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Some constructions of hyperbolic hypersurfaces in $\mathbf{P}^n(\mathbf{C})$

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Abstract.

We show some methods of constructing hyperbolic hypersurfaces in the complex projective space, which gives a hyperbolic hypersurface of degree 2^n in $P^n(\mathbf{C})$ for every $n \geq 2$. Moreover, we show that there are some hyperbolic hypersurfaces of degree d in $P^n(\mathbf{C})$ for every $d \geq 2 \times 6^n$ for each $n \geq 3$.

§1. Introcution

Since S. Kobayashi asked whether a generic hypersurface of large degree in $P^n(\mathbf{C})$ is hyperbolic or not in [8], many papers were devoted to constructing various examples of hypersurfaces in $P^n(\mathbf{C})$. In [2], R. Brody and M. Green gave an example of hyperbolic hypersurface in $P^3(\mathbf{C})$ of even degree ≥ 50. Afterwards, new types of hyperbolic hypersurfaces of degree d in $P^3(\mathbf{C})$ were given by A. Nadel in the case of $d = 6p + 3$ for $p \geq 3$ in [10], by J. El Goul for $d \geq 14$ in [7], by J. P. Demailly and by Y. T. Siu–S. K. Yeung for $d \geq 11$ in 1997 respectively. Moreover, J. P. Demailly-J. El Goul proved that a very generic hypersurface of degree at least 21 in $P^3(\mathbf{C})$ is hyperbolic in [4] and M. Shirosaki constructed a hyperbolic hypersurface of degree 10 in [11]. On the other hand, in [9], K. Masuda and J. Noguchi proved that there exists a hyperbolic hypersurface of every degree $d \geq d(n)$ for a positive integer $d(n)$ depending only on n and some concrete examples of hyperbolic hypersurfaces in $P^n(\mathbf{C})$ for $n \leq 5$.

Recently, the author constructed a family of hyperbolic hypersurfaces of degree 2^n in $P^n(\mathbf{C})$ for $n \geq 3$ in [6]. The purpose of this note is to explain the results in [6] and to give some lower estimate of $d(n)$ in the above-mentioned results given by Masuda-Noguchi. The author would like to thank J. Noguchi for useful suggestions to this work.

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§2. Construction of H-polynomials

For convenience' sake, we introduce the following terminology.

Definition 2.1. We call a homogeneous polynomial $Q(w)$ of degree d in $w = (w_0, w_1, \dots, w_n)$ an *H-polynomial* if it satisfies the conditions:

(H1) If a holomorphic map $f := (f_0 : f_1 : \cdots : f_n)$ of $\mathbf{C}$ into $P^n(\mathbf{C})$ satisfies the identity $Q(f_0, f_1, \dots, f_n) = cf_0^d$ for some $c \in \mathbf{C}$, then f is a constant.

(H2) If a holomorphic map $f := (f_1 : \cdots : f_n)$ of $\mathbf{C}$ into $P^{n-1}(\mathbf{C})$ satisfies the identity $Q(0, f_1, \dots, f_n) = cf_{n+1}^d$ for some $c \in \mathbf{C}$ and entire function f_{n+1}, then f is a constant.

Definition 2.2. We say a complex space M to be Brody hyperbolic if there is no nonconstant holomorphic map of $\mathbf{C}$ into M.

As was shown by R. Brody in [1], a compact complex manifold is Brody hyperbolic if and only if it is hyperbolic in the sense of S. Kobayashi. In the following, a compact hyperbolic space means a compact Brody hyperbolic space.

Proposition 2.3. *Let Q be an H-polynomial. Then,*

(i) $V := \{(w_0 : \cdots : w_n); Q(w_0, \dots, w_n) = 0\}$ *is hyperbolic and*

(ii) *for* $W := \{(w_1 : \cdots : w_n); Q(0, w_1, \dots, w_n) = 0\} \subset P^{n-1}(\mathbf{C})$, $P^{n-1}(\mathbf{C}) \setminus W$ *is Brody hyperbolic.*

In fact, (i) is nothing but the case $c = 0$ of (H1), and (ii) is a result of (H2) because we can find an entire function f_{n+1} such that $Q(0, f_1, \dots, f_n) = f_{n+1}^d$ if $Q(0, f_1, \dots, f_n)$ has no zeros.

For the case where $n = 2$ we have the following:

Theorem 2.4. *Let $Q(u_0, u_1, u_2)$ be a homogeneous polynomial of degree $d \geq 4$ and consider the associated inhomogeneous polynomial $\tilde{Q}(v, w) := Q(1, v, w)$. Assume that*

(C1) *the simultaneous equations $\tilde{Q}_v(v, w) = \tilde{Q}_w(v, w) = 0$ have only finitely many solutions, say $P_k := (v_k, w_k)$ $(1 \leq k \leq N)$,*

(C2) $\tilde{Q}(P_k) \neq \tilde{Q}(P_\ell)$ *for* $1 \leq k < \ell \leq N$,

(C3) $Q_{u_0}(1, v_k, w_k) \neq 0$ *for* $1 \leq k \leq N$,

(C4) $\{(u_1, u_2); Q_{u_i}(0, u_1, u_2) = 0, i = 0, 1, 2\} = \{(0, 0)\}$.

(C5) *Hessian* $\varphi := \tilde{Q}_{vv}\tilde{Q}_{ww} - \tilde{Q}_{vw}^2 \neq 0$ *at* (v_k, w_k) $(1 \leq k \leq N)$.

Then, Q is an H-polynomial.

For the proof, refer to [6].

Remark. We can show that generic homogeneous polynomials of degree $d \geq 4$ satisfy the conditions in Theorem 2.4. Here, generic homogeneous polynomials mean all polynomials in some nonempty Zariski open set in the space of all homogeneous polynomials of degree d.

For the case $n \geq 3$, we can prove the following:

Theorem 2.5. *Let $Q(u_0, u_1, \dots, u_n)$ be an H-polynomial of degree d_0 and $P(u_0, u_{n+1})$ a homogeneous polynomial of degree $d_1(\geq 3)$ such that $P(u_0, u_{n+1})$ and $\tilde{P}(w) := P(1, w)$ satisfies the conditions;*
(P1) $P(0, u_{n+1}) \not\equiv 0$,
(P2) *$\tilde{P}'(w)$ has only simple zeros $\alpha_1, \alpha_2, \dots, \alpha_{d_1-1}$,*
(P3) *$\tilde{P}(\alpha_k) \neq \tilde{P}(\alpha_\ell)$ for $1 \leq k < \ell \leq d_1 - 1$.*
For $m \geq 2$, if $d_1 := md_0$ and $2/(d_1 - 2) + 1/m < 1$, then

$$R(u_0, u_1, \dots, u_n, u_{n+1}) := P(u_0, u_{n+1}) - Q(u_0, u_1, \dots, u_n)^m$$

is an H-polynomial.

This is a slight improvement of [6, Theorem II]. We state the outline of the proof. Consider holomorphic functions f_j, some of which are nonzero, such that $R(f_0, \dots, f_{n+1}) = cf_0^{d_1}$. If $f_0 \equiv 0$, then

$$Q(0, f_1, \dots, f_n) = ef_{n+1}^{d_0}$$

for some constant e and hence f is a constant by (H2). Otherwise, setting $\varphi := f_{n+1}/f_0$ and $\tilde{Q} := Q(1, f_1/f_0, \dots, f_n/f_0)$, we have $\tilde{P}(\varphi) - c = \tilde{Q}^m$. By the assumption, $\tilde{P}(w) - c$ has at least $d_1 - 2$ simple zeros β_j and φ takes the values β_j with multiplicities at least m, whence $\Theta_\varphi(\beta_j) \geq 1 - 1/m$, where $\Theta_\varphi(\beta_j)$ denote the truncated defects of β_j. By virtue of the defect relation for meromorphic functions, we can conclude from the assumption that f is a constant. We can prove that R satisfies (H2) by the same argument as in the proof of [6, Theorem II]. We omit the details.

By Theorem 2.4 and by using Theorem 2.5 repeatedly, we can easily conclude the following:

Theorem 2.6. *For each $n \geq 2$ there is a hyperbolic hypersurfaces of degree 2^n in $P^n(\mathbf{C})$ and a hypersurface W of degree 2^n in $P^{n-1}(\mathbf{C})$ such that $P^{n-1}(\mathbf{C}) \setminus W$ is Brody hyperbolic.*

We can also construct many hyperbolic hypersurfaces in the complex projective space. For example, by Theorem 2.4, we can construct a hyperbolic hypersurface of degree 5 in $P^2(\mathbf{C})$ and, by the use of the case $m = 3$ of Theorem 2.5 repeatedly, hyperbolic hypersurfaces of degree $5 \times 3^{n-2}$ in $P^n(\mathbf{C})$, which are used later.

§3. Hyperbolic hypersurfaces of high degree

In this section, we construct some examples of hyperbolic hypersurfaces of high degrees. We first give the following:

Theorem 3.1. *Take a polynomial* $F := \sum_{i_1,\dots,i_m} a_{i_1\cdots i_m} x_1^{i_1}\cdots x_m^{i_m}$ *and consider the associated weighted homogeneous polynomial*

$$F^*(x_0, x_1, \dots, x_m) := \sum_{i_1,\dots,i_m} a_{i_1\cdots i_m} x_0^{d-i_1d_1-\cdots-i_md_m} x_1^{i_1}\cdots x_m^{i_m}$$

in $(x_0, x_1, \dots, x_n)$ *with weights* $(1, d_1, \dots, d_m)$ *for some positive integers* d_i, *where* $d := \max\{i_1d_1 + \cdots + i_md_m; a_{i_1\cdots i_m} \neq 0\}$. *Assume that*

(i) $F^*(0, x_1, \dots, x_m)$ *consits of only one monomial,*

(ii) *if* $F(\varphi_1, \dots, \varphi_m) = 0$ *for meromorphic functions* φ_i *on* $\mathbf{C}$, *then at least one of* φ_i*'s is a constant.*

Then, for arbitrary H-polynomials $Q_i(w_0, \dots, w_n)$ *of degree* d_i $(1 \leq i \leq m)$, *the hypersurface*

$$V := \left\{w = (w_0 : \dots : w_n); w_0^d F\left(Q_1(w)/w_0^{d_1}, \dots, Q_m(w)/w_0^{d_m}\right) = 0\right\}$$

in $P^n(\mathbf{C})$ *is hyperbolic.*

Proof. Consider a holomorphic map $f := (f_0 : f_1 : \cdots : f_n)$ of $\mathbf{C}$ into $V(\subset P^n(\mathbf{C}))$, where f_i are entire functions without common zeros. If $f_0 \equiv 0$, then $Q_{i_0}(0, f_1, \dots, f_n) \equiv 0$ for some i_0, whence f is a constant by (H1). Assume that $f_0 \not\equiv 0$. Then, $F(\varphi_1, \dots, \varphi_n) = 0$ for meromorphic functions $\varphi_i := Q_i(1, f_1, \dots, f_n)/f_0^{d_i}$. whence some φ_{i_0} is a constant and so f is a constant by (H1). This gives Theorem 3.1.

We give an example satisfying the assumptions of Theorem 3.1.

Proposition 3.2. *Set* $F(x, y) := x^p + y^p + x^ry^s + 1$ *for positive integers* p, r, s. *Assume that*

$$(1) \qquad p < t, \quad 6/p + 2/t < 1,$$

where $t := \min(r, s)$. *Then,* $F(x, y)$ *satisfies the assumptions* (i) *and* (ii) *of Theorem 3.1 for arbitrary positive integers* d_1 *and* d_2.

Proof. Obviously, (i) holds. To see (ii), take nonconstant meromorphic functions φ, ψ with $F(\varphi, \psi) = 0$. We write $\varphi = f_1/f_0, \psi = f_2/f_0$ with entire functions f_i such that f_1 and f_2 have no common zeros. Consider the holomorphic map $\Phi := (f_0^p : f_1^p : f_2^p) : \mathbf{C} \to P^2(\mathbf{C})$ and hyperplanes $H_j := \{w_{j-1} = 0\}$ for $j = 1, 2, 3$ and $H_4 := \{w_0 + w_1 + w_2 = 0\}$, which are in general position. Obviously, the pull-backs $\Phi^*(H_j)$ of H_j for

$j = 1, 2, 3$, considered as divisors, have no positive multiplicities smaller than p. Take a point z_0 in $f^{-1}(H_4)$. Since $f_0^p + f_1^p + f_2^p = -f_1^r f_2^s f_0^{p-(r+s)}$, if $f_0(z_0) \neq 0$, the multiplicity of $\Phi^*(H_4)$ at z_0 is at least t. Assume that $f_0(z_0) = 0$. Then, $f_1(z_0) \neq 0$ and $f_2(z_0) \neq 0$, because otherwise $\sum_{j=0}^{2} f_j(z_0)^p \neq 0$. This is impossible by the assumption $p < r + s$. In conclusion, $\Phi^*(H_4)$ has no positive multiplicities smaller than t. Then, there are constants c_0, c_1, c_2 with $(c_0, c_1, c_2) \neq (0, 0, 0)$ such that $c_0\varphi^p + c_1\psi^p + c_2 = 0$. Because, otherwise, the second main theorem for holomorphic curves in $P^n(\mathbf{C})$ gives $3(1-2/p) + (1-2/t) \leq 3$, which contradicts the assumption(cf., [5, Theorem 3.3.15]). If $c_2 = 0$, then φ and ψ are obviously constants. Otherwise, we have $c_0 f_0^p + c_1 f_1^p + c_2 f_2^p = 0$. Since $p \geq 4$ by the assumption, Φ is a constant. This gives Proposition 3.2.

By Theorem 3.1 and Proposition 3.2, we have the following:

Proposition 3.3. *Let $Q_i(w)$ be H-polynomials of degree d_i ($i = 1, 2$) in $n + 1$ variables $w = (w_0, w_1, \ldots, w_n)$ and p, r, s positive integers satisfying the condition* (1). *Then, the zero locus of the polynomial*

$$R(w) := Q_1(w)^p w_0^{d-pd_1} + Q_2(w)^q w_0^{d-pd_2} + w_0^d - Q_1(w)^r Q_2(w)^s$$

is a hyperbolic hypersurface in $P^n(\mathbf{C})$ of degree $d := rd_1 + sd_2$.

This improves Masuda-Noguchi's Theorem as follows:

Theorem 3.4. *For each $n \geq 3$ we can take a positive integer $d(n)$ such that there are hyperbolic hypersurfaces of degree d for every $d \geq d(n)$ in $P^n(\mathbf{C})$. Here, for example, we can take*

$$(2) \quad d(n) := 9(2^n + 5 \times 3^{n-2}) + 2^n(5 \times 3^{n-2} - 1) + 5 \times 3^{n-2}(2^n - 1).$$

For the proof of Theorem 3.4, we give the following Lemma:

Lemma 3.5. *Let d_1 and d_2 be mutually prime positive integers. For arbitrarily given positive integer m_0, every integer d with*

$$d \geq m_0(d_1 + d_2) + d_1(d_2 - 1) + d_2(d_1 - 1)$$

can be written as $d = rd_1 + sd_2$ with $r, s \geq m_0$.

This is easily shown by the fact that, for each number ℓ with $0 \leq \ell < d_1$, we can find integers r, s with $|r| < d_2$, $|s| < d_1$ such that $\ell = rd_1 + sd_2$.

The proof of Theorem 3.4. To this end, for each $n(\geq 3)$ we set $d_1(n) := 2^n$ and $d_2(n) := 5 \times 3^{n-2}$. As is mentioned in the previous section, we can find H-polynomials Q_1 and Q_2 of degree $d_1(n)$ and $d_2(n)$

respectively. Define $d(n)$ by (2). By Lemma 3.5, we can write every $d \geq d(n)$ as $d = rd_1(n) + sd_2(n)$ with $r, s \geq m_0 := 9$, because $d_1(n)$ and $d_2(n)$ are mutually prime. For $p := 8$ and these r, s, which satisfy the condition (1), we apply Proposition 3.3 to find a homogeneous polynomial R of degree d such that $V := \{R = 0\}$ is a hyperbolic hypersurface in $P^n(\mathbf{C})$.

References

[1] R. Brody, Compact manifolds and hyperbolicity, Trans. Amer. Math. Soc., **235**(1978), 213 – 219.

[2] R. Brody and M. Green, A family of smooth hyperbolic hypersurfaces in $\mathbf{P}_3$, Duke Math. J., **44**(1977), 873 – 874.

[3] J.-P. Demailly, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, Proc. Sympos. Pure Math., Vol. 62, Part 2, Amer. Math. Soc., Providence, RI, 1997, 285 – 360.

[4] J.-P. Demaily and J. El Goul, Hyperbolicity of generic surfaces of high degree in projective 3-space, Amer. J. Math., **122**(2000), 515 – 546.

[5] H. Fujimoto, Value distribution theory of the Gauss map of minimal surfaces in R^m, Aspect of Math. **E21**, Vieweg, 1993.

[6] H. Fujimoto, A family of hyperbolic hypersurfaces in the complex projective space, Complex Variables, **43**(2001), 273 – 283.

[7] J. El Goul, Algebraic families of smooth hyperbolic surfaces of low degree in $\mathbf{P}^3_{\mathbf{C}}$, manuscripta math., **90**(1996), 521 – 532.

[8] S. Kobayashi, Hyperbolic manifolds and holomorphic mappings, Marcel Dekker, 1970.

[9] K. Masuda and J. Noguchi, A construction of hyperbolic hypersurface of $P^n(\mathbf{C})$, Math. Ann., **304**(1996), 339 – 362.

[10] A. Nadel, Hyperbolic surfaces in $\mathbf{P}^3$, Duke. Math. J., **58**(1989), 749 – 771.

[11] M. Shirosaki, A hyperbolic hypersurface of degree 10, Kodai Math. J., **23**(2000), 376 – 379.

[12] Y. T. Siu and S. K. Yeung, Defects for ample divisors of abelian vaarieties, Schwarz lemma, and hyperbolic hypersurfaces of low degrees,Amer. J. Math., **119**(1997), 1139 – 1172.

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