# Matsumoto's legacy.
# The life and work of Professor Makoto Matsumoto

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## §1. Prologue

There is no mathematician in the world nowadays studying Finsler geometry without knowing the name of Professor Makoto Matsumoto. His lifelong work started with the study of Riemannian geometry and culminated with the foundation of the geometry of Finsler spaces, an important generalization of Riemannian geometry mentioned by B. Riemann himself.

Professor Makoto Matsumoto was born on August 1, 1920 in the old capital of Japan, the beautiful city of Kyoto, where he spent all his 85 years long life, until he passed away on January 23, 2005.

Living his youth in a Japan engaged in the Second World War, he graduated from Kyoto Daisan High school, and then entered Kyoto Imperial University (Kyoto University today), Faculty of Science, Department of Mathematics, graduating in 1944. Because of the war situation he had no choice but to work as Assistant Engineer at Fujikoshi Steel Company for one year.

Professor Makoto Matsumoto started his career in education as Assistant Professor at Doshisha Technical College in 1945, and moved to the Faculty of Engineering in Doshisha University in 1949. He was appointed as Assistant Professor at the Yoshida College of Kyoto University in 1950 and then at Faculty of Science of the same university in 1953. Further in his career he got promoted to full Professor and was appointed at the Yoshida College of Kyoto University again in 1964. He retired in 1984 and got awarded the "Professor Emeritus of Kyoto University" title. He became a professor at Gifu College of Education between 1984–1987, and then Professor at Setsunan University, Department of Engineering in the period 1987–1992.

He received the title of Doctor in Science from Kyoto University, Faculty of Science, Department of Mathematics in 1954. He was awarded

Doctor Honoris Causa of "Lajos Kossuth" University (Debrecen, Hungary) and Honorary Professor of "Al. I. Cuza" University (Iasi, Romania). He was decorated by the Japanese government in 1994 and again posthumously in 2005.

Professor Makoto Matsumoto published more than 160 papers and 8 books (see [FS2005] for a complete list). In his surroundings there was always a seminar group (Kyoto group) including T. Okada, K. Okubo, S. Hōjō and others. Even if it is impossible to overview his scientific activity in a few pages, we may divide his work into two main parts. First, he established the foundation of Finsler Geometry using the notion of fiber bundle and building the connection theory for Finsler spaces. Second, due to the lack of examples of Finsler metrics at that time, he started studying special Finsler spaces, building many concrete examples and applications not only in differential geometry, but also in Physics and Biology (see [AIM1993][1]).

For quite a long period, the only available monograph on Finsler geometry was the book of Rund ([R1959]), but this contains a lot of tensorial formulas hard to understand and to use. One of the greatest contribution of Professor Makoto Matsumoto to the development of Finsler geometry was to systematize and simplify Rund's theory in his monograph ([M1986]).

In his final years he contributed to the monograph [M2003] with a substantial survey of 440 pages on Finsler geometry in the 20-th Century. Afterwards, he was so deeply moved by receiving the impressive volumes of the book that he wrote in one of his last letters to Masao Hashiguchi: "This monograph will replace my dying body. I am confident now that in this way the Finsler geometry will continue to live forever".

One of his last manuscripts was a contribution to the monograph [OP2004], where he presented an open problem of controversy in the history of Finsler geometry: **to find an explicit example of a fundamental metric function that is Landsberg but not Berwald**. Recently, G.S. Asanov ([Asanov2006]) and Z. Shen have done some important work for solving this problem in the almost regular cathegory. See the contributions by D. Bao and by Z. Shen et al, to the present volume.

Professor Matsumoto's ideas influenced many Finsler geometers all over the world, a fact proven by his large number of joint papers with colleagues from Canada, Hungary, Romania, Korea, Poland and China ([FS2005]).

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[1]Please refer to the list of Selected Papers List of Professor Makoto Matsumoto in the present volume for the papers cited in this survey.

Professor Makoto Matsumoto visited many foreign countries, giving lectures on Finsler geometry and encountering other specialists in different fields of mathematics. For example, he visited East and West Germany in 1966, meeting there H. Heil, R. Sulanke, R. Nevanlinna, C. Ehresmann, W. Klingenberg, D. Laugwitz and others. He also visited Hungary (L. Tamássy) during the period January–May 1979, working intensively and writing in a relatively short time some important papers ([TM1979], [M1980], [MT1980]).

In 1984 Professor Makoto Matsumoto was one of the organizers (together with R. Miron from Romania) of "The Romanian-Japanese Colloquium on Finsler Geometry" held in Romania during the communist regime. Despite many difficulties the conference was a big success for the researchers in Finsler geometry from both countries, and many new ideas and future joint research projects were born on that occasion. This project occured in the year of his retirement from Kyoto University.

One of us (Shimada) met Professor Makoto Matsumoto for the first time in 1970 at the Second Symposium on Finsler Geometry, Fujisawa, Japan, participating for the first time together with his teacher H. Yasuda. Shimada received direct supervision from Professor Makoto Matsumoto during his stay as Young Researcher Fellow of the Ministry of Education in Japan at Kyoto University (1976–1977). Professor Matsumoto and Shimada worked together over the last 35 years on several important topics in Finsler geometry: Finsler spaces with 1–form metric ([MS1978]), properties of the curvature tensor of a Finsler space, Randers spaces of constant curvature, and others (see [FS2005]).

The second author of the present note (Sabau) met Professor Matsumoto in August 1996 at the Third International Conference of Tensor Society, Tsukuba, Japan. Even without any joint papers, Sabau received continuous encouragement and many useful suggestions from Professor Matsumoto during the last 9 years.

Professor Makoto Matsumoto's legacy does not contain only theorems of Finsler geometry, but also a devoted activity to educate and to help develop young scientists. He was the driving force of the Finsler geometers group in Japan and abroad for 40 years. He was also the founder of the Japanese Society of Finsler geometry and was the main chairman of the Symposium on Finsler Geometry held every year in Japan (from 1969 to present), a scientific event that gathers together mathematicians from Japan and abroad. The Symposium started at the suggestion of Professor Akitsugu Kawaguchi who always supported and encouraged this research meeting.

Professor Makoto Matsumoto used to say that the study of Finsler geometry is like "looking for old sakè skins into which new sakè can be poured". His passionate love for Finsler geometry will remain in our minds forever.

It is extremely difficult to present the scientific content of the work of Professor Makoto Matsumoto in a few pages. We will try to overview a few topics chosen by the authors of the present note. We apologize for leaving out many other important topics from this short survey.

The authors would like to thank Professors K. Okubo and M. Hashiguchi for sharing with us their memories about Professor Makoto Matsumoto.

## §2. The theory of Finsler connections

In the preface of his book [M1975] Professor Matsumoto shares with the readers the motivation of his interest in Finsler geometry. "Around 1952 Professor J. Kanitani asked me to read and present at a small research meeting one of the latest papers of Eisenhart. After reading the paper I realized that I need to read Cartan's small book on Finsler geometry also. However, Cartan's theory was for me more difficult than the initial paper, especially the definition of Finsler connections using four axioms. I realized that what I needed next to do was to study the variational problem in order to understand Cartan's theory. Starting in this way, some years later I was able to systematize and rigorously reformulate Cartan's theory using the notion of fiber bundle obtaining the Finsler geometry we have today."

At that time, Professor Matsumoto was writing his doctoral Thesis on Riemannian Geometry under the supervision of Professor J. Kanitani, the founder of the Japanese school of projective differential geometry.

According to Professor Matsumoto, a Finsler connection $F\Gamma = (\Gamma, N)$ on a differentiable manifold $M$ is composed of a connection $\Gamma$ in the fiber bundle $F(M)$, called a **Finsler bundle**, and a non-linear connection $N$ in the tangent bundle $T(M)$. In other words, a Finsler connection is equivalent to a pair $(\Gamma^h, \Gamma^v)$ of distributions in $F(M)$ given by

$$\Gamma^h = l(N), \qquad \Gamma^v = l(T^v), \tag{2.1}$$

where $l$ denotes the lift with respect to the $\Gamma$ and $T^v$ is the vertical subspace of the tangent space of $T(M)$. The following direct sum holds

$$\Gamma = \Gamma^h \oplus \Gamma^v. \tag{2.2}$$

Similar to the Riemannian case, the distributions $\Gamma^h$ and $\Gamma^v$ are spanned by the basic vector fields $B^h(v)$ and $B^v(v)$, respectively. At an arbitrary point $u = (x^i, y^i, z^i_a)$ of $F(M)$ these can be locally written as

$$\begin{aligned} B^h(v) &= z^i_a v^a \Big( \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} - z^j_b F^k_{ji} \frac{\partial}{\partial z^k_b} \Big), \\ B^v(v) &= z^i_a v^a \Big( \frac{\partial}{\partial y^i} - z^j_b C^k_{ji} \frac{\partial}{\partial z^k_b} \Big), \end{aligned}$$

where $F^k_{ji}$, $N^j_i$, $C^k_{ji}$ are functions of $(x^i, y^i)$, called the connection coefficients of the Finsler connection $F\Gamma$. From here, one can obtain the following structure equations:

$$\begin{aligned} \big[B^h(u), B^h(v)\big] &= Z(R^2(u,v)) + B^h(T(u,v)) + B^v(R^1(u,v)), \\ \big[B^h(u), B^v(v)\big] &= Z(P^2(u,v)) + B^h(C(u,v)) + B^v(P^1(u,v)), \\ \big[B^v(u), B^v(v)\big] &= Z(S^2(u,v)) + B^v(S^1(u,v)), \end{aligned}$$

where $Z(A)$ is the fundamental vector field on $F(M)$, corresponding to an element $A$ of the Lie algebra of $GL(n, R)$. From these formulas one can see that we have three curvature tensors $R^2$, $P^2$, $S^2$ and five torsion tensors $T$, $C$, $R^1$, $P^1$, $S^1$. The components of the tensor $C$ are $C^i_{jk}$ and those of the tensors $T$ and $S^1$ are

$$T \; : \quad T^i_{jk} = F^i_{jk} - F^i_{kj}, \tag{2.3}$$

$$S^1 \; : \quad S^i_{jk} = C^i_{jk} - C^i_{kj}. \tag{2.4}$$

For a given Finsler connection $F\Gamma = (\Gamma, N)$, we have two nonlinear connections $N$ and $\widetilde{N}$, where $\widetilde{N}$ is the induced nonlinear connection on $T(M)$ by $\Gamma^h$ through $L(M)$. The coefficients of $N$ are usually denoted by $N^i_j$ while the ones of $\widetilde{N}$ are denoted by $F^i_{oj}$, where $F^i_{oj} := F^i_{lj} y^l$.

Let us consider now a Finsler space $(M, F)$ with the fundamental function $F(x, y)$ and fundamental tensor $g = (g_{ij}(x, y))$. Then, the following theorem holds.

**Theorem 2.1.** *(M. Matsumoto) The Cartan connection $C\Gamma = (\Gamma, N)$ of a Finsler space $(M, F)$ is uniquely determined by the following five axioms:*

(1) *h–metrical: $B^h(v)(g) = 0$, i.e. $g_{ij|k} = 0$,*
(2) *h–torsion free: $T = 0$, i.e. $F^i_{jk} = F^i_{kj}$,*
(3) *v–metrical: $B^v(v)(g) = 0$, i.e. $g_{ij}|_k = 0$,*
(4) *v–torsion free: $S^1 = 0$, i.e. $C^i_{jk} = C^i_{kj}$,*
(5) *$N = \widetilde{N}$, $N^i_j = F^i_{0j}$.*

Professor Matsumoto studied special Finsler connections such as the induced Finsler connection, vector relative connection, Barthel connection and Cartan Y-connection. Professor Matsumoto declared the following: Cartan connection is not almighty. There is a most suitable Finsler connection for every geometrical formulation.

Matsumoto's connection theory was completed later by T. Okada ([O1982]). In this way strong tools like "the axiomatic systems for Cartan and Berwald connections fell into our hands" (M. Hashiguchi, [H2005]).

## §3. Special Finsler spaces

One of the most natural ways to look for examples of Finsler metrics is to consider deformations of Riemannian metrics. This idea leads to the notion of $(\alpha, \beta)$–metrics, stimulated by R. S. Ingarden's famous paper [I1957] on Randers spaces.

The fundamental function $F : TM \to \mathcal{R}$ of a Finsler space $F^n = (M, F)$ is called an $(\alpha, \beta)$**–metric** if $F$ is a positively 1-homogeneous function of two arguments $\alpha := \sqrt{a_{ij}(x)dx^i dx^j}$ and $\beta := b_i(x)dx^i$, i.e. a Riemannian metric and a differential 1–form, respectively.

The geometry of $(\alpha, \beta)$–metrics was intensively studied by the Japanese, Romanian and American schools of Finsler geometry ([M1992]). In particular, special cases of $(\alpha, \beta)$–metrics, Randers metric $F = \alpha + \beta$, Kropina metric $F = \alpha^2/\beta$ and the slope metric $F = \alpha^2/(\alpha - \beta)$, which is also called the **Matsumoto metric**, were paid a great deal of attention. We mention that Professor Matsumoto's earlier work leads to the classification theorem of Randers metrics of constant curvature.

Another important notion studied by Professor Matsumoto is Douglas space. A Finsler space $F^n = (M, F)$ is called a **Douglas space** if the quantities

$$D^{ij}(x, y) := G^i(x, y)y^j - G^j(x, y)y^i \tag{3.1}$$

are homogeneous polynomials of at most third order in $y^i$, where $G^i(x, y)$ are the spray coefficients of a geodesic in the Finsler space $(M, F)$.

One can remark that

$$\dot{\partial}_r \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h D^{lr} = (n + 1)D_h{}^l{}_{ij}, \tag{3.2}$$

where $D$ is the Douglas tensor of $(M, F)$.

It was proved ([BM1997]) that a Finsler space is Douglas if and only if its Douglas tensor $D$ identically vanishes.

Another important notion introduced by Professor Matsumoto is that of $C$-reducibility, which characterizes the Randers and Kropina metrics.

An $n$-dimensional Finsler space $F^n = (M, F)$ is called **$C$-reducible** if its Cartan tensor can be written in the form

$$C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j), \tag{3.3}$$

where one puts $C_i := C^r_{ir}$.

It is shown in [MH1978] that the fundamental tensor of a $C$-reducible Finsler space is of Randers or Kropina type.

In his work, Professor Matsumoto reached the following relationships among of the main special Finsler spaces. If we denote by $\mathcal{B}^n$, $\mathcal{L}^n$, $\mathcal{D}^n$ the sets of $n$-dimensional Berwald, Landsberg and Douglas spaces, respectively, the following inclusions hold

$$\mathcal{B}^n \subset \mathcal{L}^n, \qquad \mathcal{B}^n \subset \mathcal{D}^n. \tag{3.4}$$

Moreover, he also proved ([BM1996]) that if a Finsler space is a Landsberg and a Douglas space at the same time, then it must be a Berwald one, and conversely. In other words, one have

$$\mathcal{L}^n \cap \mathcal{D}^n = \mathcal{B}^n. \tag{3.5}$$

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