

Massey products of complex hypersurface complements

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Abstract.

It was shown by Kohno that all higher Massey products in the rational cohomology of a complex hypersurface complement vanish. We show that in general there exist non-vanishing triple Massey products in the cohomology with finite field coefficients.

§1. Introduction

The study of the topology of hypersurface complements is a classical subject in algebraic geometry. Most of what is known about these spaces is related to invariants of their rational homotopy type. In this paper, we attempt to show that their \mathbb{F}_p -homotopy type captures in general more information than the \mathbb{Q} -homotopy type, where \mathbb{F}_p is the prime field of p elements.

Let X be the complement to a hypersurface S in $\mathbb{C}P^d$. Then we have the following results due to Kohno [12, 13]: Massey products in $H^*(X, \mathbb{Q})$ of length ≥ 3 vanish. Moreover, the Malcev Lie algebra of $\pi_1(X)$ and the completed holonomy Lie algebra of $H^{\leq 2}(X, \mathbb{Q})$ are isomorphic. Thus, the \mathbb{Q} -completion of $\pi_1(X)$ is completely determined by the \mathbb{Q} -cohomology algebra of X . In the case when S is a hyperplane arrangement X is \mathbb{Q} -formal by Morgan [17], that is the entire \mathbb{Q} -homotopy type of X is determined by the algebra $H^*(X, \mathbb{Q})$. In this context, it seems natural to pose the following questions: Are there non-vanishing Massey products in $H^*(X, \mathbb{F}_p)$ for all primes p ? Is X a \mathbb{F}_p -formal space, particularly when X is a hyperplane arrangement complement?

Massey products are known to be obstructions to formality, see [5, 7]. So, if the answer to the first question was yes, then the space X would

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not be \mathbb{F}_p -formal. For compact Kähler manifolds the above questions were answered by Ekedhal in [6] by constructing such manifolds M with non-vanishing triple products in $H^*(M, \mathbb{F}_p)$. Thus, a compact Kähler manifold although is \mathbb{Q} -formal by [5], in general it may not be \mathbb{F}_p -formal. The case of non-compact complex algebraic varieties is already different over \mathbb{Q} from the compact case. As pointed out by Morgan in [17] such varieties may not be \mathbb{Q} -formal.

The main result of this paper settles in affirmative the existence of non-vanishing Massey products in the \mathbb{F}_p -cohomology of a hyperplane arrangement complement for all odd primes p , thus showing that arrangement complements are not \mathbb{F}_p -formal in general.

Theorem 1.1. *For every odd prime p , the complement X to the complex reflection arrangement \mathcal{A} in \mathbb{C}^3 associated with the unitary reflection group $G(p, 1, 3)$ has, modulo indeterminacy, non-vanishing Massey products in $H^2(X, \mathbb{F}_p)$.*

The cohomology operations that came to be known as Massey products were introduced by W. S. Massey in [14]. Since then, they became important tools in algebraic topology, being especially used as means of distinguishing spaces with the same cohomology but different homotopy type. In general they are rather complicated objects, since they are in fact sets of cohomology classes. But, in certain cases, they turn out to be cosets as shown by May in [16], the simplest instance being that of the Massey products of three cohomology classes. In this paper we will only consider triple Massey products of cohomology classes of degree 1 in the cohomology algebra in degrees at most 2. In fact all the computations will take place in the group cohomology of the fundamental group $\pi_1(X)$ of our hypersurface complement. By the Lefschetz-Zariski classical theorem a generic 2-dimensional section of X captures all that topological information.

The hypersurfaces S that we will be our main focus are the hyperplane arrangements. Firstly because their complements are \mathbb{Q} -formal as discussed above. Secondly because the integral cohomology of their complements is known to be torsion-free, see [19]. In general, for a hypersurface S consisting of non-linear irreducible components, $H^*(X, \mathbb{Z})$ will have torsion, and thus, at least conceptually, the chances of getting non-vanishing Massey products in $H^*(X, \mathbb{F}_p)$ are already greater. However, it is possible for a non-linear hypersurface to have torsion-free $H^*(X, \mathbb{Z})$ as long as sufficiently many components of it are hyperplanes. We will briefly consider an example of such a non-linear hypersurface that nevertheless has triple non-vanishing Massey products.

The arrangement \mathcal{A} of Theorem 1.1 is the full monomial arrangement $\mathcal{A}(r, 1, 3)$ in \mathbb{C}^3 , with $r = p \geq 3$. These arrangements are members of a series of complex reflection arrangements, $\mathcal{A}(r, 1, d)$, associated to the full monomial reflection group $G(r, 1, d)$, see [1, 4, 19]. For $r \geq 1$ and $d \geq 2$ let $\mathcal{A}(r, 1, d)$ be defined by the polynomial

$$Q = z_1 \cdots z_d \cdot \prod_{1 \leq i < j \leq d} (z_i^r - z_j^r).$$

Note that the arrangements $\mathcal{A}(1, 1, d)$ and $\mathcal{A}(2, 1, d)$ are the Coxeter arrangements of type A and respectively B .

For \mathcal{A} a complex hyperplane arrangement in \mathbb{C}^d , with complement X and group $G = \pi_1(X)$, it is known that the rings $H^{*\leq 2}(X, \mathbb{K})$ and $H^{*\leq 2}(G, \mathbb{K})$ are isomorphic for \mathbb{K} a field or \mathbb{Z} , see for example [15]. Moreover, the complement to $\mathcal{A}(r, 1, d)$ is a $K(\pi, 1)$ with π the pure braid group $P(r, 1, d)$ associated to $G(r, 1, d)$, see Orlik and Solomon [18]. Taking advantage of this, we use the cochains of G rather than those of X to compute Massey products. We will use a presentation of $P(r, 1, d)$ obtained by Cohen [2].

A key rôle in the computations is played by the so-called resonance varieties of the arrangement, see [8, 15]. The resonance variety $\mathcal{R}(\mathcal{A}, \mathbb{K})$ over a field \mathbb{K} of an arrangement \mathcal{A} is the subvariety of $H^1(X, \mathbb{K})$ encoding the vanishing cup products:

$$\mathcal{R}(\mathcal{A}, \mathbb{K}) = \{ \lambda \in H^1(X, \mathbb{K}) \mid \exists \mu \notin \mathbb{K}\lambda \text{ such that } \lambda \cup \mu = 0 \}.$$

The knowledge of the classes in $H^1(X, \mathbb{K})$ that cup zero is especially needed for calculating a triple Massey product $\langle \alpha, \beta, \gamma \rangle$ as that is well-defined only when $\alpha \cup \beta = \beta \cup \gamma = 0$. In [8], Falk gives a combinatorial recipe to detect possible essential components of $\mathcal{R}(\mathcal{A}, \mathbb{K})$. For $\mathcal{A} = \mathcal{A}(r, 1, 3)$, the classes used to define the non-vanishing Massey products belong to such components arising when $\mathbb{K} = \mathbb{F}_p$, for the special primes p dividing r . It can be shown that $\mathcal{A}(r, 1, 3)$ presents non-vanishing \mathbb{F}_p -Massey products for all primes p and all multiples r of p (multiples of 4 if $p = 2$). Here only the case $r = p$ is treated.

The paper is organized as follows. In Section 2 we define the triple Massey products of a 2-complex associated to a finitely presented group, and explain how they can be computed from the presentation. In Section 3 we introduce the monomial arrangements and give presentations by generators and relators of the fundamental groups of their complements. In Section 4 we exhibit non-vanishing triple Massey products in the \mathbb{F}_p -cohomology of the complements to 3-dimensional monomial

arrangements, for p an odd prime. We also present a non-linear arrangement of curves in $\mathbb{C}P^2$ whose complement has non-vanishing triple Massey products over \mathbb{F}_2 . In the last section we pose some further questions that we intend to explore elsewhere.

§2. Massey products of CW -complexes

The results on Massey products that we need may be found in the works of Porter [20], Turaev [21], and Fenn and Sjerve [9, 10]. In these papers the Massey products of 1-cohomology classes are computed in terms of the so-called Magnus coefficients, via the free calculus of Fox. Unless otherwise specified, all the homology and cohomology groups will have coefficients in \mathbb{F}_p , the integers modulo a prime p .

Definition 2.1. Let X be a space of the homotopy type of a CW -complex. If α, β, γ in $H^1(X)$ are such that $\alpha \cup \beta = \beta \cup \gamma = 0$ then the triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined as follows: Choose representative 1-cocycles α', β', γ' and cochains x, y in $C^1(X)$ such that $dx = \alpha' \cup \beta'$ and $dy = \beta' \cup \gamma'$. Then $z = \alpha' \cup y + x \cup \gamma'$ is a 2-cocycle. The cohomology classes $z \in H^2(X)$ constructed in this way are only determined up to $\alpha \cup H^1(X) + H^1(X) \cup \gamma$, and they form a set denoted by $\langle \alpha, \beta, \gamma \rangle$.

As pointed out by May [16], the indeterminacy is a vector space, and so $\langle \alpha, \beta, \gamma \rangle$ can be thought of as a coset modulo $\alpha \cup H^1(X) + H^1(X) \cup \gamma$. The triple Massey product $\langle \alpha, \beta, \gamma \rangle$ is said to be vanishing if this coset is trivial.

In this paper X will always be a $K(G, 1)$ for G a finitely presented group. We will identify from now on the cohomology of X with that of G .

Let $G = \langle x_1, \dots, x_n \mid R_1, \dots, R_m \rangle$ be a presentation for $G = \pi_1(X)$. Assume that R_l is a commutator and that the presentation is minimal. By Hopf's formula the homology classes of the relators R_l form a basis in $H_2(G, \mathbb{Z}) = \mathbb{Z}^m$. Moreover, the generators x_i determine a basis of $H_1(G, \mathbb{Z}) = \mathbb{Z}^n$. Let e_i be the dual basis in $H^1(G, \mathbb{Z}) = \mathbb{Z}^n$.

Let F be the free group on x_1, \dots, x_n . If w is a word in F then its Fox derivative $\partial_j(w)$ is computed by the following rules: $\partial_j(1) = 0$, $\partial_j(x_i) = \delta_{i,j}$, and $\partial_j(uv) = \partial_j(u)\epsilon(v) + u\partial_j(v)$, where $\epsilon: \mathbb{Z}F \rightarrow \mathbb{Z}$ is the augmentation of the group ring $\mathbb{Z}F$.

Let $I = (i_1, \dots, i_q)$ be a multi-index with i_j taking values in $1, \dots, n$. The Magnus I -coefficient of a word w is defined by $\epsilon_I^{(0)}(w) = \epsilon \partial_I(w)$, where $\partial_I(w) = \partial_{i_1} \dots \partial_{i_q}(w)$. The \mathbb{F}_p -valued Magnus coefficients $\epsilon_I^{(p)}(w)$ of w are defined simply by taking integers modulo the prime p . Most of the time we will drop the reference to it and simply write $\epsilon_I(w)$ for

$\epsilon_F^{(p)}(w)$. We will usually refer to $\epsilon_{i,j}(w)$ and $\epsilon_{i,j,k}(w)$ as a double, and respectively triple Magnus coefficient.

The following result [9, 10, 20, 21] will be used to compute triple Massey products. Let α, β, γ be cohomology classes in $H^1(G)$ such that $\alpha \cup \beta = \beta \cup \gamma = 0$. Denote by (\cdot, \cdot) the Kronecker pairing between cohomology and homology. Then we have:

Theorem 2.2. *The Massey product $\langle \alpha, \beta, \gamma \rangle$ of $\alpha = \sum \alpha_i e_i$, $\beta = \sum \beta_j e_j$, $\gamma = \sum \gamma_k e_k$ contains ξ , where:*

$$(\xi, R_l) = \sum_{1 \leq i,j,k \leq n} \alpha_i \beta_j \gamma_k \cdot \epsilon_{i,j,k}(R_l).$$

From now on by $\langle \alpha, \beta, \gamma \rangle$ we will understand the coset of the class ξ modulo the indeterminacy $\alpha \cup H^1(G) + H^1(G) \cup \gamma$. The Massey product $\langle \alpha, \beta, \gamma \rangle = \xi$ is functorial with respect to maps of spaces, as shown by Fenn and Sjerve in [9, 10].

The following formulae can readily deduced from the definitions and they will be used to compute the Magnus coefficients of a commutator word.

$$(2.1) \quad \epsilon_{k,l}([u, v]) = \epsilon_k(u)\epsilon_l(v) - \epsilon_k(v)\epsilon_l(u), \text{ where } [u, v] = uvu^{-1}v^{-1}.$$

$$(2.2) \quad \begin{aligned} \epsilon_{k,l,m}([u, v]) = & \epsilon_k(u)\epsilon_{l,m}(v) - \epsilon_m(u)\epsilon_{k,l}(v) + \epsilon_{k,l}(u)\epsilon_m(v) - \epsilon_k(v)\epsilon_{l,m}(u) + \\ & (\epsilon_k(v)\epsilon_l(u) - \epsilon_k(u)\epsilon_l(v)) \cdot (\epsilon_m(u) + \epsilon_m(v)). \end{aligned}$$

We will also need formulae for products of conjugated generators:

$$(2.3) \quad \epsilon_k(x_{i_1}^{w_1} \cdots x_{i_j}^{w_j}) = \sum_{a=1}^j \epsilon_k(x_{i_a}) = \sum_{a=1}^j \delta_{k,i_a}.$$

$$(2.4) \quad \epsilon_{k,l}(x_{i_1}^{w_1} \cdots x_{i_j}^{w_j}) = \sum_{a=1}^j (\epsilon_k(w_a)\delta_{l,i_a} - \epsilon_l(w_a)\delta_{k,i_a}) + \sum_{1 \leq a < b \leq j} \delta_{k,i_a} \delta_{l,i_b},$$

where $x^a = axa^{-1}$ and $\delta_{i,j}$ is Kronecker's delta.

§3. Monomial arrangements and their groups

We introduce in this section our main examples of hypersurfaces whose complements have non-vanishing Massey products in the \mathbb{F}_p -cohomology. They are the complex reflection arrangements $\mathcal{A}(r, 1, d)$ associated with the monomial reflection group $G(r, 1, d)$. Their complements are $K(\pi, 1)$ spaces and for all practical purposes we will identify their cohomology with that of their fundamental groups. We will describe here the group presentations that will be used in the Massey products computation.

3.1. Arrangements groups

We start with a brief overview of the fundamental group of hyperplane complements. Let \mathcal{A} be a hyperplane arrangement in the affine space \mathbb{C}^d and X its complement. Let us recall now the most salient features of the fundamental group $G = \pi_1(X)$ as a finitely presentable group. For all the details see [19]. First G is generated by the meridians γ_H around each hyperplane $H \in \mathcal{A}$. Each codim 2 intersection $H_{i_1} \cap \dots \cap H_{i_n}$ of hyperplanes in \mathcal{A} determines $n - 1$ relations: $g_1 g_2 \dots g_n = g_2 \dots g_n \cdot g_1 = \dots = g_n \cdot g_1 \dots g_{n-1}$, where g_j is some conjugate of the generator $x_j = g_{H_j}$. We denote by $[g_1, \dots, g_n]$ the family of commutator relators $[g_1 \dots g_i, g_{i+1} \dots g_n]$, with $1 \leq i < n$.

Thus we are lead to compute the Magnus coefficients of relators in families of the form: $[x_{i_1}^{w_1}, \dots, x_{i_n}^{w_n}]$. Note that the indices i_j are all distinct. Denote by $R_{I,w}^j$ the commutator $[x_{i_1}^{w_1} \dots x_{i_j}^{w_j}, x_{i_{j+1}}^{w_{j+1}} \dots, x_{i_n}^{w_n}]$.

The Magnus coefficients of order 2 of $R_{I,w}^j$ are given by:

$$(3.1) \quad \epsilon_{k,l} \left(R_{I,w}^j \right) = \sum_{a=1}^j \sum_{b=j+1}^n (\delta_{k,i_a} \delta_{l,i_b} - \delta_{k,i_a} \delta_{l,i_b}).$$

It is easily seen that:

$$\epsilon_{k,l} \left(R_{I,w}^j \right) = \begin{cases} 1 & \text{if } k = i_a \text{ and } l = i_b, \\ & \text{for some } 1 \leq a \leq j \text{ and } j + 1 \leq b \leq n \\ -1 & \text{if } k = i_b \text{ and } l = i_a, \\ & \text{for some } 1 \leq a \leq j \text{ and } j + 1 \leq b \leq n \\ 0 & \text{otherwise.} \end{cases}$$

The Magnus coefficients of order 3 of $R_{I,w}^j$ are given by:

$$(3.2) \quad \epsilon_{k,l,m} \left(R_{I,w}^j \right) = \begin{cases} \epsilon_m(w_b) & \text{if } k = i_a, \\ & l = i_b, m \notin I \\ -\epsilon_m(w_a) & \text{if } k = i_b, \\ & l = i_a, m \notin I \\ \epsilon_l(w_b) + \epsilon_k(w_{a'}) - \epsilon_l(w_a) + \delta_{a \leq a'} & \text{if } k = i_a, \\ & l = i_{a'}, m = i_b \\ \epsilon_l(w_a) + \epsilon_k(w_{b'}) - \epsilon_l(w_b) + \delta_{b \leq b'} & \text{if } k = i_b, \\ & l = i_{b'}, m = i_a \\ \epsilon_k(w_b) - \epsilon_m(w_b) - 1 & \text{if } k = i_a, \\ & l = i_b, m = i_{a'} \\ \epsilon_l(w_{b'}) - \epsilon_l(w_a) - \epsilon_m(w_b) + \delta_{b \leq b'} - 1 & \text{if } k = i_a, \\ & l = i_b, m = i_b \\ \epsilon_k(w_a) - \epsilon_m(w_a) + 1 & \text{if } k = i_b, \\ & l = i_a, m = i_{b'} \\ \epsilon_l(w_{a'}) - \epsilon_l(w_b) - \epsilon_m(w_a) + \delta_{a \leq a'} + 1 & \text{if } k = i_b, \\ & l = i_a, m = i_{a'} \\ 0 & \text{otherwise.} \end{cases}$$

where always $1 \leq a, a' \leq j$ and $j + 1 \leq b, b' \leq n$.

3.2. Monomial arrangements

We introduce now our main class of examples. For $r \geq 1$ and $d \geq 2$ let $\mathcal{A}(r, 1, d)$ be the arrangement defined by:

$$Q = z_1 \cdots z_d \cdot \prod_{1 \leq i < j \leq d} (z_i^r - z_j^r).$$

The complement of $\mathcal{A}(r, 1, d)$ is a $K(\pi, 1)$ with π the pure braid group $P(r, 1, d)$ associated to the full monomial complex reflection group $G(r, 1, d)$, see [1, 18]. The group $P(r, 1, d)$ admits an iterated semidirect product structure: $P(r, 1, d) = F_{n_i} \rtimes \cdots \rtimes F_{n_1}$, where $n_i = (i - 1)r + 1$ for $1 \leq i \leq r$, as shown in [1].

A presentation for $P(r, 1, d)$ was obtained by Cohen in [2]. Following that paper, let us first describe the codim 2 intersections among the hyperplanes of $\mathcal{A}(r, 1, d)$:

$$(3.3) \quad H_i \cap H_{i,j}^{(1)} \cap \dots \cap H_{i,j}^{(r-1)} \cap H_j \cap H_{i,j}^{(r)}$$

$$(3.4) \quad H_k \cap H_{i,j}^{(q)}, \text{ if } k \neq i \text{ or } k > j$$

$$(3.5) \quad H_{i,j}^{(q)} \cap H_{k,l}^{(s)}, \text{ if } i, j, k, l \text{ distinct,}$$

$$(3.6) \quad H_{i,j}^{(q)} \cap H_{j,k}^{(s)} \cap H_{i,k}^{(t)}, \text{ if } t = q + s \pmod{r},$$

where $H_i = \{z_i = 0\}$, $1 \leq i \leq 3$, and $H_{i,j}^{(q)} = \{z_i = \zeta^q z_j\}$, where $\zeta = \exp(2\pi i/r)$, $1 \leq i < j \leq 3$ and $1 \leq q \leq r$.

We focus now on the case $d = 3$. In [2] a presentation of $P(r, 1, 3)$ is given having $3r + 3$ generators, say x_1, \dots, x_{3r+3} , and $2r^2 + 6r + 3$ relators. We group these relators in nine families corresponding to the types of the codimension 2 intersections.

$$(3.7)$$

$$A = [x_{3r+1}, x_1, \dots, x_{r-1}, x_{3r+2}, x_r],$$

$$(3.8)$$

$$B = [x_{3r+1}, x_{2r+1}, \dots, x_{3r-1}, x_{3r+3}, x_{3r}],$$

$$(3.9)$$

$$C = [x_{3r+2}, x_{r+1}^{x_r x_{3r+1} x_1 x_2 \dots x_{r-1} x_{3r+1}^{-1}}, \dots, x_{2r-1}^{x_r x_{3r+1} x_1 x_{3r+1}^{-1}}, x_{3r+3}, x_{2r}],$$

$$(3.10)$$

$$D_{1,s} = [x_{3r+1}, x_{r+i}], 1 \leq s \leq r,$$

$$(3.11)$$

$$D_{2,s} = [x_{3r+3}, x_i], 1 \leq s \leq r,$$

$$(3.12)$$

$$D_{3,s} = [x_{3r+2}^{x_i x_{i+1} \dots x_{r-1}}, x_{2r+i}], 1 \leq s \leq r,$$

$$(3.13)$$

$$T_s = [x_s, x_{2r+s}, x_{2r}], 1 \leq s \leq r,$$

$$(3.14)$$

$$U_{t,s} = [x_s, x_{2r-t}, x_{2r+s-t}^{x_{2r-t+1} \dots x_{2r-1}}], 1 \leq t < s \leq r,$$

$$(3.15)$$

$$V_{s,t} = [x_s^{x_{3r+1}}, x_{2r-t}, x_{3r+s-t}^{x_{2r-t+1} \dots x_{2r-1}}], 1 \leq s \leq t < r.$$

Now, recall that the notation $R = [x_{i_1}^{w_1}, \dots, x_{i_n}^{w_n}]$ stands for the following set of commutators: $\{R^j = [x_{i_1}^{w_1} \dots x_{i_j}^{w_j}, x_{i_{j+1}}^{w_{j+1}} \dots x_{i_n}^{w_n}] \mid 1 \leq j <$

$n\}$. Thus, in agreement with the notations of (3.7)-(3.15), the relators in $P(r, 1, 3)$ will be denoted by: A^j, B^j, C^j , where $j = 1, \dots, r + 1$, and $D_{1,s}, D_{2,s}, D_{3,s}$, where $j = 1$ and is omitted, and finally $T_s^j, U_{t,s}^j, V_{s,t}^j$, where $j = 1, 2$.

§4. Non-vanishing triple Massey products

In this section we will present non-vanishing triple Massey products in the \mathbb{F}_p cohomology of certain hypersurface complements. All such products will be of the form $\langle \alpha, \alpha, \beta \rangle$ with α and β linearly independent. The main example will consist of the monomial arrangements introduced in the previous section. We will also give an example of a non-linear arrangement of curves with the desired non-vanishing property.

4.1. Resonance varieties

We first determine the vanishing cup products in $H^2(X, \mathbb{F}_p)$, for X the complement of a monomial arrangement \mathcal{A} , using an invariant of a cohomology ring introduced by Falk in [8]. The resonance variety $\mathcal{R}(\mathcal{A}, \mathbb{F}_p)$ of an arrangement \mathcal{A} is the subvariety of $H^1(X, \mathbb{F}_p)$ defined by:

$$\mathcal{R}(\mathcal{A}, \mathbb{F}_p) = \{ \lambda \in H^1(X, \mathbb{K}) \mid \exists \mu \notin \mathbb{K}\lambda \text{ such that } \lambda \cup \mu = 0 \}.$$

In [8] it is shown how one can construct components of $\mathcal{R}(\mathcal{A}, \mathbb{F}_p)$ from the so-called neighborly partitions of the arrangement \mathcal{A} . The neighborly partitions of the monomial arrangements $\mathcal{A} = \mathcal{A}(r, 1, 3)$ have been determined in [4]. The most interesting for us is the partition $\Pi = (H_3, H_{12}^{(i)} \mid H_2, H_{13}^{(j)} \mid H_1, H_{23}^{(k)})$ giving rise to a component C_Π of $\mathcal{R}(\mathcal{A}, \mathbb{F}_p)$ having the following equations:

$$\begin{aligned} \lambda_1 + \dots + \lambda_r &= \lambda_{r+1} + \dots + \lambda_{2r-1} = \lambda_{2r+1} + \dots + \lambda_{3r-1} = 0 \\ \lambda_i + \lambda_{2r} + \lambda_{2r+i} &= 0, \quad 1 \leq i \leq r, \\ (4.1) \quad \lambda_i + \lambda_{2r-j} + \lambda_{2r+i-j} &= 0, \quad 1 \leq j < i \leq r, \\ \lambda_i + \lambda_{2r-j} + \lambda_{3r+i-j} &= 0, \quad 1 \leq i \leq j < r, \\ \lambda_{3r+1} &= \lambda_{3r+2} = \lambda_{3r+3} = 0 \end{aligned}$$

It is easily seen that $\dim C_\Pi = 3$ if p divides r (or 4 divides r , if $p = 2$), and $\dim C_\Pi = 2$, otherwise.

4.2. Massey products of monomial arrangements

We prove here the main result, showing that, in general, Massey products in the positive characteristic cohomology of a hypersurface

complement may not vanish modulo indeterminacy, although over the rationals they always do so.

Theorem 4.1. *For every odd prime p the complement X of the arrangement $\mathcal{A}(p, 1, 3)$ in \mathbb{C}^3 of degree $3p + 3$ has non-vanishing triple Massey products in $H^2(X, \mathbb{F}_p)$.*

Proof. We will show that a certain triple product $\langle \alpha, \alpha, \beta \rangle$ does not vanish modulo its indeterminacy. The cohomology classes α and β are given in coordinates by

$$a : \alpha_i = 1, \alpha_{r+i} = -1, \alpha_{2r+i} = \alpha_{3r+1} = \alpha_{3r+2} = \alpha_{3r+3} = 0,$$

and respectively by

$$\beta : \beta_i = 0, \beta_{r+i} = 1, \beta_{2r+i} = -1, \beta_{3r+1} = \beta_{3r+2} = \beta_{3r+3} = 0,$$

where $1 \leq i \leq r$. Clearly the points α and β satisfy the equations (4.1), so they belong to C_{II} , and moreover $\alpha \cup \beta = 0$. Using (3.2) we can express $\langle \alpha, \alpha, \beta \rangle$ in the basis of $H^2(X, \mathbb{F}_p)$ given by the duals of the relators (3.7)-(3.15), abusing the notation for the sake of simplicity.

(4.2)

$$\begin{aligned} \langle \alpha, \alpha, \beta \rangle = & \sum_{j=1}^p (j-1)C^j + (p-1)C^{p+1} - \sum_{s=1}^p T_s^2 + \sum_{1 \leq t < s \leq p} t U_{t,s}^1 + \\ & \sum_{1 \leq t < s \leq p} U_{t,s}^2 + \sum_{1 \leq s \leq t < p} t V_{s,t}^1 + \sum_{1 \leq s \leq t < p} V_{s,t}^2. \end{aligned}$$

Next, using (3.1), we obtain the indeterminacy $\alpha \cup H^1(X) + H^1(X) \cup \beta$. If $a = \sum a_i e_i$ and $b = \sum b_i e_i$ are arbitrary classes in $H^1(X)$ then we

find the following expression for $\alpha \cup a + b \cup \beta$:

$$\begin{aligned}
 & \left(\sum_{s=1}^p a_s + a_{3p+1} + a_{3p+2} \right) \left(\sum_{j=1}^p (j-1)A^j + (p-1)A^{p+1} \right) + \\
 & \left(\sum_{s=1}^p b_{2p+s} + b_{3p+1} + b_{3p+3} \right) \left(\sum_{j=1}^p (j-1)B^j + (p-1)B^{p+1} \right) - \\
 & \left(\sum_{s=1}^p (a_{p+s} + b_{p+s}) + a_{3p+2} + a_{3p+3} + b_{3p+2} + b_{3p+3} \right) \\
 & \left(\sum_{j=1}^p (j-1)C^j + (p-1)C^{p+1} \right) + (a_{3p+1} + b_{3p+1}) \sum_{s=1}^p D_{1,s} - \\
 (4.3) \quad & a_{3p+3} \sum_{s=1}^p D_{2,s} - b_{3p+2} \sum_{s=1}^p D_{3,s} + \sum_{s=1}^p (a_s + a_{2p+s} + a_{2p}) (T_s^1 + T_s^2) + \\
 & \sum_{s=1}^p (b_s + b_{2p+s} + b_{2p}) T_s^2 + \sum_{1 \leq t < s \leq p} (a_s + a_{2p-t} + a_{2p+s-t}) U_{t,s}^1 - \\
 & \sum_{1 \leq t < s \leq p} (b_s + b_{2p-t} + b_{2p+s-t}) U_{t,s}^2 + \sum_{1 \leq s \leq t < p} (a_s + a_{2p-t} + a_{3p+s-t}) V_{s,t}^1 - \\
 & \sum_{1 \leq s \leq t < p} (b_s + b_{2p-t} + b_{3p+s-t}) V_{s,t}^2.
 \end{aligned}$$

We want to show that the triple Massey product $\langle \alpha, \alpha, \beta \rangle$ does not vanish modulo indeterminacy. Suppose that it does vanish, and so there exist a and b in $H^1(X)$ such that $\langle \alpha, \alpha, \beta \rangle$ is of the form $\alpha \cup a + b \cup \beta$. This leads to the following set of equations over \mathbb{F}_p :

$$(4.4) \quad \sum_{s=1}^p a_s + a_{3p+1} + a_{3p+2} = \sum_{s=1}^p b_{2p+s} + b_{3p+1} + b_{3p+3} = 0,$$

$$(4.5) \quad \sum_{s=1}^p (a_{p+s} + b_{p+s}) + a_{3p+2} + a_{3p+3} + b_{3p+2} + b_{3p+3} = -1,$$

$$(4.6) \quad a_{3p+1} + b_{3p+1} = a_{3p+3} = b_{3p+2} = 0,$$

$$(4.7) \quad a_s + a_{2p+s} + a_{2p} = 0, \quad b_s + b_{2p+s} + b_{2p} = -1,$$

$$(4.8) \quad a_s + a_{2p-t} + a_{2p+s-t} = t, \quad b_s + b_{2p-t} + b_{2p+s-t} = -1,$$

$$(4.9) \quad a_s + a_{2p-t} + a_{3p+s-t} = t, \quad b_s + b_{2p-t} + b_{3p+s-t} = -1,$$

where the ranges of the indices in (4.7), (4.8), and (4.9) are those in (4.3).

Now from (4.7), (4.8), and (4.9) we can readily see that we must have:

$$\sum_{s=1}^p a_s = \sum_{s=1}^p a_{p+s} = \sum_{s=1}^p a_{2p+s} = 0 \text{ and } \sum_{s=1}^p b_s = \sum_{s=1}^p b_{p+s} = \sum_{s=1}^p b_{2p+s} = 0.$$

From these equations combined with (4.4), (4.5) and (4.6) we obtain:

$$a_{3p+1} + a_{3p+2} = b_{3p+1} + b_{3p+3} = a_{3p+1} + b_{3p+1} = 0, \quad a_{3p+2} + b_{3p+3} = -1.$$

But this system of equations clearly has no solution.

Q.E.D.

Remark 4.2. We will show elsewhere that in fact any triple Massey product in $H^2(P, \mathbb{F}_p)$ of the form $\langle \alpha, \alpha, \beta \rangle$ with α and β (not proportional) in $C_\Pi \subset \mathcal{R}(P, \mathbb{F}_p) \subset H^1(P, \mathbb{F}_p)$ does not vanish modulo the indeterminacy $\alpha \cup H^1(P, \mathbb{F}_p) + H^1(P, \mathbb{F}_p) \cup \beta$, if $p \mid r$ (or $4 \mid r$, if $p = 2$), where $P = P(r, 1, 3)$. Thus it will follow that for every prime p and multiple $N \geq 3$ of p (of 4 if $p = 2$) there exists a line arrangement \mathcal{A} in \mathbb{C}^2 of degree $3N + 3$ whose complement X has non-vanishing triple Massey products in $H^2(X, \mathbb{F}_p)$.

4.3. Curves with non-linear components

Let $\mathcal{C} = Q_2 \cup T_1 \cup T_2 \cup T_3$ be the curve in $\mathbb{C}P^2$ of degree 5, consisting of a smooth irreducible curve Q_2 of degree 2 and three lines T_1, T_2, T_3 tangent to Q_2 . As explained by Kaneko, Tokunaga and Yoshida in [11], this curve is related with the discriminant of a certain crystallographic group, thus is of the same nature as the above reflection arrangements. In [11] a presentation for the fundamental group of the complement to \mathcal{C} in $\mathbb{C}P^2$ is determined:

$$\pi_1(\mathbb{C}P^2 \setminus \mathcal{C}) = \langle x_1, x_2, x_3 \mid [x_3 x_i x_3, x_i], i = 1, 2, [x_3 x_1 x_3^{-1}, x_2] \rangle.$$

An easy computation with double Magnus coefficients shows that all \mathbb{F}_2 cup products $e_i \cup e_j$ vanish except for $e_1 \cup e_2$. Moreover, by computing triple Magnus coefficients we can see that the Massey products $\langle \alpha, \alpha, \beta \rangle$ over \mathbb{F}_2 do not vanish, if $\alpha \notin \mathbb{F}_2 \cdot (e_1 + e_2 + e_3)$.

Remark 4.3. It is possible to generalize this example to a curve $\mathcal{C} = Q_d \cup T_1 \cup \dots \cup T_n$ of degree $d + n$, where Q_d is a smooth irreducible curve of degree $d \geq 2$ and $T_1 \dots T_n$ are $n \geq d + 1$ tangent lines to Q . Then the complement X of \mathcal{C} will have non-vanishing triple Massey products of the form $\langle \alpha, \alpha, \beta \rangle$ in $H^2(X, \mathbb{F}_p)$, for every prime p dividing d .

§5. Further questions

We end the paper by raising a few questions:

- (1) Note that the above arrangements exhibiting non-vanishing Massey products do not admit linear equations over the reals! Is it true that real complexified arrangements never have non-vanishing Massey products? Computational evidence suggest that in this case Massey products in $H^2(X, \mathbb{F}_p)$ indeed all vanish.
- (2) All non-orientable matroids realizable over some $\mathbb{Q}(\alpha)$ lead to complex arrangements with non-vanishing Massey products?
- (3) Is there an analogue of Kohno's result over \mathbb{F}_p ? Is it true that non-vanishing of higher Massey products over \mathbb{F}_p implies that the \mathbb{F}_p -completion of $\pi_1(X)$ is not isomorphic to the completed holonomy algebra of $H^{\leq 2}(X, \mathbb{F}_p)$?
- (4) Do non-linear curves (with enough cohomology) always present non-vanishing Massey products?
- (5) Are there any good criteria for \mathbb{F}_p -formality of X ? In this context, what is the rôle played by the lower p -central series of $\pi_1(X)$?

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