related to topological entropy. Hartman sequences can be interpreted as binary coding sequences of ergodic group rotations which are known to have entropy 0. A consequence is that the complexity function of an arbitrary Hartman sequence has a subexponential growth rate. A less obvious fact is that this upper bound is best possible if arbitrary Hartman sequences are allowed (Theorem 8). For more special classes one can get sharper results. In this context Steineder's theorem (Theorem 9, a result from [38]) is the most remarkable one. It opens an interesting new connection between symbolic dynamics and convex geometry.

The dynamical aspects are recalled in a more systematic manner in Section 8. In particular I am interested in the role of shift spaces for the coding of group rotations and in the spectrum as an invariant.

The topic of Section 9 is an alternative way of more topological flavour for obtaining invariants which relate group rotations with properties of Hartman sequences. Such a method has been developed in [45] and generalized to arbitrary Hartman functions on LCA groups in [28]. The main idea is to recover from the Hartman function a metric inducing a precompact group topology. The underlying group rotation can then be obtained by completing this metric.

It is obvious that Hartman functions are closely related to almost periodicity and related concepts. Recall that a continuous function on a topological group G is almost periodic if and only if it can be extended to a continuous function on the Bohr compactification of G. (Here we take this as the definition for almost periodicity.) Hence every almost periodic function is a Hartman function while the converse does not hold. If one replaces the Bohr compactification by the weak almost periodic compactification and considers functions which can be extended continuously to this bigger compactification one gets the notion of weak almost periodicity. It defines a further algebra of functions with a unique invariant mean. Connections among such spaces of functions and related questions touching harmonic analysis have been investigated in [29] and are discussed in Section 10.

§2. Invariant probability measures and means on groups

The research on the topics presented here started with the interest in natural invariant probability measures μ on groups G, for instance the additive group $G = \mathbb{Z}$ of integers. Thus we are interested in set functions μ assigning a number $\mu(A) \in [0,1]$ to certain subsets $A \subseteq G$ in such a way that $\mu(Ag)$, $Ag = \{ag : a \in A\}$, is defined for all $g \in G$ whenever $\mu(A)$ is defined; similarly for gA. If $\mu(A)$ is defined we say that A is measurable with respect to μ and require that $\mu(gA) = \mu(A) = \mu(Ag)$ as well as $\mu(\emptyset) = 0$ and $\mu(G) = 1$.

In the case of a compact group G the Haar measure has this property and is defined for all Borel sets $A \subseteq G$, which form a sufficiently large σ-algebra. But for noncompact G we cannot expect σ-additivity: Consider $G = \mathbb{Z}$ and the two cases $\mu(\{0\}) = \mu(\{k\}) = 0$ or > 0 for all $k \in \mathbb{Z}$. By σ-additivity we would get either $\mu(G) = 0$ or $\mu(G) = \infty$, contradicting our requirement $\mu(G) = 1$. So we cannot expect more than finite additivity of μ. Therefore we will use the term measure also in the finitely additive case.

In fact, for a satisfactory integration theory, it is desirable to have a Boolean algebra $\mathbf{B}$ of measurable sets: Measurability of sets A_i should be equivalent to measurability of their characteristic functions $\mathbf{1}_{A_i}$. We want to have an algebra of measurable functions, hence $\mathbf{1}_{A_1} \cdot \mathbf{1}_{A_2} = \mathbf{1}_{A_1 \cap A_2}$ and therefore $A_1 \cap A_2$ should be measurable whenever A_i are. Furthermore $A \in \mathbf{B}$ is intended to imply $G \setminus A \in \mathbf{B}$ by considering $\mathbf{1}_{G \setminus A} = \mathbf{1}_G - \mathbf{1}_A$. A fortiori $A_i \in \mathbf{B}$ implies $A_1 \cup A_2 = G \setminus ((G \setminus A_1) \cap (G \setminus A_2))$. Hence we assume that $\mathbf{B}$ is a Boolean algebra and μ is a finitely invariant probability measure μ defined for all $A \in \mathbf{B}$. Then, similar to the Riemann integrable functions in real analysis, there is an algebra $\mathcal{R}$ of functions including all $\mathbf{1}_A$, $A \in \mathbf{B}$, and a positive linear functional (integral) m, called an invariant mean on $\mathcal{R}$, such that $m(\mathbf{1}_A) = \mu(A)$ for all $A \in \mathbf{B}$. Conversely it is obvious that every invariant mean (defined on a sufficiently big algebra) induces a finitely additive invariant measure on a Boolean set algebra. As a classical monograph on invariant means we refer to [19].

The most (too) immodest question is: Given a group G, is there a natural invariant finitely additive probability measure μ on G which is defined for all $A \subseteq G$? In most interesting cases the answer is no, but by different reasons, depending on G. For amenable groups G, by definition, there exists such a μ (called an invariant mean), but typically it is far from unique. (Note that all abelian groups are amenable. We will illustrate this later for $G = \mathbb{Z}$.) On the other hand there are groups which are not amenable.

The classical example of a group which is not amenable is the group $F = F(x, y)$ having two free generators x and y. Thus each $w \in F$ can be uniquely represented by a reduced group word. F is the disjoint union of the four subsets $F_x, F_{x^{-1}}, F_y$ and $F_{y^{-1}}$ containing those w which start with the letter x, x^{-1}, y resp. y^{-1}, plus the singleton $\{\emptyset\}$ containing the empty word $\emptyset$. Note that we have the set equation

$$x^{-1}F_x = F_x \cup F_y \cup F_{y^{-1}} \cup \{\emptyset\}$$

where the union is pairwise disjoint. Suppose we have an invariant mean μ defined for all subsets. Then additivity and invariance of μ applied to the above equation yield

$$\mu(F_x) = \mu(x^{-1}F_x) = \mu(F_x) + \mu(F_y) + \mu(F_{y^{-1}}) + \mu(\{\emptyset\})$$

and thus $\mu(F_y) = \mu(F_{y^{-1}}) = \mu(\{\emptyset\}) = 0$. By symmetry we also get $\mu(F_x) = \mu(F_{x^{-1}}) = 0$ and therefore, again by additivity, $\mu(G) = 0$, contradiction. Thus there is no invariant mean on F defined on the whole power set, i.e. F indeed is not amenable. (This observation is one of the main ingredients for the celebrated Banach-Tarski paradox, cf. for instance the comprehensive textbook [42] or the elementary introduction [46].)

Let us now turn to the group $G = \mathbb{Z}$. For each $N \in \mathbb{N}$ and $A \subseteq \mathbb{Z}$ define

$$\mu_N(A) = \frac{|A \cap [-N, N]|}{2N+1}.$$

All μ_N are finitely additive probability measures defined on the whole power set $\mathbf{P}(\mathbb{Z})$, but they are not invariant. To overcome this problem note that for every $A \subseteq \mathbb{Z}$ and fixed $k \in \mathbb{Z}$ we have

$$\lim_{N \to \infty} (\mu_N(A) - \mu_N(A + k)) = 0.$$

It follows that every accumulation point μ_0 of the μ_N, $N \in \mathbb{N}$, in the compact space $[0,1]^{\mathbf{P}(\mathbb{Z})}$ of all functions $\mathbf{P}(\mathbb{Z}) \to [0,1]$ is an invariant measure. By compactness, such a μ_0 exists and coincides with the density

$$\mathrm{dens}(A) = \lim_{N\to\infty} \frac{|A \cap [-N,N]|}{2N+1}$$

whenever the limit exists. Otherwise $\mu_0(A)$ depends on the special choice of the accumulation point. We illustrate this by considering the set

$$A_0 = \pm \bigcup_{n\in\mathbb{N}} ((2n)!, (2n+1)!] \cap \mathbb{Z}.$$

Note that $\lim_{n\to\infty} \mu_{(2n+1)!}(A_0) = 1$ while $\lim_{n\to\infty} \mu_{(2n)!}(A_0) = 0$. Hence limit measures μ_1 of the first sequence take the value $\mu_1(A_0) = 1$, limit measures μ_2 of the second one take the value $\mu_2(A_0) = 0$. Considering appropriate convex combinations $\mu = c\mu_1 + (1-c)\mu_2$ one can achieve every $c \in [0,1]$ as value $\mu(A_0)$ for an invariant probability measure μ. Note that obvious refinements of this method allow the construction of invariant measures with a lot of prescribed properties. This shows that invariant means on $\mathbb{Z}$ are not unique in a rather extreme way. (In fact a description of all invariant measures is possible by invoking the Stone-Čech compactification.)

But there is hope to get uniqueness by restriction to an appropriate subsystem of sets $A \subseteq \mathbb{Z}$. Our construction above suggests to consider the system $\mathbf{D}$ of all A such that $\mathrm{dens}(A)$ exists. The disadvantage is that this system is not a Boolean algebra. As an example take the sets $A_1 = 2\mathbb{Z}$ of all even integers and the set $A_2 = (2\mathbb{Z} \cap A_0) \cup ((2\mathbb{Z}+1) \setminus A_0)$ of all even numbers inside A_0 and all odd numbers outside A_0. Then $\mathrm{dens}(A_1) = \mathrm{dens}(A_2) = \frac{1}{2}$ exists (even uniformly in the shift) but neither $A_1 \cap A_2$ nor $A_1 \cup A_2$ have a density.

The consequence of this reasoning is to look for a nontrivial Boolean algebra of subsets of G contained in $\mathbf{D}$. We are going to do this in the next section.

§3. Group compactifications and Hartman sets

We have to start with basic notions concerning topological groups, especially group compactifications. For the reader interested in more background we refer to the classical textbook [22] on harmonic analysis, the more recent one [12] on topological groups and the article [41] on group compactifications.

Let G be any (topological or discrete) group. We assume all topological spaces to satisfy the Hausdorff separation axiom. We try to relate

G to a compact group C where we can use the Haar measure μ_C which is the unique invariant Borel probability measure on C. A brief inspection suggests to consider continuous homomorphisms $\iota : G \to C$ such that the image of G is dense in C.

Definition 1. *Let G be a topological group, C a compact group and $\iota : G \to C$ a continuous homomorphism such that $\overline{\iota(G)} = C$. Then the pair (C, ι) is called a (group) compactification of G.*

A first example which will play a major role is given by $G = \mathbb{Z}$ (additive group of integers), $C = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the one dimensional torus which can be identified with the multiplicative unit circle in the complex plane whenever convenient) and $\iota = \chi_\alpha : k \mapsto k\alpha$ with an irrational $\alpha \in \mathbb{R}/\mathbb{Z}$.

Note that we can as well take a rational $\alpha = \frac{p}{q}$ (let p and q have gcd 1) if we take for C the finite cyclic subgroup of order q instead of the whole group $\mathbb{T}$. Thus in the definition of a compactification we do not require ι to be injective.

It is tempting to define a measure for subsets $S \subseteq G$ by putting $\mu(S) = \mu_C(M)$ whenever $S = \iota^{-1}(M)$. But $\mu(S)$ is not well defined in this way without further restrictions. As an example take $G = \mathbb{Z}$, $C = \mathbb{T}$, $\iota = \chi_\alpha$ with α irrational and any $S \subseteq \mathbb{Z}$. Then the countable set $M_0 = \iota(S)$ has Haar measure $\mu_C(M_0) = 0$. For every $c \in [0,1]$ we can take any set $X_c \subseteq \mathbb{T}$ with $\mu_C(X_c) = c$ and consider $M_c = M_0 \cup X_c \setminus \iota(\mathbb{Z} \setminus S)$ in order to obtain $S = \iota^{-1}(M_c)$ and $\mu_C(M_c) = c$. What is the most natural choice for M?

The first guess is to take $M = \overline{\iota(S)}$. But note that this still does not work for all $S \subseteq \mathbb{Z}$. Consider in our example with irrational α the sets $S_0 = 2\mathbb{Z}$ of even and $S_1 = 2\mathbb{Z} + 1$ of odd integers. In both cases we would get $M = \overline{\iota(S_i)} = \mathbb{T}$, hence $\mu(S_0) = \mu(S_1) = 1$ and finally $\mu(\mathbb{Z}) = \mu(S_0) + \mu(S_1) = 2$, contradiction.

In fact it is not surprising that there are problems; otherwise we would have found a natural measure defined for all $S \subseteq \mathbb{Z}$. We have to restrict the system of measurable S. In which way?

We want to avoid the critical situation that $S = \iota^{-1}(M_1) = \iota^{-1}(M_2)$ and $\mu_C(M_1) \neq \mu_C(M_2)$. For this reason we have to restrict the allowed choices for M_i to a class of sets where $\mu_C(M_1 \Delta M_2) = 0$ for the symmetric difference whenever $\iota^{-1}(M_1) = \iota^{-1}(M_2)$. This works if we take the class of continuity (Jordan measurable) sets.

Definition 2. *A subset $M \subseteq C$ of a topological space C with Borel measure μ_C is called a continuity set or Jordan measurable set if $\mu_C(\partial M) = 0$, i.e. if the topological boundary ∂M of M is a zero set.*

If C is a compact group we write $\mathbf{J} = \mathbf{J}(C)$ for the Boolean set algebra of continuity sets.

If furthermore (C, ι) is a compactification of G then all preimages $\iota^{-1}(M) \subseteq G$, $M \in \mathbf{J}(C)$, are called (C, ι)-measurable. The Boolean set algebra of all such sets is denoted by $\mathbf{H}(C, \iota)$.

Indeed, pick two continuity sets M_i with $\mu_C(M_1 \Delta M_2) \neq 0$. Without loss of generality we may assume that $\mu_C(M_1 \setminus M_2) > 0$. The difference set again is a continuity set an thus contains a nonempty open set O of positive Haar measure in the group C. Since $\iota(G)$ is dense in C the preimages $S_i = \iota^{-1}(M_i)$ cannot coincide. This shows that $\mu_{(C,\iota)}(\iota^{-1}(M)) = \mu_C(M)$ is well defined if we restrict ourselves to $M \in \mathbf{J}(C)$.

For given $S \subseteq G$ the value $\mu_{(C,\iota)}$ might depend on the special choice of the compactification (C, ι). We show that this is not the case; only the systems $\mathbf{H}(C, \iota)$ vary but not the measures of sets whenever defined. To see this we have to relate different compactifications to each other.

Definition 3. *Let (C_i, ι_i), $i = 1, 2$, be two compactifications of G. We write $(C_1, \iota_1) \leq (C_2, \iota_2)$ if there is a continuous homomorphism $\pi : C_2 \to C_1$ such that $\iota_1 = \pi \circ \iota_2$. We call the compactifications equivalent and write $(C_1, \iota_1) \cong (C_2, \iota_2)$ if π can be taken to be an algebraic and topological isomorphism.*

Example: Take $G = \mathbb{Z}$, $(C_1, \iota_1) = (\mathbb{T}, \chi_\alpha)$ and $(C_2, \iota_2) = (\mathbb{T}, \chi_\beta)$ with irrational β. Then $\iota : k \mapsto (k\alpha, k\beta)$ gives rise to a compactification (C, ι) which is a common upper bound of both (C_i, ι_i) since $\pi_i : (x_1, x_2) \mapsto x_i$, $i = 1, 2$ shows $(C_i, \iota_i) \leq (C, \iota)$. Note that in the case that $\alpha, \beta \in \mathbb{T}$ are linearly independent over $\mathbb{Z}$ we get $C = \mathbb{T}^2$. Furthermore we see that this product construction is possible for any two compactifications.

Remark: Given the group G, we may consider (C, ι) as an object of the category $\mathcal{COMP}(G)$ of all (group) compactifications of G. As (unique) morphism $(C_2, \iota_2) \to (C_1, \iota_1)$ we take $\pi : C_2 \to C_1$ establishing $(C_1, \iota_1) \leq (C_2, \iota_2)$ whenever it exists. Composition of morphisms is the usual composition of maps.

Consider the situation of Definition 3 and pick any $M_1 \in \mathbf{J}(C_1)$. Using the uniqueness of the Haar measures involved it is an easy exercise to show that $M_2 = \pi^{-1}(M_1) \in \mathbf{J}(C_2)$ and $\mu_{C_2}(M_2) = \mu_{C_1}(M_1)$. Furthermore it is clear that $S = \iota_1^{-1}(M_1) = \iota_2^{-1}(M_2)$. This shows that for $(C_1, \iota_1) \leq (C_2, \iota_2)$ we have $\mathbf{H}(C_1, \iota_1) \subseteq \mathbf{H}(C_2, \iota_2)$ and that $\mu_{(C_2,\iota_2)}$ extends $\mu_{(C_1,\iota_1)}$.

But also in the case that two compactifications are not comparable we easily see $\mu_{(C_1,\iota_1)}(S) = \mu_{(C_1,\iota_2)}(S)$ whenever $S \in \mathbf{H}(C_1,\iota_1) \cap \mathbf{H}(C_2,\iota_2)$. We only have to take a common upper bound (C,ι) as in the above example to obtain $\mu_{(C_1,\iota_1)}(S) = \mu_{(C,\iota)}(S) = \mu_{(C_1,\iota_2)}(S)$.

We can generalize the product construction giving a common upper bound for two given compactifications to an arbitrary family of compactifications (C_i,ι_i), $i \in I$. By Tychonoff's theorem $P = \prod_{i\in I} C_i$ is a compact group. $\iota : G \to P$, $g \mapsto (\iota_i(g))_{i\in I}$ is a continuous homomorphism. Hence (C,ι) with $C = \overline{\iota(G)}$ is a compactification of G. One can think of (C,ι) as an inverse limit or take the following point of view. The projections $\pi_{i_0} : C \to C_{i_0}$, $(x_i)_{i\in I} \mapsto x_{i_0}$, show $(C_{i_0},\iota_{i_0}) \leq (C,\iota)$ for all $i_0 \in I$. (C,ι) together with all π_i, $i \in I$, is a product of the (C_i,ι_i), $i \in I$, in the sense of theory of categories. (This means, by definition, that for every $(C',\iota') \in \mathcal{COMP}(G)$ and every family of morphisms $\pi_i' : C' \to C_i$ showing $(C_i,\iota_i) \leq (C',\iota')$ there is a unique morphism $\varphi : C' \to C$ with $\pi_i \circ \varphi = \pi_i'$ for all $i \in I$. In our situation this is the case with $\varphi : c' \mapsto (\pi_i'(c'))_{i\in I}$.) In order to get a universally big compactification, i.e. a universal object in $\mathcal{COMP}(G)$, we would like to allow all possible compactifications among the (C_i,ι_i). Of course this is not possible because the involved class is not a set. To overcome this difficulty note that each compactification can, up to equivalence, be realized on a set X of bounded cardinality $|X| \leq 2^{2^{|G|}}$. (Consider any compactification (C,ι) of G. By the Hausdorff separation axiom every $c \in C$ is uniquely determined by its neighbourhood filter F_c. Since $\iota(G)$ is dense the same holds if we restrict the filter to $\iota(G)$. Thus there cannot be more points in C than filters on G. So $|C| \leq 2^{2^{|G|}}$.) All compactifications (C,ι) with $C \subseteq X$ form a set and we can apply the product construction to all of them. In this way we get a common upper bound which has to be the maximal compactification up to equivalence.

Definition 4. *Let G be a topological group. The maximal compactification of G (which, as a universal object in $\mathcal{COMP}(G)$, is uniquely determined up to equivalence) is called the Bohr compactification of G and denoted by (bG,ι_b).*

It follows that

$$\mathbf{H}(G) = \mathbf{H}(bG,\iota_b) = \bigcup_{(C,\iota)} \mathbf{H}(C,\iota),$$

where the union is taken over all compactifications of G, defines a Boolean set algebra with a canonical invariant finitely additive probability measure $\mu_G = \mu_{(bG,\iota_b)}$.

Definition 5. *A subset $H \subseteq G$ is called a Hartman (measurable) set if it is a member of the Boolean set algebra $\mathbf{H}(G)$. μ_G is called the Hartman measure on $\mathbf{H}(G)$.*

Our abstract construction of the Bohr compactification works in a very general setting (even for other classes of topological algebraic structures) but does not give a clear idea how it looks like. There are examples of infinite groups G where bG is the one element group. Of course these examples are not interesting in our context. But for an important class of topological groups, namely for locally compact abelian groups (briefly denoted LCA-groups), Pontrjagin's duality theory yields a nice description. In particular $\iota_b : G \to bG$ is injective in this case. A sketch of the argument is as follows.

Let G be an LCA-group and $\hat{G}$ its dual. Each $\chi \in \hat{G}$ defines a compactification (C_χ, χ) with $C_\chi = \overline{\chi(G)} \subseteq \mathbb{T}$. Let (C, ι) be the product of all such compactifications. We claim that this is (up to equivalence) already the Bohr compactification. It suffices to show $(C', \iota') \leq (C, \iota)$ for an arbitrary compactification (C', ι') of G. Since C' is a compact group, its dual $\hat{C'}$ is a discrete group. Pontrjagin's duality asserts that $\varphi : g' \mapsto (\chi'(g'))_{\chi' \in \hat{C'}}$ defines an isomorphism between C' and a closed subgroup C'' of $\mathbb{T}^{\hat{C'}}$. Note that every $\chi' \in \hat{C'}$ induces a $\chi = \chi' \circ \iota' \in \hat{G}$. It follows that there is a (unique) continuous homomorphism $\pi : C \to C'$ such that $\varphi \circ \pi(c) = (y_{\chi'})_{\chi' \in \hat{C'}}$ with $c = (x_\chi)_{\chi \in \hat{G}} \in \hat{C}$ and $y_{\chi'} = x_{\chi' \circ \iota'}$, $\chi' \in \hat{C'}$.

§4. Hartman functions and sequences

The notion of continuity or Jordan measurable sets is closely related to the Riemann integral. A subset M of the real interval $[a, b]$ is a continuity set if and only if its characteristic function $\mathbf{1}_M$ is integrable in the Riemann sense. This characterization extends for arbitrary bounded complex valued functions $f : [a, b] \to \mathbb{C}$ to the well known criterion: f is integrable in the Riemann sense if and only if the set of its discontinuities has Lebesgue measure 0. For real valued f a further equivalent condition is that f can be approximated arbitrarily well in the sense of the integral from above and below by continuous functions. These characterizations can be used as definitions in a more general setting.

Definition 6. *Let X be a compact space with a complete and regular Borel probability measure μ and $f : X \to \mathbb{R}$ bounded. We call f Riemann integrable if the following equivalent conditions hold.*

(1) *The set of discontinuities of f has measure 0.*

(2) *For each $\varepsilon > 0$ there are continuous $f_i : X \to \mathbb{R}$, $i = 1, 2$, such that $f_1 \le f \le f_2$ and $\int_X f_2 - f_1 d\mu < \varepsilon$.*

A complex valued $f : X \to \mathbb{C}$ is called Riemann integrable if its real and imaginary part are both Riemann integrable.

A proof of the equivalence of conditions (1) and (2) also for the non metrizable case is contained in [40] (Lemma 2).

The natural extension of the measure theory for Hartman sets presented in the previous section to an integration theory of complex valued functions works in an obvious way.

Definition 7. *Let G be a topological group and (C, ι) a compactification of G. Let $\mathcal{R}(C)$ denote the set of all complex valued Riemann integrable functions on C with respect to μ_C. Every $f = F \circ \iota$ with $F \in \mathcal{R}(C)$ is called a Hartman function on G, represented by F in (C, ι). $m_G(f) = \int_C F d\mu_C$ is called the mean value of f. The set of all Hartman functions on G with a representation in (C, ι) is denoted by $\mathcal{H}(C, \iota)$. The union of all $\mathcal{H}(C, \iota)$, where (C, ι) runs through all compactifications of G, coincides with $\mathcal{H}(bG, \iota_b)$ and is called the algebra of Hartman functions on G and denoted by $\mathcal{H}(G)$.*

Modifications of the arguments presented in the previous section show that: $(C_1, \iota_1) \le (C_2, \iota_2)$ by $\pi : C_2 \to C_1$ and $F_1 \in \mathcal{R}(C_1)$ implies $F_2 = F_1 \circ \pi \in \mathcal{R}(C_2)$ and $\int_{C_1} F_1 d\mu_{C_1} = \int_{C_2} F_2 d\mu_{C_2}$; hence $\mathcal{H}(C_1, \iota_1) \subseteq \mathcal{H}(C_2, \iota)$ and $m_G(f)$ indeed do not depend on the special representation of f.

Recall that almost periodic functions $f : G \to \mathbb{C}$ can be characterized by the property that they can be extended to the Bohr compactification in a continuous way. Thus Hartman functions are a generalization of almost periodic functions in the same sense as Riemann integrable functions are generalizations of continuous functions, cf. also Section 10. We will be particularly interested in the following special case of Hartman functions.

Definition 8. *A Hartman function $f : \mathbb{Z} \to \{0, 1\}$ on the group $G = \mathbb{Z}$ taking only the binary values 0 and 1 is also called a (two-sided infinite) Hartman sequence, written as $\mathbf{a} = (a_n)_{n \in \mathbb{Z}}$ with $a_n = f(n)$.*

§5. Historic remarks on Hartman uniform distribution

The notions of Hartman sets, functions and sequences appeared under different names in the work of the Polish mathematician Stanisław Hartman (1914-1992), cf. [20] and [21], who worked mainly in harmonic analysis. He used the Bohr compactification to define a natural concept of uniform distribution (u.d.) of sequences on groups which are not necessarily compact.

Definition 9. *A (one-sided infinite) sequence* $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ *taking values in a topological group is called Hartman uniformly distributed (H-u.d.) if its image* $(\iota_b(x_n))_{n \in \mathbb{N}}$ *is uniformly distributed in* bG *where* (bG, ι_b) *denotes the Bohr compactification.*

By the same reason as for Hartman sets, namely by the availability of Pontrjagin's duality, this concept works very well on LCA-groups. Using the description of the Bohr compactification given at the end of Section 3 one easily can transfer the Weyl criterion for u.d. sequences on compact groups to H-u.d. sequences on LCA groups.

Theorem 1. (Weyl criterion for Hartman uniform distribution) *Let* G *be an LCA group and* $x_n \in G$, $n \in \mathbb{N}$. *Then the sequence* $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ *is H-u.d. if and only if*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \chi(x_i) = 0$$

for all characters $\chi \in \hat{G}$ *different from the constant character.*

In [26] (Definition 5.6, p. 295) this criterion was taken as the definition of Hartman uniform distribution.

Examples: It follows directly from Theorem 1 that, for $G = \mathbb{Z}$, the sequence $x_n = n$, $n \in \mathbb{N}$, is H-u.d. The same holds for every sequence $\mathbf{x} = b_1 b_2 \ldots$ which is the concatenation of blocks $b_n = a_n, a_n + 1, \ldots, a_n + l_n - 1$, $a_n \in \mathbb{Z}$, of increasing length l_n. Similar constructions are possible for the group $G = \mathbb{R} \cong \hat{\mathbb{R}}$ if one takes appropriate blocks which are finite arithmetic progressions of increasing lengths and decreasing step lengths.

Much deeper examples are due to Wierdl ([44]) asserting that for instance the sequences $[n \log n]$ and $[n^{\frac{2}{3}}]$ are H-u.d. In [34] instead of *Hartman uniformly distributed sequences* the term *ergodic sequences* is used, but the results show that both notions are equivalent. The authors use powerful methods from ergodic theory and harmonic analysis. Very recent results in this direction are contained in [9]. In [8] H-u.d. sequences are called *homogeneously distributed.*

Hartman could characterize those G which admit H-u.d. sequences.

Theorem 2. *(Hartman, cf.* [21] *or* [26], *p. 298, Theorem 5.12) An LCA group G admits H-u.d. distributed sequences if and only if $|\hat{G}| \leq 2^{\aleph_0}$, the cardinality of the continuum.*

It is a classical fact in the theory of uniform distribution that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_X f d\mu.$$

holds whenever $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ is a sequence uniformly distributed with respect to the measure μ and f is integrable in the Riemann sense. This translates to the corresponding statement on Hartman-u.d.

Theorem 3. *Let G be a topological group, $\mathbf{x}$ a H-u.d. sequence taking values in G and $f : G \to \mathbb{C}$ a Hartman function. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = m_G(f).$$

Let us consider the special case $G = \mathbb{Z}$, the H-u.d. sequence $x_n = n$ and the Hartman function (sequence) $f = \mathbf{1}_H$ corresponding to an arbitrary Hartman set $H \in \mathbf{H}(\mathbb{Z})$. Then we observe

$$\lim_{n\to\infty} \frac{1}{n}|H \cap (0, n]| = \mathrm{dens}(H).$$

Thus Hartman sets in $\mathbb{Z}$ have a density which, according to classical facts in uniform distribution (or unique ergodicity) is in fact uniform in $k \in \mathbb{Z}$ if we consider $\lim_{n\to\infty} \frac{1}{n}|H \cap (k, n+k]|$.

§6. Number-theoretic aspects for $G = \mathbb{Z}$

First we give some very simple examples of Hartman sets, sequences or functions on $G = \mathbb{Z}$.

Periodic sequences: Since every periodic function is almost periodic it is a Hartman function as well. We can see this more explicitly in the following way. Let $f : \mathbb{Z} \to \mathbb{C}$ be periodic with period length m, i.e. $f(k+m) = f(k)$ for all $k \in \mathbb{Z}$. Then we consider the compactification (C_m, ι_m) defined by $C_m = \mathbb{Z}/m\mathbb{Z}$ and $\iota_m(k) = k + m\mathbb{Z}$. By the periodicity of f there is a unique $F : C_m \to \mathbb{C}$ such that $f = F \circ \iota_m$. Since C_m is a finite group with discrete topology, all complex valued functions on C_m are continuous, in particular F. Hence $f \in \mathcal{H}(\mathbb{Z})$. It follows immediately that a Hartman function is periodic iff it has a representation in a finite compactification. Thus arithmetic progressions are very special examples of Hartman sets. The mean value of f is given by

$$m_{\mathbb{Z}}(f) = \frac{1}{m} \sum_{k=0}^{m-1} f(k).$$

The set $\mathbb{P}$ of primes: Let us consider the compactification $(\widehat{\mathbb{Z}}, \widehat{\iota})$ of $\mathbb{Z}$ also known as the profinite completion of $\mathbb{Z}$, defined as the product of all finite compactifications, i.e. $\widehat{\iota} : k \mapsto (\iota_m(k))_{m \geq 1}$, $\widehat{\mathbb{Z}} = \overline{\widehat{\iota}(\mathbb{Z})} \subseteq \prod_{m \in \mathbb{N}} C_m$. Note that $\widehat{\mathbb{Z}}$ carries the structure of a compact topological ring since all C_m are rings. If one restricts the indices m in the product to all powers p^e, $e \in \mathbb{N}$, of a fixed prime $p \in \mathbb{P}$ then one gets $(\mathbb{Z}_p, \pi_p \circ \widehat{\iota})$, $\pi_p : (x_m)_{m \geq 1} \mapsto (x_{p^e})_{e \in \mathbb{N}}$, where $\mathbb{Z}_p$ denotes the compact ring of p-adic integers. $\widehat{\mathbb{Z}}$ is the product of all $\mathbb{Z}_p$, $p \in \mathbb{P}$.

We are going to show that the set $\mathbb{P} \subseteq \mathbb{Z}$ is a Hartman set of measure 0. It suffices to prove that, for arbitrary $\varepsilon > 0$, we can find a closed set $M_\varepsilon \subseteq \widehat{\mathbb{Z}}$ such that $\mu_{\widehat{\mathbb{Z}}}(M_\varepsilon) < \varepsilon$ and $\widehat{\iota}(\mathbb{P}) \subseteq M_\varepsilon$. Note that $(C_m, \iota_m) \leq (\widehat{\mathbb{Z}}, \widehat{\iota})$ for every $m \in \mathbb{N}$. Hence it suffices to find a number $m = m(\varepsilon) \in \mathbb{N}$ and a set $M'_\varepsilon \subseteq C_m$ such that $\iota_m(\mathbb{P}) \subseteq M'_\varepsilon$ and $|M'_\varepsilon| < \varepsilon |C_m| = \varepsilon m$. Consider $m = \prod_{i=1}^r p_i$ for sufficiently large $r = r(\varepsilon)$ where $p_1 = 2 < p_2 = 3 < \ldots$ denotes the sequence of all primes in their natural order. Let M'_ε be the set $C_m^* \cup \{0\}$ of all prime remainder classes modulo m plus the single element 0 and φ the Euler function which counts the prime remainder classes modulo m. Then $\iota_m(\mathbb{P}) \subseteq M'_\varepsilon$ and

$$|M'_\varepsilon| = \varphi(m) + 1 = \prod_{i=1}^r (p_i - 1) + 1.$$

Since the Euler product $\prod_{i=1}^r (1 - \frac{1}{p_i})$ tends to 0 for $r \to \infty$ we get

$$|M'_\varepsilon| = m \prod_{i=1}^r (1 - \frac{1}{p_i}) + 1 < \varepsilon m$$

for r sufficiently large.

Range of integer polynomials: Let $f \in \mathbb{Z}[x]$ be a polynomial with integer coefficients. Then its set $f(\mathbb{Z})$ of values turns out to be a Hartman set. If f is a constant polynomial this is trivial since singletons in $\mathbb{Z}$ always are Hartman sets of measure 0. If $f(x) = ax + b$ is of degree 1 we obtain an arithmetic progression which is a Hartman set of measure $|\frac{1}{a}|$ (cf. periodic sequences). For degree ≥ 2 the argument has to be different, showing that $f(\mathbb{Z})$ is a set of Hartman measure 0. Let us sketch a proof for the very special case $f(x) = x^2$. For each odd $p \in \mathbb{P}$ there are only $\frac{p+1}{2}$ squares in C_p, having a proportion close to $\frac{1}{2}$. Considering C_m similar to the example above one can make this proportion arbitrarily small for sufficiently large k. For arbitrary polynomials one has to use stronger tools on polynomial functions with integer coefficients. In [15] the following generalization has been proved:

Theorem 4. *Let R be a ring of algebraic integers and $f \in R[x]$ a polynomial of degree ≥ 2. Then $f(R)$ is Hartman measurable (R considered as the additive group) with $\mu_R(f(\mathbb{Z})) = 0$.*

A common feature of the examples up to now was that the results can be obtained by using just compactifications induced by characters χ_α with rational α. Such additive characters of $\mathbb{Z}$ can be considered as ring homomorphisms as well. Of course one can take advantage of this fact if one investigates sets of mainly algebraic nature as in the above examples or in [33]. As in the case of group compactifications there is a maximal ring compactification. For commutative rings R or rings R with unit element one can show that this ring compactification can be obtained as the profinite completion. For $R = \mathbb{Z}$ this is the product in $\mathcal{COMP}(\mathbb{Z})$ of all finite compactifications which are given exactly by all (C_m, ι_m), $m = 1, 2, \ldots$. Hence in this case it suffices to consider countable products of finite structures. It follows that the maximal ring compactification of $R = \mathbb{Z}$ (similar for a big class of rings) is metrizable. Also Mauclaire's approach (cf. for instance [30]) is mainly based on this kind of compactification.

Now we turn to compactifications (C, χ_α) where α is allowed to be irrational.

Beatty sequences: Fix any $\alpha \in [0, 1)$ where we identify $[0, 1)$ with $\mathbb{T}$ in the obvious way whenever convenient. We will have in mind mainly irrational α, although our notation will be applicable to rational α as well. Let us consider very special continuity sets $M \subseteq \mathbb{T}$, namely half open segments (connected subsets) of the circle group $\mathbb{T}$ of Haar measure α. It is easy to see that the induced Hartman set $H = \{k \in \mathbb{Z} : \ k\alpha \in M\}$ can also be written as a generalized arithmetic progression, also called a Beatty sequence, consisting of the integers $[k\beta + \gamma]$, $k \in \mathbb{Z}$, where $\beta = \alpha^{-1}$ and γ is appropriate. For irrational α the corresponding binary Hartman sequence $\mathbf{1}_H$ is a so-called Sturmian sequence. Sturmian sequences are usually defined by the property that they are not ultimately periodic and, for each n, there occur exactly $n + 1$ binary words (cf. for instance [13], Chapter 6, or [5]). We will return to related complexity questions in the next chapter.

Here we emphasize that Sturmian (Beatty) sequences are understood very well by means of the continued fraction expansion of α (or, equivalently, β; for details we refer to [36] or [13]). Thus the following result from [36] gives a rather satisfactory description of the class $\mathbf{H}(\mathbb{Z})$ of Hartman sets in $\mathbb{Z}$.

Theorem 5. *Let* $\mathbf{B}_0$ *denote the system of all Beatty sequences (also allowing rational* α*), considered as subsets of* $\mathbb{Z}$*. The system of all finite unions of finite intersections of members of* $\mathbf{B}_0$ *forms a Boolean set algebra* $\mathbf{B} \subseteq \mathbf{H}(\mathbb{Z})$*. A subset* $H \subseteq \mathbb{Z}$ *is a Hartman set, i.e.* $H \in \mathbf{H}(\mathbb{Z})$*, if and only if it can be approximated by sets from* $\mathbf{B}$ *in the following sense: For arbitrary* $\varepsilon > 0$ *there are* $B_i \in \mathbf{B}$ *such that* $B_1 \subseteq H \subseteq B_2$ *and* $\mu_{\mathbb{Z}}(B_2 \setminus B_1) < \varepsilon$*. Thus* $\mathbf{H}(\mathbb{Z})$ *can be considered to be the* $\mu_{\mathbb{Z}}$*-complete (in the above sense) Boolean algebra generated by all Beatty sequences.*

The main ideas of the proof are:

- Every continuity set in the one-dimensional torus $\mathbb{T}$ can be approximated in the sense of the measure by finite unions of segments.
- An arbitrary Hartman set is induced by a continuity set $M \subseteq b\mathbb{Z}$. Continuity subsets of $b\mathbb{Z}$ can be approximated by finite Boolean combinations of the one-dimensional components in the product representation of bG (cf. section 3) and all sets which can be approximated in this way are Hartman sets.

Ideas behind Theorem 5 could be used in [36] to generalize results from [14] to arbitrary Boolean combinations of Beatty sequences.

Hartman sets and Hartman uniformly distributed sequences are complementary in the following sense.

Theorem 6. *Let* $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$*,* $x_n \in \mathbb{Z}$*, be a Hartman uniformly distributed sequence which, as a subset* $H = \{x_n : n \in \mathbb{N}\}$ *of* $\mathbb{Z}$*, is a Hartman set. Then* $dens(H) = 1$*.*

Sketch of proof: Every H-u.d. sequence meets every Hartman set with an asymptotic frequency equal to the measure and hence to the density of the set. Apply this to the given sequence $\mathbf{x}$ and the set H to obtain $\mathrm{dens}(H) = 1$.

It follows that the H-u.d. sequences $[n \log n]$ and $[n^{\frac{3}{2}}]$ of Wierdl mentioned in Section 4 are not Hartman sets (though having a uniform density 0).

Lacunary sequences: In contrast to Wierdl's sequences there are stronger conditions on the growth rate which are sufficient for a sequence to be Hartman. Recall that a sequence of positive integers $k_1 < k_2 < \dots$ with $\frac{k_{n+1}}{k_n} \geq q$ for all $n \in \mathbb{N}$ and some $q > 1$ is called lacunary.

Theorem 7. *The members of any lacunary sequence of integers form a Hartman set.*

Of course the Hartman measure of such sets is 0. A proof of Theorem 7 can be found in [36].

§7. Combinatorial and geometric aspects

We still stick to the case $G = \mathbb{Z}$ and focus on the complexity of Hartman sequences. We have to introduce some notation. Let A be a fixed finite set which we call alphabet. Every tuple $w = (b_0, \dots, b_{n-1}) \in A^n$ is called a (finite) word over A of length n. We say that w occurs in $\mathbf{a} = (a_n)_{n\in\mathbb{Z}}$, $a_n \in A$, if there is some $k \in \mathbb{Z}$ such that $(a_k, \dots, a_{k+n-1}) = w$. In this situation we call k an occurrence of w in $\mathbf{a}$. Let $P_{\mathbf{a}}(n) \in \{1, \dots, |A|^n\}$ denote the number of different words of length n occurring in $\mathbf{a}$. The function $P_{\mathbf{a}} : n \mapsto P_{\mathbf{a}}(n)$ is called the complexity function of $\mathbf{a}$. Clearly $P_{\mathbf{a}}(n)$ is monotonically non decreasing in n. It is not difficult to see that a sequence $\mathbf{a}$ is periodic if and only if $P_{\mathbf{a}}$ is bounded. Let us now turn to those sequences with minimal unbounded complexity function.

We recall from Section 6 that a Sturmian sequence $\mathbf{a}$ can be written as a special coding sequence for a group rotation, namely as $\mathbf{1}_H$ for a set of the type

$$H = \{k \in \mathbb{Z} : \ k\alpha \in M\}$$

where α is irrational and $M \subseteq \mathbb{T}$ is a half open segment of the circle of Haar measure α. It is easy to see that $P_{\mathbf{a}}(n) = n + 1$ for all $n \in \mathbb{N}$. The converse is true as well: Every binary sequence $\mathbf{a}$ with complexity function $P_{\mathbf{a}}(n) = n + 1$ which is not ultimately periodic comes from a rotation of the circle in the above described manner (cf., for instance, chapter 6 in [13]). What is the maximal complexity of a Hartman sequence?

The complexity of a coding sequence is closely related to the topological entropy of the underlying dynamical system which in the case of a Hartman sequence is an ergodic group rotation (we will treat this aspect more carefully in the next section) having entropy 0, in the topological as well as in the measure theoretic sense. By using the variational principle (cf. for instance [43]) this fact can be used to obtain

$$\lim_{n\to\infty} \frac{1}{n} \log(P_{\mathbf{a}}(n)) = 0$$

for every Hartman sequence $\mathbf{a}$. A direct proof is given in [39]. The more interesting result from [39] in this context is that this upper bound for $P_{\mathbf{a}}$ is best possible in the following sense.

Theorem 8. *Let $\varphi : \mathbb{N} \to \mathbb{N}$ satisfy $\varphi(n) \leq 2^n$ and have subexponential asymptotic growth rate, i.e.*

$$\lim_{n\to\infty} \frac{1}{n} \log(\varphi(n)) = 0.$$

Consider any compactification (C, ι) of $\mathbb{Z}$ with infinite C. Then there is a continuity set $M \subseteq C$ such that the induced Hartman sequence $\mathbf{a} = \mathbf{1}_H$ with $H = \iota^{-1}(M)$ satisfies $P_{\mathbf{a}}(n) \geq \varphi(n)$ for all n.

In other words: There are Hartman sequences with complexity arbitrarily close to exponential growth rate with a representation in (C, ι) for any given infinite compactification (C, ι).

The set M which has to be constructed in the proof, though having a topological boundary of measure 0, is geometrically rather complicated. So it is natural to ask for the complexity of $\mathbf{a}$ for the case that (C, ι) and $M \subseteq C$ have a certain prescribed structure.

In [39] the following case has been investigated: $C = \mathbb{T}^s$, $\iota : k \mapsto kg$ ($g = (\alpha_1, \dots, \alpha_s)$ a topological generator of $\mathbb{T}^s$, i.e. with the α_i together with 1 linearly independent over $\mathbb{Q}$), and $M = \{x + \mathbb{Z}^s : x \in [0, \rho]^s\} \subseteq \mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s$, $0 < \rho < 1$, an s-dimensional cube embedded into $\mathbb{T}^s$. Under certain number theoretic restrictions which are fulfilled in most cases the resulting Hartman sequence $\mathbf{a}$ has a complexity function of polynomial growth rate $P_{\mathbf{a}}(n) \sim cn^s$ with a certain constant c. A formula for c could be given, but it was not clear how to interpret this quantity. The solution was given by Steineder in his thesis [38] by proving the following more general theorem:

Theorem 9. *Let $g = (\alpha_1, \dots, \alpha_s)$ be a topological generator of $C = \mathbb{T}^s$, $\iota : k \mapsto kg$, and M an s-dimensional convex polytope considered as a subset of $\mathbb{T}^s$. Under certain technical number theoretic independence conditions (which can be made explicit and are fulfilled, for fixed M, by almost all choices of g, in the measure theoretic sense as well as in the sense of Baire categories) the complexity function of the Hartman sequence $\mathbf{a} = \mathbf{1}_H$, $H = \iota^{-1}(M) = \{k \in \mathbb{Z} : kg \in M\}$, satisfies the asymptotic formula $P_{\mathbf{a}}(n) \sim cn^s$ (for $n \to \infty$) where c is the volume of the projection body $\pi(M)$ of M.*

We have to explain the notion of the projection body $\pi(M)$ of a convex body $M \subseteq \mathbb{R}^s$ in the s-dimensional euclidean space. (A standard reference on convex geometry is [37].) For $x \in \mathbb{R}^s$ and $r \in \mathbb{R}$ let us consider the hyperplane $h_{x,r} = \{y \in \mathbb{R}^s : \ x \cdot y = r\}$. Every convex body $M \subseteq \mathbb{R}^s$ is uniquely determined by its support function $h_M : S^{s-1} \to \mathbb{R}$, $S^{s-1} \subseteq \mathbb{R}^s$ the unit sphere, defined by

$$h_M(x) = \sup\{r : \ h_{x,r} \cap M \neq \emptyset\}$$

for every vector $x \in S^{s-1}$ of unit length. Let λ denote the $s-1$-dimensional measure and π_x the projection in $\mathbb{R}^s$ along x onto a hyperplane orthogonal to x. Then the so-called projection body $\pi(M)$ of M is defined by its support function in such a way that

$$h_{\pi(M)} = \lambda(\pi_x(M)).$$

We conjecture that generalizations of this connection between the complexity of Hartman sequences and convex geometry to more general M (for instance such with smooth boundary) hold, but rigorous proofs seem to be highly nontrivial.

§8. Dynamical aspects

Compared to Sturmian sequences, Hartman sequences seem to be rather general objects. But in the broad context of dynamical systems and ergodic theory they are very special. In this section we are interested in the dynamical context. (I refer to [11], [43] or [25] as examples of general textbooks about ergodic theory and dynamical systems.) There are far reaching theories making use of very powerful methods from ergodic theory in order to obtain striking results in number theory or combinatorics. The most famous one might be Furstenberg's ergodic proof of Szemerédi's theorem, cf. [16]. His textbook [17] gives a broad introduction into this area where current research is extremely intensive (cf. [3] and [18] as very recent examples). In the context of Hartman sequences unique ergodicity (corresponding to uniform density) is an important feature which might lead to interesting generalizations in future research. Here we only present standard basics for our very special context.

Every compactification (C, ι) of $G = \mathbb{Z}$ induces a dynamical system, namely the ergodic group rotation $T : C \to C$, $T(x) = gx$ where $g = \iota(1)$ is a topological generator of C. In the sense of spectral analysis our situation is quite simple.

Consider the space $L^2(C)$ of all $f : C \to \mathbb{C}$ such that $\int_C |f|^2 d\mu_C < \infty$ and the operator $U_T : f \mapsto f \circ T$ on $L^2(C)$. U_T is unitary since the rotation T is measure preserving. Let $\chi \in \hat{C}$ be any character of C. Then one directly computes $U_T(\chi)(x) = \chi(gx) = \chi(g)\chi(x)$, i.e. χ is an eigenfunction of U_T with eigenvalue $\chi(g)$. Since the characters form an orthonormal base for L^2 this completely describes the spectral properties. The eigenfunctions are given by the members of $\hat{C}$ and the corresponding eigenvalues are forming the subgroup $\{\chi(g) : \ \chi \in \hat{C}\}$ of the circle group $\mathbb{T}$. One says that the system (C, T) has discrete spectrum. It is a classical fact that every ergodic system with discrete spectrum is measure theoretically isomorphic to an ergodic group rotation. The construction above shows that every subgroup S of $\mathbb{T}$ can occur as discrete spectrum of a group rotation. One just has to start with the compactification corresponding to $\iota_S : k \mapsto (k\alpha)_{\alpha \in S}$.

We now relate Hartman sequences induced by a continuity set $M \subseteq C$ to shift spaces of symbolic dynamics. Let us therefore consider two sided infinite binary sequences as members of the compact space $\{0,1\}^{\mathbb{Z}}$. On this space the shift σ is defined by $\sigma(\mathbf{a}) = \sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$. For a given Hartman sequence $\mathbf{a}$ let $X_{\mathbf{a}}$ denote the shift closure of $\mathbf{a}$, i.e. the topological closure of the shift orbit $\{\sigma^k(\mathbf{a}) : \ k \in \mathbb{Z}\}$ of $\mathbf{a}$. To each $\mathbf{x} \in X_{\mathbf{a}}$ we try to associate a point $\varphi(\mathbf{x}) \in C$ in a natural way. By definition $\mathbf{x}$ can be approximated by a sequence $\sigma^{k_n}(\mathbf{a})$. Note that $\sigma^k(\mathbf{a})$ is the binary coding sequence of the system (C, T) if we take as a starting point g^k instead of $0 \in C$. There exists an accumulation point of the g^{k_n}, $n \in \mathbb{N}$, in the compact space C. Under certain conditions (which will become clear in Section 9) this accumulation point is unique and therefore a limit. Then we define

$$\varphi(\mathbf{x}) = \lim_{n\to\infty} g^{k_n},$$

and the mapping $\varphi : X_{\mathbf{a}} \to C$ is an almost conjugation between the uniquely ergodic systems $(X_{\mathbf{a}}, \sigma)$ and (C, T). More explicitly this means the following:

(1) $T \circ \varphi = \varphi \circ \sigma$.
(2) φ is continuous.
(3) φ is measure preserving with respect to the unique invariant measures μ on $X_{\mathbf{a}}$ respectively μ_C on C.
(4) Though φ is not injective, the set of points $\mathbf{x} \in X_{\mathbf{a}}$ such that there exists an $\mathbf{y} \neq \mathbf{x}$ with $\varphi(\mathbf{x}) = \varphi(\mathbf{y})$ has measure 0 and is meager in the sense of Baire categories.

It follows from these connections how the Hartman sequence $\mathbf{a}$ contains information about the underlying group rotation. One just has to consider its orbit closure $X_{\mathbf{a}}$ under the shift σ. The resulting system $(X_{\mathbf{a}}, \sigma)$ is uniquely ergodic. This property can be expressed in terms of a uniform density. Consider the finite binary word $w = (b_0, \ldots, b_{k-1})$ and let $[w]$ denote the cylinder set of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in X_{\mathbf{a}}$ such that $(x_0, \ldots, x_{k-1}) = (b_0, \ldots, b_{k-1})$. Then the limit

$$\lim_{n\to\infty} \frac{1}{n} \left|\{l \in \mathbb{N} : \ 0 \le l \le n-1, \ \sigma^{m+l}(\mathbf{x}) \in [w]\}\right|$$

exists, is uniform (in $m \in \mathbb{Z}$ and $\mathbf{x} \in X_{\mathbf{a}}$) and coincides with $\mu([w])$. Thus spectral analysis of the system $(X_{\mathbf{a}}, \sigma)$ can successfully be applied to obtain the corresponding information about the underlying group rotation (C, T) as well. In particular (C, T) is determined by the discrete group of eigenvalues of the system.

As mentioned above one has to assume certain conditions on C and the continuity set M inducing the Hartman sequence $\mathbf{a}$. We are going to investigate this aspect more carefully by presenting another approach which is rather topological than functional analytic and can (similar to the classical approach via spectral analysis which we do not explain further here) be extended to arbitrary Hartman functions.

§9. Topological aspects

We know that every Hartman function f on the topological group G has a representation in the Bohr compactification bG. Since in interesting cases bG is not metrizable, i.e. a huge object in the sense of topological weight, it is desirable to have a representation in a smaller compactification. We are going to present a topological construction and compare it with the spectral analysis.

Although much can be done in the nonabelian case as well we assume in the following that G is abelian in order to avoid complications caused by distinguishing left and right translates. Sometimes we even will require G to be LCA.

Assume that $f = F \circ \iota$ is a representation of the Hartman function f by the Riemann integrable function F on the compactification (C, ι) of the topological group G. For each $c \in C$ (similarly for $g \in G$) we consider the translation operator τ_c defined by $\tau_c \circ F(x) = F(x + c)$ and the nonnegative function $||.||_F : C \to [0, \infty)$ defined by

$$||c||_F = \int_C |F - \tau_c \circ F| d\mu_C(x)$$

which is continuous on C and thus induces an almost periodic function $||.||_f$ on G defined by

$$||g||_f = ||\iota(g)||_F$$

for $g \in G$. Since $\iota(G)$ is dense in C, $||.||_F$ is uniquely determined by $||.||_f$. The analogue statement holds for the invariant pseudometrics d_f and d_F defined by $d_F(c_1, c_2) = ||c_1 - c_2||_F$ and $d_f(g_1, g_2) = ||g_1 - g_2||_f = ||\iota(g_1 - g_2)||_F$.

It might happen that d_F is not a metric. This occurs whenever the zero set $Z_F = \{c \in C : ||c||_F = 0\}$ contains nonzero elements $c \in C$. Let us call F aperiodic whenever $Z_F = \{0\}$, otherwise periodic. Note that this definition of periodicity is not pointwise but up to sets of measure 0. It is an easy exercise to check that $Z_F \subseteq C$ is a closed subgroup. We consider the corresponding compactification $(C_f, \iota_f) \leq (C, \iota)$ with $C_f = C/Z_F$ and $\iota_f : g \mapsto \iota(g) + Z_F \in C_f$. By construction we may consider $d_F(c_1 + Z_F, c_2 + Z_F) = d_F(c_1, c_2)$ to be defined on C_f. Then d_F is a metric and, by continuity of d_F and compactness of C_f, induces the topology on C_f. Note that by the above remarks d_F is uniquely determined by d_f. The space (C_f, d_F) can be considered to be the metric completion of (G, d_f). This reasoning has the following consequence.

Assume that we know that $f : G \to \mathbb{C}$ is a Hartman function on G. Then f has a representation by a Riemann integrable F_b on the Bohr compactification. The above paragraph gives the lower bound $(C_f, \iota_f) \leq (C, \iota)$ for every compactification (C, ι) of G where a representation of f might be possible. More information is given by the following result.

Theorem 10. *Let $f : G \to \mathbb{C}$ be a Hartman function on the LCA group G and let the Riemann integrable function $F : C \to \mathbb{C}$ be a representation of f in the compactification (C, ι) of G, i.e. $f = F \circ \iota$. Then the following assertions hold.*

(1) $(C_f, \iota_f) \leq (C, \iota)$.
(2) $(C_f, \iota_f) \cong (C, \iota)$ *if and only if F is aperiodic.*
(3) *There exist aperiodic F on C if and only if C is metrizable.*
(4) *f has an almost representation in (C_f, ι_f) in the sense that there is a Riemann integrable F_1 on C_f such that $m_G(|f - F_1 \circ \iota_f|) = 0$.*
(5) *f has a representation in a metrizable compactification whenever $\hat{G}$ is separable.*
(6) *f has a representation in (C_f, ι_f) whenever f is almost periodic, i.e. whenever f has a continuous extension to the Bohr compactification.*

For the special case $G = \mathbb{Z}$ and $f = \mathbf{1}_H$ a Hartman sequence, the above construction and statements (1), (2) and (3) have been presented in [45]. Sander (cf. [35]) has obtained results in the direction of (1) in purely number theoretic terms. The general case as well as statements

(4) and (6) are due to Maresch, cf. [28]. In both papers [45] and [28] filters generalizing metrics have been used as invariants. This method has been used in [1] and [2] to generalize results from [7] resp. [6]. (Example from [2]: Among all compact abelian groups C exactly the metrizable ones have the property that each countable subgroup S of C is characterized by a sequence of characters $\chi_n \in \hat{G}$, $n \in \mathbb{N}$, in the sense that for $\alpha \in C$ $\lim_{n\to\infty} \chi_n(\alpha) = 0 \in \mathbb{T}$ iff $\alpha \in S$.) Furthermore, [28] investigates the connections with generalized Fourier coefficients of a Hartman function f. Statement (5) (as well as refinements) are contained in [29].

Recalling the case $G = \mathbb{Z}$ and the dynamical aspects discussed in Section 8 we see from (3) that Hartman sequences give a satisfactory coding of group rotations exactly for metrizable groups.

We add a further remark on the case $G = \mathbb{Z}$ which is of particular interest. In this case, for any Hartman function f the mean $m_{\mathbb{Z}}(f)$ is given by the limit of Cesàro means, namely

$$m_{\mathbb{Z}}(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=l}^{l+n-1} f(i),$$

where the limit is uniform in l. Thus for a given Hartman function f one can construct the compactification (C_f, ι_f) in terms of the limits $m_{\mathbb{Z}}(|f - \tau_k \circ f|)$, $k \in \mathbb{Z}$, of the corresponding Cesàro means.

§10. Connections to harmonic analysis

In this last section we investigate further the problem of representing complex valued functions on G by certain functions on compactifications, this time not necessarily on group compactifications.

The biggest compactification among all topological spaces (not restricted by further algebraic compatibility conditions) is the Stone-Čech compactification $(\beta G, \iota_\beta)$ which exists for arbitrary completely regular spaces G. βG contains the homeomorphic copy $\iota_\beta(G)$ of G and has the universal property that every continuous function $f : G \to X$ to any compact space X has a unique continuous extension F, i.e. $f = F \circ \iota_\beta$. $\beta\mathbb{Z}$ has been used by Indlekofer for applications in probabilistic number theory, cf. for instance [24]. Applications of $\beta\mathbb{Z}$ (similarly for arbitrary discrete semigroups instead of $\mathbb{Z}$) of more combinatorial flavour are due to the fact that every semigroup operation on $\mathbb{Z}$ can be extended to $\beta\mathbb{Z}$ in a natural way (though one looses continuity in one component). The standard textbook in this context is [23].

Here we are mainly interested in compactifications with a natural invariant mean. An interesting example is the weakly almost periodic compactification (wG, ι_w) which is maximal in the class of all semitopological semigroup compactifications of G: A semigroup S which is at the same time a topological space is called a semitopological semigroup if its binary operation is continuous in each component but not necessarily simultaneously. Then (S, ι) is called a semitopological semigroup compactification of G if $\iota : G \to S$ is a continuous semigroup homomorphism with $\iota(G)$ dense in S. Indeed there is a unique invariant Borel probability measure μ_S on S. A continuous function $f : G \to \mathbb{C}$ is weakly almost periodic, in symbols $f \in \mathcal{W}(G)$, if and only if there is a continuous $F : wG \to \mathbb{C}$ such that $f = F \circ \iota_w$. Recall that f is called almost periodic, in symbols $f \in \mathcal{AP}(G)$ if the same holds with (bG, ι_b) instead of (wG, ι_w). Using $(bG, \iota_b) \leq (wG, \iota_w)$ one easily concludes $\mathcal{AP}(G) \subseteq \mathcal{W}(G) \cap \mathcal{H}(G)$. Note that $\mathcal{W}$ and $\mathcal{H}$ arise from $\mathcal{AP}$ by different ways of generalizations, $\mathcal{H}$ by relaxing the regularity condition on representations and $\mathcal{W}$ by allowing bigger compactifications. There is an algebra $\mathcal{WH}(G)$ of functions with unique invariant mean which we call the algebra of weak Hartman functions on G and which contains both, $\mathcal{W}(G)$ and $\mathcal{H}(G)$. By definition, $f \in \mathcal{WH}(G)$ if there is an $F : wG \to \mathbb{C}$ which is Riemann integrable with respect to the invariant measure μ_{wG} and satisfies $f = F \circ \iota_w$.

The paper [29] is mainly devoted to the comparison of spaces of functions of this and similar type. Here we mention only a few of the main results. One of them expresses that $\mathcal{H} \cap \mathcal{W}$ is rather small. This can be made precise by the following statement.

Theorem 11. *Let $f \in \mathcal{H}(G)$ have the representation $f = F \circ \iota$ in the group compactification (C, ι). Assume that F has a generalized jump discontinuity $c \in C$, i.e. there are open sets $O_1, O_2 \subseteq C$ with $c \in \overline{O_1} \cap \overline{O_2}$ where the values of F are separated in the sense that $\overline{F(O_1)} \cap \overline{F(O_2)} = \emptyset$. Then f is not weakly almost periodic.*

In particular this implies that for every infinite LCA group G there is an abundance of Hartman functions which are not weakly almost periodic. Hartman functions even need not be measurable. This follows from the following result. Recall that a function $f : G \to \mathbb{C}$ is said to vanish at infinity if for every $\varepsilon > 0$ there is a compact subset $K \subseteq G$ such that $|f(x)| < \varepsilon$ whenever $x \in G \setminus K$. Note that we did not assume f to be continuous.

Theorem 12. *If G is an LCA group which is not compact. Then every function vanishing at infinity is a Hartman function.*

Hence indeed every $f = \mathbf{1}_H$ with $H \subseteq [0,1] \subseteq G = \mathbb{R}$, $\emptyset \neq H \neq [0,1]$, is an example for such an $f \in \mathcal{H} \setminus \mathcal{W}$, even if H is not measurable.

Conversely there are weakly periodic functions which are not Hartman functions. Examples for $G = \mathbb{Z}$ are obtained if one takes ergodic sequences like $([n \log n])$ where the gaps between two members tend to infinity. As mentioned in Section 6 as a consequence of Theorem 6, ergodicity implies that such sequences are not Hartman while weak almost periodicity follows from Theorem 4.2 in [4]. As a further standard reference on weak almost periodicity we mention [10].

Though the mentioned results indicate that the spaces $\mathcal{H}$ and $\mathcal{W}$ are very different, both spaces contain the space $\mathcal{C}_0$ of continuous functions vanishing at infinity. But even examples of $f \in (\mathcal{H} \cap \mathcal{W}) \setminus (\mathcal{AP} + \mathcal{C}_0)$ can be constructed. For more details we have to refer to [29].

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Institute of Discrete Mathematics and Geometry
University of Technology
Vienna
Austria