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Some Comments about Itô's Construction Procedure

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For K. Itô on his 88th birthday

Abstract.

This article reviews Itô's procedure for constructing the Markov process generated by variable coefficient Lévy–Khinchine operators. In particular, it examines conditions under which Itô's procedure succeeds but more analytic procedures appear to fail.

§0 Introduction

In his famous memoir [1], Itô dealt with the construction of Markov processes corresponding to variable coefficient Lévy–Khinchine operators. His method rests on the ability to represent of the action of Lévy–Khinchine operator L with diffusion coefficient $x \rightsquigarrow a(x)$, drift coefficient $x \rightsquigarrow b(x)$, and Lévy measure $x \rightsquigarrow M(x,\,\cdot\,)$ on a $\varphi \in C^2_c(\mathbb{R}^n;\mathbb{R})$ in the form

$$
\begin{aligned}
L\varphi(x) = \tfrac{1}{2}\sum_{i,j=1}^{n}\left(\sigma(x)\sigma(x)^\top\right)_{ij}\partial_i\partial_j\varphi(x) + \sum_{i=1}^{n} c(x)_i\partial_i\varphi(x)\\
+\int_{\mathbb{R}^n\setminus\{0\}}\Big(\varphi\big(x+F(x,y)\big)-\varphi(x)\\
-\mathbf{1}_{[0,1]}\big(|y|\big)\big(F(x,y),\mathrm{grad}_x\varphi\big)_{\mathbb{R}^n}\Big)\,M(dy),
\end{aligned}
\tag{0.1}
$$

for appropriate functions $\sigma : \mathbb{R}^n \longrightarrow \mathrm{Hom}(\mathbb{R}^n;\mathbb{R}^n)$, $c : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and Lévy measure M. In order for his method to have a chance of working, these functions must be at least (Borel) measurable, and, in practice, they must be much better than that. Indeed, apart from refinements (cf. [6]), which are important but of restricted

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applicability, what one needs is that σ and c be uniformly Lipschitz continuous and that F satisfy conditions of the form:

$$
\begin{aligned}
&\lim_{r\searrow 0}\sup_{x\in\mathbb{R}^n}\frac{1}{1+|x|^2}\int_{0<|y|\le r}|F(x,y)|^2\,\frac{dy}{|y|^{n+1}}=0\\
(0.2)\qquad &\sup_{x\in\mathbb{R}^n}\frac{1}{1+|x|^2}\int_{0<|y|\le R}|F(x,y)|^2\,\frac{dy}{|y|^{n+1}}<\infty\\
&\sup_{x_2\neq x_1}\frac{1}{|x_2-x_1|^2}\int_{0<|y|\le R}\big|F(x_2,y)-F(x_1,y)\big|^2\,\frac{dy}{|y|^{n+1}}<\infty
\end{aligned}
$$

for each $R\in(0,\infty)$. Under these conditions it is possible to carry out (cf. §3.1 and §4.1 in [3]) Itô's procedure for constructing the Markov process corresponding to $x\rightsquigarrow\big(a(x),b(x),M(x.\,\cdot\,)\big)$ by transforming the paths of the Lévy process whose continuous part is standard Brownian motion and whose Lévy part is the symmetric Cauchy process whose Lévy measure is $M_0(dy)=\mathbf{1}_{\mathbb{R}^n\setminus\{0\}}(y)\frac{dy}{|y|^{n+1}}$.

Assuming that $x\rightsquigarrow a(x)$, $x\rightsquigarrow b(x)$, and $x\rightsquigarrow M(x,\,\cdot\,)$ are measurable, it is always possible (cf. §3.2, and especially Theorem 3.2.5, in [3]) to make measurable choices of $x\rightsquigarrow\sigma$ and $(x,y)\rightsquigarrow F(x,y)$ so that (0.1) holds. In addition, it is well-known (cf. §3.2.1 in [3]) that the non-negative definite, symmetric square root of $x\rightsquigarrow a(x)$ will be uniformly Lipschitz if either $x\rightsquigarrow a(x)$ is uniformly Lipschitz and uniformly positive definite or a and its second derivatives a are uniformly bounded. On the other hand, it is much less clear what smoothness properties of $x\rightsquigarrow M(x,\,\cdot\,)$ will guarantee that F can be chosen so that (0.2) holds. Because it is the Lévy term which poses the greatest challenge to traditional analytic techniques, it may be of interest to investigate how successful Itô's theory is with it, and that is what we will be doing here.

§1 Basic Result

In this section we will show how to construct an F satisfying (0.2) when $x\rightsquigarrow M(x,\,\cdot\,)$ can be expressed in the form

$$
(1.1)\qquad M(x,\Gamma)=\omega_{n-1}\int_{\mathbb{S}^{n-1}}\left(\int_{(0,\infty)}\mathbf{1}_\Gamma(r\omega)\beta(x,\omega,r)\,dr\right)\mu(d\omega)
$$

for $\Gamma\in\mathcal{B}_{\mathbb{R}^n\setminus\{0\}}$, where ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $\mu\in\mathbf{M}_1(\mathbb{S}^{n-1})$ (i.e., μ is a Borel probability measure on $\mathbb{S}^{n-1}$), and $\beta:\mathbb{R}^n\times\mathbb{S}^{n-1}\times(0,\infty)\longrightarrow(0,\infty)$ is a measurable function with the properties that

$$\int_{(0,\infty)} \beta(x,\omega,r)\,dr = \infty \quad \text{for all } (x,\omega) \in \mathbb{R}^n \times \mathbb{S}^{n-1}, \tag{1.2}$$

$$\inf_{(x,\omega)} \int_{[1,\infty)} \beta(x,\omega,s)\,ds > 0 \ \& \ \sup_{(x,\omega)} \int_{(0,\infty)} \frac{r^2\beta(x,\omega,r)}{1+r^2}\,dr < \infty, \tag{1.3}$$

$$\lim_{R\searrow 0} \sup_{(x,\omega)} \int_{(0,R]} r^2\beta(x,\omega,r)\,dr = 0 = \lim_{R\to\infty} \sup_{(x,\omega)} \int_{[R,\infty)} \beta(x,\omega,s)\,ds, \tag{1.4}$$

and, for each $(\omega, r) \in \mathbb{S}^{n-1} \times (0,\infty)$, $\beta(\,\cdot\,,\omega,r)$ has a continuous derivative which satisfies

$$\sup_{(x,\omega)} \int_{(0,R]} \frac{\left(\int_{[r,\infty)} \left|\operatorname{grad}_x \beta(\,\cdot\,,\omega,s)\right| ds\right)^2}{\beta(x,\omega,r)}\,dr < \infty \tag{1.5}$$

for each $R \in (0,\infty)$.

The construction of F in this case can be carried out as follows. First, one determines $\rho : \mathbb{R}^n \times \mathbb{S}^{n-1} \times (0,\infty) \longrightarrow (0,\infty)$ so that

$$\int_{[\rho(x,\pm 1,r),\infty)} \beta(x,\pm 1,s)\,ds = \frac{2\mu(\{\pm 1\})}{r} \quad \text{when } n = 1$$

and

$$\int_{[\rho(x,\omega,r),\infty)} \beta(x,\omega,s)\,ds = \frac{1}{r} \qquad \text{when } n \geq 2.$$

Second, $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ is chosen so that $f(\omega) = \omega$ when $n = 1$ and, when $n \geq 2$, f is a measurable map with the property that $f_*\lambda_{\mathbb{S}^{n-1}} = \omega_{n-1}\mu$, where $f_*\lambda_{\mathbb{S}^{n-1}}$ denotes the pushforward under f of the standard surface measure $\lambda_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$. (The existence of such an f is assured by Theorem 3.2.5 in [3].) Finally, one takes $F(x, r\omega) = \rho\big(x, f(\omega), r\big) f(\omega)$.

To see that this F does the job, begin by observing that, by construction,

$$\int_{\mathbb{R}^n\setminus\{0\}} \varphi\circ F(x,y)\,\frac{dy}{|y|^{n+1}} = \int_{\mathbb{S}^{n-1}} \left(\int_{(0,\infty)} \varphi\circ F(x,r\omega)\frac{dr}{r^2}\right)\lambda_{\mathbb{S}^{n-1}}(d\omega)$$
$$= \int_{\mathbb{S}^{n-1}} \left(\int_{(0,\infty)} \varphi\big(rf(\omega)\big)\beta\big(x,f(\omega),r\big)\,dr\right)\lambda_{\mathbb{S}^{n-1}}(d\omega)$$
$$= \int_{\mathbb{R}^n\setminus\{0\}} \varphi(y)\,M(x,dy)$$

for any $\varphi \in C\big(\mathbb{R}^n\setminus\{0\};[0,\infty)\big)$. That is,

$$\text{(1.6)}\qquad \int_{\mathbb{R}^n\setminus\{0\}} \varphi\circ F(x,y)\,M_0(dy) = \int_{\mathbb{R}^n\setminus\{0\}} \varphi(y)\,M(x,dy)$$

for $\varphi \in C\big(\mathbb{R}^n\setminus\{0\};[0,\infty)\big)$. Thus, if $\psi \in C^\infty\big(B_{\mathbb{R}^n}(0,1);\mathbb{R}^n\big)$ satisfies $|\psi(y)-y| \le C|y|^2$ for some $C<\infty$ and we adopt

$$\text{(1.7)}\quad K_M\varphi(x) = \int_{\mathbb{R}^n\setminus\{0\}} \Big(\varphi(x+y)-\varphi(x)-\big(\psi(y),\operatorname{grad}_x\varphi\big)_{\mathbb{R}^n}\Big)\,M(x,dy)$$

as the operator associated with $x \rightsquigarrow M(x,\,\cdot\,)$, then $K_M\varphi(x)$ is equal to

$$\sum_{i=1}^n c_i(x)\partial_i\varphi(x) + \int_{\mathbb{R}^n\setminus\{0\}} \Big(\varphi\big(x+F(x,y)\big)-\varphi(x) - \mathbf{1}_{[0,1]}(|y|)\big(F(x,y),\operatorname{grad}_x\varphi\big)_{\mathbb{R}^n}\Big)\,M_0(dy)$$

where $c:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$\text{(1.8)}\qquad c(x) = \int_{\mathbb{R}^n\setminus\{0\}} \Big(\mathbf{1}_{[0,1]}(|y|)F(x,y) - \psi\big(F(x,y)\big)\Big)\,M_0(dy).$$

Next, we need to check that F satisfies (0.2) and c is uniformly Lipschitz. To this end, first observe that, by (1.6),

$$\int_{0<|y|\le r} |F(x,y)|^2\frac{dy}{|y|^{n+1}} = \int_{\{y:0<|y|\le R(r)\}} |y|^2\,M(x,dy)$$
$$\le \sup_\omega \int_{\{0<|y|\le R(r)\}} r^2\beta(x,\omega,r)\,dr,$$

where $R(r) \equiv \sup_{(x,\omega)}\rho(x,\omega,r)$. Since, by the second part of (1.4), $R(r)<\infty$ for all $r\in(0,\infty)$, we know that the second line of (0.2) holds.

At the same time, from second part of (1.3), we know that $R(r) \searrow 0$ as $r \searrow 0$, and so the first line of (0.2) also holds.

Turning to the last line of (0.2), observe that

$$\partial_x F(\,\cdot\,, r\omega) = \frac{\int_{[\rho(x,f(\omega),r),\infty)} \partial_x \beta(\,\cdot\,, f(\omega), s)\, ds}{\beta(x, f(\omega), \rho(x, f(\omega), r)} f(\omega),$$

and therefore, by (1.6), that

$$\int_{0<|y|\le R} \left|F(x_2, y) - F(x_1, y)\right|^2 \frac{dy}{|y|^{n+1}}$$
$$\le |x_2 - x_1|^2 \sup_{(x,\omega)} \int_{(0,R(r)]} \frac{\left(\int_{[r,\infty)} |\mathrm{grad}_x \beta(\,\cdot\,, \omega, s)|\, ds\right)^2}{\beta(x, \omega, r)}\, dr.$$

Hence the third line of (0.2) follows from (1.5).

Finally, we must check that the c in (1.8) is uniformly Lipschitz continuous. But, since

$$c(x) = \int_{0<|y|\le 1} \Big(\psi\big(F(x,y)\big) - F(x,y)\Big)\, M_0(dy) + \int_{|y|\ge 1} \psi\big(F(x,y)\big)\, M_0(dy),$$

we can use the first part of (1.3) and the same line of reasoning as above to see that there is an $r \in (0, \infty)$ and a $C < \infty$ for which

$$|\partial_x c| \le C \sup_\omega \int_{(0,r]} \rho(x, \omega, s) \big|\partial_x \rho(\,\cdot\,, \omega, s)\big|\, \frac{ds}{s^2}$$
$$\le C \sup_\omega \int_{(0,R]} s \left(\int_{[s,\infty)} |\partial_x \beta(\,\cdot\,, \omega, \sigma)|\, d\sigma \right) ds$$
$$\le C \sup_\omega \sqrt{\int_{(0,R]} s^2 \beta(x, \omega, s)\, ds} \sqrt{\int_{(0,R]} \frac{\left(\int_{[s,\infty)} |\partial_x \beta(\,\cdot\,, \omega, \sigma)|\, d\sigma\right)^2}{\beta(x, \omega, s)}\, ds}.$$

Hence, by the second part of (1.3) and (1.5), it is clear that c is uniformly Lipschitz.

By the results in §3.1 of [3], we can now say that when M is given by (1.1) with a β satisfying (1.2)–(1.5), then Itô's construction leads to a Markov process which corresponds to the operator K_M in (1.7) in the sense that, starting at each $x \in \mathbb{R}^n$, the process solves the martingale problem (cf. §3 below) for K_M on $C^2_c(\mathbb{R}^n; \mathbb{R})$.

An Example: In order to demonstrate that Itô's theory can handle situations which defy more analytic methodology, consider the case when

$$\beta(x,\omega,r) = \alpha(x,\omega)r^{-1-\lambda(x,\omega)},$$

where $\alpha : \mathbb{R}^n \times \mathbb{S}^{n-1} \longrightarrow [\alpha_1, \alpha_2]$ and $\lambda : \mathbb{R}^n \times \mathbb{S}^{n-1} \longrightarrow [\lambda_1, \lambda_2]$ are measurable functions, $0 < \alpha_1 \le \alpha_2 < \infty$, and $0 < \lambda_1 \le \lambda_2 < 2$. Assuming that $\alpha(\,\cdot\,,\omega)$ and $\lambda(\,\cdot\,,\omega)$ are continuously differentiable for each ω and that $(x,\omega) \rightsquigarrow \mathrm{grad}_x\alpha(\,\cdot\,,\omega)$ and $(x,\omega) \rightsquigarrow \mathrm{grad}_x\lambda(\,\cdot\,,\omega)$ are bounded, one can easily verify that β's of this sort satisfy (1.2)–(1.5). The reason why traditional analytic approaches would have difficulties with an operator K_M of the form in (1.7) when M is given by (1.1) with these β's is that, unless λ is independent of x, K_M will have no principal part. For this reason, perturbative techniques, like those on which standard pseudodifferential arguments (cf. [2]) depend, do not apply.

Remark: It is reasonable to ask whether there is any advantage to be gained by considering reference Lévy measures other than $M_0(dy) = \mathbf{1}_{\mathbb{R}^n\setminus\{0\}}\frac{dy}{|y|^{n+1}}$. However, at least so far as the considerations in this and the next sections[1)] are concerned, the answer seems to be *no*. Indeed, without any change in the proof, one can show that Itô's procedure works when M_0 in (0.2) is replaced by any Lévy measure M and the conditions there are replaced by

$$\lim_{N\to\infty}\sup_{x\in\mathbb{R}^n}\frac{1}{1+|x|^2}\int_{\Delta_N}|F(x,y)|^2\,M(dy) = 0$$

$$\sup_{x\in\mathbb{R}^n}\frac{1}{1+|x|^2}\int_{\Gamma_\epsilon}|F(x,y)|^2\,M(dy) < \infty \tag{1.9}$$

$$\sup_{x_2\neq x_1}\frac{1}{|x_2-x_1|^2}\int_{\Gamma_\epsilon}\left|F(x_2,y)-F(x_1,y)\right|^2 M(dy) < \infty,$$

where, for each $N \ge 1$, $0 \in \Delta_N \in \mathcal{B}_{\mathbb{R}^n}$ satisfies $M(\mathbb{R}^n \setminus \Delta_N) < \infty$, and, for each $\epsilon > 0$, $0 \in \Gamma_\epsilon \in \mathcal{B}_{\mathbb{R}^n}$ satisfies $M(\mathbb{R}^n \setminus \Gamma_\epsilon) < \epsilon$. However, because one can always find a measurable $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n$ such that $M = f_*M_0$, one can easily check that if M and F satisfies $F(x,\,\cdot\,)_*M = M(x,\,\cdot\,)$ and the conditions in (1.9), then $(x,y) \rightsquigarrow \tilde{F}(x,y) \equiv F\big(x,f(y)\big)$ will satisfy $\tilde{F}(x,\,\cdot\,)_*M_0 = M(x,\,\cdot\,)$ and (1.9) with $\tilde{F}$ and M_0 replacing F and M and $f^{-1}(\Delta_N)$ and $f^{-1}(\Gamma_\epsilon)$ replacing Δ_N and Γ_ϵ.

1) See the concluding Remark in §3 for a consideration in which there is an advantage to allowing more general reference Lévy measures.

§2 Some Extensions

It is important to note that there are situations in which it is impossible to construct an F which satisfies (1.9) for any choice of Lévy measure M, even though $x \rightsquigarrow M(x,\cdot)$ is as smooth as one could hope. For example, consider the seemingly trivial case in which $n = 1$ and $M(x,dy) = \alpha(x)\delta_1(dy)$, where $\alpha : \mathbb{R} \longrightarrow [1,2]$ is smooth and δ_1 is the unit point mass at 1. Clearly, if M is a Lévy measure and $F : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \longrightarrow \mathbb{R}$ satisfies $F(x,\cdot)_*M = M(x,\cdot)$, then, for each x, $F(x,\cdot) \in \{0,1\}$ M-almost everywhere and $M\big(\Gamma(x)\big) = \alpha(x)$ when $\Gamma(x) \equiv \{y : F(x,y) = 1\}$. Thus, for each $\epsilon > 0$,

$$\begin{aligned}\int_{\Gamma_\epsilon} \Big(F(x_1,y) - F(x_0,y)\Big)^2 M(dy) &\geq \alpha_\epsilon(x_1) + \alpha_\epsilon(x_0) - 2\alpha_\epsilon(x_1) \wedge \alpha_\epsilon(x_0) \\ &= \big|\alpha_\epsilon(x_1) - \alpha_\epsilon(x_0)\big|,\end{aligned}$$

where $\alpha_\epsilon(x) \equiv M\big(\Gamma(x) \cap \Gamma_\epsilon\big) \nearrow \alpha(x)$ uniformly as $\epsilon \searrow 0$. In particular, the only way that the third line of (1.9) could hold is that α_ϵ be constant for each $\epsilon > 0$, which means that α itself would have to be constant. Of course, one can object that this example is a little ridiculous since it is easy to carry out Itô's construction whenever $x \rightsquigarrow M(x,\mathbb{R}^n)$ is bounded, even if the third line of (1.9) fails. On the other hand, one can overcome this objection by considering $M(x,\cdot) = \sum_{m=0}^\infty \alpha_m(x)\delta_{3^{-m}}$ where each $\alpha_m \in C^\infty_{\mathrm{b}}\big(\mathbb{R};(0,\infty)\big)$ satisfies $\|\alpha_m\|_{C^1_{\mathrm{b}}(\mathbb{R};\mathbb{R})} \leq C8^m$. Proceeding as before, we know that $M\big(\Gamma_m(x)\big) = \alpha_m(x)$ and, M-almost everywhere, $F(x,\cdot) = \sum_{m=0}^\infty 3^{-m}\mathbf{1}_{\Gamma_m(x)}$, where $\Gamma_m(x) \equiv \{y : F(x,y) = 3^{-m}\}$. Hence, if $M_\epsilon(dy) = \mathbf{1}_{\Gamma_\epsilon}(y)M(dy)$, then

$$\begin{aligned}&\int_{\Gamma_\epsilon} \big|F(x_1,y) - F(x_0,y)\big|^2 M(dy) \\ &= \sum_{m=0}^\infty 9^{-m}\Big(M_\epsilon\big(\Gamma_m(x_1)\big) + M_\epsilon\big(\Gamma_m(x_1)\big) - 2M_\epsilon\big(\Gamma_m(x_0)\cap\Gamma_m(x_1)\big)\Big) \\ &\qquad - 2\sum_{m=0}^\infty 3^{-m}\sum_{n>m} 3^{-n}\Big(M_\epsilon\big(\Gamma_m(x_0)\cap\Gamma_n(x_1)\big) + M_\epsilon\big(\Gamma_n(x_0)\cap\Gamma_m(x_1)\big)\Big) \\ &\geq \sum_{m=0}^\infty 9^{-m}\Big(M_\epsilon\big(\Gamma_m(x_1)\big) + M_\epsilon\big(\Gamma_m(x_1)\big) - 2M_\epsilon\big(\Gamma_m(x_0)\cap\Gamma_m(x_1)\big)\Big) \\ &\qquad - \frac{2}{3}\sum_{m=0}^\infty 9^{-m}\Big(M_\epsilon\big(\Gamma_m(x_0)\cap\Gamma_m(x_1)\complement\big) + M_\epsilon\big(\Gamma_m(x_1)\cap\Gamma_m(x_0)\complement\big)\Big)\end{aligned}$$

$$= \frac{1}{3}\sum_{m=0}^{\infty} 9^{-m}\Big(M_\epsilon\big(\Gamma_m(x_0)\cup\Gamma_m(x_1)\big) - M_\epsilon\big(\Gamma_m(x_0)\cap\Gamma_m(x_1)\big)\Big)$$

$$\geq \frac{1}{3}\sum_{m=0}^{\infty} 9^{-m}\big|M_\epsilon\big(\Gamma_m(x_1)\big) - M_\epsilon\big(\Gamma_m(x_1)\big)\big|.$$

Hence, by the same argument as the one just used, the third line of (1.9) can hold only if each of the α_m's is constant.

In view of the preceding example, it is interesting to note that the problems encountered there disappear if the measure has a sufficiently strong absolutely continuous part. To be more precise, let $\beta : \mathbb{R}^n \times \mathbb{S}^{n-1} \longrightarrow (0,\infty)$ be a function which satisfies the conditions in (1.2)–(1.4), and suppose that $(x,\omega) \rightsquigarrow \mu(x,\omega,\,\cdot\,)$ is a measurable map from $\mathbb{R}^n \times \mathbb{S}^{n-1}$ into measures on $(0,\infty)$ such that

$$(2.1)\qquad \begin{gathered} \mu(x,\omega,dr) = \beta(x,\omega,r)\,dr + \nu(x,\omega,dr) \quad \text{where} \\ \sup_{(x,\omega)} \int_{(0,\infty)} \frac{r^2}{1+r^2}\,\nu(x,\omega,dr) < \infty \\ \lim_{R\searrow 0}\sup_{(x,\omega)} \int_{(0,R]} r^2\,\nu(x,\omega,dr) = 0 = \lim_{R\to\infty}\sup_{(x,\omega)} \nu\big(x,\omega,[R,\infty)\big). \end{gathered}$$

Further, assume that $x \rightsquigarrow \beta(x,\omega,r)$ and $x \rightsquigarrow \nu\big(x,\omega,[r,\infty)\big)$ are continuously differentiable for each $(x,r) \in \mathbb{R}^n \times (0,\infty)$. Finally, choose $\eta \in C_{\mathrm{c}}^\infty\big((0,\infty);[0,\infty)\big)$ with total integral 1, set

$$\beta_\epsilon(x,\omega,r) = \beta(x,\omega,r) + \int_{(0,\infty)} \eta_\epsilon(s-r)\,\nu(x,\omega,ds) \quad \text{for } \epsilon \in (0,1],$$

where $\eta_\epsilon(s) \equiv \epsilon^{-1}\eta\left(\frac{s}{\epsilon}\right)$, and assume that, for each $R \in (0,\infty)$,

$$(2.2)\qquad \sup_{\substack{(x,\omega)\\ \epsilon\in(0,1]}} \int_{(0,R]} \frac{\Big(\int_{[r,\infty)} \big|\mathrm{grad}_x\beta_\epsilon(\,\cdot\,,\omega,s)\big|\,ds\Big)^2}{\beta(x,\omega,r)}\,dr < \infty.$$

Next, given a probability measure μ on $\mathbb{S}^{n-1}$, define $\rho_\epsilon : \mathbb{R}^n \times \mathbb{S}^{n-1} \longrightarrow (0,\infty)$ and $F_\epsilon : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ relative to β_ϵ by the prescription used in §1 (cf. the discussion preceding (1.6)). By the arguments used in §1, we know that, when F and β are replaced throughout by F_ϵ and β_ϵ, then everything in (0.2) as well as the Lipschitz continuity of the associated c_ϵ in (1.8) can be controlled in terms of quantity in (1.5). But clearly the quantity in (1.5) is dominated uniformly for $\epsilon \in (0,1]$ by the quantity in (2.2). At the same time, if

$$(2.3) \qquad M(x,\Gamma) = \int_{\mathbb{S}^{n-1}} \left(\int_{(0,\infty)} \mathbf{1}_\Gamma(r\omega)\, \mu(x,\omega,dr) \right) \mu(d\omega),$$

then $F_\epsilon \longrightarrow F$ where $F(x,\,\cdot\,)_* M = M(x,\,\cdot\,)$. Hence, when $x \rightsquigarrow M(x,\,\cdot\,)$ is given by (2.3) for any $\mu \in \mathbf{M}_1(\mathbb{S}^{n-1})$ and a $(x,\omega) \in \mathbb{R}^n \times \mathbb{S}^{n-1} \longmapsto \mu(x,\omega,\,\cdot\,) \in \mathbf{M}_1\big((0,\infty)\big)$ which satisfies (2.1) and (2.2), then a choice of F satisfying (0.2) is available.

§3 Uniqueness for the Martingale Problem

Suppose that (cf. (1.7))

$$(3.1) \qquad L\varphi(x) = \frac{1}{2} \sum_{i,j=1}^{n} a(x)_{ij} \partial_i \partial_j \varphi(x) + \sum_{i=1}^{n} b(x)_i + K_M \varphi(x)$$

for $\varphi \in C^2_c(\mathbb{R}^n;\mathbb{R})$, where $x \rightsquigarrow a(x)$ and $x \rightsquigarrow b(x)$ are continuous mappings into, respectively, non-negative definite, symmetric $n \times n$-matrices and $\mathbb{R}^n$ and $x \rightsquigarrow M(x,\,\cdot\,)$ takes its values in Lévy measures and satisfies

$$\sup_{|x|\le R} \int_{\mathbb{R}^n \setminus \{0\}} \frac{|y|^2}{1+|y|^2}\, M(x,dy) < \infty \quad \text{for all } R \in (0,\infty).$$

Let $D\big([0,\infty);\mathbb{R}^n\big)$ be the space of right continuous paths $p : [0,\infty) \longrightarrow \mathbb{R}^n$ which possess a left limit $p(t-)$ at each $t \in (0,\infty)$, and use $\mathcal{B}_t$ to denote the σ-algebra over $D\big([0,\infty);\mathbb{R}^n\big)$ generated by $p \rightsquigarrow p(\tau)$ for $\tau \in [0,t]$. We will say that $\mathbb{P} \in \mathbf{M}_1\big(D\big([0,\infty);\mathbb{R}^n\big)\big)$ *solves the martingale problem for* L if

$$\left(\varphi\big(p(t)\big) - \int_0^t L\varphi\big(p(\tau)\big)\, d\tau, \mathcal{B}_t, \mathbb{P} \right) \quad \text{is a martingale}$$

for all $\varphi \in C^2_c(\mathbb{R}^n;\mathbb{R})$. If, in addition, $\mathbb{P}\big(p(0) = x\big) = 1$, then we will say that $\mathbb{P}$ *solves the martingale problem for* L *starting from* x.

In §3.1.5 of [3], it is shown that when L admits a representation of the form in (0.1) with uniformly Lipschitz continuous σ and b and an $x \rightsquigarrow F(x,\,\cdot\,)$ satisfying (0.2), Itô's construction leads to a solution to the martingale problem for L starting from x. On the other hand, there are lots of other ways in which one might go about constructing solutions to this martingale problem. (In fact, even if one restricts ones attention to Itô's method, there is are lots of choices of σ and F, and each one gives rise to a different construction.) Thus, it is of some importance to determine conditions which guarantee that there is only one solution to the martingale problem for a given L starting from a given x.

Under the condition that $M = 0$, the problem of determining when uniqueness holds for the martingale problem was studied systematically in Chapter 6 of [4]. The methods used there are of two types. Methods of the first type work by duality and yield (cf. Theorem 6.3.2 in *op cit*) uniqueness for solutions to the martingale problem as a consequence of existence of solutions to the evolution equation

$$\partial_t u = Lu \quad \text{with } u(0,\,\cdot\,) = \varphi \tag{3.2}$$

for sufficiently many φ's. This duality method is quite powerful and leads to the most refined results obtained in [4]. For example, when $M = 0$ and a and b have two bounded, continuous derivatives, it is shown in §3.2 of [4] that (3.2) admits classical solutions for $\varphi \in C^2_c(\mathbb{R}^n;\mathbb{R})$, and this is more than enough to check uniqueness for the associated martingale problem. In §4.2 of [3], this sort of reasoning is extended to situations with $M \neq 0$, when the quantities σ and b entering (0.1) have two bounded, continuous derivatives and $x \rightsquigarrow F(x,\,\cdot\,)$ has continuous second derivatives which satisfies appropriate (cf. $\mathbf{(H2)}^2$ in *op cit*) mean square bounds. For example, these conditions are often met by F's of the sort constructed in §1. Unfortunately, they seem unlikely to hold for situations requiring the extension introduced in §2.

The second method introduced in [4] is more directly dependent on Itô's theory. Namely, when $M = 0$, it is shown there (cf. Theorem 5.3.2 in *op cit*) that any solution to the martingale problem can be realized as the solution of an Itô stochastic integral equation. Thus, when $M = 0$ and σ and b are Lipschitz continuous, uniqueness for the martingale problem comes quite easily as a consequence of Itô's theory. (This is the result which was refined in [6].) In this concluding section, we will examine possible extensions of this line of reasoning to the case when $M \neq 0$.

Suppose that $\mathbb{P}$ solves the martingale problem for L starting from x. Using the techniques developed in §1 of [5], one can make an Itô decompostion of the paths p into their "continuous" and "discontinuous parts" parts. More precisely, given $p \in D([0,\infty);\mathbb{R}^n)$, a $\Gamma \in \mathcal{B}_{[0,\infty)} \times \mathcal{B}_{\mathbb{R}^n}$ with $\big([0,\infty)\times\{0\}\big) \cap \overline{\Gamma} = \emptyset$, define $\nu(\Gamma;p)$ to be the number of $\tau \in (0,\infty)$ such that $\big(\tau, p(\tau) - p(\tau-)\big) \in \Gamma$. One can then show that there exists a measurable map $p \in D\big([0,\infty);\mathbb{R}^n\big) \longmapsto p_1 \in D\big([0,\infty);\mathbb{R}^n\big)$ such that (cf. (1.7))

$$(3.4)\quad p_1(t) = \lim_{r\searrow 0} \iint_{[0,t]\times B_{\mathbb{R}^n}(0,r)\complement} \Big(y\nu\big(d\tau\times dy;p\big) - \psi(y)d\tau \times M\big(p(\tau),dy\big)\Big),$$

uniformly for t's in compacts, in $\mathbb{P}$-probability. Moreover, if $p_0 = p - p_1$, then

(**a**) $p_0 \in C\big([0,\infty);\mathbb{R}^n\big)$ $\mathbb{P}$-almost surely,

(**b**) for each $\varphi \in C^2_c(\mathbb{R}^n;\mathbb{R})$,

$$\left(\varphi\big(p_0(t)\big) - \int_0^t L_0\varphi\big(p(\tau)\big)\,d\tau, \mathcal{B}_t, \mathbb{P}\right)$$

is a martingale, where

$$L_0\varphi(x) = \frac{1}{2}\sum_{i,j=1}^n a(x)_{ij}\partial_i\partial_j\varphi(x) + \sum_{i=1}^n b(x)_i\partial_i\varphi(x),$$

(**c**) for each $\varphi \in C^2_c(\mathbb{R}^n;\mathbb{R})$,

$$\left(\varphi\big(p_1(t)\big) - \int_0^t K_M\varphi\big(p(\tau)\big)\,d\tau, \mathcal{B}_t, \mathbb{P}\right)$$

is a martingale.

In spite of the obvious ambiguity in this decomposition, we will call p_0 and p_1 the *continuous part* and the *discontinuous part* of p.

Given a measurable $x \rightsquigarrow \sigma(x)$ satisfying $a(x) = \sigma(x)\sigma(x)^\top$, one can start from (**b**) above and, by mimicking the procedure in Theorem 5.3.2 of [4], produce a Brownian motion β such that

$$(3.5)\qquad p_0(t) = x + \int_0^t \sigma\big(p(\tau)\big)\,d\beta(\tau) + \int_0^t b\big(p(\tau)\big)\,d\tau, \quad t \in [0,\infty).$$

There are technical difficulties which arise when σ becomes degenerate, and these necessitate the introduction of a larger probability space, one which is big enough to support a full blown Brownian motion. However, as is explained in the theorem just cited, the resolution of such difficulties is well understood. On the other hand, it is not so clear how to treat the analogous difficulties for the discontinuous part p_1. Specifically, given a measurable $F : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and a Lévy measure[2)] M for

[2)] For reasons which will be explained below in the concluding Remark, it is best to allow general reference Lévy measures here rather than always taking M_0.

which $F(x,\,\cdot\,)_*M = M(x,\,\cdot\,)$, it is not clear in general how one can use F to produce a Lévy process from which p_1 can be re-constructed via Itô's procedure. Nonetheless, when F is *non-degenerate* in the sense that, for each $x \in \mathbb{R}^n$, $F(x,\,\cdot\,)$ is one-to-one from $\mathbb{R}^n \setminus \{0\}$ onto itself, then one can construct such a Lévy process. Namely, take $F^{-1}(x,\,\cdot\,)$ to be the inverse[3] of $F(x,\,\cdot\,)$, and set

$$q_r(t,p) = \iint_{[0,t]\times B_{\mathbb{R}^n}(0,r)\complement} \Phi^{-1}(\tau,y;p)\Big(\nu(d\tau\times dy;p) - d\tau \times M\big(p(\tau),dy\big)\Big),$$

where $\Phi^{-1}(\tau,y;p) \equiv F^{-1}\big(p(\tau-),y\big)$. Then one can show that there exists a $\{\mathcal{B}_t : t \geq 0\}$-progressively measurable $p \in D\big([0,\infty);\mathbb{R}^n\big) \longrightarrow q(\,\cdot\,,p) \in D\big([0,\infty);\mathbb{R}^n\big)$ such that, as $r \searrow 0$, $q_r(\,\cdot\,,p) \longrightarrow q(\,\cdot\,,p)$ uniformly on compacts in $\mathbb{P}$-probability. Moreover, the $\mathbb{P}$-distribution of $p \rightsquigarrow q(\,\cdot\,,p)$ is that of the Lévy process corresponding to M in the sense that, for each $\xi \in \mathbb{R}^n$,

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big[e^{\sqrt{-1}(\xi,q(1,p))_{\mathbb{R}^n}}\Big]\\ &= \exp\left[\int_{\mathbb{R}^n\setminus\{0\}}\Big(e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^n}} - 1 - \mathbf{1}_{[0,1]}(|y|)(\xi,y)_{\mathbb{R}^n}\Big)M(dy)\right].\end{aligned}$$

In addition, it should be clear that, $\mathbb{P}$-almost surely, $\nu\big(\,\cdot\,;q(\,\cdot\,,p)\big) = \Phi^{-1}(\,\cdot\,;p)_*\nu(\,\cdot\,;p)$. In particular, if $\Phi(\tau,y;q) \equiv F\big(q(\tau-),y\big)$, then

$$\nu(\,\cdot\,;p) = \Phi\big(\,\cdot\,;q(\,\cdot\,,p)\big)_*\nu\big(\,\cdot\,;q(p)\big) \quad \mathbb{P}\text{-almost surely,}$$

and so (cf. (3.4)) $p_1(t)$ is equal to

$$\begin{aligned}\lim_{r\searrow 0}\iint_{[0,t]\times B_{\mathbb{R}^n}(0,r)} &\Big(F\big(p(\tau-),y\big)\nu\big(d\tau\times dy;q(\,\cdot\,;p)\big)\\ &\qquad - \psi(y)\big(F(p(\tau),y)\big)_{\mathbb{R}^n}\,d\tau\times M(dy)\Big).\end{aligned}$$

Hence, after putting this together with (3.5), the path p can be recovered via Itô's procedure from a Lévy process for which

$$\xi \rightsquigarrow \exp\left[-\frac{|\xi|^2}{2} + \int_{\mathbb{R}^n\setminus\{0\}}\Big(e^{\sqrt{-1}(\xi,y)_{\mathbb{R}^n}} - 1 - \mathbf{1}_{[0,1]}(|y|)(\xi,y)_{\mathbb{R}^n}\Big)\,M(dy)\right]$$

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[3] By a famous theorem due to C. Kuratowski, this inverse will be Borel measurable with respect to (x,y).

is the characteristic function of the distribution at time 1.

These considerations yield the following uniqueness theorem.

Theorem. *Suppose that $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $b : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are uniformly Lipschitz continuous functions, M is a Lévy measure, and the $F : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is a measurable function which satisfies the conditions in (1.9). Further, assume that, for each $x \in \mathbb{R}^n$, $F(x, \cdot)$ is one-to-one from $\mathbb{R}^n \setminus \{0\}$ onto itself. Then, for each starting point, Itô's construction yields the one and only solution to the martingale problem for operator L described in (3.1) when $a(x) = \sigma(x)\sigma(x)^\top$ and $M(x, \cdot) = F(x, \cdot)_* M$ for all $x \in \mathbb{R}^n$.*

Remark: As distinguished from our earlier results, there is an advantage to allowing reference Lévy measures other than M_0 when applying the preceding theorem. For instance, suppose that $(x, \omega, r) \rightsquigarrow \mu(x, \omega, r)$ satisfies the conditions in (2.1) and (2.2), and let $x \rightsquigarrow M(x, \cdot)$ be given by (2.3) for some $\mu \in \mathbf{M}_1(\mathbb{S}^{n-1})$. Then the function F which was constructed so that $M(x, \cdot) = F(x, \cdot)_* M_0$ need not be one-to-one and onto because the map $f : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^{n-1}$ may fail to be. On the other hand, if we take $M(dy) = \frac{1}{|y|^{n+1}}\mu(dy)$, then the construction given in §1 does not require the use of f and leads to an F which is one-to-one and onto and satisfies $M(x, \cdot) = F(x, \cdot)_* M$ as well as the conditions in (1.9).

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