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The Canonical Contact Form

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Abstract.

The structure group and the involutive differential system that characterize the pseudo-group of contact transformations on a jet space are determined.

§1. Introduction

The canonical form on the coframe bundle over a smooth manifold originally arose as the natural generalization of the canonical form on the cotangent bundle, which plays an essential role in Hamiltonian mechanics, [19, §III.7]. The coframe bundle $\mathcal{F}^*M \to M$ forms a principal $\mathrm{GL}(m)$ bundle over the m-dimensional manifold M. The canonical form on the coframe bundle serves to characterize the diffeomorphism pseudo-group of the manifold, or, more correctly, its lift to the coframe bundle. Indeed, the invariance of the canonical form forms an involutive differential system, whose general solution, guaranteed by the Cartan-Kähler Theorem, is the lifted diffeomorphism pseudo-group. Kobayashi, [11], introduces a vector-valued canonical form on the higher order frame bundles over the manifold. He demonstrates that the components of the canonical form constitute an involutive differential system that characterizes the higher order lifts of the diffeomorphism group.

The geometrical study of differential equations relies on the jet space first introduced by Ehresmann, [6]. In the jet bundle framework, the pseudo-group of contact transformations, [13], [16], assumes the role of the diffeomorphism pseudo-group. Contact transformations are characterized by the fact that they preserve the contact ideal generated by the contact forms on the jet bundle. Thus, the characterization of the contact pseudo-group by an involutive differential system should rely on

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a "canonical contact form" constructed on a suitable principal bundle lying over the jet bundle. This canonical contact form should play the same basic role in the study of the geometry of jet bundles and differential equations that the canonical form over the coframe bundle plays in the ordinary differential geometry of manifolds and submanifolds. In [21], Yamaguchi uses the theory of exterior differential systems to conduct a detailed investigation of the contact geometry of higher order jet space, but does not provide a general construction of the required principal bundle or canonical form. This is more complicated than the frame bundle construction, since the definition of a contact transformation via the contact ideal does not directly yield an involutive differential system; see [4], [16]. One must apply the Cartan procedure of absorption and normalization of torsion in order to reduce the original structure group to the appropriate involutive version, and this in turn will yield the "minimal, involutive" version of the canonical contact form.

A crucial theorem, due to Bäcklund, [2], demonstrates that every contact transformation is either a prolonged point transformation, or, in the case of a single dependent variable, a prolonged first order contact transformation; see also [16], [20]. This allows us to restrict the structure group associated with the contact pseudo-group to one of block upper triangular form, but this still is not enough to produce an involutive differential system, and further normalizations must be imposed. In this paper, we find the complete system of normalizations, thereby constructing an involutive differential system on a certain principal bundle over the jet bundle that characterizes the contact pseudo-group.

A significant source of applications of this construction can be found in a variety of equivalence problems defined on the jet bundle, including differential equations, variational problems, and others. In such situations, one needs to incorporate the contact structure into the problem via the contact forms. The canonical contact form will provide the minimal lift that can be imposed on the contact component of the lifted coframe, and thus help avoid normalizations that are universally valid for all contact transformations. Examples include equivalence problems for differential equations, for differential operators, and for variational problems. See [10], [16] for typical problems and applications. Additional applications to the method of moving frames developed by Mark Fels and the author, [8], [9], [17], will appear elsewhere.

§2. Contact Forms on Jet Bundles

We will work with the smooth category of manifolds and maps throughout this paper. Let $E \to X$ be a smooth vector bundle over a p-dimensional base manifold X, with q-dimensional fibers. We use $x = (x^1, \ldots, x^p)$ to denote local coordinates on X, and $u = (u^1, \ldots, u^q)$ to denote the fiber coordinates, so that sections of E are prescribed by smooth functions $u = f(x)$. Let $\mathrm{J}^n = \mathrm{J}^n E$ denote the n^{th} jet bundle of E, with associated local coordinates $z^{(n)} = (x, u^{(n)}) = (\ldots x^i \ldots u^\alpha_J \ldots)$, where the derivative coordinates u^α_J are indexed by unordered multi-indices $J = (j_1, \ldots, j_k)$, with $1 \leq j_\kappa \leq p$, of orders $0 \leq k = \#J \leq n$. Given a (local) section $f\colon X \to E$, we let $\mathrm{j}_n f\colon X \to \mathrm{J}^n$ denote its n-jet, which forms a section of the n^{th} order jet bundle.

Definition 2.1. A differential form θ on the jet space J^n is called a *contact form* if it is annihilated by all jets: $(\mathrm{j}_n f)^* \theta = 0$.

The space of contact forms on J^n forms differential ideal $\mathcal{I}^{(n)}$, called the *contact ideal*, over J^n.

Theorem 2.2. *In local coordinates, every contact one-form on* J^n *can be written as a linear combination of the* basic contact forms

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i, \qquad \alpha = 1, \ldots, q, \quad 0 \leq \#J < n. \tag{2.1}$$

These one-forms constitute a basis for the contact ideal $\mathcal{I}^{(n)}$.

For instance, in the case of one independent and one dependent variable, the basic contact forms are

$$\begin{aligned} \theta_0 &= du - u_x\, dx, \\ \theta_1 &= du_x - u_{xx}\, dx, \\ \theta_2 &= du_{xx} - u_{xxx}\, dx, \quad \ldots \end{aligned} \tag{2.2}$$

In (2.1), we call $\#J$ the *order* of the contact form θ^α_J. The reader should note that the contact forms on J^n have orders at most $n-1$.

Lemma 2.3. *A section* $F\colon X \to \mathrm{J}^n$ *locally coincides with the n-jet of a section* $f\colon X \to E$, *meaning* $F = \mathrm{j}_n f$ *on an open subset of* X, *if and only if* F *annihilates all the contact forms on* J^n:

$$F^* \theta^\alpha_J = 0, \qquad \alpha = 1, \ldots, q, \quad 0 \leq \#J < n. \tag{2.3}$$

Definition 2.4. A local diffeomorphism $\Psi\colon \mathrm{J}^n \to \mathrm{J}^n$ defines a *contact transformation* of order n if it preserves the contact ideal, meaning that if θ is any *contact form* on J^n, then $\Psi^*\theta$ is also a contact form.

Definition 2.5. The $(n+k)^{\text{th}}$ order *prolongation* of the contact transformation $\Psi^{(n)}$ is the unique contact transformation

$\Psi^{(n+k)}\colon \mathrm{J}^{n+k} \longrightarrow \mathrm{J}^{n+k}$ satisfying $\pi^{n+k}_n \circ \Psi^{(n+k)} = \Psi^{(n)} \circ \pi^{n+k}_n$, where $\pi^{n+k}_n\colon \mathrm{J}^{n+k} \longrightarrow \mathrm{J}^n$ is the usual projection.

In local coordinates, a local diffeomorphism Ψ defines a contact transformation if and only if

$$\Psi^*\theta^\alpha_J = \sum_{\beta,K} A^{\alpha,K}_{J,\beta}\,\theta^\beta_K, \tag{2.4}$$

for suitable coefficient functions $A^{\alpha,K}_{J,\beta}\colon \mathrm{J}^n \longrightarrow \mathbb{R}$. There are nontrivial constraints on these coefficients resulting from Bäcklund's Theorem, [2].

Theorem 2.6. *If the number of dependent variables is greater than one, $q > 1$, then every contact transformation is the prolongation of a point transformation $\psi\colon E \to E$. If $q = 1$, then every n^{th} order contact transformation is the prolongation of a first order contact transformation $\psi\colon \mathrm{J}^1 \to \mathrm{J}^1$.*

Remark. Interestingly, if one restricts to a submanifold of the jet space defined by system of differential equations, additional "internal" higher order contact transformations can exist; see [1] for a Bäcklund-style classification of these transformations.

§3. The Prolonged General Linear Group

There are two fundamental transformation groups that lie at the foundation of the geometric characterization of contact transformations. The first is the standard prolongation of the general linear group, [12, p. 139], [14]. Let $\mathrm{GL}(p)$ denote the general linear group on $\mathbb{R}^p$ consisting of all real, invertible, $p \times p$ matrices. Let $\mathcal{D}_0(p)$ denote the space of all diffeomorphisms $\varphi\colon \mathbb{R}^p \longrightarrow \mathbb{R}^p$ preserving the origin, so $\varphi(0) = 0$. We let $\mathrm{j}_n\varphi(0)$ denote the n-jet (or n^{th} order Taylor expansion) of the diffeomorphism at the origin.

Definition 3.1. The n^{th} prolongation of the general linear group $\mathrm{GL}(p)$ is the group

$$\mathrm{GL}^{(n)}(p) = \{\mathrm{j}_n\varphi(0) \mid \varphi \in \mathcal{D}_0(p)\}. \tag{3.1}$$

The group multiplication is given by composition of diffeomorphisms, so that if $S = \mathrm{j}_n\varphi(0)$, $T = \mathrm{j}_n\psi(0)$, then $S \cdot T = \mathrm{j}_n(\varphi \circ \psi)(0)$.

Note that the one-jet of a diffeomorphism φ at 0 is uniquely determined by its Jacobian matrix $D\varphi(0)$, which can be viewed as an invertible matrix in $\mathrm{GL}(p)$, and, in this way, we identify $\mathrm{GL}(p) = \mathrm{GL}^{(1)}(p)$.

The most convenient method of representing the elements of $\mathrm{GL}^{(n)}(p)$ is via formal Taylor polynomials. We introduce coordinates $t = (t^1, \ldots, t^p)$ in a neighborhood of $0 \in M$. We then identify a group element $S \in \mathrm{GL}^{(n)}(p)$ with the vector-valued Taylor polynomial† $\mathbf{S}(t) = (S^1(t), \ldots, S^p(t))^T$ of any smooth diffeomorphism $\varphi(x)$ that represents it, so

$$S^i(t) = \sum_{1 \le \#J \le n} S^i_J \frac{t^J}{J!}, \quad \text{where} \quad S^i_J = \frac{\partial^{\#J} \varphi^i}{\partial x^J}(0), \quad i = 1, \ldots, p. \tag{3.2}$$

Note that there is no constant (order 0) term in the Taylor polynomial (3.2) since we are assuming that $\varphi(0) = 0$; moreover the first order Taylor coefficients $\left(S^i_j\right)$ form an invertible $p \times p$ matrix, whereas the higher order coefficients can be arbitrary. Therefore, $\mathrm{GL}^{(n)}(p)$ forms a Lie group of dimension

$$p^{(n)} = p\left[\binom{p+n}{n} - 1\right]. \tag{3.3}$$

The group multiplication is then given by formal composition of polynomials, so that $U = R \cdot S$ if and only if the corresponding polynomials satisfy

$$\mathbf{U}(t) = \mathbf{R}(\mathbf{S}(t)) \bmod n, \tag{3.4}$$

where $\bmod n$ means that we truncate the resulting polynomial to order n. The explicit formulae can be identified with the Faà di Bruno formula, [7, p. 222], [14], for the derivatives of the composition of two functions.

Example 3.2. In the one-dimensional situation, $p = 1$, the Taylor polynomial of a diffeomorphism $\varphi \to \mathbb{R} \to \mathbb{R}$ that fixes $0 = \varphi(0)$ takes the form

$$\mathbf{S}(t) = s_1 t + \frac{1}{2} s_2 t^2 + \frac{1}{3!} s_3 t^3 + \frac{1}{4!} s_4 t^4 + \cdots, \tag{3.5}$$

with the coefficients $s_1, s_2, \ldots$ representing the derivatives $s_k = \varphi^{(k)}(0)$ of our diffeomorphism $\varphi\colon \mathbb{R} \to \mathbb{R}$ fixing $0 = \varphi(0)$. The composition formula (3.4) gives the explicit rules

$$u_1 = r_1 s_1, \qquad u_2 = r_1 s_2 + r_2 s_1^2, \qquad u_3 = r_1 s_3 + 3 r_2 s_1 s_2 + r_3 s_1^3,$$
$$u_4 = r_1 s_4 + r_2(4 s_1 s_3 + 3 s_2^2) + 6 r_3 s_1^2 s_2 + r_4 s_1^4,$$

†We use a formal variable t here instead of x for later clarity.

and so on. As in [5, §3.4], the one-dimensional Faà di Bruno formula is

$$u_k = \sum_{m=1}^{k} r_m B_k^m(s_1, \ldots, s_k), \tag{3.6}$$
$$\text{where} \qquad B_k^m(s_1, \ldots, s_k) = \sum_{\Sigma I = k} \frac{s_{i_1} s_{i_2} \cdots s_{i_m}}{I!(\# I)!}$$

is a Bell polynomial, [3], [18, §2.8]. The sum in (3.6) is over all unordered multi-indices $I = (i_1, \ldots, i_m)$ with $1 \leq i_\nu \leq k$, $\sum I = i_1 + \cdots + i_m = k$, and where $J = \# I$ denotes the "repetition" multi-index of I, so that $j_r = \#\{i_\nu = r\}$ indicates the number of times that the integer r appears in the multi-index I.

We can explicitly realize $\mathrm{GL}^{(n)}(p)$ as a matrix Lie group, namely a subgroup of $\mathrm{GL}(p^{(n)})$, as follows. The space of vector-valued Taylor polynomials $\mathbf{x}(t)$ of degree at most n without constant term, $\mathbf{x}(0) = 0$, can be identified with $\mathbb{R}^{p^{(n)}}$. Given $S \in \mathrm{GL}^{(n)}(p)$, we define $\rho(S) \in \mathrm{GL}(p^{(n)})$ by

$$\rho(S)\mathbf{x}(t) = \mathbf{x}(\mathbf{S}(t)), \tag{3.7}$$

where $\mathbf{S}(t)$ is the Taylor polynomial (3.2) corresponding to S. The explicit formulae for the *Faà di Bruno injection* ρ can be found in [14, p. 503].

Example 3.3. In the one-dimensional situation described in example 3.2, we identify a fourth order Taylor polynomial (3.5) with its coefficient vector (s_1, s_2, s_3, s_4). The corresponding matrix is

$$\rho(S) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0 & s_1^2 & 3s_1 s_2 & 4s_1 s_3 + 3s_2^2 \\ 0 & 0 & s_1^3 & 6s_1^2 s_2 \\ 0 & 0 & 0 & s_1^4 \end{pmatrix}.$$

The reader may enjoy verifying that this forms a subgroup of $\mathrm{GL}(4)$. The k^{th} order version has $\rho(S)$ equal to the upper triangular matrix with entries given by the Bell polynomials $B_j^i(s_1, \ldots, s_j)$ for $i \leq j$.

We next determine the left and right-invariant Maurer-Cartan forms on the prolonged general linear group. These will be found by adapting the usual formulae

$$\mu_L = A^{-1} \cdot dA, \qquad \mu_R = dA \cdot A^{-1}, \tag{3.8}$$

valid for matrix Lie groups $G \subset \mathrm{GL}(n)$, [16]. In our case, the Maurer-Cartan forms will appear as the coefficients of a formal "Taylor" polynomial

$$\sigma(t) = \sum_{1 \leq \#J \leq n} \frac{t^J}{J!} \sigma_J, \tag{3.9}$$

where each σ_J is a p vector of one-forms defined on the group $\mathrm{GL}^{(n)}(p)$. Using (3.8) and the multiplication rule (3.4) for the group, we deduce that the right-invariant Maurer-Cartan form polynomial is given by

$$\widetilde{\sigma}(t) = d\mathbf{S}\big[\mathbf{S}^{-1}(t)\big] \bmod n, \tag{3.10}$$

obtained by composing the formal inverse series (or inverse Taylor polynomial) $\mathbf{S}^{-1}(t)$ and the formal series of basis one-forms

$$d\mathbf{S}(t) = \sum_{1 \leq \#J \leq n} \frac{t^J}{J!} dS_J, \tag{3.11}$$

on the group. On the other hand, the left-invariant Maurer-Cartan form polynomial can be found by first computing the differential of the composition

$$d\big[\mathbf{T}(\mathbf{S}(t))\big] = D\mathbf{T}(\mathbf{S}(t)) \cdot d\mathbf{S}(t), \tag{3.12}$$

of the two power series with respect to the coefficients of $\mathbf{S}$. Here $D\mathbf{T}(t) = (\partial \mathbf{T}^i / \partial t^j)$ denotes the Jacobian matrix series associated with $\mathbf{T}(t)$. Replacing $\mathbf{T}$ in (3.12) by the inverse of $\mathbf{S}(t)$ and truncating produces the left-invariant Maurer-Cartan form polynomial:

$$\sigma(t) = D\mathbf{S}^{-1}(\mathbf{S}(t)) \cdot d\mathbf{S}(t) \bmod n = D\mathbf{S}(t)^{-1} \cdot d\mathbf{S}(t) \bmod n, \tag{3.13}$$

where $D\mathbf{S}(t)^{-1}$ is the inverse of the Jacobian matrix of $\mathbf{S}(t)$.

Example 3.4. For the one-dimensional situation considered above we have

$$\begin{aligned}
\mathbf{S}(t) &= s_1 t + \frac{1}{2} s_2 t^2 + \frac{1}{3!} s_3 t^3 + \frac{1}{4!} s_4 t^4 + \cdots, \\
d\mathbf{S}(t) &= t\, ds_1 + \frac{1}{2} t^2\, ds_2 + \frac{1}{3!} t^3\, ds_3 + \frac{1}{4!} t^4\, ds_4 + \cdots, \\
\mathbf{S}^{-1}(t) &= \frac{1}{s_1} t - \frac{s_2}{2 s_1^3} t^2 - \frac{s_1 s_3 - 3 s_2^2}{6 s_1^5} t^3 - \frac{s_1^2 s_4 - 10 s_1 s_2 s_3 + 12 s_2^3}{24 s_1^7} t^4 + \cdots .
\end{aligned}$$

Therefore, the right-invariant Maurer-Cartan forms on $\mathrm{GL}^{(n)}(1)$ are obtained as the coefficients of the "Maurer-Cartan polynomials"

$$\begin{aligned}
\widetilde{\boldsymbol{\sigma}}(t) = d\mathbf{S}\left[\frac{1}{\mathbf{S}(t)}\right] &= \widetilde{\sigma}_1 t + \frac{1}{2}\widetilde{\sigma}_2 t^2 + \frac{1}{3!}\widetilde{\sigma}_3 t^3 + \frac{1}{4!}\widetilde{\sigma}_4 t^4 + \cdots \\
&= \frac{ds_1}{s_1} t + \frac{s_1\, ds_2 - s_2\, ds_1}{2s_1^3} t^2 - \frac{s_1^2\, ds_3 - 3s_1 s_2\, ds_2 - (s_1 s_3 - 3s_2^2)\, ds_1}{6s_1^5} t^3 \\
&\quad + \frac{1}{24 s_1^7}\Big\{ s_1^3\, ds_4 - 6s_1^2 s_2\, ds_3 - (4s_1^2 s_3 - 15 s_1 s_2^2)\, ds_2 \\
&\qquad - (s_1^2 s_4 - 10 s_1 s_2 s_3 + 15 s_2^3)\, ds_1 \Big\} t^4 + \cdots .
\end{aligned}$$

The left-invariant Maurer-Cartan form polynomial (3.10) for $\mathrm{GL}^{(n)}(1)$ is

$$\begin{aligned}
\boldsymbol{\sigma}(t) = \frac{d\mathbf{S}(t)}{\mathbf{S}'(t)} &= \sigma_1 t + \frac{1}{2}\sigma_2 t^2 + \frac{1}{3!}\sigma_3 t^3 + \frac{1}{4!}\sigma_4 t^4 + \cdots \\
&= \frac{ds_1}{s_1} t + \frac{s_1\, ds_2 - 2s_2\, ds_1}{2s_1^2} t^2 \\
&\quad + \frac{s_1^2\, ds_3 - 3s_1 s_2\, ds_2 - 3(s_1 s_3 - 2s_2^2)\, ds_1}{6s_1^3} t^3 \\
&\quad + \frac{1}{24 s_1^4}\Big\{ s_1^3\, ds_4 - 4s_1^2 s_2\, ds_3 - 6s_1(s_1 s_3 - 2s_2^2)\, ds_2 \\
&\qquad - 4(s_1^2 s_4 - 6 s_1 s_2 s_3 + 6 s_2^3)\, ds_1 \Big\} t^4 + \cdots .
\end{aligned}$$

Let $\rho(\boldsymbol{\sigma}) = \rho(\mathbf{S})^{-1} d\rho(\mathbf{S})$ denote the corresponding left Maurer-Cartan matrix, (3.8). In view of (3.7), (3.13), it acts on the column vector $\mathbf{x}$ according to the power series formulation

$$\begin{aligned}
[\rho(\boldsymbol{\sigma})\mathbf{x}](t) = \rho(\mathbf{S})^{-1}\, d[\rho(\mathbf{S})\mathbf{x}](t) &= \rho(\mathbf{S})^{-1}\, d[\mathbf{x}(\mathbf{S}(t))] \\
&= \rho(\mathbf{S})^{-1}\left(\sum_{i=1}^{p} \frac{\partial \mathbf{x}}{\partial t^i}[\mathbf{S}(t)]\, d\mathbf{S}(t) \right) \\
&= \sum_{i=1}^{p} \frac{\partial \mathbf{x}}{\partial t^i}(t)\, d\mathbf{S}^i[\mathbf{S}^{-1}(t)] \\
&= \sum_{i=1}^{p} \frac{\partial \mathbf{x}}{\partial t^i}(t)\, \boldsymbol{\sigma}^i(t).
\end{aligned}$$

Example 3.5. For the one-dimensional version, we have

$$\mathbf{x}(t) = x_1 t + \frac{1}{2} x_2 t^2 + \frac{1}{3!} x_3 t^3 + \frac{1}{4!} x_4 t^4 + \cdots,$$
$$\mathbf{x}'(t) = x_1 + x_2 t + \frac{1}{2!} x_3 t^2 + \frac{1}{3!} x_4 t^3 + \cdots.$$

Therefore,

$$\begin{aligned}\rho(\boldsymbol{\sigma})\mathbf{x}(t) &= \mathbf{x}'(t)\boldsymbol{\sigma}(t)\\ &= (\sigma_1 x_1)t + \frac{1}{2}(\sigma_2 x_1 + 2\sigma_1 x_2)t^2\\ &\quad + \frac{1}{3!}(\sigma_3 x_1 + 3\sigma_2 x_2 + 3\sigma_1 x_3)t^3\\ &\quad + \frac{1}{4!}(\sigma_4 x_1 + 4\sigma_3 x_2 + 6\sigma_2 x_3 + 4\sigma_1 x_4)t^4 + \cdots,\end{aligned}$$

and hence the Maurer-Cartan form matrix for $\mathrm{GL}^{(n)}(1)$ is

$$\rho(\boldsymbol{\sigma}) = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \cdots \\ 0 & 2\sigma_1 & 3\sigma_2 & 4\sigma_3 & \cdots \\ 0 & 0 & 3\sigma_1 & 6\sigma_2 & \cdots \\ 0 & 0 & 0 & 4\sigma_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.14}$$

The (i, j) entry of the full $n \times n$ matrix is

$$\rho(S)^i_j = \begin{cases} \binom{j}{i-1}\sigma_{i-j+1}, & i \leq j, \\ 0, & i > j. \end{cases} \tag{3.15}$$

§4. The Leibniz Group

Besides the prolonged general linear group that provides the structure group for jets of diffeomorphisms, we also require a structure group related to the multiplication of jets.

Definition 4.1. The *Leibniz group* $\mathrm{L}^{(n)}(p, q)$ is the Lie group consisting of all n-jets of smooth maps $\Psi\colon \mathbb{R}^p \to \mathrm{GL}(q)$ at the point 0, so

$$\mathrm{L}^{(n)}(p, q) = \{\mathrm{j}_n\Psi(0) \mid \Psi\colon \mathbb{R}^p \to \mathrm{GL}(q)\}. \tag{4.1}$$

The group law is induced by matrix multiplication $\Phi(x) \cdot \Psi(x)$ of the smooth maps.

Given a vector bundle $E \to X$ over a p-dimensional base with q-dimensional fiber, there is an induced representation

$$\tau(L^{(n)}) \cdot z^{(n)} = \mathrm{j}_n[\Psi(x) \cdot f(x)] \qquad \text{whenever} \qquad L^{(n)} = \mathrm{j}_n \Psi(x), \quad z^{(n)} = \mathrm{j}_n f(x) \tag{4.2}$$

of $\mathrm{L}^{(n)}(p, q)$ on the jet fiber $\mathrm{J}^n E$. As with the prolonged general linear group, we identify the elements of the Leibniz group with their Taylor series. Thus, the group element $L^{(n)} = \mathrm{j}_n \Psi(0)$ is identified with the n^{th} order truncation of the power series

$$\mathbf{L}(t) = \sum_{1 \le \# J \le n} \frac{t^J}{J!} L_J, \tag{4.3}$$

where each L_J is a $q \times q$ matrix. The entries $(L_J)^\alpha_\beta$ can be identified with the Taylor coefficients $\partial^k \Psi^\alpha_\beta / \partial x^J(0)$ for the corresponding matrix entry of $\Psi(x)$. Identifying a point $z^{(n)} \in \mathrm{J}^n$ with the corresponding n^{th} order Taylor polynomial $\mathbf{z}(t)$, the action of the Leibniz group is given by

$$[\tau(L^{(n)})\mathbf{z}](t) = \mathbf{L}(t) \cdot \mathbf{z}(t) \bmod n. \tag{4.4}$$

Example 4.2. In the one-dimensional version, $\mathrm{GL}(1) \simeq \mathbb{R}^*$ is just the set of nonzero reals, and so the maps $\Psi \colon \mathbb{R} \to \mathrm{GL}(1)$ are scalar-valued. The Leibniz group is induced by multiplication of Taylor series, and so the product of

$$\mathbf{L}(t) = l_0 + l_1 t + \frac{1}{2} l_2 t^2 + \frac{1}{3!} l_3 t^3 + \cdots,$$
$$\mathbf{M}(t) = m_0 + m_1 t + \frac{1}{2} m_2 t^2 + \frac{1}{3!} m_3 t^3 + \cdots,$$

is given by truncating the product series

$$\mathbf{L}(t) \cdot \mathbf{M}(t) = l_0 m_0 + (l_0 m_1 + l_1 m_0) t + \frac{1}{2}(l_0 m_2 + 2 l_1 m_1 + l_2 m_0) t^2 + \frac{1}{3!}(l_0 m_3 + 3 l_1 m_2 + 3 l_2 m_1 + l_3 m_0) t^3 + \cdots,$$

at order n. The action (4.4) on a series

$$\mathbf{z}(t) = z_0 + z_1 t + \frac{1}{2} z_2 t^2 + \frac{1}{3!} z_3 t^3 + \cdots \tag{4.5}$$

is the same — just replace the m's by z's. Therefore, the matrix representation (4.2) of an element of $\mathrm{L}^{(4)}(1,1)$ is

$$\tau(L^{(4)}) = \begin{pmatrix} l_0 & l_1 & l_2 & l_3 & l_4 \\ 0 & l_0 & 2l_1 & 3l_2 & 4l_3 \\ 0 & 0 & l_0 & 3l_1 & 6l_2 \\ 0 & 0 & 0 & l_0 & 4l_1 \\ 0 & 0 & 0 & 0 & l_0 \end{pmatrix}.$$

The matrix of Maurer-Cartan forms on the Leibniz group are found using the usual formula (3.8), which becomes

$$\boldsymbol{\lambda} = \sum_J \frac{t^J}{J!}\lambda_J, \tag{4.6}$$

where each λ_J is a $q \times q$ matrix of one-forms. We have

$$\boldsymbol{\lambda}(t) = \tau(L)^{-1}\, d\mathbf{L}(t) = \mathbf{L}(t)^{-1}\, d\mathbf{L}(t) = d \log \mathbf{L}(t). \tag{4.7}$$

Let $\tau(\boldsymbol{\lambda}) = \tau(\mathbf{L})^{-1} d\tau(\mathbf{L})$ denote the corresponding Maurer-Cartan matrix. In view of (4.4), it acts on the column vector $\mathbf{z}$ according to the power series formulation

(4.8)
$$[\tau(\boldsymbol{\lambda})\mathbf{z}](t) = \tau(\mathbf{L})^{-1} d[\tau(\mathbf{L})\mathbf{z}](t) = \tau(\mathbf{L})^{-1} d[\mathbf{L}(t)] \cdot \mathbf{z}(t) = \boldsymbol{\lambda}(t) \cdot \mathbf{z}(t).$$

Example 4.3. For the one-dimensional version, we have

$$\begin{aligned} \mathbf{L}(t) &= l_0 + l_1 t + \frac{1}{2} l_2 t^2 + \frac{1}{3!} l_3 t^3 + \cdots, \\ d\mathbf{L}(t) &= dl_0 + t\, dl_1 + \frac{1}{2} t^2\, dl_2 + \frac{1}{3!} t^3\, dl_3 + \cdots, \\ \mathbf{L}(t)^{-1} &= \frac{1}{l_0} - \frac{l_1}{l_0^2} t - \frac{l_0 l_2 - 2l_1^2}{2l_0^3} t^2 - \frac{l_0^2 l_3 - 6 l_0 l_1 l_2 + 6 l_1^3}{6 l_0^4} t^3 + \cdots. \end{aligned}$$

Therefore, the Maurer-Cartan form series for $\mathrm{L}^{(n)}(1,1)$ is

$$\begin{aligned} \boldsymbol{\lambda}(t) &= d \log \mathbf{L}(t) = \lambda_0 + \lambda_1 t + \frac{1}{2}\lambda_2 t^2 + \frac{1}{3!}\lambda_3 t^3 + \cdots, \\ &= \frac{dl_0}{l_0} + \frac{l_0\, dl_1 - l_1\, dl_0}{l_0^2} t + \frac{l_0^2\, dl_2 - 2 l_0 l_1\, dl_1 - (l_0 l_2 - 2l_1^2)\, dl_0}{2l_0^3} t^2 \\ &\quad + \frac{1}{6l_0^4}\Big\{ l_0^3\, dl_3 - 3l_0^2 l_1\, dl_2 - 3(l_0^2 l_2 - 2 l_0 l_1^2)\, dl_1 \\ &\qquad - (l_0^2 l_3 - 6 l_0 l_1 l_2 + 6 l_1^3)\, dl_0 \Big\} t^3 + \cdots. \end{aligned}$$

Given $\mathbf{z}(t)$ as in (4.5), equation (4.8) implies that

$$\begin{aligned}\tau(\boldsymbol{\lambda})\mathbf{z}(t) &= \boldsymbol{\lambda}(t)\mathbf{z}(t)\\ &= \lambda_0 z_0 + (\lambda_1 z_0 + \lambda_0 z_1)t + \frac{1}{2}(\lambda_2 z_0 + 2\lambda_1 z_1 + \lambda_0 z_2)t^2\\ &\quad + \frac{1}{3!}(\lambda_3 z_0 + 3\lambda_2 z_1 + 3\lambda_1 z_2 + \lambda_0 z_3)t^3 + \cdots .\end{aligned}$$

Thus, the Maurer-Cartan form matrix for $\mathrm{L}^{(n)}(1,1)$ is

$$\tau(\boldsymbol{\lambda}) = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \dots \\ 0 & \lambda_0 & 2\lambda_1 & 3\lambda_2 & 4\lambda_3 & \dots \\ 0 & 0 & \lambda_0 & 3\lambda_1 & 6\lambda_2 & \dots \\ 0 & 0 & 0 & \lambda_0 & 4\lambda_1 & \dots \\ 0 & 0 & 0 & 0 & \lambda_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.9}$$

The (i,j) entry of the full $n \times n$ matrix is

$$\tau(L)^i_j = \begin{cases} \binom{j}{i} \lambda_{i-j}, & i \le j, \\ 0, & i > j. \end{cases} \tag{4.10}$$

Note the remarkable similarity between the Maurer-Cartan form matrices for the prolonged general linear group, (3.14), and for the Leibniz group, (4.9)! The Leibniz version forms a “Pascal upper triangular matrix”, whereas the prolonged version is obtained by throwing away the main diagonal of the Pascal matrix.

§5. The Contact Group

We are now in a position to describe the structure group for the pseudo-group of contact transformations on the jet bundle J^n.

Definition 5.1. The n^{th} order *contact group* is the semidirect product group

$$\mathrm{C}^{(n)}(p,q) = \mathrm{GL}^{(n-1)}(p) \ltimes \mathrm{L}^{(n-1)}(p,q). \tag{5.1}$$

The group acts on a Taylor series $\mathbf{z}(t)$ according to

$$\psi(\mathbf{S},\mathbf{L}) \cdot \mathbf{z}(t) = \mathbf{L}(t) \cdot \mathbf{z}(\mathbf{S}^{-1}(t)), \tag{5.2}$$

and then truncating to order n. Therefore, the group multiplication in $\mathrm{C}^{(n)}(p,q)$ is given, in series form, by

$$(5.3)\qquad \big(\mathbf{S}(t),\mathbf{L}(t)\big)\cdot\big(\mathbf{T}(t),\mathbf{M}(t)\big)=\big(\mathbf{S}(\mathbf{T}(t)),\mathbf{L}(t)\cdot\mathbf{M}(\mathbf{S}^{-1}(t))\big).$$

The Maurer-Cartan form matrix for the contact structure group is given by the "difference" between the two Maurer-Cartan form matrices, so $\psi(\boldsymbol{\sigma},\boldsymbol{\lambda})=\rho(\boldsymbol{\sigma})-\tau(\boldsymbol{\lambda})$. Thus, we find

$$[\psi(\sigma,\lambda)\mathbf{z}](t)=\boldsymbol{\lambda}(t)\cdot\mathbf{z}(t)-\sum_{i=1}^{p}\frac{\partial\mathbf{z}}{\partial t^i}(t)\boldsymbol{\sigma}^i(t).$$

Note that the prolonged general linear group acts trivially on the zero$^{\text{th}}$ order coefficient in the power series for $\mathbf{z}$. In the one-dimensional version, we have

(5.4)

$$\psi(\boldsymbol{\lambda},\boldsymbol{\sigma})=\begin{pmatrix}\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \dots\\ 0 & \lambda_0-\sigma_1 & 2\lambda_1-\sigma_2 & 3\lambda_2-\sigma_3 & 4\lambda_3-\sigma_4 & \dots\\ 0 & 0 & \lambda_0-2\sigma_1 & 3\lambda_1-3\sigma_1 & 6\lambda_2-4\sigma_1 & \dots\\ 0 & 0 & 0 & \lambda_0-3\sigma_1 & 4\lambda_1-6\sigma_2 & \dots\\ 0 & 0 & 0 & 0 & \lambda_0-4\sigma_1 & \dots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

We now introduce the infinite power series of basis contact forms

$$(5.5)\qquad \boldsymbol{\theta}^\alpha(t)=\sum_{0\le\#J}\frac{t^J}{J!}\theta^\alpha_J,\qquad \alpha=1,\dots,q,$$

in the variable u^α, and let $\boldsymbol{\theta}(t)=(\boldsymbol{\theta}^1(t),\dots,\boldsymbol{\theta}^q(t))^T$ be the associated column vector-valued series of contact forms. Note that the contact forms on J^n are obtained by truncating the series $\boldsymbol{\theta}(t)$ at order $n-1$ and *not* at order n.

We are now able to introduce the goal of our investigations.

Definition 5.2. The *canonical contact form* is the vector-valued series of one-forms

$$(5.6)\qquad \boldsymbol{\vartheta}(t)=\psi(\mathbf{L},\mathbf{S})\boldsymbol{\theta}(t)=\mathbf{L}(t)\cdot\boldsymbol{\theta}(\mathbf{S}^{-1}(t)),$$

where $\mathbf{L}(t)$ and $\mathbf{S}(t)$ are the associated group series.

Example 5.3. In the one-dimensional situation, the canonical contact form is composed of the following linear combinations of contact forms:

$$
\begin{aligned}
\vartheta_0 &= l_0\theta_0, \\
\vartheta_1 &= \frac{l_0}{s_1}\theta_1 + l_1\theta_0, \\
\vartheta_2 &= \frac{l_0}{s_1^2}\theta_2 + \frac{2s_1^2 l_1 - s_2 l_0}{s_1^3}\theta_1 + l_2\theta_0, \\
\vartheta_3 &= \frac{l_0}{s_1^3}\theta_3 + \frac{3s_1^2 l_1 - 3s_2 l_0}{s_1^4}\theta_2 \\
&\quad + \frac{3s_1^4 l_2 - 3s_1^2 s_2 l_1 - (s_1 s_3 - 3s_2^2) l_0}{s_1^5}\theta_1 + l_3\theta_0.
\end{aligned} \tag{5.7}
$$

Remark. We can compute ϑ_k by repeatedly applying the (formal) differential operator $\mathcal{D} = (1/s_1)D_x$ to ϑ_0, using the identifications $\mathcal{D}(l_j) = l_{j+1}$, $\mathcal{D}(s_j) = s_{j+1}/s_1$. A proof of this observation is left to the reader.

Theorem 5.4. *The canonical contact form of order n defines an involutive differential system. The equivalence maps preserving the canonical contact form are the lifts of contact transformations on* J^n.

The structure equations are found as follows. The usual contact form structure equations

$$
d\theta_I^\alpha = \sum_{i=1}^{p} \theta_{I,i}^\alpha \wedge dx^i,
$$

can be rewritten in series form

$$
d\boldsymbol{\theta}(t) = \boldsymbol{\theta}'(t) \wedge d\mathbf{x} = \sum_{i=1}^{p} \frac{\partial \boldsymbol{\theta}}{\partial t^i} \wedge dx^i, \qquad \alpha = 1, \ldots, q. \tag{5.8}
$$

Here $\boldsymbol{\theta}'(t) = \left(\partial\theta^\alpha/\partial t^i\right)$ is the formal $q \times p$ Jacobian matrix of $\boldsymbol{\theta}(t)$ with respect to t. Therefore, using (5.8), we can compute

$$
\begin{aligned}
d\boldsymbol{\vartheta}(t) &= \boldsymbol{\lambda}(t) \wedge \boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t) \wedge \boldsymbol{\sigma}(t) + \mathbf{L}(t) d\boldsymbol{\theta}(\mathbf{S}^{-1}(t)) \\
&= \boldsymbol{\lambda}(t) \wedge \boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t) \wedge \boldsymbol{\sigma}(t) + \mathbf{L}(t)\boldsymbol{\theta}'(\mathbf{S}^{-1}(t)) \wedge d\mathbf{x}.
\end{aligned} \tag{5.9}
$$

In the language of the Cartan equivalence method, cf. [10], [16], the first two terms in (5.9) form the group components of the structure equations, while the third term is the torsion.

On the other hand, using the definition (5.6), we can compute

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{\vartheta}(t) &= \mathbf{L}'(t)\cdot\boldsymbol{\theta}(\mathbf{S}^{-1}(t)) + \mathbf{L}(t)\cdot\boldsymbol{\theta}'(\mathbf{S}^{-1}(t))\cdot\frac{\partial}{\partial t}[\mathbf{S}^{-1}(t)]\\ &= \mathbf{L}'(t)\cdot\mathbf{L}(t)^{-1}\cdot\boldsymbol{\vartheta}(t) + \mathbf{L}(t)\cdot\boldsymbol{\vartheta}'(\mathbf{S}^{-1}(t))\cdot\big[\mathbf{S}'(\mathbf{S}^{-1}(t))\big]^{-1},\end{aligned}$$

the last equality following from the chain rule. Therefore,

$$\begin{aligned}d\boldsymbol{\vartheta}(t) &= \boldsymbol{\lambda}(t)\wedge\boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t)\wedge\boldsymbol{\sigma}(t)\\ &\qquad + \big[\boldsymbol{\vartheta}'(t) - \mathbf{L}'(t)\cdot\mathbf{L}(t)^{-1}\cdot\boldsymbol{\vartheta}(t)\big]\wedge\mathbf{S}'(\mathbf{S}^{-1}(t))\,d\mathbf{x}.\end{aligned}\tag{5.10}$$

Most of the torsion terms can therefore be absorbed by suitably modifying the Maurer-Cartan forms $\boldsymbol{\lambda}(t)$ and $\boldsymbol{\sigma}(t)$; the only exceptions are the constant terms multiplying $\boldsymbol{\vartheta}'(t)$; this is because $\boldsymbol{\sigma}(t)$ does not contain any constant terms, i.e., $\boldsymbol{\sigma}(0) = 0$. If we define the modified Maurer-Cartan forms to be

$$\begin{aligned}\widetilde{\boldsymbol{\lambda}}(t) &= \boldsymbol{\lambda}(t) + \mathbf{L}'(t)\cdot\mathbf{L}(t)^{-1}\cdot\mathbf{S}'(\mathbf{S}^{-1}(t))\,d\mathbf{x},\\ \widetilde{\boldsymbol{\sigma}}(t) &= \boldsymbol{\sigma}(t) + \big[\mathbf{S}'(\mathbf{S}^{-1}(t)) - \mathbf{S}'(0)\big]\,d\mathbf{x},\end{aligned}\tag{5.11}$$

we can rewrite the structure equations (5.10) in the "semi-absorbed form"

$$d\boldsymbol{\vartheta}(t) = \widetilde{\boldsymbol{\lambda}}(t)\wedge\boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t)\wedge\widetilde{\boldsymbol{\sigma}}(t) + \boldsymbol{\vartheta}'(t)\wedge\mathbf{S}'(0)\,d\mathbf{x}.\tag{5.12}$$

We now complete the canonical contact form to a coframe on J^n by including the additional p one-forms

$$\boldsymbol{\xi} = \mathbf{S}'(0)d\mathbf{x} + \mathbf{a}\boldsymbol{\vartheta}(0) + \mathbf{B}\boldsymbol{\vartheta}'(0).\tag{5.13}$$

Here $\mathbf{a} = (a^i_\alpha)$ is a $p\times q$ matrix and $\mathbf{B} = (b^{ik}_\alpha)$ a $p\times p\times q$ tensor of parameters. In components, (5.13) reads

$$\xi^i = \sum_{j=1}^p S^i_j\,dx^j + \sum_{\alpha=1}^q a^i_\alpha\vartheta^\alpha + \sum_{\alpha=1}^q\sum_{k=1}^p b^{ik}_\alpha\vartheta^\alpha_k,$$

$$\text{where}\qquad \vartheta^\alpha = \boldsymbol{\vartheta}^\alpha(0),\qquad \vartheta^\alpha_k = \frac{\partial\boldsymbol{\vartheta}^\alpha}{\partial t^k}(0),$$

are the lifted zero$^{\text{th}}$ and first order contact forms, which can be written as linear combinations of the ordinary zero$^{\text{th}}$ and first order contact forms θ^α, θ^α_k via (5.6). Bäcklund's Theorem implies that the $\mathbf{x}$ coordinates depend only on x, u, and, if $q = 1$, first order derivatives of u. This implies that the first order contact form coefficients in (5.13) must vanish, $\mathbf{B} = 0$, when $q > 1$. (Alternatively, one can use a particular unabsorbable torsion term to justify this normalization.) We therefore use (5.13) to rewrite the structure equations (5.12) in the fully absorbed form

$$d\boldsymbol{\vartheta}(t) = \widehat{\boldsymbol{\lambda}}(t) \wedge \boldsymbol{\vartheta}(t) + \boldsymbol{\vartheta}'(t) \wedge \widehat{\boldsymbol{\sigma}}(t) + \boldsymbol{\vartheta}'(t) \wedge \boldsymbol{\xi}, \tag{5.14}$$

where the modified Maurer-Cartan forms are now

$$\begin{aligned} \widehat{\boldsymbol{\lambda}}(t) &= \widetilde{\boldsymbol{\lambda}}(t) - \mathbf{a} \cdot \boldsymbol{\vartheta}'(t), \\ \widehat{\boldsymbol{\sigma}}(t) &= \widetilde{\boldsymbol{\sigma}}(t) + \mathbf{a} \cdot [\boldsymbol{\vartheta}(t) - \boldsymbol{\vartheta}(0)] + \mathbf{B} \cdot [\boldsymbol{\vartheta}'(t) - \boldsymbol{\vartheta}'(0)]. \end{aligned} \tag{5.15}$$

(Again, note that $\widehat{\boldsymbol{\sigma}}(0) = \boldsymbol{\sigma}(0) = 0$, so that this modification is allowed.) The only term in (5.12) which remains unaccounted for is

$$\boldsymbol{\vartheta}'(t) \wedge \mathbf{B} \cdot \boldsymbol{\vartheta}'(t),$$

but this vanishes because either $q = 1$, in which case the wedge product of the two scalar one-forms $\boldsymbol{\vartheta}'(t)$ is zero, or $q > 1$, in which case, by Bäcklund's Theorem, $\mathbf{B} = 0$. In fact, this is the essential torsion component that provides the equivalence method proof of this part of Bäcklund's Theorem, cf. [16]. Equation (5.14) provides the main constituent of the structure equations for the contact pseudo-group.

We also need to compute the remaining structure equations for the one-forms (5.13). We find

$$d\boldsymbol{\xi} = \boldsymbol{\sigma}'(0) \wedge \boldsymbol{\xi} + \boldsymbol{\alpha} \wedge \boldsymbol{\vartheta}(0) + \boldsymbol{\beta} \wedge \boldsymbol{\vartheta}'(0) + \mathbf{a} \cdot\, d\boldsymbol{\vartheta}(0) + \mathbf{B}\, d\boldsymbol{\vartheta}'(0), \tag{5.16}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are the Maurer-Cartan forms corresponding to the additional group parameters $\mathbf{a}$, $\mathbf{B}$. Note that $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ do not depend on t. Differentiating (5.14) with respect to t, and recalling $\boldsymbol{\sigma}(0) = 0$, we find

$$\begin{aligned} d\boldsymbol{\vartheta}(0) &= \widehat{\boldsymbol{\lambda}}(0) \wedge \boldsymbol{\vartheta}(0) + \boldsymbol{\vartheta}'(0) \wedge \boldsymbol{\xi}, \\ d\boldsymbol{\vartheta}'(0) &= \widehat{\boldsymbol{\lambda}}'(0) \wedge \boldsymbol{\vartheta}(0) + \widehat{\boldsymbol{\lambda}}(0) \wedge \boldsymbol{\vartheta}'(0) + \boldsymbol{\vartheta}'(0) \wedge \widehat{\boldsymbol{\sigma}}'(0) + \boldsymbol{\vartheta}''(0) \wedge \boldsymbol{\xi}. \end{aligned} \tag{5.17}$$

Moreover, according to (5.11), (5.15), for any constant (column) vector $\mathbf{z} \in \mathbb{R}^p$,

$$\begin{aligned} \widehat{\boldsymbol{\sigma}}'(0) \cdot \mathbf{z} &= \widetilde{\boldsymbol{\sigma}}'(0) \cdot \mathbf{z} + \big[\mathbf{a} \cdot \boldsymbol{\vartheta}'(0) + \mathbf{B} \cdot \boldsymbol{\vartheta}''(0)\big] \cdot \mathbf{z} \\ &= \boldsymbol{\sigma}'(0) + \mathbf{S}''(0)\big(\mathbf{S}'(0)^{-1} \cdot \mathbf{z}, \mathbf{x}\big) + [\mathbf{a} \cdot \boldsymbol{\vartheta}'(0) + \mathbf{B} \cdot \boldsymbol{\vartheta}''(0)] \cdot \mathbf{z}. \end{aligned}$$

Wedging the result with $\boldsymbol{\xi}$, and using (5.13), (5.17) and, we find

$$\widehat{\boldsymbol{\sigma}}'(0) \wedge \boldsymbol{\xi} = \boldsymbol{\sigma}'(0) \wedge \boldsymbol{\xi} + \boldsymbol{\pi} \wedge \boldsymbol{\vartheta}(0) + \boldsymbol{\vartheta}'(0) \wedge \boldsymbol{\varpi} + \mathbf{B} \cdot \boldsymbol{\vartheta}''(0) \wedge \boldsymbol{\xi}, \tag{5.18}$$

for certain one-forms $\boldsymbol{\pi}$, $\boldsymbol{\varpi}$, whose precise form is not hard to find, but which is unimportant. Note that we used the fact that the extra term

$$\mathbf{S}''(0)\big(\mathbf{S}'(0)^{-1}\cdot\boldsymbol{\xi},\mathbf{S}'(0)^{-1}\cdot\boldsymbol{\xi}\big)=0$$

vanishes by symmetry of second order derivatives. Finally, substituting (5.18), (5.17) into (5.16), we conclude that

$$d\boldsymbol{\xi}=\widehat{\boldsymbol{\sigma}}'(0)\wedge\boldsymbol{\xi}+\widehat{\boldsymbol{\alpha}}\wedge\boldsymbol{\vartheta}(0)+\widehat{\boldsymbol{\beta}}\wedge\boldsymbol{\vartheta}'(0), \tag{5.19}$$

where $\widehat{\boldsymbol{\sigma}}'(0)$ are the order 1 terms of our earlier modified Maurer-Cartan forms (5.15), while $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\beta}}$ are suitably modified one-forms corresponding to the additional structure group parameters $\mathbf{a}$, $\mathbf{B}$. Note particularly that (5.19) contains no essential torsion. Equations (5.14) and (5.19) form the complete structure equations for the contact pseudo-group on the infinite jet bundle.

There is one final item to deal with when working on a finite jet bundle J^n. Since the contact forms which are well-defined on J^n have orders at most $n-1$, we must include $q\binom{p+n-1}{n}$ additional one-forms to complete the coframe on J^n. These will clearly be the basis forms du_J^α, $\#J=n$, which must be lifted appropriately. (See [15] for more details.) However, we can most simply accomplish this as follows: First, truncate the canonical contact form series $\boldsymbol{\vartheta}(t)$ at order n. The resulting lifted contact form will depend on $(n+1)^{\mathrm{st}}$ order derivatives of u. These can be eliminated, while retaining the proper lift, by adding in a suitable multiple of the base forms dx^i. Thus, the lifted coframe on J^n consists of the one-forms (5.13) along with the modified canonical contact form

$$\widehat{\boldsymbol{\vartheta}}(t)=\boldsymbol{\vartheta}(t)+\mathbf{e}(t^n)\cdot\boldsymbol{\xi}\ \mathrm{mod}\ n, \tag{5.20}$$

where $\mathbf{e}=(e_I^\alpha)$ is a $q\times\binom{p+n-1}{n}$ matrix of additional group parameters. The corresponding truncated structure equations are now

$$d\widehat{\boldsymbol{\vartheta}}(t)=\widehat{\boldsymbol{\lambda}}(t)\wedge\widehat{\boldsymbol{\vartheta}}(t)+\widehat{\boldsymbol{\vartheta}}'(t)\wedge\widehat{\boldsymbol{\sigma}}(t)+\boldsymbol{\varepsilon}(t^n)\wedge\boldsymbol{\xi}+\widehat{\boldsymbol{\vartheta}}'(t)\wedge\boldsymbol{\xi}\ \mathrm{mod}\ n. \tag{5.21}$$

This completes our proof.

Example 5.5. The structure equations for the one-dimensional situation are as follows:

$$
\begin{aligned}
d\vartheta_0 &= \lambda_0 \wedge \vartheta_0 + \xi \wedge \vartheta_1, \\
d\vartheta_1 &= \lambda_1 \wedge \vartheta_0 + (\lambda_0 - \sigma_1) \wedge \vartheta_1 + \xi \wedge \vartheta_2, \\
d\vartheta_2 &= \lambda_2 \wedge \vartheta_0 + (2\lambda_1 - \sigma_2) \wedge \vartheta_1 + (\lambda_0 - 2\sigma_1) \wedge \vartheta_2 + \xi \wedge \vartheta_3, \\
d\vartheta_3 &= \lambda_3 \wedge \vartheta_0 + (3\lambda_2 - \sigma_3) \wedge \vartheta_1 + (3\lambda_1 - 3\sigma_2) \wedge \vartheta_2 \\
&\qquad + (\lambda_0 - 3\sigma_1) \wedge \vartheta_3 + \xi \wedge \vartheta_4, \\
&\vdots \qquad\qquad\qquad \vdots \\
d\vartheta_{n-1} &= \sum_{i=0}^{n-1} \left[\binom{n-1}{i} \lambda_{n-1-i} - \binom{n-1}{i-1} \sigma_{n-i} \right] \wedge \vartheta_i + \xi \wedge \widetilde{\vartheta}_n, \\
d\widetilde{\vartheta}_n &= \sum_{i=0}^{n} \left[\binom{n}{i} \lambda_{n-i} - \binom{n}{i-1} \sigma_{n+1-i} \right] \wedge \vartheta_i + \varepsilon \wedge \xi, \\
d\xi &= \sigma_1 \wedge \xi + \varphi \wedge \vartheta_0 + \psi \wedge \vartheta_1.
\end{aligned}
\tag{5.22}
$$

Here $\lambda_0, \ldots, \lambda_n$ are the Leibniz Maurer-Cartan forms, $\sigma_1, \ldots, \sigma_n$ the prolonged general linear group Maurer-Cartan forms, and ε, φ, ψ the three additional Maurer-Cartan forms, corresponding to the truncated or non-canonical part of the lifted coframe.

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