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# Lie Algebras, Geometric Structures and Differential Equations on Filtered Manifolds

**Tohru Morimoto**

## Introduction

Since Lie, Klein and Cartan, there has been a great deal of progress in understanding deep relations between groups, geometry and differential equations.

In this paper we give a survey on some recent development made by systematic studies from the view point of nilpotent geometry on transformation groups (or rather Lie algebras), geometric structures and differential equations, placing ourselves on filtered manifolds.

A filtered manifold is a differential manifold $M$ endowed with a filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ consisting of subbundles $\mathfrak{f}^p$ of the tangent bundle $TM$ such that

i) $\mathfrak{f}^p \supset \mathfrak{f}^{p+1}$,
ii) $\mathfrak{f}^0 = 0, \quad \bigcup_{p\in\mathbb{Z}} \mathfrak{f}^p = TM$,
iii) $[\underline{\mathfrak{f}}^p, \underline{\mathfrak{f}}^q] \subset \underline{\mathfrak{f}}^{p+q}$ for all $p,\ q \in \mathbb{Z}$,

where $\underline{\mathfrak{f}}^p$ denotes the sheaf of the germs of sections of $\mathfrak{f}^p$.

This notion of a filtered manifold has arisen from a fundamental paper of Tanaka [Tan70] on differential systems that he elaborated, inspired by the deep work of Cartan, especially by [Car10].

Of fundamental importance is the fact that, to a filtered manifold $(M, \mathfrak{f})$, there is associated at each point $x$ of $M$ the nilpotent graded Lie algebra $gr\mathfrak{f}_x$, as the first order approximation at $x$ to the filtered manifold, where $gr\mathfrak{f}_x = \bigoplus \mathfrak{f}^p_x/\mathfrak{f}^{p+1}_x$. It should be remarked that if the filtration is trivial (i.e., $\mathfrak{f}^{-1} = TM$) then $gr\mathfrak{f}_x$ is nothing but the tangent space $T_xM$ regarded as an abelian Lie algebra.

To study various objects on filtered manifolds by letting the tangent nilpotent Lie algebras play the usual rôle of the tangent spaces may be

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called nilpotent geometry or nilpotent analysis. The generalization from the abelian to the nilpotent allows one to develop more refined theories than the anterior from much wider perspectives.

We now describe the contents of this paper.

In Section 0 we give basic definitions and notation about filtered manifolds.

In Section 1 we study transformation groups on filtered manifolds, confining ourselves only to the algebraic parts of transitive infinitesimal transformation groups.

We know well about the structures of transitive filtered Lie algebras (algebraic abstractions of transitive transformation groups) through the work of Guillemin-Sternberg [GS64], Singer-Sternberg [SS65], Kobayashi-Nagano [KN66] and others.

If a transitive transformation group on a filtered manifold preserves the tangential filtration, then the transitive Lie algebra corresponding to the transformation group admits a natural filtration compatible with the tangential filtration and more refined than the usual filtration, the former deriving from the "weighted" Taylor expansion and the latter from the usual Taylor expansion. This leads to the notion of a transitive filtered Lie algebra of depth $\mu(\geq 1)$ introduced in [Mor88]. If $\mu = 1$ it reduces to the usual one.

The fundamental problem is to understand how the structure of a transitive filtered Lie algebra $(L, \{L^p\})$ can be determined from the truncated structure $L/L^k$ which consists of finite dimensional data, or certain information up to finite orders. This is a prototype of the problems that we encounter in the study of geometric structures and differential equations.

By extending the theory of Guillemin-Sternberg to the transitive filtered Lie algebras of depth greater than one, we have a fairly complete answer to the above algebraic problem.

We shall introduce the notion of weighted involutivity by using generalized Spencer cohomology groups. This notion plays a fundamental rôle not only in the study of filtered Lie algebras, but also in the study of geometric structures and differential equations.

In Section 2 we study geometric structures on filtered manifold and explain our general method to treat equivalence problems.

The key concept is what we introduced as C-fibre in [Mor83] and as tower in [Mor93] (the latter is a refinement of the former to apply to filtered manifolds).

Roughly speaking, it is a principal fibre bundle $P$ over a manifold $M$ with structure group $G$ endowed with a 1-form $\theta$ taking values in a vector space $E$ which defines an absolute parallelism on $P$ and satisfies some natural conditions. In particular, we assume $E$ is a $G$-module containing the Lie algebra $\mathfrak{g}$ of $G$ as a $G$-submodule and $\theta$ is an equivariant map.

It should be noticed that $P$ is not necessarily finite dimensional and that $E$ is not necessarily a Lie algebra. The reader who is familiar with Cartan connections will notice that if $E$ is a Lie algebra containing $\mathfrak{g}$ the tower introduced above is just a principal fibre bundle with a Cartan connection. Therefore the notion of a tower is a generalization of that of a Cartan connection. However, we might rather say that the former precedes (and is more basic than) the latter; the notion of a tower seems to represent some general heuristic ideas of Cartan which appeared in his papers of infinite groups [Car04], [Car05], [Car08], [Car09] much earlier than his notion of éspace généralisé.

The category of the towers has an advantage that it is well adapted to deal with all possible (virtual) symmetries of geometric structures. Moreover, the notion of differentiation is geometrically well represented by the associated filtration on the bundle $P$ (and on the group $G$). If a tower $(P, M, G)$ on a filtered manifold $(M, \mathfrak{f})$ is compatible with the tangential filtration $\mathfrak{f}$, it admits another natural filtration on $P$ and on $G$ associated with $\mathfrak{f}$, which is a filtration deriving from "weighted order", and it is this filtration that plays an important rôle when we treat towers on filtered manifolds.

In the framework of tower we shall construct a unified scheme to treat geometric structures on filtered manifolds. Any geometric structure on a filtered manifold may be regarded as a tower or as a truncated tower on a filtered manifold. Given a geometric structure on a filtered manifold, we shall show the general procedure to find the invariants of the structure. When we study geometric structures, it is important to distinguish the difference between the intransitive and the transitive, and that between infinite type and finite type. It should be remarked that a structure of infinite type in the usual sense can be of finite type with respect to the weighted filtration associated with the filtered manifold.

To treat a transitive geometric structure on a filtered manifold of infinite type, we introduce a notion of weighted involutivity and clarify the procedure to find all the invarints of the structure, which is exactly a geometrical version of the procedure to determine a transitive filtered Lie algebra from a truncated Lie algebra.

Though we can also treat the intransitive infinite cases, we will not enter into the discussion rather complicated.

For a geometric structure of finite type, it is important in applications to construct a Cartan connection associated to it. But the construction has been usually difficult and technical. Our method also answers this question. We shall give a criterion (probably best possible) for the existence of a Cartan connection and a unified algorithm to construct the Cartan connection.

In Section 3 we study general systems of non-linear partial differential equations on filtered manifolds.

Now we just recall the development of general theories on analytic systems of non-linear partial differential equations. As seen easily, any system of differential equations can be brought to an exterior differential system. Cartan established for the first time a general existence theorem in the framework of exterior differential systems [Car04]. He introduced the notion of a Pfaff system in involution and obtained the solutions by successive use of the Caucy-Kowalevski theorem, which was generalized to any involutive exterior differential system by Kähler [Käh34] as well known as the Cartan-Kähler theorem.

The modern theory of these systems was initiated by Kuranishi to establish the so-called Cartan-Kuranishi prolongation theorem [Kur57]. Using Ehresman's theory of jet, Spencer introduced fundamental tools to treat systems in jet formulation (cf. [Spe69]). In this framework Goldschmidt [Gol67] and Quillen [Qui64] established a formal theory of systems of partial differential equations clarifying in modern language the notion of involutivity; the sufficient condition for the existence of formal solutions. Malgrange [Mal72] gave an elegant proof for the existence of analytic solutions to an involutive analytic system by using the "privileged neighbourhood theorem" of Grauert.

Now in studying differential equations on a filtered manifold, it is the notion of weighted orders for differential operators associated with the filtered manifold that will play the principal rôle.

We shall first introduce the notion of a weighted jet bundle for a vector bundle on a filtered manifold, and establish a formal theory in terms of weighted jet bundles analogously to the formal theory of Goldschmidt. We introduce a notion of weighted involutivity for the system, which gives sufficient condition in order that the formal solutions can be constructed in some regular way by "weighted Taylor expansion".

Next we consider the problem of convergence. Since a weightedly involutive system is in general not involutive in the ordinary sense, we cannot expect in general the existence of analytic solutions.

Without loss of generality we will work on a standard filtered manifold, namely a nilpotent Lie group $N$ whose Lie algebra $\mathfrak{n}$ is graded: $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$.

We first establish the following theorem:

*For a weightedly involutive analytic system, there exists a formal solution satisfying a certain type of estimate, called Gevrey estimate.*

The Gevrey estimate is expressed in terms of the filtration of the filtered manifold and is a little weaker than the analyticity estimate. We can then define the class of formal Gevrey functions on a filtered manifold.

For the proof of the above theorem we employ Malgrange's method after generalizing the privileged neighbourhood theorem to the universal enveloping algebra of a nilpotent Lie algebra $\mathfrak{n}$.

We then study geometric properties of formal Gevrey functions on a graded nilpotent Lie group $N$, which leads to the following remarkable theorem:

*If the Lie algebra $\mathfrak{n}$ is generated by $\mathfrak{n}_1$ (Hörmander condition), then the formal Gevrey function on $N$ are analytic.*

Combining the above theorems, we finally establish the following existence theorem (a generalization of the Cartan-Kähler theorem):

*Consider an analytic system of non-linear partial differential equations of weighted order $k$ on a graded nilpotent Lie group $N$ with a Lie algebra $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$. Suppose that the Lie algebra $\mathfrak{n}$ is generated by $\mathfrak{n}_1$, and that the system is weightedly involutive. Then there exists an analytic solution for any prescribed weighted $k$-jet solution.*

It should be noted that the class of the weightedly involutive systems is much larger than that of the ordinary involutive systems and contains a wide class of differential equations with singularities.

Our primary purpose of this paper is to try to make clear intrinsic relations underlying three objects; Lie algebras, geometric structures, and differential equations on filtered manifolds. We, therefore, will not enter into the details of each subjects, and not intend to give a complete proof of each statement, referring for them to our papers ([Mor83], [Mor88], [Mor90], [Mor93], [Mor95], [Mor0x]), on which our discussions are mainly based.

## §0. Filtered manifolds

0.1. Definitions.

A *tangential filtration* $\mathfrak{f}$ on a differentiable manifold $M$ is a sequence $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ of subbundles of the tangent bundle $TM$ of $M$ such that the following conditions are satisfied:

i) $\mathfrak{f}^p \supset \mathfrak{f}^{p+1}$,
ii) $\mathfrak{f}^0 = 0, \quad \bigcup_{p\in\mathbb{Z}} \mathfrak{f}^p = TM$,
iii) $[\underline{\mathfrak{f}}^p, \underline{\mathfrak{f}}^q] \subset \underline{\mathfrak{f}}^{p+q}, \quad$ for all $\quad p,\ q \in \mathbb{Z}$,

where $\underline{\mathfrak{f}}^p$ denotes the sheaf of the germs of sections of $\mathfrak{f}^p$.

A *filtered manifold* is a differentiable manifold $M$ equipped with a tangential filtration $\mathfrak{f}$. We shall denote the filtered manifold by $(M, \mathfrak{f})$ or often by the bold letter $\mathbb{M}$ and its tangential filtration by $\{\mathfrak{f}^p\}$, $\{\mathfrak{f}^p TM\}$ or $\{T^p\mathbb{M}\}$.

An isomorphism of a filtered manifold $\mathbb{M}$ onto a filtered manifold $\mathbb{M}'$ is a diffeomorphism $\varphi\colon M \to M'$ such that $\varphi_* T^p\mathbb{M} = T^p\mathbb{M}'$ for all $p \in \mathbb{Z}$, where $\varphi_*$ denotes the differential of $\varphi$.

If $\mathbb{M}$ is a filtered manifold, by definition there is an integer $\mu \geq 0$ such that $T^{-\mu}\mathbb{M} = TM$. The minimum of such integers is called the depth of $\mathbb{M}$.

Let $\mathbb{M}$ be a filtered manifold. The tangential filtration $\{T^p\mathbb{M}\}$ defines on each tangent space $T_xM$, $x \in M$, the induced filtration $\{T^p_x\mathbb{M}\}$. We denote by $T_x\mathbb{M}$ this filtered vector space $(T_xM, \{T^p_x\mathbb{M}\})$. Now by setting

$$gr_p T_x\mathbb{M} = T^p_x\mathbb{M}/T^{p+1}_x\mathbb{M},$$

we form a graded vector space:

$$grT_x\mathbb{M} = \bigoplus_{p\in\mathbb{Z}} gr_pT_x\mathbb{M}.$$

This vector space carries a natural bracket operation induced from the Lie bracket of vector fields: For $\xi \in gr_pT_x\mathbb{M}$, $\eta \in gr_qT_x\mathbb{M}$, take local cross-sections $X$, $Y$ of $T^p\mathbb{M}$, $T^q\mathbb{M}$ resp. such that $\xi \equiv X_x (\bmod\ T^{p+1}_x\mathbb{M})$, $\eta \equiv Y_x (\bmod\ T^{q+1}_x\mathbb{M})$, and define

$$[\xi, \eta] \equiv [X, Y]_x \qquad \bmod T^{p+q+1}_x\mathbb{M}.$$

It is then easy to see that this bracket operation is well defined and makes $grT_x\mathbb{M}$ a Lie algebra. Clearly we have:

i) $[gr_pT_x\mathbb{M}, gr_qT_x\mathbb{M}] \subset gr_{p+q}T_x\mathbb{M}$,

ii) $gr_pT_x\mathbb{M} = 0 \qquad$ for $p \geq 0$.

This graded Lie algebra $grT_x\mathbb{M}$ is called the *symbol algebra* of $\mathbb{M}$ at $x$ ([Tan70]), and may be consider as the tangent space (algebra) at $x$ of the filtered manifold $\mathbb{M}$.

We say that a filtered manifold $\mathbb{M}$ is *regular* of type $\mathfrak{m}$ if the symbol algebras $grT_x\mathbb{M}$ are all isomorphic to a graded Lie algebra $\mathfrak{m}$.

0.2. Some examples.

1) Trivial filtration. A differentiable manifold $M$ itself may be regarded as a filtered manifold equipped with the trivial filtration defined by $\mathfrak{f}^p_{tr}TM = TM$ for $p < 0$ and $\mathfrak{f}^q_{tr}TM = 0$ for $q \geq 0$. The symbol algebras $grT_x\mathbb{M}$ of this trivial filtered manifold is nothing but the tangent space $T_xM$ regarded as an abelian Lie algebra with trivial gradation.

2) Standard filtered manifold. Let $\mathfrak{n}$ be a finite-dimensional nilpotent Lie algebra endowed with a gradation $\mathfrak{n} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{n}_p$ such that

i) $[\mathfrak{n}_p, \mathfrak{n}_q] \subset \mathfrak{n}_{p+q}$,
ii) $\mathfrak{n}_p = 0 \qquad p \geq 0$.

Let $N$ be a Lie group whose Lie algebra is $\mathfrak{n}$. Set $\mathfrak{n}^p = \bigoplus_{i\geq p} \mathfrak{n}_i$ and identify $N \times \mathfrak{n}^p$ with a left invariant subbundle of $TN$, then $\{N \times \mathfrak{n}^p\}_{p\in\mathbb{Z}}$ is a tangential filtration on $N$. The filtered manifold $\mathbb{N} = (N, \{N \times \mathfrak{n}^p\})$ is called a *standard filtered manifold* of type $\mathfrak{n}$.

3) Tangential filtration derived from a regular differential system [Tan70]. Let $D$ be a differential system on a differentiable manifold $M$, that is, a subbundle of the tangent bundle of $M$. Then there is associated a sequence of subsheaves $\{\mathcal{D}^p\}_{p<0}$ of $\underline{TM}$, called the derived systems of $D$, which is defined inductively by:

$$\begin{cases} \mathcal{D}^{-1} = \underline{D}, \\ \mathcal{D}^{p-1} = \mathcal{D}^p + [\mathcal{D}^p, \mathcal{D}^{-1}] \qquad (p < 0). \end{cases}$$

It then holds that:

$$[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q} \qquad \text{for} \quad p,\ q < 0.$$

Now suppose that the derived systems $\mathcal{D}^p$ are all vector bundles, that is, there are subbundles $D^p \subset TM$ such that $\underline{D^p} = \mathcal{D}^p$ for all $p < 0$ (in this case the differential system $D$ is called regular [Tan70]). Then there exists a minimum integer $\mu \geq 1$ such that $D^p = D^{-\mu}$ for all $p \leq -\mu$. Setting

$$\mathfrak{f}^p TM = \begin{cases} 0 & (p \geq 0) \\ D^p & (-1 \geq p \geq -\mu) \\ TM & (p \leq -\mu - 1), \end{cases}$$

we have a filtered manifold $(M, \mathfrak{f})$ derived from the regular differential system $D$. If $D^{-\mu} = TM$, we say that the tangential filtration $\mathfrak{f}$ is generated by the differential system $D$. If $D^{-\mu} \subsetneq TM$, then $D^{-\mu}$ is completely integrable and defines a foliation on $M$. In particular, if $D$ is completely integrable the filtered manifold $\mathbb{M}$ is nothing but a foliated manifold.

If a filtered manifold $\mathbb{M}$ (or $\mathbb{M}'$) is derived from a differential system $D$ on $M$ (resp. $D'$ on $M'$), then $\mathbb{M}$ and $\mathbb{M}'$ are isomorphic if and only if $(M, D)$ and $(M', D')$ are isomorphic, that is, there is a diffeomorphism $\varphi\colon M \to M'$ such that $\varphi_* D = D'$.

4) Higher order contact manifold (cf. [Yam82]). Let $\pi\colon M \to N$ be a fibred manifold. Let $J^k(M,N)$ be the bundle of $k$-jets of cross-sections of $\pi$. On this jet bundle we have a sequence of canonical differential systems $\{D^p\}$ called the higher order contact structure. In local coordinates it is expressed as follows: Let $(x^1,\ldots,x^n)$, $(x^1,\ldots,x^n,y^1,\ldots,y^m)$ be local coordinates of $N$ and $M$ respectively. Then $(x^1,\ldots,x^n,\ldots,p^i_\alpha,\ldots)$, where $p^i_\alpha = \frac{\partial^{|\alpha|}y^i}{\partial x^\alpha}$ with $\alpha = (\alpha_1,\ldots,\alpha_n)$, $|\alpha| \le k$, gives a local coordinate system of $J^k(M,N)$ called a canonical coordinates system. Put

$$\omega^i_\alpha = dp^i_\alpha - \sum_{j=1}^{n} p^i_{\alpha+1_j}\, dx^j$$

for $|\alpha| \le k-1$, with $\alpha + 1_j = (\alpha_1,\ldots,\alpha_j+1,\ldots,\alpha_n)$, and define $D^p (p \le -1)$ by the following Pfaff equations:

$$D^p\colon \quad \omega^i_\alpha = 0 \qquad (i = 1,\ldots,n \quad |\alpha| \le k+p).$$

It is easy to see that $D^p$ are well-defined subbundles of $TJ^k(M,N)$ and satisfy:

i) $\underline{D}^{p-1} = \underline{D}^p + [\underline{D}^p, \underline{D}^{-1}]$,
ii) $D^p = TJ^k(M,N) \qquad \text{for} \quad p \le -k-1.$

We thus obtain a canonical tangential filtration $\{D^p\}$ on $J^k(M,N)$ of depth $k+1$ generated by $D^{-1}$. It should be noted that if $\dim M = n+1$, $\dim N = n$ and $k = 1$ then $J^1(M,N)$ is a contact manifold having $D^{-1}$ as its contact structure.

0.3. We shall often use the following notation and terminologies throughout this paper without explicit mention in each place.

For filtered objects (vector spaces $V$, $W$, Lie groups $G$ etc.) we denote by $\{\mathfrak{f}^p\}$ not only their filtrations but also the induced filtrations defined naturally on various associated spaces, for instance:

$$\mathfrak{f}^p(V \oplus W) = \mathfrak{f}^pV \oplus \mathfrak{f}^pW$$
$$\mathfrak{f}^p(V \otimes W) = \sum_{r+s=p} \mathfrak{f}^rV \otimes \mathfrak{f}^sW$$
$$\mathfrak{f}^p(G/\mathfrak{f}^hG) = \mathfrak{f}^pG/\mathfrak{f}^pG \cap \mathfrak{f}^hG$$
$$\mathfrak{f}^p \operatorname{Hom}(V,W) = \{\alpha \in \operatorname{Hom}(V,W) \mid \alpha(\mathfrak{f}^iV) \subset \mathfrak{f}^{i+p}W \quad \forall i\}$$
$$\mathfrak{f}^pGL(V) = \{\alpha \in GL(V) \mid \alpha - 1_V \in \mathfrak{f}^p \operatorname{Hom}(V,V)\}.$$

When we take a quotient space, for instance $\mathfrak{f}^0G/\mathfrak{f}^pG$, we often write it simply as $\mathfrak{f}^0G/\mathfrak{f}^p$.

If $\mathfrak{v} = \bigoplus \mathfrak{v}_p$, $\mathfrak{w} = \bigoplus \mathfrak{w}_p$ are graded vector spaces, we set

$$\mathrm{Hom}(\mathfrak{v}, \mathfrak{w})_p = \{\alpha \in \mathrm{Hom}(\mathfrak{v}, \mathfrak{w}) \mid \alpha(\mathfrak{v}_i) \subset \mathfrak{w}_{i+p} \quad \forall i\},$$

that is, the set of all linear maps of degree $p$.

## §1. Transitive Lie algebras on filtered manifolds

1.1. Let us begin with the following definition:

**Definition 1.1.** A transitive filtered Lie algebra (TFLA) of depth $\mu(\geq 1)$ is a Lie algebra $L$ equipped with a filtration $\{L^p\}_{p\in\mathbb{Z}}$ satisfying:

i) $L = L^{-\mu}$
ii) $L^p \supset L^{p+1}$
iii) $[L^p, L^q] \subset L^{p+q}$
iv) $\dim L^p/L^{p+1} < \infty$
v) $\bigcap_{p\in\mathbb{Z}} L^p = 0$
vi) $L^{p+1} = \{x \in L^p; [x, L^a] \subset L^{p+a+1}$ for all $a < 0\}$ for $p \geq 0$.

To justify the above definition, some remarks are in order.

*Remark 1.* The so-called continuous groups that Lie studied are in modern language the pseudo-groups of transformations on manifolds which are defined by systems of partial differential equations. In other words, a continuous group of Lie is a pseudo-group of transformations that leave invariant certain geometric structure on a manifold. To study such pseudo-groups, in particular, infinite dimensional ones to which there are no good global representatives such as finite dimensional Lie groups, it is usually more convenient to study infinitesimal objects, namely Lie algebra subsheaf $\mathcal{L}$ of the Lie algebra sheaf $\underline{TM}$ of the germs of local vector fields on a manifold $M$. The Lie algebra sheaf $\mathcal{L}$ is said to be transitive if the evaluation map $\mathcal{L}_x \to T_xM$ is surjective for all $x \in M$, where $\mathcal{L}_x$ denotes the stalk of $\mathcal{L}$ at $x$.

Let $\mathcal{L}$ be a Lie algebra sheaf on $M$. We can associate to each $x$ the formal algebra $L_x$ defined as follows [SS65]: Let, for $k \geq 0$, $\mathfrak{f}^k\mathcal{L}_x$ be the subalgebra of $\mathcal{L}_x$ consisting of all germs $[X]_x$ at $x$ of sections $X$ of $\mathcal{L}$ such that $X$ vanishes at $x$ to order $k$. Then put $L_x = \operatorname{proj\,lim}_{k\to\infty} \mathcal{L}_x/\mathfrak{f}^k\mathcal{L}_x$. The formal algebra $L_x$ has a natural filtration $\{L_x^p\}_{p\in\mathbb{Z}}$, where we put $L_x^p = \operatorname{proj\,lim}_{k\to\infty} \mathfrak{f}^p\mathcal{L}_x/\mathfrak{f}^{p+k}\mathcal{L}_x$, and $L_x^q = L_x$ for $q < 0$. It is then easy to see that $\{L_x^p\}_{p\in\mathbb{Z}}$ satisfies all the conditions of Definition 1.1 with $\mu = 1$ except (vi), which is also satisfied if $\mathcal{L}$ is transitive. Thus the formal algebra $L_x$ of a transitive Lie algebra sheaf $\mathcal{L}$ is a transitive filtered Lie algebra of depth 1. It is well-known that under the category of analyticity and under certain regularity condition $\mathcal{L}$ is locally uniquely determined by its formal algebra $L_x$ at a point $x$. In these contexts, and in particular, in connection with the classification of simple infinite Lie algebras, the transitive filtered Lie algebras of depth 1 were well studied ([GS64], [SS65], [KN66], [Hay70], etc.).

*Remark 2.* Suppose now we are given a transitive Lie algebra sheaf $\mathcal{L}$ on a filtered manifold $(M, \mathfrak{f})$ and suppose that $\mathcal{L}$ leaves invariant the filtration $\mathfrak{f}$. Then we can introduce on its formal algebra $L_x$ another filtration more refined than the original one and well adapted to the underlying filtration $\mathfrak{f}$: We define for $q \leq 0$ $\hat{\mathcal{L}}_x^q$ to be the subspace of $\mathcal{L}_x$ consisting of all germs $[X]_x$ at $x$ of sections $X$ of $\mathcal{L}$ such that $X_x \in \mathfrak{f}_x^q$ and set $\hat{L}_x^q$ to be the its image on $L_x$. For $p > 0$ we define $\hat{L}_x^p$ by the condition (vi) of Definition 1.1, replacing $L^k$ by $\hat{L}_x^k$. In this way we have a transitive filtered Lie algebra $\{\hat{L}_x^p\}$ of depth $\mu$. (If we use the notion of weighted jet bundle introduced in Section 3, we can better understand the meaning of this new filtration.)

The remarks above will motivate to study the transitive filtered Lie algebras of depth greater than 1, which not only leads us to a natural generalization of Guillemin-Sternberg but also becomes a good guide to studying geometric structures and differential equations on filtered manifolds.

In this section we study the transitive filtered Lie algebras of depth greater than 1, and we shall see how a filtered Lie algebra can be constructed from finite dimensional data i.e., its truncated Lie algebra.

Let $L$ be a transitive filtered Lie algebra of depth $\mu$. Let $grL = \bigoplus_{p \in \mathbb{Z}} gr_p L$ be its associated graded Lie algebra, where $gr_p L = L^p / L^{p+1}$. Then $grL$ is a transitive graded Lie algebra of depth $\mu$ in the following sense:

**Definition 1.2.** A graded Lie algebra $g = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is called transitive graded Lie algebra (TGLA) of depth $\mu$ if it satisfies the following conditions:

i) $\mathfrak{g}_p = 0$ for $p < -\mu$
ii) $\dim \mathfrak{g}_p < \infty$
iii) For $i \geq 0$, $x_i \in \mathfrak{g}_i$, if $[x_i, \mathfrak{g}_-] = 0$ then $x_i = 0$,

where we set $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$.

Let us recall the notion of prolongation concerning a TGLA ([GS64], [Tan70]). For this we first give the following:

**Definition 1.3.** Let $k$ be an integer or $\infty$. A truncated graded Lie algebra of order $k$ is a graded vector space $\mathfrak{g}(k) = \bigoplus_{p \leq k} \mathfrak{g}_p$ equipped with a bracket operation (skew-symmetric bilinear map)

$$[\ ,\ ]\colon \mathfrak{g}_p \times \mathfrak{g}_q \to \mathfrak{g}_{p+q}$$

defined partially for $p$, $q$, $p+q \leq k$, satisfying the partial Jacobi identity:

$$\mathfrak{S}[[x_p, y_q], z_r] = 0$$

for $x_p \in \mathfrak{g}_p$, $y_q \in \mathfrak{g}_q$, $z_r \in \mathfrak{g}_r$, whenever $p$, $q$, $p+q$, $q+r$, $r+p$, $p+q+r \leq k$, where $\mathfrak{S}$ denotes the cyclic sum in $x_p$, $y_q$, $z_r$. If moreover the conditions (1) (2) (3) of Definition 1.2 are satisfied, $\mathfrak{g}(k)$ is called truncated transitive graded Lie algebra (truncated TGLA) of order $k$ of depth $\mu$.

Note that a truncated TGLA of order $\infty$ is just a TGLA. If $\mathfrak{g}(k) = \bigoplus_{p\leq k} \mathfrak{g}_p$ is a truncated TGLA of order $k$, then for each integer $l \leq k$, $\bigoplus_{p\leq l} \mathfrak{g}_p$ becomes a truncated TGLA of order $l$ with respect to the induced bracket operation, which we will denote by $\mathrm{Trun}_l\, \mathfrak{g}(k)$. Morphisms of truncated TFLA's can be defined in the natural manner. In particular, a homomorphism $\varphi\colon \mathfrak{h}(k) \to \mathfrak{g}(k)$ will be called an embedding if $\varphi$ induces an isomorphism of $\mathfrak{h}_- = \bigoplus_{p<0} \mathfrak{h}_p$ onto $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$. Note that an embedding is necessarily injective.

Now let us define the prolongation of a truncated TGLA $\mathfrak{g}(k) = \bigoplus_{p\leq k} \mathfrak{g}_p$ of order $k \geq -1$. Put $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$ and define $\mathrm{Der}_{k+1}\, \mathfrak{g}(k)$ to be the vector space consisting of all $\alpha \in \mathrm{Hom}(\mathfrak{g}_-, \mathfrak{g}(k))$ such that

$$\begin{cases} \alpha(\mathfrak{g}_p) \subset \mathfrak{g}_{p+k+1} \quad (p<0) \\ \alpha([x,y]) = [\alpha(x), y] + [x, \alpha(y)], \quad \text{for } x,\ y \in \mathfrak{g}_- \end{cases}$$

and we set

$$p\mathfrak{g}(k) = \mathfrak{g}(k) \oplus \mathrm{Der}_{k+1}\, \mathfrak{g}(k).$$

It is then easy to see that there exists a unique bracket operation on $p\mathfrak{g}(k)$ which makes $p\mathfrak{g}(k)$ into a truncated TGLA of order $k+1$ such that $\mathrm{Trun}_k(p\mathfrak{g}(k)) = \mathfrak{g}(k)$ and $[\alpha, x] = \alpha(x)$ for $\alpha \in \mathrm{Der}_{k+1}\, \mathfrak{g}(k)$, $x \in \mathfrak{g}_-$.

Iterating this construction, we obtain a truncated TGLA $p^i\mathfrak{g}(k)$ $(= p(p^{i-1}\mathfrak{g}(k))$ of order $k+i$, and a TGLA $p^\infty\mathfrak{g}(k)$ $(= \mathrm{inj}\lim p^i\mathfrak{g}(k))$. Thus we have:

**Proposition 1.1** (Tanaka). *For a truncated TGLA $\mathfrak{g}(k)$ of order $k$, there exists, uniquely up to isomorphism, a truncated TGLA $p^i\mathfrak{g}(k)$ of order $k+i (0 \leq i \leq \infty)$ which satisfies the following conditions*:

i) $\mathrm{Trun}_k(p^i\mathfrak{g}(k)) = \mathfrak{g}(k)$

ii) *If* $\mathfrak{h}(k+i)$ *is a truncated TGLA of order* $k+i$ *and if there is an embedding* $\psi_k\colon \mathrm{Trun}_k\,\mathfrak{h}(k+i) \to \mathfrak{g}(k)$, *then there exists a unique embedding* $\psi_{k+i}\colon \mathfrak{h}(k+i) \to p^i\mathfrak{g}(k)$ *such that* $\psi_{k+i}\,|_{\mathrm{Trun}_k\,\mathfrak{h}(k+i)} = \psi_k$.

The truncated TGLA $p^i\mathfrak{g}(k)$ is called the prolongation of $\mathfrak{g}(k)$. We will often denote $p^\infty\mathfrak{g}(k)$ by $\mathrm{Prol}\,\mathfrak{g}(k)$. We say also that a TGLA $\mathfrak{g}$ is the prolongation of $\mathrm{Trun}_k\,\mathfrak{g}$ if $\mathfrak{g} = \mathrm{Prol}\,\mathrm{Trun}_k\,\mathfrak{g}$. Note that, by the proposition above, $\mathfrak{g}$ can be always identified with a graded subalgebra of $\mathrm{Prol}\,\mathrm{Trun}_k\,\mathfrak{g}$.

1.2. Generalized Spencer cohomology groups. Now we define a cohomology group associated with a TGLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$. We set

$$\mathfrak{m} = \mathfrak{g}_- = \bigoplus_{p<0}\mathfrak{g}_p,$$

which is a nilpotent subalgebra of $\mathfrak{g}$, and consider the cohomology group associated with the adjoint representation of $\mathfrak{m}$ on $\mathfrak{g}$, namely the cohomology group $H(\mathfrak{m},\mathfrak{g}) = \bigoplus H^p(\mathfrak{m},\mathfrak{g})$ of the cochain complex $(C(\mathfrak{m},\mathfrak{g}) = \oplus C^p(\mathfrak{m},\mathfrak{g}), \partial)$, where

$$C^p(\mathfrak{m},\mathfrak{g}) = \mathrm{Hom}\Big(\bigwedge^p \mathfrak{m},\mathfrak{g}\Big)$$

and the coboudary operator $\partial\colon \mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g}) \to \mathrm{Hom}(\bigwedge^{p+1}\mathfrak{m},\mathfrak{g})$ is defined by

$$\begin{aligned}(\partial\omega)(X_1,\dots,X_{p+1}) &= \sum_{i=1}^{n+1}(-1)^{i-1}[X_i,\omega(X_1,\dots,\hat{X}_i,\dots,X_{p+1})]\\ &+ \sum_{1\le i<j\le p+1}(-1)^{i+j}\omega([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1})\end{aligned}$$

for $\omega \in \mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g})$, $X_1,\dots,X_{p+1}\in\mathfrak{m}$. Since both $\mathfrak{m}$ and $\mathfrak{g}$ are graded, we can define a bigradation $\bigoplus H^p_r(\mathfrak{m},\mathfrak{g})$ of $H(\mathfrak{m},\mathfrak{g})$ as follows: Denote by $\mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g})_r$ the set of all homogeneous $p$-cochain $\omega$ of degree $r$ (i.e., $\omega(\mathfrak{g}_{a_1}\wedge\cdots\wedge\mathfrak{g}_{a_p}) \subset \mathfrak{g}_{a_1+\cdots+a_p+r}$ for any $a_1,\dots,a_p<0$), and set $C_r(\mathfrak{m},\mathfrak{g}) = \mathrm{Hom}(\bigwedge\mathfrak{m},\mathfrak{g})_r = \bigoplus_p \mathrm{Hom}(\bigwedge^p\mathfrak{m},\mathfrak{g})_r$. Note that $\partial$ preserves the degree. Hence $C_r(\mathfrak{m},\mathfrak{g})$ is a subcomplex and the direct sum decomposition

$$C(\mathfrak{m},\mathfrak{g}) = \bigoplus C_r(\mathfrak{m},\mathfrak{g})$$

yields the decomposition of the cohomology group:

$$H(\mathfrak{m},\mathfrak{g}) = \bigoplus H_r(\mathfrak{m},\mathfrak{g}) = \bigoplus H^p_r(\mathfrak{m},\mathfrak{g}).$$

This cohomology group $H(\mathfrak{m},\mathfrak{g})$ was introduced by Tanaka [Tan79] with another gradation:

$$H^{s,p}(\mathfrak{m},\mathfrak{g}) = H^p_{s+p-1}(\mathfrak{m},\mathfrak{g}).$$

It should be remarked that if the depth $\mu = 1$, then $\mathfrak{m}$ is abelian and $H^{s,p}(\mathfrak{m},\mathfrak{g})$ is known as the Spencer cohomology group. The following theorem generalizes the well-known result in the case $\mu = 1$ to the case of arbitrary $\mu$ [Mor88].

**Theorem 1.1.** *Let $\mathfrak{g}$ be a TGLA of depth $\mu$. Then there exists an integer $r_0$ such that $H_r(\mathfrak{m},\mathfrak{g}) = 0$ for all $r \geq r_0$.*

The proof is based on the fact that the universal enveloping algebra of a finite dimensional Lie algebra is Noetherian.

For a concrete criterion in terms of quasi-regular bases for the vanishing of the cohomology group, see [Mor91].

1.3. Truncated transitive filtered Lie algebras. Let $A$ be a vector space. For $\alpha$, $\beta \in \operatorname{Hom}(\bigwedge^2 A, A)$ define $\alpha \circ \beta \in \operatorname{Hom}(\bigwedge^3 A, A)$ by

$$(\alpha \circ \beta)(x,y,z) = \mathfrak{S}\alpha(\beta(x,y),z),$$

where $\mathfrak{S}$ denotes the cyclic sum in $x$, $y$, $z \in A$. Define then a quadratic map

$$J\colon \operatorname{Hom}\Big(\bigwedge^2 A, A\Big) \to \operatorname{Hom}\Big(\bigwedge^3 A, A\Big)$$

by $J(\gamma) = \gamma\circ\gamma$ for $\gamma \in \operatorname{Hom}(\bigwedge^2 A, A)$. Note that to define a Lie algebra structure on $A$ is equivalent to taking a $\gamma \in \operatorname{Hom}(\bigwedge^2 A, A)$ satisfying $J(\gamma) = 0$.

Now if $A$ is endowed with a descending filtration $\{A^p\}_{p\in\mathbb{Z}}$, then $\operatorname{Hom}(\bigwedge^r A, A)$ has the natural filtration $\{\operatorname{Hom}(\bigwedge^r A, A)^k\}$, where $\operatorname{Hom}(\bigwedge^r A, A)^k$ consists of all $\alpha \in \operatorname{Hom}(\bigwedge^r A, A)$ satisfying $\alpha(A^{p_1}\wedge\cdots\wedge A^{p_r}) \subset A^{p_1+\cdots+p_r+k}$ for any $(p_1,\ldots,p_r) \in \mathbb{Z}^r$. Let us introduce on $\operatorname{Hom}(\bigwedge^r A, A)^0$ another filtration $\{I^k \operatorname{Hom}(\bigwedge^r A, A)^0\}_{k\in\mathbb{Z}}$ by defining $I^k \operatorname{Hom}(\bigwedge^r A, A)^0$ to be the subspace of $\operatorname{Hom}(\bigwedge^r A, A^k)^0$ which consists of all $\alpha \in \operatorname{Hom}(\bigwedge^r A, A)^0$ such that

$$\alpha(A^{p_1}\wedge\cdots\wedge A^{p_r}) \subset A^{p_1^*+\cdots+p_r^*+k}$$

for any $(p_1,\ldots,p_r) \in \mathbb{Z}^r$, where we set $p^* = \operatorname{Min}\{p,0\}$.

It is easy to check that if $\alpha - \beta \in I^k \operatorname{Hom}(\bigwedge^2 A, A)^0$ for $\alpha$, $\beta \in \operatorname{Hom}(\bigwedge^2 A, A)^0$ then $J(\alpha)-J(\beta) \in I^k \operatorname{Hom}(\bigwedge^3 A, A)^0$. Therefore if we put

$$\left[\operatorname{Hom}\Big(\bigwedge^r A, A\Big)^0\right]^{[k]} = \operatorname{Hom}\Big(\bigwedge^r A, A\Big)^0 \Big/ I^{k+1} \operatorname{Hom}\Big(\bigwedge^r A, A\Big)^0,$$

we have the induced map

$$J\colon \left[\operatorname{Hom}\left(\bigwedge^2 A, A\right)^0\right]^{[k]} \to \left[\operatorname{Hom}\left(\bigwedge^3 A, A\right)^0\right]^{[k]}$$

defined by $J\alpha^{[k]} = (J(\alpha))^{[k]}$ for $\alpha \in \operatorname{Hom}(\bigwedge^2 A, A)^0$, where $\beta^{[k]}$ denotes the equivalence class of $\beta \in \operatorname{Hom}(\bigwedge^r A, A)^0$ modulo $I^{k+1}\operatorname{Hom}(\bigwedge^r A, A)^0$.

**Definition 1.4.** A truncated filtered Lie algebra of order $k$ is a vector space $A$ endowed with a descending filtration $\{A^p\}_{p\in\mathbb{Z}}$ and a truncated bracket $\gamma^{[k]} \in [\operatorname{Hom}(\bigwedge^2 A, A)^0]^{[k]}$ satisfying the following conditions:

i) $A^{k+1} = 0$
ii) $J(\gamma^{[k]}) = 0$ (truncated Jacobi identity)

Note that if $A(k) = (A, \{A^p\}, \gamma^{[k]})$ is a truncated filtered Lie algebra then $grA$ has the induced structure of truncated graded Lie algebra, which will be denoted by $grA(k)$.

**Definition 1.5.** A truncated filtered Lie algebra $A(k)$ is called a truncated transitive filtered Lie algebra (truncated TFLA) if $grA(k)$ is transitive.

Note that a truncated TFLA of order $\infty$ is just a TFLA.

If $A(k)$ is a truncated TFLA of order $k(\leq \infty)$ then for each $l \leq k$, we have a truncated TFLA of order $l$ denoted by $\operatorname{Trun}_l A(k)$ by passage to the quotient $A(k)/A^{l+1}$.

Homomorphisms of truncated TFLA's are defined in the natural manner. Note that a homomorphism $\varphi\colon A(k) \to B(k)$ of truncated TFLA's gives rise to a homomorphism $gr\varphi\colon grA(k) \to grB(k)$ of truncated GLA's. We say $\varphi$ is an embedding if so is $gr\varphi$.

For a truncated TFLA $A(k)$, the cohomology group $H^p_r(grA(k))$ will be defined to be $H^p_r((\operatorname{Prol} grA(k))_-, \operatorname{Prol} grA(k))$.

1.4. Now we are in a position to state main structure theorems:

**Theorem 1.2.** *Let $A(k)$ be a truncated TFLA of order $k \geq 0$. Assume that*

$$H^2_r(grA(k)) = 0 \quad \textit{and} \quad H^3_s(grA(k)) = 0$$

*for $r \geq k+1$ and $s \geq \operatorname{Max}\{k+1, 2\}$. Then there exists, uniquely up to isomorphism, a complete TFLA $L$ such that*

$$\operatorname{Trun}_k L = A(k) \quad \textit{and} \quad grL = \operatorname{Prol} grA(k).$$

**Theorem 1.3.** *Let $L$ be a complete TFLA and $k$ a non-negative integer such that*

$$H_r^1(grL) = H_r^2(grL) = 0 \quad for \quad r \geq k+1.$$

*If there is an embedding $\psi_k\colon \mathrm{Trun}_k\, K \to \mathrm{Trun}_k\, L$ for a TFLA $K$, then there exists an embedding $\varphi\colon K \to L$ such that $\mathrm{Trun}_k\, \varphi = \psi_k$ Moreover, two such embeddings differ by an inner automorphism of $L$ which fixes $\mathrm{Trun}_k\, L$.*

**Theorem 1.4.** *Let $L_1$ and $L_2$ be complete TFLA's and $k$ a non-negative integer such that*

$$H_r^1(grL_i) = H_r^2(grL_i) = 0 \ \ for \ \ i = 1,\ 2 \ \ and \ \ r \geq k+1.$$

*Then $L_1$ and $L_2$ are isomorphic if and only if so are $\mathrm{Trun}_k\, L_1$ and $\mathrm{Trun}_k\, L_2$.*

**Corollary 1.5.** *If $L$ is a complete TFLA satisfying*

$$H_r^1(grL) = H_r^2(grL) = 0 \ \ for \ \ r \geq 1,$$

*then $L$ is graded, that is, isomorphic to the completion of $grL$.*

For the proofs see [Mor88]. The above theorems as well as their proofs clarify how a TFLA is constructed step by step from a truncated TFLA of lower order.

## §2. Geometric structures on filtered manifolds

2.0. The main problem that we discuss in this section is the equivalence problem: It is, given two geometric structures, to obtain criteria to decide whether they are (locally) equivalent. For this problem, geometrically, the main task is to determine the complete invariants of a given structure.

The general equivalence problem was first posed by Lie to find the differential invariants under the action of a finite or infinite dimensional Lie group and was investigated during the last quarter of the 19th century by Lie himself, Halphen, Tresse, Wilczynski etc., mainly for various classes of differential equations.

At the beginning of the 20th century Cartan invented a powerful method for the equivalence problem by combining Lie group theory and the method of moving frames, and applied it to his extensive work, especially to his brilliant work in 1900s on the theory of infinite groups and on geometric studies of differential equations. Later in 1920s he also introduced the notion of éspace généralisé, what is nowadays called Cartan connection. He wrote:

*In the wake of the movement of ideas which followed the general theory of relativity, I was led to introduce the notion of new geometries, more general than Riemannian geometry, and playing with respect to the different Klein geometries the same role as the Riemannian geometries play with respect to Euclidean space. The vast synthesis that I realized in this way depends of course on the ideas of Klein formulated in his celebrated Erlangen programme while at the same time going far beyond it since it includes Riemannian geometry, which had formed a completely isolated branch of geometry, within the compass of a very general scheme in which the notion of group still plays a fundamental role.* (E. Cartan [Car31] p. 58. The translation is borrowed from [Sha97].)

However, it took a rather long time until after the second world war that Cartan's fundamental ideas came to be rigorously formulated and developed into modern theory by the work of Ehresmann, Chern, Kuranishi, Spencer and others. In particular, Chern [Che54], Guillemin, Singer and Sternberg ([GS64], [SS65]) formulated the equivalence problem as that of $G$-structures and clarified many of Cartan's ideas.

Meanwhile, inspired by the deep work of Cartan, Tanaka elaborated skilled methods to construct Cartan connections through his studies on conformal and projective connections, on $CR$ geometry and on geometry of differential equations. In particular, he developed fundamental work on differential systems. ([Tan62], [Tan70], [Tan76] and [Tan79].)

Pursuing a more complete treatment of equivalence problem, Morimoto introduced the notion of $C$-fibre [Mor83] and then extended this as that of tower on filtered manifolds [Mor93], which gives us wider perspectives to develop a unified theory on the equivalence problem, in particular, to lead us to a basic notion of weighted involutivity and to a general criterion for the existence of Cartan connection.

In this section we will introduce the notion of a tower and explain the method to treat the equivalence problem, laying emphasis on the conceptual aspects and referring to [Mor93] for technical details.

2.1. Let us begin with some reflection on differentiation. Let $f$ be a function of $x_1, \dots, x_n$. The partial derivatives $\frac{\partial f}{\partial x_i}$ are nothing but the coefficients of $df$:

$$df = f_1\, dx_1 + \cdots + f_n\, dx_n.$$

If we have in mind a certain geometric structure and if we have no reason to choose a special system of coordinates but a certain family of coframes $\omega_1, \dots, \omega_n$ invariantly associated to the structure, it will be better to differentiate with respect to the coframes:

$$df = f_1\omega_1 + \cdots + f_n\omega_n.$$

But then $f_i$ will be a function not only of $(x_1, \ldots, x_n)$ but also of new parameters $\lambda_1, \ldots, \lambda_l$ if coframes $\omega_1, \ldots, \omega_n$ depend on $l$-parameters $\lambda_1, \ldots, \lambda_l$.

Next if we have a distinguished family of coframes of the space $x_1, \ldots, x_n$, $\lambda_1, \ldots, \lambda_l$, say, $\omega_1, \ldots, \omega_n$, $\pi_1, \ldots, \pi_l$, where $\pi_p$ are 1-forms expressed in terms of $x_1, \ldots, x_n$, $\lambda_1, \ldots, \lambda_l$ and new parameters $\mu_1, \ldots,$ $\mu_m$, we get second order derivatives with respect to these coframes:

$$df_i = \sum f_{ij}\omega_j + \sum f_{i;p}\pi_p,$$

where $f_{ij}$, $f_{i;p}$ are now functions of $x$, $\lambda$, $\mu$.

If each family of coframes is taken in an invariant way, then the parameter spaces $\{\lambda_1, \ldots, \lambda_l\}$, $\{\lambda_1, \ldots, \lambda_l, \mu_1, \ldots, \mu_m\}$ will all form Lie groups.

Iterating this procedure to higher orders, we may arrive to a space to which we need no longer to add new parameters, or we have to continue the procedure infinitely.

This leads us to consider as a model of the space in which we finally arrive after the above procedure the following objects: $(P, M, G, E, \rho, \theta)$, where $P$ is a principal fibre bundle over a manifold $M$ with the structure group $G$ equipped with an absolute parallelism, that is, a 1-form $\theta$ taking values in a vector space $E$ such that $\theta_z \colon T_zP \to E$ is an isomorphism for all $z$.

It will be natural to assume that there is a representation $\rho$ of $G$ on $E$ and satisfies the following conditions:

(T1) $$R_a^*\theta = \rho(a)^{-1}\theta \quad \text{for } a \in G,$$

where $R_a$ denotes the right translation by $a$.

(T2) There is an exact sequence of $G$-modules:

$$0 \to \mathfrak{g} \to E,$$

where the Lie algebra $\mathfrak{g}$ of $G$ is regarded as a $G$-module by the adjoint action of $G$ on $\mathfrak{g}$.

(T3) $$\theta(\widetilde{A}) = A \quad \text{for } A \in \mathfrak{g},$$

where $\widetilde{A}$ denotes the vector field on $P$ induced by the right translations $\{R_{\exp tA}\}$.

The structure $(P, M, G, E, \rho, \theta)$ as above, with some additional assumptions mentioned later on, will be called a *tower* on $M$ with (algebraic) *skeleton* $(G, E, \rho)$.

If $E$ happens to be a finite dimensional Lie algebra $\mathfrak{l}$ containing $\mathfrak{g}$ as a Lie subalgebra, then the tower is just a principal fibre bundle with a Cartan connection. If moreover the structure function $\gamma$ defined by the structure equation;

$$d\theta + \frac{1}{2}\gamma(\theta \wedge \theta) = 0$$

is constant, then the tower represents locally a homogeneous space: $L \to L/G$, where $L$ is a Lie group with Lie algebra $\mathfrak{l}$ and $\theta$ is the Maurer-Cartan form of $L$.

It should be noted that towers $P$ (and hence $G$, $E$) may be infinite dimensional.

**Example.** The infinite order frame bundle $\mathcal{F}^\infty(M)$ of a differentiable manifold $M$ is defined to be the set of all infinite order jet $j_0^\infty f$, where $f\colon \mathbb{R}^n \to M$ is a local diffeomorphism from a neighbourhood of the origin $0 \in \mathbb{R}^n$ into $M$. This is a principal fibre bundle (of infinite dimension) over $M$ of which the structure group $G^\infty(\mathbb{R}^n)$ is the group consisting of all $j_0^\infty g$, where $g\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a local diffeomorphism with $g(0) = 0$. Let $L = J_0^\infty T\mathbb{R}^n$ denote the set of all $\infty$-jet at 0 of local vector fields on a neighbourhood of 0 in $\mathbb{R}^n$, i.e., the Lie algebra of all formal vector fields at 0. Since the tangent space of $\mathcal{F}^\infty(M)$ at $j_0^\infty f$ may be identified with $J_{f(0)}^\infty TM$, the map

$$j(f_*^{-1})\colon J_{f(0)}^\infty TM \to J_0^\infty T\mathbb{R}^n$$

defines a $L$-valued 1-form $\theta$ on $\mathcal{F}^\infty(M)$.

What is important is that every tower has a natural filtration which represents implicitly the notion of "differentiation". Let $(P, M, G, \theta, E)$ be a tower. Then there is a canonical filtration $\mathfrak{f}_{tr}$ of $G$ and $\mathfrak{g}$ (and then on the tangent bundle of $P$) defined inductively by the following exact sequences:

$$1 \longrightarrow \mathfrak{f}_{tr}^{p+1}G \longrightarrow G \longrightarrow GL(E/\mathfrak{f}_{tr}^p\mathfrak{g}) \tag{2.1}$$

with $\mathfrak{f}_{tr}^0 G = G$ and $\mathfrak{f}_{tr}^p\mathfrak{g}$ denoting the Lie algebra of $\mathfrak{f}_{tr}^p G$. This is, so to speak, the filtration according to the Taylor expansion.

It is natural to assume the action of $G$ is *formally effective* in the following sense:

$$\bigcap_{p \in \mathbb{Z}} \mathfrak{f}_{tr}^p G = \{e\}. \tag{T4}$$

Now let us turn our attention to filtered manifolds. A tower $(P, M, G, \theta)$ with skeleton $(E, G, \rho)$ is called a tower on a filtered manifold $\mathbb{M}$ if there is a filtration $\{\mathfrak{f}^p E/\mathfrak{g}\}$ of $E/\mathfrak{g}$ invariant under the action of $G$ and if $\theta$ preserves the filtrations, that is, for all $z \in P$ the map $T_{\pi(z)}M \to E/\mathfrak{g}$ induced by $\theta_z$ preserves the filtrations.

In this case the skeleton $(E, G, \rho)$ leaves invariant the filtration $\{\mathfrak{f}^p E/\mathfrak{g}\}$, and we can introduce another filtration $\{\mathfrak{f}^p G\}$ of $G$ by the following exact sequence:

$$(2.2) \qquad 1 \longrightarrow \mathfrak{f}^{p+1}G \longrightarrow G \longrightarrow GL(E/\mathfrak{f}^p\mathfrak{g})/\mathfrak{f}^{p+1}GL(E/\mathfrak{f}^p\mathfrak{g}),$$

with $\mathfrak{f}^0 G = G$, where the filtration $\{\mathfrak{f}^k\}$ of $E/\mathfrak{f}^p\mathfrak{g}$ is the induced one from that of $E/\mathfrak{g}$ for $k \leq 0$ and from $\mathfrak{f}^k\mathfrak{g}$ for $k \geq 0$, and the filtration $\mathfrak{f}^j GL(E/\mathfrak{f}^p\mathfrak{g})$ is the natural one induced from that of $E/\mathfrak{f}^p\mathfrak{g}$.

This filtration derives from the notion of weighted order of the filtered manifold and will be used exclusively in studying the towers on the filtered manifold.

So far we have not specified the category in which possibly infinite dimensional objects $P$, $G$, $E$ are considered. By virtue of the filtrations introduced above, we see the proper category is that of projective limits of finite dimensional objects, where we can freely speak of principal bundles, Lie groups, their Lie algebras, differential forms etc. as in the finite dimensional case.

2.2. To define the morphisms of towers on filtered manifolds, we make the following convention:

We choose once for all one filtered vector space $\mathbb{V} = (V, \{V^p\})$ for each isomorphic class of filtered vector spaces, for instance, $V = \mathbb{R}^{\dim V}$, $V^p = \mathbb{R}^{\dim V^p}$ with fixed standard inclusions, and we assign to each filtered manifold $\mathbb{M}$ one such filtered vector space $\mathbb{V}$ called the type (filtered vector space) of $\mathbb{M}$ such that $T_x\mathbb{M}$ is isomorphic to $\mathbb{V}$ for all $x \in M$. Thus two filtered manifolds are of the same type if they are isomorphic.

If $P$ is a tower on a filtered manifold $\mathbb{M}$ with skeleton $(E, G)$, then $E/\mathfrak{g}$ is isomorphic to the type space $\mathbb{V}$ of $\mathbb{M}$. We shall always fix one splitting $\mathbb{V} \to E$ for the skeleton $(E, G)$ so that we have:

$$(\mathrm{T5}) \qquad E = V \oplus \mathfrak{g}$$

Now let $\mathbb{M}$ and $\mathbb{M}'$ be filtered manifolds of the same type $\mathbb{V}$. Let $P(\mathbb{M}, G, \theta)$ and $P(\mathbb{M}', G', \theta')$ be towers on $\mathbb{M}$ and $\mathbb{M}'$ respectively. A morphism of towers $P$ to $P'$ is a homomorphism of principal fibre bundles $\varphi \colon P \to P'$ with the induced diffeomorphism $\varphi^{(-1)} \colon M \to M'$ and the induced Lie homomorphism $\iota\colon G \to G'$ such that

$$\varphi^*\theta' = \iota_* \circ \theta,$$

where $\iota_*$ denotes the induced map:

$$\iota_*(= \mathrm{Id}_V + \iota_*)\colon E(= V \oplus \mathfrak{g}) \to E'(= V \oplus \mathfrak{g}').$$

A morphism $\varphi$ will be referred to as an isomorphism if $\varphi$ is a diffeomorphism, and as an embedding if $M = M'$ and $\varphi^{(-1)} = \mathrm{Id}_M$.

The category of towers has the following remarkable properties:

**Proposition 2.1.** *For a filtered manifold $\mathbb{M}$ there exists a universal tower on $\mathbb{M}$ such that any tower on $\mathbb{M}$ is uniquely embedded in the universal tower.*

We denote the universal tower of the filtered manifold $\mathbb{M}$ of type $\mathbb{V}$ by $(\mathcal{R}(\mathbb{M}), M, G(\mathbb{V}), \theta_{\mathcal{R}})$ and its skeleton by $(E(\mathbb{V}), G(\mathbb{V}))$.

Using the filtration introduced by (2.2), we set:

$$\mathcal{R}^{(k)}(\mathbb{M}) = \mathcal{R}(\mathbb{M})/\mathfrak{f}^{k+1},$$

the quotient bundle by the action of $\mathfrak{f}^{k+1}G(\mathbb{V})$. It is a principal bundle over $M$ with structure group $G^{(k)}(\mathbb{V}) = G(\mathbb{V})/\mathfrak{f}^{k+1}$ and is referred to as the *non-commutative frame bundle* of $\mathbb{M}$ of (weighted) order $k+1$.

We say that a principal subbundle $P^{(k)}$ of $\mathcal{R}^{(k)}(\mathbb{M})$ is *adapted* if there exists a tower $P$ on $\mathbb{M}$ such that $P^{(k)} = P/\mathfrak{f}^{k+1}$. We then have:

**Proposition 2.2.** *For an adapted subbundle $P^{(k)}$ of $\mathcal{R}^{(k)}(\mathbb{M})$ there exists a unique universal tower $\mathcal{R}P^{(k)}$ such that $\mathcal{R}P^{(k)}/\mathfrak{f}^{k+1} = P^{(k)}$ and any tower $Q$ on $\mathbb{M}$ is embedded in $\mathcal{R}P^{(k)}$ if $Q/\mathfrak{f}^{k+1} \subset P^{(k)}$.*

We have also:

**Proposition 2.3.** *If $f\colon \mathbb{M} \to \mathbb{M}'$ is an isomorphism of filtered manifolds, it canonically induces isomorphisms $\mathcal{R}f\colon \mathcal{R}(\mathbb{M}) \to \mathcal{R}(\mathbb{M}')$ and $\mathcal{R}^{(k)}f\colon \mathcal{R}^{(k)}(\mathbb{M}) \to \mathcal{R}^{(k)}(\mathbb{M}')$. Moreover if $P^{(k)}$ is an adapted subbundle of $\mathcal{R}^{(k)}(\mathbb{M})$, then $(\mathcal{R}^{(k)}f)P^{(k)}$ is also adapted and $(\mathcal{R}f)(\mathcal{R}P^{(k)}) = \mathcal{R}((\mathcal{R}^{(k)}f)P^{(k)})$.*

The tower $\mathcal{R}P^{(k)}$ is called the universal tower prolonging $P^{(k)}$ or the universal prolongation of $P^{(k)}$. We set

$$\#P^{(k)} = \mathcal{R}P^{(k)}/\mathfrak{f}^{k+2}$$

and call it also the prolongation of $P^{(k)}$.

The above universal properties completely characterize the inductive construction of $\mathcal{R}^{(p)}(\mathbb{M})$ and $\mathcal{R}P^{(k)}/\mathfrak{f}^{p+1}$ to obtain $\mathcal{R}(\mathbb{M})$ and $\mathcal{R}P^{(k)}$ as their projective limits.

The first order frame bundle $\mathcal{R}^{(0)}(\mathbb{M})$ of a filtered manifold $\mathbb{M}$ of type $\mathbb{V}$ is given as follows: Let $\hat{\mathcal{R}}^{(0)}(\mathbb{M})$ be the set of all linear frames

$$\hat{z}\colon \mathbb{V} \to T_x\mathbb{M}$$

preserving the filtrations. It is a principal fibre bundle over $\mathbb{M}$ whose structure group is $\mathfrak{f}^0 GL(\mathbb{V})$, the group of all filtration preserving linear isomorphisms. Then we see

$$\mathcal{R}^{(0)}(\mathbb{M}) = \hat{\mathcal{R}}^{(0)}(\mathbb{M})/\mathfrak{f}^1,$$

namely, the quotient bundle by the action of $\mathfrak{f}^1 GL(\mathbb{V})$, which is a principal fibre bundle over $\mathbb{M}$ with structure group $G^{(0)}(\mathbb{V}) = \mathfrak{f}^0 GL(\mathbb{V})/\mathfrak{f}^1$.

In other word, $\mathcal{R}^{(0)}(\mathbb{M})$ is a the set of all isomorphisms of graded vector spaces

$$z\colon gr\mathbb{V} \to grT_x\mathbb{M}.$$

Then the inductive construction can be carried out by the following properties:

(1) Every subbundle of $\mathcal{R}^{(0)}(\mathbb{M})$ is adapted.

(2) For $k > 0$, a subbundle $P^{(k)}$ of $\mathcal{R}^{(k)}(\mathbb{M})$ is adapted if and only if so is $P^{(k-1)} = P^{(k)}/\mathfrak{f}^k$ and $P^{(k)}$ is a subbundle of $\#P^{(k-1)}$.

(3) To every adapted subbundle $(P^{(k)}, G^{(k)})$ of $\mathcal{R}^{(k)}(\mathbb{M})$, there is associated the principal fibre bundle $(\hat{\#}P^{(k)}, \hat{\#}G^{(k)})$ over $\mathbb{M}$ consisting of filtration preserving linear isomorphisms

$$\hat{z}^{k+1}\colon V \oplus \mathfrak{g}^{(k)} \to T_{z^k}P^{(k)} \quad (\text{with } z^k \in P^{(k)})$$

such that

i) $\hat{z}^{k+1}(A) = \widetilde{A}_{z^k}$ for $A \in \mathfrak{g}^{(k)}$,

ii) $[\hat{z}^k] = z^k$, where $[\hat{z}^k]$ denotes the equivalence class modulo $\mathfrak{f}^{k+1}$ and $\hat{z}^k$ is defined by the commutative diagramme:

$$\begin{array}{ccc}
V \oplus \mathfrak{g}^{(k)} & \xrightarrow{\hat{z}^{k+1}} & T_{z^k}P^{(k)} \\
\downarrow & & \downarrow{\scriptstyle \pi_*} \\
V \oplus \mathfrak{g}^{(k-1)} & \xrightarrow{\hat{z}^{k}} & T_{z^{k-1}}P^{(k-1)}
\end{array}$$

(4) For an adapted subbundle $P^{(k)}$ of $\mathcal{R}^k(\mathbb{M})$,

$$\#P^{(k)} = \hat{\#}P^{(k)}/\mathfrak{f}^{k+2}.$$

Then $\mathcal{R}(\mathbb{M})$ and $\mathcal{R}P^{(k)}$ are obtained by

$$\mathcal{R}(\mathbb{M}) = \operatorname*{proj\,lim}_{i\to\infty} \#^i\mathcal{R}^{(0)}(\mathbb{M}), \quad \mathcal{R}P^{(k)} = \operatorname*{proj\,lim}_{i\to\infty} \#^i P^{(k)}.$$

We remark that if $M$ is a trivial filtered manifold then $\mathcal{R}(M)$ has a system of local coordinates $(x^i_{j_1,\dots,j_m})$ with $1 \le i,\ j_1,\dots,j_m \le \dim M$, $m = 0,\ 1,\ 2,\dots$, (the introduction of new variables which stand for the higher order derivatives, but without any commutation relations), while the usual infinite order frame bundle $\mathcal{F}^\infty(M)$ is embedded in $\mathcal{R}(M)$ by the equation $x^i_{j_{\sigma(1)},\dots,j_{\sigma(m)}} = x^i_{j_1,\dots,j_m}$ for all permutations $\sigma$. This is the reason why $\mathcal{R}(M)$ is called the non-commutative frame bundle of $M$ and has a great advantage for studying curved spaces.

For instance, if the filtration is not trivial, since the filtered manifold $\mathbb{M}$ itself may not be even locally trivial, we have a priori no counter-part to $\mathcal{F}(M)$ associated to $\mathbb{M}$, but we always have the non-commutative frame bundle $\mathcal{R}(\mathbb{M})$ which is large enough to contain all curved structures.

2.3. Structure functions.

We now introduce the structure function of a tower. Let $(P, \mathbb{M}, G, \theta)$ be a tower on a filtered manifold $\mathbb{M}$ with skeleton $(E, G)$. Since $\theta$ defines an absolute parallelism on $P$, there exists a unique $\mathrm{Hom}(\bigwedge^2 E, E)$-valued function $\gamma$ on $P$ which satisfies the following structure equation:

$$d\theta + \frac{1}{2}\gamma(\theta,\theta) = 0. \tag{2.3}$$

This function $\gamma$, referred to as the structure function of the tower $P$, has the following properties:

**Proposition 2.4.** *Let $\gamma$ be the structure function of a tower $(P, \mathbb{M}, G, \theta)$. Then*

i) $\gamma(z)(A, X) = A \cdot X$ *for* $z \in P$, $A \in \mathfrak{g}$, $X \in E$.
ii) $\gamma(za)(X, Y) = a^{-1}\gamma(z)(aX, aY)$ *for* $z \in P$, $a \in G$, $X$, $Y \in E$.
iii) *If* $\varphi \colon P \longrightarrow P'$ *is a morphism of towers then* $\varphi^*\gamma' = \gamma$, *where* $\gamma'$ *denotes the structure function of* $P'$.

If we denote also $\rho$ the natural representation of $G$ on $\mathrm{Hom}(\bigwedge^2 E, E)$, then the above formula (2) is written as

$$R_a^*\gamma = \rho(a)^{-1}\gamma.$$

Since $E = V \oplus \mathfrak{g}$, we have the direct sum decomposition:

$$\mathrm{Hom}\Big(\bigwedge^2 E, E\Big) = \mathrm{Hom}\Big(\bigwedge^2 V, E\Big) \oplus \mathrm{Hom}(\mathfrak{g} \otimes E, E).$$

Note that $\mathrm{Hom}(\bigwedge^2 V, E)$ is a $G$-invariant subspace of $\mathrm{Hom}(\bigwedge^2 E, E)$, while $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ is not invariant. Let $\beta$ denote the element of $\mathrm{Hom}(\mathfrak{g} \otimes E, E)$ given by the action of $\mathfrak{g}$ on $E$:

$$\beta(A, X) = A \cdot X \quad \text{for } A \in \mathfrak{g},\ X \in E.$$

Then the representation $\rho$ induces the affine representation of $G$ on the affine subspace $\beta + \mathrm{Hom}(\bigwedge^2 V, E)$, and the structure function $\gamma$ is a $G$-equivariant map from $P$ to the affine space $\beta + \mathrm{Hom}(\bigwedge^2 V, E)$. The $\mathrm{Hom}(\bigwedge^2 V, E)$-valued function $c$ given by

$$\gamma = \beta + c \tag{2.4}$$

is therefore the crucial part of $\gamma$ and also called the structure function of $P$.

We next introduce the structure function of a truncated tower. Let $(P^{(k)}, \mathbb{M}, G^{(k)})$ be a truncated tower, that is, an adapted subbundle of $\mathcal{R}^{(k)}(\mathbb{M})$. Let $(P, \mathbb{M}, G, \theta)$ be any tower prolonging $P^{(k)}$, i.e., $P/\mathfrak{f}^{k+1} = P^{(k)}$ and $\gamma$ its structure function. Let $\{\mathfrak{f}^p \mathrm{Hom}(\bigwedge^2 E, E)\}$ be the natural induced filtration. First of all we see that the structure function of a tower takes values in $\mathfrak{f}^0 \mathrm{Hom}(\bigwedge^2 E, E)$.

To define the structure function of $P^{(k)}$ we put

$$\mathrm{Hom}\Big(\bigwedge^2 E, E\Big)^{[k]} = \mathfrak{f}^0 \mathrm{Hom}\Big(\bigwedge^2 E, E\Big)/I^{k+1}. \tag{2.5}$$

Here $\{I^k\}$ is the filtration of $\mathrm{Hom}(\bigwedge^2 E,\ E)$, the same filtration as we used for truncated Lie algebras (cf. §1, 1.3):

$$\alpha \in I^k \mathrm{Hom}\Big(\bigwedge^2 E, E\Big) \iff \alpha(\mathfrak{f}^p E \wedge \mathfrak{f}^q E) \subset \mathfrak{f}^{p^*+q^*+k} E \qquad \forall p,\ q \in \mathbb{Z},$$

where $p^* = p$ for $p < 0$ and $p^* = 0$ for $p \geq 0$. Note that

$$\mathrm{Hom}\Big(\bigwedge^2 E, E\Big)^{[k]} \cong \mathfrak{f}^0 \mathrm{Hom}\Big(\bigwedge^2 E^{(k)}, E^{(k)}\Big)/I^{k+1}.$$

Note also that $G$ acts on $\mathrm{Hom}(\bigwedge^2 E, E)^{[k]}$ with $\mathfrak{f}^{k+1}G$ acting trivially. Hence the structure function $\gamma$ of $P$ induces a map $\gamma^{[k]}$ which makes the following diagramme commutative:

$$\begin{array}{ccc}
P & \xrightarrow{\ \gamma\ } & \mathrm{Hom}(\bigwedge^2 E, E) \\
\downarrow & & \downarrow \\
P^{(k)} & \xrightarrow{\ \gamma^{[k]}\ } & \mathrm{Hom}(\bigwedge^2 E, E)^{[k]}
\end{array}$$

Moreover, with respect to the induced representation $\rho^{[k]}$ of $G^{(k)}$ on $\mathrm{Hom}(\bigwedge^2 E, E)^{[k]}$, we have

$$\gamma^{[k]}(za) = \rho^{[k]}(a^{-1})\gamma^{[k]}(z) \qquad \text{for } z \in P^{(k)},\ a \in G^{(k)}.$$

It should be noted that $\gamma^{[k]}$ does not depend of the choice of the tower $P$ but only on $P^{(k)}$. We can also decompose $\gamma^{[k]}$ as

$$\gamma^{[k]} = \beta^{[k]} + c^{(k)}, \tag{2.6}$$

where $c^{(k)}$ is the $\mathfrak{f}^0 \mathrm{Hom}(\bigwedge^2 V, E^{(k-1)})/\mathfrak{f}^{k+1}$-component and $\beta^{[k]}$ the projection of $\beta$.

The function $\gamma^{[k]}$ as well as $c^{(k)}$ will be referred to as the structure function of $P^{(k)}$. Summarizing the above discussion, we have:

**Proposition 2.5.** *The structure function $\gamma^{[k]}$ of a truncated tower $(P^{(k)}, \mathbb{M}, G^{(k)})$ is a $G^{(k)}$-equivariant map*

$$\gamma^{[k]} \colon P^{(k)} \longrightarrow \mathrm{Hom}\Big(\bigwedge^2 E, E\Big)^{[k]} \cong \mathfrak{f}^0 \mathrm{Hom}\Big(\bigwedge^2 E^{(k)}, E^{(k)}\Big)/I^{k+1},$$

*and if $\varphi^{(k)} \colon P^{(k)} \longrightarrow P'^{(k)}$ is an adapted homomorphism then*

$$(\varphi^{(k)})^* \gamma'^{[k]} = \gamma^{[k]}.$$

Let $(P, M, \pi; \theta)$ be a tower with skeleton $(E, G)$. Let us see what the tower $P$ looks like when the structure function $\gamma$ is constant.

Assume that $\gamma$ is constant. Applying the exterior differentiation to the structure equation (2.3), we have

$$\gamma(\gamma(\theta, \theta), \theta) = 0,$$

which implies $\gamma(\in \mathrm{Hom}(\bigwedge^2 E, E))$ satisfies the Jacobi identify:

$$\mathfrak{S}\gamma(\gamma(x, y), z) = 0, \quad x,\ y,\ z \in E.$$

Hence the filtered vector space $E$, endowed with the bracket operation given by $\gamma$, becomes a Lie algebra. Moreover, as easily seen, it is a transitive filtered Lie algebra. Thus,

**Proposition 2.6.** *If the structure function $\gamma$ of a tower $P$ with skeleton $(E, G)$ is constant, then $(E, \gamma)$ is a transitive filtered Lie algebra.*

Thus a tower $(P, M, G; \theta)$ with constant structure function $\gamma$ is an analogue of a homogeneous space $\tilde{G}/G$ with $\tilde{G}$ a Lie group possibly infinite dimensional and $G$ its closed Lie subgroup.

We have also:

**Proposition 2.7.** *If the structure function $\gamma^{[k]}$ of a truncated tower $(P^{(k)}, M, G^{(k)})$ is constant, then $(E^{(k)}, \gamma^{[k]})$ is a truncated transitive filtered Lie algebra.*

It should be noted that the constancy of the structure function $\gamma$ or $\gamma^{[k]}$ has a strong effect to reduce the "size" of $G$ or $G^{(k)}$ (hence $P$ and $P^{(k)}$) just as the passage from a tensor algebra to an enveloping algebra.

2.4. Equivalence problems.

Without much loss of generality, we may define a geometric structure of weighted order $k$ on a filtered manifold $\mathbb{M}$ to be an adapted subbundle $P^{(k)}$ of $\mathcal{R}(\mathbb{M})^{(k)}$, which will be alternatively called a truncated tower of order $k$ on $\mathbb{M}$.

Two geometric structures $(P^{(k)}, \mathbb{M}, G^{(k)})$ and $(P'^{(k)}, \mathbb{M}', G'^{(k)})$ are said to be isomorphic (or equivalent) if there exists an isomorphism $\varphi^{(k)} \colon P^{(k)} \longrightarrow P'^{(k)}$ of adapted subbundles. This is equivalent to saying that there exists an isomorphism $f \colon \mathbb{M} \longrightarrow \mathbb{M}'$ of filtered manifolds such that the lift $\mathcal{R}^{(k)} f \colon \mathcal{R}^{(k)}(\mathbb{M}) \longrightarrow \mathcal{R}^{(k)}(\mathbb{M}')$ sends $P^{(k)}$ onto $P'^{(k)}$.

We say that $P^{(k)}$ and $P'^{(k)}$ are locally isomorphic (or locally equivalent) at $(x, x') \in M \times M'$ if there exist neighbourhoods $U$, $U'$ of $x$, $x'$ respectively and an isomorphism of filtered manifolds $f \colon U \longrightarrow U'$ such that $f(x) = x'$ and that

$$\mathcal{R}^{(k)} f(P^{(k)}|_U) = P'^{(k)}|_{U'}.$$

Given a geometric structure of order $k+1$ on a filtered manifold $\mathbb{M}$, that is, a truncated tower $(P^{(k)}, \mathbb{M}, G^{(k)})$. The general procedure to find the invariants of $P^{(k)}$ proceeds as follows:

Since the structure function $\gamma^{[k]} \colon P^{(k)} \longrightarrow \mathrm{Hom}(\bigwedge^2 E, E)^{[k]}$ is a $G^{(k)}$-equivariant map, the image $\gamma^{[k]}$ decomposes into $G^{(k)}$-orbits. Suppose that it consists of a single $G^{(k)}$-orbit. Then choose a $\mathring{\gamma} \in \gamma^{[k]}(P^{(k)})$, and reduce $P^{(k)}$ to obtain $Q^{(k)} = (\gamma^{[k]})^{-1}(\mathring{\gamma})$. Note that a different choice of $\mathring{\gamma}$ yields a conjugate subbundle.

Note also that $Q^{(k)}$ may not be adapted. If it is not adapted we take an $l (< k)$ such that $Q^{(l)} = Q^{(k)}/\mathfrak{f}^{l+1}$ is adapted. (If $Q^{(k)} \to P^{(l-1)}$ is surjective then $Q^{(l)}$ is adapted. $Q^{(0)}$ is always adapted.)

Next we prolong $Q^{(l)}$ to get $\#Q^{(l)}$ and iterate this procedure.

In the course of the procedure, if the image of a structure function happens to contain more than one orbits, the geometric structure is intransitive. To treat the intransitive cases, we have to generalize our formulation to treat principal bundles whose structure groups $G^{(k)}$

may vary with parameters. For detailed discussion we refer to [Kis79], [Mor83] and [Mor93].

If the structure is transitive, then the finiteness theorem (Theorem 1.1) of generalized Spencer cohomology group assures that after a finite number of prolongation and reduction we will arrive at what we call a (weightedly) involutive truncated tower:

**Definition 2.1.** An adapted subbundle $(P^{(k)}, M, G^{(k)})$ of $\mathcal{R}^{(k)}(\mathbb{M})$ (namely, a truncated tower) is called weightedly involutive if the following conditions are satisfied:

i) The structure function $\gamma^{[k]}$ is constant.

ii) $H^2(grE^{(k)})_r = 0 \quad \text{for } r \geq k+1.$

Note that, in the definition above, since $\gamma^{[k]}$ is constant, $grE^{(k)}$ becomes a transitive truncated graded Lie algebra, so that it makes sense to speak of the cohomology group $H(\operatorname{Prol} grE^{(k)})$, which is denoted simply by $H(grE^{(k)})$.

We shall often use the adjective "involutive" in the extended sense of "weightedly involutive" if it is clear from the context.

Then we have:

**Theorem 2.1.** *For an involutive truncated tower $P^{(k)}$, we can construct, in a natural manner, a tower $P$ with constant structure function such that $P/\mathfrak{f}^{k+1} = P^{(k)}$.*

In fact, by the vanishing of the cohomology group, it can be shown that the image of the structure function $\gamma^{[k+1]}$ of $\#P^{(k)}$ consists of a single orbit, moreover that the reduction $P^{(k+1)} = (\gamma^{[k+1]})^{-1}(\overset{\circ}{\gamma})$ is adapted and involutive for any $\overset{\circ}{\gamma} \in \operatorname{Im} \gamma^{[k+1]}$. Iterating this, we obtain a tower $P$ with constant structure function: $P = \operatorname{proj}\lim_l P^{(k+l)}$

Thus, after we reach an involutive tower the prolongation and reduction procedure proceeds automatically and there appear no essentially new invariants.

It should be remarked that the way of constructing $P^{(k+l)}$ from $P^{(k)}$ just correspond to the way in which truncated transitive Lie algebra $(E^{(k+l)}, \gamma^{(k+l)})$ is algebraically constructed from $(E^k, \gamma^{(k)})$.

Now, assuming the analyticity, we solve the local equivalence problem of involutive truncated towers.

**Theorem 2.2.** *Let $\mathbb{M}$ and $\mathbb{M}'$ be filtered manifolds of type $\mathbb{V}$, and let $(P^{(k)}, \mathbb{M}, G^{(k)})$ and $({P'}^{(k)}, \mathbb{M}', {G'}^{(k)})$ be involutive subbundle of $\mathcal{R}^{(k)}(\mathbb{M})$ and $\mathcal{R}^{(k)}(\mathbb{M}')$ with structure functions $\gamma^{[k]}$ and ${\gamma'}^{[k]}$ respectively. Then under the assumption of analyticity the following two conditions are equivalent:*

1) $G^{(k)} = G'^{(k)}$ *and* $\gamma^{[k]} = \gamma'^{[k]}$
2) *For any* $(p, p') \in P^{(k)} \times P'^{(k)}$, *there exist open neighbourhoods* $U$ *and* $U'$ *of* $\pi(p)$ *and* $\pi'(p')$ *respectively* ($\pi$ *and* $\pi'$ *denote the projections* $P^{(k)} \to M$ *and* $P'^{(k)} \to M'$), *and a filtration preserving analytic homomorphism* $\varphi\colon U \to U'$ *such that* $(\mathcal{R}^{(k)}\varphi)(P^{(k)}|_U) = P'^{(k)}|_{U'}$ *and that* $(\mathcal{R}^{(k)}\varphi)(p) = p'$.

To prove the theorem, it might seem rather natural to use the theory of differential equations on filtered manifolds that we shall discuss in the next section. However, the usual Cartan Kähler theorem suffices to prove it, since the structure is actually transitive and has no singularities in this case.

2.5. Cartan connections.

2.5.0. What we nowadays call Cartan connection was first introduced by E. Cartan as "éspace généralisé". It is a curved space modeled after a homogeneous space. Let us recall the definition.

Let $\mathfrak{l}$ be a Lie algebra and $\mathfrak{k}$ a Lie subalgebra of $\mathfrak{l}$. Let K be a Lie group with Lie algebra $\mathfrak{k}$ equipped with a representation $\rho\colon K \to GL(\mathfrak{l})$ such that the differential $\rho_*\colon \mathfrak{k} \to \mathfrak{gl}(\mathfrak{g})$ coincides with the adjoint representation of $\mathfrak{k}$ on $\mathfrak{l}$. By abuse of notation this representation $\rho$ will be denoted by $Ad$.

Let $P(M, K)$ be a principal fibre bundle over a manifold $M$ with structure group $K$. A Cartan connection in $P$ of type $(\mathfrak{l}, K)$ is a 1-form $\theta$ on $P$ with values in $\mathfrak{l}$ satisfying the following conditions:

i) $\theta\colon T_zP \to \mathfrak{l}$ is an isomorphism for all $z \in P$.
ii) $R_a^*\theta = Ad(a)^{-1}\theta$ for $a \in K$.
iii) $\theta(\widetilde{A}) = A$ for $A \in \mathfrak{k}$.

We know various examples of Cartan connections hitherto obtained: Riemannian, conformal, projective (cf. [Kob72]), or strongly pseudo-convex CR-structures [Tan62], and more generally certain geometric structures associated with simple graded Lie algebras [Tan79].

It then naturally arises the following question: Given a geometric structure $\Gamma$ on a manifold $M$, is it possible to construct a principal bundle over $M$ and a Cartan connection $\theta$ in $P$ in such a way that $(P, \theta)$ is canonically associated with $\Gamma$?

First of all it should be remarked that our frame bundle $\mathcal{R}(M)$ has the universal property also for the Cartan connections: Assume that the pair $(\mathfrak{l}, K)$ is formally effective (see (T4) in §2, 2.1). By choosing a complementary subspace $V$ of $\mathfrak{l}$ to $\mathfrak{k}$ we can view $(\mathfrak{l}, K, Ad)$ as a skeleton over $V$. Then it is clear that a Cartan connection $(P, M, K, \theta)$ of type

$(\mathfrak{l}, K)$ is a tower over $M$. Hence, by Proposition 2.1, there exists a unique embedding $\iota\colon P \to \mathcal{R}(M)$ such that $\iota^*\theta_{\mathcal{R}} = \theta$.

Thus the problem of finding a Cartan connection is reduced to the problem of constructing, for a given tower $Q$ (or truncated tower $Q^{(k)}$ of order $k$) on a filtered manifold $(M, F)$, a sub-tower $P(M, \theta, E)$ in a canonical way so that $E$ becomes a Lie algebra.

In the next subsections we will give a general criterion and a unified method to construct Cartan connections.

2.5.1. First we need to introduce reduced frame bundles. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ be a graded Lie algebra. We say that a filtered manifold $\mathbb{M}$ is regular (of type $\mathfrak{m}$) if the symbol algebra $grT_x\mathbb{M}$ $(x \in \mathbb{M})$ are all isomorphic (to $\mathfrak{m}$) as graded Lie algebras.

Let $(\mathcal{R}^{(0)}(\mathbb{M}), \mathbb{M}, G^{(0)}(\mathbb{V}))$ be the first order frame bundle of $\mathbb{M}$. Then, we have immediately:

**Proposition 2.8.** *A filtered manifold is regular if and only if the structure function of $\mathcal{R}^{(0)}(\mathbb{M})$ takes its values in a single $G^{(0)}(\mathbb{V})$-orbit.*

Given a filtered manifold $\mathbb{M}$ regular of type $\mathfrak{m}$, we shall identify $\mathfrak{m}$ with the type filtered vector space $\mathbb{V}$ and also with $gr\mathbb{V}$ (as filtered vector space or as graded vector space). Let $\gamma^{[0]} = \beta^{[0]} + c^{(0)}$ be the structure function of $\mathcal{R}^{(0)}(\mathbb{M})$. Then $c^{(0)}$ may be considered as taking values in $\mathrm{Hom}(\bigwedge^2 \mathfrak{m}, \mathfrak{m})_0$. Let $c^{(0)}_{\mathfrak{m}}$ be the bilinear map which defines the bracket operation of $\mathfrak{m}$. We set

$$\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m}) = \{z \in \mathcal{R}^{(0)}(m) \mid c^{(0)}(z) = c^{(0)}_{\mathfrak{m}}\}.$$

Then $\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$ is a principal subbundle of $\mathcal{R}^{(0)}(\mathbb{M})$. Its structure group, denoted by $G^{(0)}(\mathfrak{m})$, consists of all automorphisms of the graded Lie algebra $\mathfrak{m}$. In other words, $\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$ is nothing but the set of all isomorphisms $z\colon \mathfrak{m} \to grT_x\mathbb{M}$ of graded Lie algebras.

We shall denote by $\mathcal{R}(\mathbb{M}, \mathfrak{m})$ the universal tower $\mathcal{R}\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$ prolonging $\mathcal{R}^{(0)}(\mathbb{M}, \mathfrak{m})$, and by $(E(\mathfrak{m}), G(\mathfrak{m}))$ its skeleton. Hence $E(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{g}(\mathfrak{m})$, where $\mathfrak{g}(\mathfrak{m})$ is the Lie algebra of $G(\mathfrak{m})$. We set $\mathcal{R}^{(k)}(\mathbb{M}, \mathfrak{m}) = \mathcal{R}(\mathbb{M}, \mathfrak{m})/\mathfrak{f}^{k+1}$ and call it the reduced frame bundle of $\mathbb{M}$ of order $k+1$.

Now let us examine the structure function of $\mathcal{R}(\mathbb{M}, \mathfrak{m})$. We define a bilinear map

$$\beta_{\mathfrak{m}} = [\ ,\ ]\colon E(\mathfrak{m}) \times E(\mathfrak{m}) \to E(\mathfrak{m})$$

by

$$\begin{cases} [u,v] = [u,v]_{\mathfrak{m}} & \text{(the bracket of } \mathfrak{m}) \\ [A,B] = [A,B]_{\mathfrak{g}(\mathfrak{m})} & \text{(the bracket of } \mathfrak{g}(\mathfrak{m})) \\ [A,x] = Ax & \text{(the action of } \mathfrak{g}(\mathfrak{m}) \text{ on } E(\mathfrak{m})) \end{cases}$$

for $u,\ v \in \mathfrak{m},\ x \in E(\mathfrak{m})$, and $A,\ B \in \mathfrak{g}(\mathfrak{m})$. Note that this bracket does not satisfy the Jacobi identity. Recall that $G(\mathfrak{m})$ has the natural representation on $\mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$, and note that the subspace $\mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$ is $G(\mathfrak{m})$-invariant. Moreover it is easy to see that the equivalence class of $\beta_{\mathfrak{m}} \mod \mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$ is fixed by the induced action of $G(\mathfrak{m})$ on the quotient space. Hence $G(\mathfrak{m})$ has the induced affine representation on the affine space $\beta_{\mathfrak{m}} + \mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$. Hence:

**Proposition 2.9.** *The structure function $\gamma_{\mathcal{R}(\mathbb{M},\mathfrak{m})}$ of $\mathcal{R}(\mathbb{M},\mathfrak{m})$ is a $G^{(k)}$-equivariant map from $\mathcal{R}(\mathbb{M},\mathfrak{m})$ to the affine space*

$$\beta_{\mathfrak{m}} + \mathfrak{f}^1 \mathrm{Hom}\Big(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m})\Big).$$

We therefore write

$$\gamma_{\mathcal{R}(\mathbb{M},\mathfrak{m})} = \beta_{\mathfrak{m}} + \hat{c} \tag{2.7}$$

with $\hat{c}$ an $\mathfrak{f}^1 \mathrm{Hom}(\bigwedge^2 \mathfrak{m}, E(\mathfrak{m}))$-valued function on $\mathcal{R}(\mathbb{M},\mathfrak{m})$.

In applications, most of the first order geometric structures are defined as subbundles $P^{(0)}$ of the reduced frame bundle $\mathcal{R}^{(0)}(\mathbb{M},\mathfrak{m})$. Thus the prolongation $\mathcal{R}P^{(0)}$ is contained in $\mathcal{R}(\mathbb{M},\mathfrak{m})$ as an adapted subbundle. Clearly the structure function of an adapted subbundle of $\mathcal{R}(\mathbb{M},\mathfrak{m})$ satisfies the same properties as in Proposition 3.5.3.

2.5.2. Criterion for the existence of Cartan connections.

Now we can formulate our problem for geometric structures of first order as follows:

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{m}_p$ be a graded Lie algebra and $G^{(0)}$ a Lie subgroup of $G^{(0)}(\mathfrak{m})$. Let $\mathbb{M}$ be a filtered manifold regular of type $\mathfrak{m}$. Given a principal subbundle $(P^{(0)}, M, G^{(0)})$ of $\mathcal{R}^{(0)}(\mathbb{M},\mathfrak{m})$ with structure group $G^{(0)}$. We ask whether there exists a Cartan connection $(P,\theta)$ naturally associated with $P^{(0)}$.

A little more generally, let us pose the problem for higher order geometric structures. Consider a transitive graded Lie algebra $\mathfrak{l} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{l}_p$ and write:

$$\mathfrak{l} = \mathfrak{m} \oplus \mathfrak{k},$$

where we put:

$$\mathfrak{m} = \bigoplus_{p<0} \mathfrak{l}_p, \quad \mathfrak{k} = \bigoplus_{p\geq 0} \mathfrak{l}_p.$$

Let $G(\mathfrak{m})$ be the structure group of $\mathcal{R}(\mathbb{M}, \mathfrak{m})$ and $\mathfrak{g}(\mathfrak{m})$ its Lie algebra. Then from the universal property of $G(\mathfrak{m})$ it turns out that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}(\mathfrak{m})$.

Let $\nu$ be an integer ($\geq 0$) such that $\mathfrak{l}$ is a prolongation of $\bigoplus_{p\leq\nu} \mathfrak{l}$, that is, $H^1_r(\mathfrak{m}, \mathfrak{l}) = 0$ for $r \geq \nu + 1$.

Let $G(\mathfrak{m})^{(\nu)}$ be the structure group of $\mathcal{R}^{(k)}(\mathbb{M}, \mathfrak{m})$, i.e. $G(\mathfrak{m})^{(\nu)} = G(\mathfrak{m})/\mathfrak{f}^{\nu+1}$ and $\mathfrak{g}(\mathfrak{m})^{(\nu)}$ its Lie algebra. Then $\mathfrak{k}^{(\nu)} = \mathfrak{k}/\mathfrak{f}^{\nu+1} = \bigoplus_{p=0}^{\nu} \mathfrak{l}_p$ is a subalgebra of $\mathfrak{g}(\mathfrak{m})^{(\nu)}$.

Now given a Lie subgroup $K^{(\nu)}$ of $G^{(\nu)}(\mathfrak{m})$ with Lie algebra $\mathfrak{k}^{(\nu)}$. Let $K$ denotes the maximal subgroup of $G(\mathfrak{m})$ such that its Lie algebra is $\mathfrak{k}$ and that $K/\mathfrak{f}^{\nu+1} = K^{(\nu)}$, which we call the prolongation of $K^{(\nu)}$. Note that $K$ acts on $\mathfrak{l}$ naturally.

Now we consider the following complex:

$$\cdots \to \operatorname{Hom}(\mathfrak{m}, \mathfrak{l}) \xrightarrow{\partial} \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{l}\Big) \xrightarrow{\partial} \operatorname{Hom}\Big(\bigwedge^3 \mathfrak{m}, \mathfrak{l}\Big) \xrightarrow{\partial} \cdots$$

Since $\mathfrak{m}$ is a Lie subalgebra of $\mathfrak{l}$, the coboundary operator $\partial$ is defined as usual. Note that the group $K$ acts on $\operatorname{Hom}(\bigwedge^\bullet \mathfrak{m}, \mathfrak{l})$ and preserves the filtration $\{\mathfrak{f}^p \operatorname{Hom}(\bigwedge^\bullet \mathfrak{m}, \mathfrak{l})\}$.

It being prepared,

**Definition 2.2.** We say that a Lie subgroup $K^{(\nu)} \subset G^{(\nu)}(\mathfrak{m})$ satisfies the condition (C) if there exists a subspace

$$W \subset \mathfrak{f}^1 \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{l}\Big)$$

such that

i) $\mathfrak{f}^1 \operatorname{Hom}(\bigwedge^2 \mathfrak{m}, \mathfrak{l}) = W \oplus \partial \mathfrak{f}^1 \operatorname{Hom}(\bigwedge^1 \mathfrak{m}, \mathfrak{l})$,

ii) $W$ is stable under the actions of $K$ on $\mathfrak{f}^1 \operatorname{Hom}(\bigwedge^2 \mathfrak{m}, \mathfrak{l})$.

If $P^{(\nu)}(\mathbb{M}, K^{(\nu)})$ is an adapted subbundle of $\mathcal{R}^{(\nu)}(\mathbb{M}, \mathfrak{m})$, then the structure function $\gamma^{[\nu]}$ is, according to the decomposition (2.7), written as:

$$\gamma^{[\nu]} = [\beta_{\mathfrak{m}}]^{[\nu]} + \hat{c}^{(\nu)},$$

where $\hat{c}^{(\nu)}$ takes values in

$$\mathfrak{f}^1 \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{k}^{(\nu)}\Big) \Big/ I^{\nu+1} \Big(= \mathfrak{f}^1 \operatorname{Hom}\Big(\bigwedge^2 \mathfrak{m}, \mathfrak{m} + \mathfrak{k}^{(\nu)}\Big) \Big/ \mathfrak{f}^{\nu+1}\Big).$$

Now we can state:

**Theorem 2.3.** *The notation being as above, let $\mathfrak{l}(= \mathfrak{m} + \mathfrak{k})$ be a transitive graded Lie algebra with $H^1_r(\mathfrak{m}, \mathfrak{l}) = 0$ for $r \geq \nu + 1$. Let $K^{(\nu)}$ be a Lie subgroup of $G^{(\nu)}(\mathfrak{m})$ and $K$ its prolongation. Suppose that $K^{(\nu)}$ satisfies the condition* (C). *Then for any filtered manifold $\mathbb{M}$ regular of type $\mathfrak{m}$ and for each adapted subbundle $P^{(\nu)}$ of $\mathcal{R}^{(\nu)}(\mathbb{M}, \mathfrak{m})$ with structure group $K^{(\nu)}$ such that its structure function $\hat{c}^{(\nu)}$ takes value in $W^{(\nu)}(= W/I^{\nu+1})$, we can construct a tower $P \subset \mathcal{R}(\mathbb{M}, \mathfrak{m})$ in such a way that*

i) *$P$ is a tower on $\mathbb{M}$ with skeleton $(\mathfrak{l}, K)$,*
ii) *The structure function $\hat{c}$ of $P$ takes values in $W$.*
iii) *The assignment $P^{(\nu)} \rightarrowtail P$ is compatible with equivalences.*

*Thus $(P, \theta)$ is a Cartan connection of type $(\mathfrak{l}, K)$ associated with $P^{(\nu)}$, where $\theta$ is the canonical form of $P$.*

*Proof.* The construction of $P$ proceeds by induction. Assume that we have constructed for an $l(\geq \nu)$ an adapted subbundle $P^{(l)}(\mathbb{M}, K^{(l)})$ of $\mathcal{R}^{(l)}(\mathbb{M}, \mathfrak{m})$ with structure group $K^{(l)} = K/\mathfrak{f}^{l+1}$ so that $P^{(l)}/\mathfrak{f}^{\nu+1} = P^{(\nu)}$ and that the structure function $\hat{c}^{(l)}$ takes values in $W^{(\nu)}$. We put

$$P^{(l+1)} = \{z \in \#P^{(l)} \mid \hat{c}^{(l+1)}(z) \in W^{(l+1)}\},$$

where $\#P^{(l)}$ is the prolongation of $P^{(l)}$ and $\hat{c}^{(l+1)}$ its structure function. Passing to the limit, we obtain: $P = \operatorname{proj\,lim} P^{(l)}$. $\square$

The following proposition is useful in application.

**Proposition 2.10.** *The notation being as above, let $\mathfrak{l}(= \mathfrak{m} + \mathfrak{k})$ be a transitive graded Lie algebra with $H^1_r(\mathfrak{m}, \mathfrak{l}) = 0$ for $r \geq \nu + 1$. Let $K^{(\nu)}$ be a Lie subgroup of $G^{(\nu)}(\mathfrak{m})$ and $K$ its prolongation and let $K^{(0)} = K/\mathfrak{f}^1$. Assume that $\mathfrak{l}$ is finite dimensional and that there exists a positive definite symmetric bilinear form*

$$(\ ,\ ) \colon \mathfrak{l} \times \mathfrak{l} \to \mathbb{R}$$

*satisfying*:

i) $(\mathfrak{l}_p, \mathfrak{l}_q) = 0$ *if $p \neq q$.*
ii) *There exists $\tau \colon \mathfrak{k} \to \mathfrak{l}$ such that*

$$\begin{cases} \tau(\mathfrak{l}_p) \subset \mathfrak{l}_{-p} & \text{for } p \geq 0 \\ ([A, x], y) = (x, [\tau(A), y]) & \text{for } x,\ y \in \mathfrak{l},\ A \in \mathfrak{k}. \end{cases}$$

iii) *There exists $\tau_0 \colon K^{(0)} \to K^{(0)}$ such that*

$$(ax, y) = (x, \tau_0(a)y) \qquad \text{for } x,\ y \in \mathfrak{l},\ a \in K^{(0)}.$$

*Then $K^{(\nu)}$ satisfies the condition* (C).

The proof is same as that given in [Mor93], though we give here a statement in a little more general form so as to apply to higher order structures(cf. [CS00]).

If the Lie algebra $\mathfrak{l}$ is semi-simple the conditions of Proposition 2.10 are satisfied. Thus Theorem 2.3 together with Proposition 2.10 covers all the existence theorems for Cartan connection hitherto known.

Proposition 2.10 also applies to the case where $\mathfrak{l}$ is given as a semi-direct product of a simple Lie algebra and its irreducible representation, which appears in the geometry of holonomic systems of differential equations ([Tan82], [Tan89], [DKM99]). For further detailed geometric studies based on Cartan connections, see [SY98], [Yam93], [Yam99], [Yat92].

## §3. Differential equations on filtered manifolds

3.0. In this section we develop a general study of differential equations on filtered manifolds. Let $\mathbb{M}$ be a filtered manifold. Recalling that the filtration $\mathfrak{f}$ satisfies

$$TM = \mathfrak{f}^{-\mu} \supset \mathfrak{f}^{-\mu+1} \supset \cdots \supset \mathfrak{f}^{-1} \supset \mathfrak{f}^{0} = 0,$$

we say that a local vector field $X$ on $\mathbb{M}$ is of *weighted order* $\leq k$ if $X$ is a section of $\mathfrak{f}^{-k}$. The minimum of such $k$ is called the weighted order of $X$ and denoted by w-ord$X$. A differential operator $P$ on $\mathbb{M}$ is said to be of *weighted order* $\leq k$ if $P = \sum X_1 \cdots X_r$ (locally) for local vector fields $X_1, \ldots, X_r$ and if $\sum \text{w-ord}\, X_i \leq k$. The minimum of such $k$ is called the weighted order of $P$ and denoted by w-ord $P$. Since it is only under the inequality that we actually use the notion of order, we shall often say, by abuse of terminology, that w-ord $P = k$ if w-ord $P \leq k$. This notion of weighted order, which well accords with the filtration of a filtered manifold, was rather implicit and disguised into algebraic or geometric appearances when we studied transformation groups and geometric structures on filtered manifolds in the preceding sections, but will become explicit and play a fundamental rôle in this section.

3.1. Formal theory. We shall explain rapidly how to treat a system of differential equations on a filtered manifold by using weighted orders and introduce the notion of weighted involutivity, a sufficient condition for the system to be formally integrable. A detailed account together with some geometric applications will be treated in [Mor0x].

Let us first introduce the notion of a *weighted jet bundle.* Consider a filtered vector bundle $(E, \mathfrak{f})$ over a filtered manifold $(M, \mathfrak{f})$, that is, a vector bundle $E$ over $M$ of finite rank equipped with a filtration consisting of subbundles $\mathfrak{f} = \{E^p\}_{p\in\mathbb{Z}}$ and satisfying:

i) $E^p \supset E^{p+1}$

ii) There exist integers $\nu_I$, $\nu_T$ such that $E^{\nu_I} = E$, $E^{\nu_T+1} = 0$.

Let $\underline{E}$ denote the sheaf of local sections of $E$ and $\underline{E}_a$ the stalk over $a \in M$. First we define a filtration $\{\mathfrak{f}^k \underline{E}_a\}$ of $\underline{E}_a$ by setting $\mathfrak{f}^k \underline{E}_a$ to be the subspace of $\underline{E}_a$ consisting of $s \in \underline{E}_a$ such that

$$(P\langle \alpha^i, s\rangle)(a) = 0$$

for any differential operator $P$ and any section $\alpha^i$ of the annihilating bundle $(E^{i+1})^\perp$ of $E^{i+1}$ whenever

$$\text{w-ord}\, P + i < k.$$

We then define:

$$\mathfrak{J}^k E = \bigcup_{a\in M} \mathfrak{J}^k_a E, \quad \mathfrak{J}^k_a E = \underline{E}_a / \mathfrak{f}^{k+1} \underline{E}_a.$$

We denote by $\mathfrak{j}^k$ and $\mathfrak{j}^k_a$ the natural projections $\underline{E} \to \mathfrak{J}^k E$ and $\underline{E}_a \to \mathfrak{J}^k_a E$ respectively. It is easy to see that $\mathfrak{J}^k E$ is a vector bundle over $M$.

There is a natural filtration of $\mathfrak{J}^k E$ defined by $\mathfrak{f}^l \mathfrak{J}^k E = 0$ for $l \geq k+1$ and by the following exact sequences for $l \leq k$:

$$0 \longrightarrow \mathfrak{f}^{l+1}\mathfrak{J}^k E \longrightarrow \mathfrak{J}^k E \xrightarrow{\pi_{kl}} \mathfrak{J}^l E \longrightarrow 0,$$

where $\pi_{kl}$ are the natural projections.

The vector bundle $\mathfrak{J}^k E$ equipped with this filtration will be called the weighted jet bundle of order $k$ of $(E, \mathfrak{f})$ over $(M, \mathfrak{f})$.

Note that if $E^{\nu_I} = E$ then

$$\mathfrak{J}^{\nu_I - 1} E = 0, \quad \mathfrak{J}^{\nu_I} E = E^{\nu_I}/E^{\nu_I+1}, \quad \mathfrak{f}^{\nu_I}\mathfrak{J}^k E = \mathfrak{J}^k E.$$

Note also that if the filtrations of $M$ and $E$ are trivial, that is $T^{-1}M = TM$, $E = E^0 \supset E^1 = 0$, then the weighted jet bundle $\mathfrak{J}^k E$ reduces to the usual jet bundle $J^k E$. But it should be noted that $\mathfrak{J}^l \mathfrak{J}^k E$ and $J^l J^k E$ are different since the former respects the filtration of $\mathfrak{J}^k E$ but the latter does not.

The subbundle $\mathfrak{f}^k \mathfrak{J}^k E$ is called the *symbol* of $\mathfrak{J}^k E$. Let us describe it more explicitly. For $x \in M$, let $grT_x\mathbb{M}$, $grE_x$ be the associated graded Lie algebra, graded vector space of $(T_x M, \mathfrak{f})$ and of $(E_x, \mathfrak{f})$ respectively, and $U(grT_x\mathbb{M})$ the universal enveloping algebra of $grT_x\mathbb{M}$. Let $U_l$ denote the set of all homogeneous elements of degree $l$ ($\deg\xi = \sum p_i$ if $\xi = A_1 \cdots A_m$ with $A_i \in gr_{p_i}T_x\mathbb{M}$.) We denote $\mathrm{Hom}(U(grT_x\mathbb{M}), grE_x)_k$ the set of all linear map $f\colon U \to grE_x$ of degree $k$, namely $f(U_l) \subset gr_{l+k}E_x$. Then we have the following fundamental exact sequence of bundle maps:

$$0 \longrightarrow \mathrm{Hom}(U(grT\mathbb{M}), grE)_k \longrightarrow \mathfrak{J}^k E \longrightarrow \mathfrak{J}^{k-1}E \longrightarrow 0. \tag{3.1}$$

If the filtrations of $M$ and $E$ are trivial the above exact sequence reduces to the well-known one:

$$0 \longrightarrow E \otimes S^k T^* M \longrightarrow J^k E \longrightarrow J^{k-1}E \longrightarrow 0.$$

Now some elementary properties are in order:

(1) As easily seen, the map $\mathfrak{j}_x^k\colon \underline{E}_x \to \underline{\mathfrak{J}^k E}_x$ preserves the filtration, that is

$$\mathfrak{j}_x^k(\mathfrak{f}^{l+1}\underline{E}_x) \subset \mathfrak{f}^{l+1}\underline{\mathfrak{J}^k E}_x$$

for $l \in \mathbb{Z}$. Hence we have the bundle map:

$$\iota\colon \mathfrak{J}^l E \to \mathfrak{J}^l \mathfrak{J}^k E. \tag{3.2}$$

(2) If $\varphi\colon \mathbb{E} \to \mathbb{F}$ is a bundle map of degree $r$, that is, $\varphi(E^p) \subset F^{p+r}$ for all $p$, then we have the bundle map:

$$\mathfrak{j}^k\varphi\colon \mathfrak{J}^k E \to \mathfrak{J}^{k+r}F.$$

Now we are going to study differential equations on a filtered manifold, confining our discussion to the linear case for the sake of simplicity. It is not difficult to extend the following formulation to the non-linear case.

Let $\mathbb{E}$, $\mathbb{F}$ be filtered vector bundles over a filtered manifold $\mathbb{M}$. A bundle map (of degree $r$)

$$\varphi\colon \mathfrak{J}^k E \to F$$

is a linear differential operator of weighted order $k$ and the kernel of $\varphi$, denoted by $R$, is a system of linear differential equations. A section $s$ of $E$ is a solution of $R$ if $\varphi(\mathfrak{j}^k s) = 0$.

Without loss of generality we may assume that $\varphi$ is of degree 0 and $E^{k+1} = F^{k+1} = 0$.

If $\varphi\colon \mathfrak{J}^k E \to F$ is a bundle map of degree 0, it induces bundle maps for $i \leq k$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & \mathfrak{J}^k E & \xrightarrow{\varphi} & F & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^i & \longrightarrow & \mathfrak{J}^i E & \xrightarrow{\varphi^i} & F^{(i)} & \longrightarrow & 0,
\end{array}$$

where we set $F^{(i)} = F/F^{i+1}$. We call $\varphi^i$ (or $R^i$) differential operator (or equation) associated with $\varphi$ (or $R$ resp.).

By the following commutative diagram we define $\sigma(R^i)$ and call it the *symbol* of degree $i$ of $R$:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \sigma(R^i) & \longrightarrow & \mathrm{Hom}(U, grE)_i & \longrightarrow & gr_i F \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R^i & \longrightarrow & \mathfrak{J}^i E & \xrightarrow{\varphi^i} & F^{(i)} \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & R^{i-1} & \longrightarrow & \mathfrak{J}^{i-1} E & \xrightarrow{\varphi^{i-1}} & F^{(i-1)} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
\end{array}$$

A bundle map $\varphi\colon \mathfrak{J}^k E \to F$ of degree 0 gives rise to bundle maps,

$$p^l\varphi\colon \mathfrak{J}^l E \to \mathfrak{J}^l F \tag{3.3}$$

for all $l \in \mathbb{Z}$, called the prolongation of $\varphi$, defined by the following commutative diagram:

$$\begin{array}{ccc}
\mathfrak{J}^l \mathfrak{J}^k E & \xrightarrow{\mathrm{j}^l\varphi} & \mathfrak{J}^l F \\
\uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \mathrm{Id}} \\
\mathfrak{J}^l E & \xrightarrow{p^l\varphi} & \mathfrak{J}^l F.
\end{array}$$

We put $\bar{\varphi}^l = p^l\varphi$ and $\bar{R}^l = \mathrm{Ker}\,\bar{\varphi}^l$, and call $\bar{\varphi} = \{\bar{\varphi}^l\}_{l\in\mathbb{Z}}$ or $\bar{R} = \{\bar{R}^l\}l \in \mathbb{Z}$ the prolongation of $\varphi$ or $R$. We have:

1) For $i \leq l$, $\bar{R}^i$ is the associated equations of order $i$ to $\bar{R}^l$.
2) If $l \geq k$, the prolongation of $\bar{R}^l$ coincides with $\bar{R}$.
3) For $i \leq k$, $R^i \supset \bar{R}^i$.

Before entering to the study of symbols of $\bar{R}$, we just have a look at the following graded spaces:

$$\mathrm{Hom}(U, grE) = \bigoplus_p \mathrm{Hom}(U, grE)_p,$$

whose dual space is identified with:

$$U \otimes (grE)^* = \bigoplus_q (U \otimes (grE)^*)_q,$$

where

$$(U \otimes (grE)^*)_q = \bigoplus_{a+b=q} U_a \otimes (grE)^*_b,$$

and

$$(grE)^*_b = (gr_{-b}E)^*.$$

In particular,

$$\mathrm{Hom}(U, grE)^*_p \cong (U \otimes (grE)^*)_{-p}.$$

$\mathrm{Hom}(U, grE)$ is a right $U$-graded module and $U \otimes (grE)^*$ a left $U$-graded module by means of the formula:

$$\langle t, \xi(\eta \otimes \alpha) \rangle = \langle t\xi, \eta \otimes \alpha \rangle$$

for $t \in \mathrm{Hom}(U, grE)$, $\xi \in U$, $\eta \otimes \alpha \in U \otimes (grE)^*$, and we have

$$\mathrm{Hom}(U, grE)_p U_q \subset \mathrm{Hom}(U, grE)_{p+q}.$$

Now we set

$$\mathfrak{s}_l = \begin{cases} \sigma(R^l) & \text{for } l \leq k \\ \mathrm{Hom}(U, grE)_k & \text{for } l > k, \end{cases} \qquad \bar{\mathfrak{s}}_l = \sigma(\bar{R}^l)$$

and let $\mathfrak{s}_l^\perp$ and $\bar{\mathfrak{s}}_l^\perp$ be the null spaces of $\mathfrak{s}_l$ and $\bar{\mathfrak{s}}_l$ in $(U \otimes (grE)^*)_{-l}$ respectively, and

$$\mathfrak{s} = \bigoplus \mathfrak{s}_l, \quad \bar{\mathfrak{s}} = \bigoplus \bar{\mathfrak{s}}_l$$

$$\mathfrak{s}^\perp = \bigoplus \mathfrak{s}_l^\perp, \quad \bar{\mathfrak{s}}^\perp = \bigoplus \bar{\mathfrak{s}}_l^\perp.$$

Then we have:

**Proposition 3.1.**

1) $\bar{\mathfrak{s}}^{\perp}$ *is a left* $U$*-module generated by* $\mathfrak{s}^{\perp}$; $\bar{\mathfrak{s}}^{\perp} = U\mathfrak{s}^{\perp}$.
2) $\bar{\mathfrak{s}}$ *is a right* $U$*-module. Furthermore,* $t \in \bar{\mathfrak{s}}_l$ *for* $t \in \mathrm{Hom}(U, grE)_l$ *if and only if* $t \in \mathfrak{s}_l$ *and if* $t\xi \in \bar{\mathfrak{s}}_{l+\deg\xi}$ *for any* $\xi \in U$.

Since $\bar{\mathfrak{s}} = \bigoplus \bar{\mathfrak{s}}_l$ is a right $U$-module, we can consider the following differential chain complex:

$$\mathrm{Hom}\Big(\bigwedge^p grT\mathbb{M}, \bar{\mathfrak{s}}\Big)_r \xrightarrow{\partial} \mathrm{Hom}\Big(\bigwedge^{p+1} grT\mathbb{M}, \bar{\mathfrak{s}}\Big)_r$$

defined by:

$$\begin{aligned}(\partial\omega)&(X_1, \ldots, X_{p+1})\\ &= \sum (-1)^i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})X_i\\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1})\end{aligned}$$

for $\omega \in \mathrm{Hom}(\bigwedge^p grT\mathbb{M}, \bar{\mathfrak{s}})_r$ and $X_1, \ldots, X_{p+1} \in grT\mathbb{M}$.

The cohomology group $H^p_r(grT\mathbb{M}, \bar{\mathfrak{s}})$ is the generalized Spencer cohomology group. We can deduce from Theorem 1.1 the following:

**Theorem 3.1.** *For any* $x_0 \in M$*, there exists a neighbourhood* $\mathcal{U}$ *of* $x_0$ *and an integer* $r_0$ *such that*

$$H^p_r(gr_x T\mathbb{M}, \bar{\mathfrak{s}}_x) = 0$$

*for* $r \geq r_0$ *and all* $p$*, and for all* $x \in \mathcal{U}$.

Now we have:

**Theorem 3.2.** *Given a differential operator of weighted order* $k$

$$0 \to R^k \to \mathfrak{J}^k E \xrightarrow{\varphi} F.$$

*Let* $\bar{R}^i$ *be the* $i$*th prolongation. Assume that there exists an integer* $l_0 (\geq k)$ *which satisfies the following conditions*:

i) $\bar{R}^i$ *are vector bundles (i.e., rank constant) for* $i \leq l_0$.
ii) $\bar{R}^{l_0} \to \bar{R}^{l_0-1} \to \cdots \to \bar{R}^{l_0-\mu}$ *are all surjective.*
iii) $H^2_r(grT\mathbb{M}, \bar{\mathfrak{s}}) = 0$ *for* $r \geq l_0 + 1$.

*Then for any* $l > l_0$ *it holds*:

i) $\bar{R}^l$ *is a vector bundle.*

ii) $\bar{R}^l \to \bar{R}^{l-1}$ *is surjective.*

*In particular,* $\bar{R}^\infty (= \operatorname{proj}\lim \bar{R}^l) \to \bar{R}^{l_0}$ *is surjective. Therefore the equation* $R^k$ *has a formal solution for any prescribed* $l_0$*-jet in* $\bar{R}^{l_0}$.

This theorem gives a criterion for the existence of formal solutions. We say that the equations $\bar{R}$ is *weightedly involutive* if the conditions of Theorem 3.2 hold.

This theorem can be extended to the non-linear systems of differential equations.

For a single equation the criterion of weighted involutivity is easy: It is weightedly involutive if one of the highest order terms does not vanish.

3.2. By the preceding discussion, we have shown that any weightedly involutive system of differential equations on a filtered manifold has formal solutions. Now we turn to the problem of convergence under the category of analyticity. First we shall show that a weightedly involutive system has not always an analytic solution, but does have a formal solution satisfying a certain estimate, namely a formal Gevrey solution. Next, studying geometric properties of formal Gevrey functions, we shall show that if the filtered manifold satisfies the Hörmander condition then the formal Gevrey functions proves to be analytic functions, which, in turn, establish a general existence theorem of analytic solutions for a weightedly involutive analytic system of differential equations on a filtered manifold.

For our purpose, we may assume without loss of generality that our filtered manifold is a standard one, that is, a nilpotent Lie group $N$ whose Lie algebra $\mathfrak{n}$ is graded: $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$ (for convenience's sake we reverse the gradation) and that the filtration of $E$ is trivial. Choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{n}$ such that $\{X_{d(p-1)+1}, \ldots, X_{d(p)}\}$ is a basis of $\mathfrak{n}_p$, where $d(p) = \sum_{i=1}^{p} \dim \mathfrak{n}_i$. We define a weight function

$$w\colon \{1, \ldots, n\} \to \{1, \ldots, \mu\}$$

by the condition: $X_i \in \mathfrak{n}_{w(i)}$ for all $i$. For $I = (i_1, \ldots, i_l) \in \{1, \ldots, n\}^l$, we set

$$X_I = X_{i_1} \cdots X_{i_l},\ w(X_I) = w(I) = \sum_{a=1}^{l} w(i_a).$$

We will regard $X_I$ as a left invariant differential operator on $N$ of weighted order $w(I)$. For a function $F$ in a neighbourhood of a point $o \in N$, the values $\{(X_I F)(o) : w(I) \leq k\}$ determine the weighted $k$-jet $\mathfrak{j}_o^k F$ of $F$ at $o$.

Now a general system of non-linear differential equations of weighted order $k$ on $N$ can be written as:

$$\text{(E)} \qquad \Phi(x, (X_I F)(x)) = 0, \quad w(I) \leq k, \ x \in N,$$

where $\Phi$ and $F$ are vector valued functions taking values in some vector spaces, say, $W$ and $V$ respectively.

We say a weighted $l$-jet $F^l$ at $o \in N$ $(l \geq k)$ is an $l$-jet solution of (E) if

$$X_J \Phi(x, (X_I F)(x)) \mid_{x=o} = 0$$

for all $J$ such that $w(J) \leq l - k$. We say also that an $l$-jet solution is *strongly prolongeable* (with respect to the weighted order) if for all $m$-jet solution $F^m (m \geq l)$ such that $\mathrm{j}^l_o F^m = F^l$ there exists $(m+1)$-jet solution $F^{m+1}$ satisfying $\mathrm{j}^m_o F^{m+1} = F^m$.

We remark that if the system (E) is weightedly involutive then any $k$-jet solution is strongly prolongeable.

**Example 3.1.** Let $N = \mathbb{R}^2$ be an abelian Lie algebra with non-trivial gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1 = \langle \frac{\partial}{\partial x} \rangle, \mathfrak{n}_2 = \langle \frac{\partial}{\partial t} \rangle$ in terms of coordinates $(x, t)$ of $\mathbb{R}^2$.

Regarding $N$ as a standard filtered manifold, we consider on $N$ the following equation:

$$\text{(3.4)} \qquad \left( \frac{\partial}{\partial t} - a(x,t) \frac{\partial^2}{\partial x^2} \right) F = b \left( x, t, \frac{\partial F}{\partial x} \right).$$

The left-hand side of this equation is homogeneous of weighted order 2 with respect to this filtration and weightedly involutive since the coefficient of $\frac{\partial}{\partial t}$ does not vanish $(= 1)$. Hence any 2-jet solution is strongly prolongeable in the weighted sense. But if the function $a(x,t)$ vanishes at a point then the equation is not Kowalevskayan and is not involutive in the usual sense around the point. Therefore jet solutions are not always strongly prolongeable in the ordinary sense.

**Example 3.2.** Let $(x, y, z)$ be coordinates of $\mathbb{R}^3$ and consider the vector fields:

$$X = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z}.$$

Since $[X, Y] = Z$ and the other brackets are trivial, $\{X, Y, Z\}$ span a graded Lie algebra $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_1$ is spanned by $X$, $Y$ and $\mathfrak{h}_2$ by $Z$. We identify $\mathbb{R}^3$ with the Heisenberg group $H$ with Lie algebra $\mathfrak{h}$. Consider a differential equation:

$$\text{(3.5)} \qquad (Z + aX^2 + bXY + cY^2)F = h(x, y, z, XF, YF),$$

where $a$, $b$, $c$, are functions of $x$, $y$, $z$. Since the left-hand side of the equation is homogeneous of weighted order two, the equation is weightedly involutive. But it is not involutive in the usual sense at the points where $a$, $b$, $c$ simultaneously vanish.

Now a fundamental problem is, given an $l_0$-jet solution $F^{l_0}$ with $l_0 \geq k$ of (E), to find an analytic solution $F$ of (E) such that $\mathfrak{j}_o^{l_0} F = F^{l_0}$.

However, there are not always such analytic solutions as seen for the equation (3.4) in Example 3.1. In fact, it is easy to choose function $a(x,t)$ and $b$ satisfying $a(0,0) = 0$ in such a way that there exists an $l$-jet solution at $(0,0)$ which cannot be prolonged to any analytic solution.

Now we introduce the formal Gevrey functions on a graded nilpotent Lie group $N$. A formal function $F$ at $o \in N$ is called *formal Gevrey* if there exist positive constants $C$, $\rho$ such that

$$|(X_I F)(o)| \leq C w(I)!\, \rho^{w(I)} \quad \text{for all multi-index } I. \tag{3.6}$$

The first fundamental theorem we obtain is the following:

**Theorem 3.3.** *Given an equation* (E). *Assume that* $\Phi(x, y_I)$ *is formal Gevrey with respect to* $x$ *and analytic with respect to* $y_I$ *at* $(o, y_I^0)$. *If* $F^k = (y_I^0) \in \mathfrak{J}^k(N \times V)$ *is a* $k$*-jet solution of* (E) *and strongly prolongeable in the weighted sense, there exists a formal Gevrey solution* $F$ *of* (E) *such that* $\mathfrak{j}_o^k F = F^k$.

In the case of Example 3.1 above, the theorem asserts that for any prescribed 2-jet solution there exists a formal solution of (3.4) satisfying the estimate:

$$\left| \left( \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial t}\right)^j F \right)(o) \right| \leq C(i+2j)!\, \rho^{i+2j},$$

which means that $F$ is analytic in $x$ but Gevrey of order 2 in $t$. In this case it is rather easy to find such a solution by direct calculation. However, already in the case of Example 3.2, it is hard to find a formal Gevrey solution for a prescribed jet solution by direct calculation. In general, difficulties occur in finding a good algorithm to determine inductively an $(l+1)$-jet solution from a given $l$-jet solution since $F^{l+1}$ is not uniquely determined from $F^l$.

To prove the theorem, following the method of Malgrange [Mal72], we employ the privileged neighbourhood theorem [Mal77]. In order that we generalize the privileged neighbourhood theorem to the universal enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$ as follows:

Any element of $U(\mathfrak{n})$ being a linear combinations of $X_I = X_{i_1} \cdots X_{i_l}$, we set

$$U_a(\mathfrak{n}) = \left\{ P = \sum c_I X_I \in U(\mathfrak{n}) \mid w(X_I) = a \right\}$$

and $\hat{U}(\mathfrak{n}) = \widehat{\bigoplus} U_a(\mathfrak{n})$ be the completion of $U(\mathfrak{n})$.

For $\rho = (\rho_1, \ldots, \rho_n)$ with $\rho_i > 0$, define a pseudo-norm $|\cdot|_\rho$ of $\hat{U}(\mathfrak{n})$: If $P_a \in U_a(\mathfrak{n})$ we set

$$|P_a|_\rho = \inf \sum |c_I| \rho^{w(I)},$$

where the infimum is taken over all expressions $P_a = \sum_{w(I)=a} c_I X_I$ of $P_a$. For $P = \sum P_a \in \hat{U}(\mathfrak{n})$ we put $|P|_\rho = \sum |P_a|_\rho$ and set

$$\mathcal{U}(\rho) = \{P \in \hat{U}(\mathfrak{n}) \mid |P|_\rho < \infty\}$$

$$\mathcal{U} = \bigcup_{\rho > 0} \mathcal{U}(\rho).$$

We see that $\mathcal{U}$ is noetherian and that

$$|PQ|_\rho \leq |P|_\rho |Q|_\rho \quad \text{for } P,\ Q \in \mathcal{U}(\rho).$$

Now let

$$u \colon \mathcal{U}^{m'} = (\mathcal{U} \times \cdots \times \mathcal{U}) \longrightarrow \mathcal{U}^m$$

be a left $\mathcal{U}$-linear map. We say after Malgrange [Mal77] that an $\mathbb{R}$-linear map $\lambda \colon \mathcal{U}^m \to \mathcal{U}^{m'}$ is a scission of $u$ if $u\lambda u = u$ and that $\lambda$ is adapted to a polydisk $\mathbb{P}(\rho) = \{(x_1, \ldots, x_n); |x_i| < \rho_i\}$ if there exists $C > 0$ such that

$$|\lambda P|_\rho \leq |P|_\rho$$

for all $P \in \mathcal{U}^m$. Then we have:

**Theorem 3.4** (A non-commutative version of privileged neighbourhood theorem). *Let $u \colon \mathcal{U}^{m'} \to \mathcal{U}^m$ be a left $\mathcal{U}$-linear map. Then one can find a scission $\lambda$ of $u$ having the following property: The set of all $\mathbb{P}(\rho)$ to which $\lambda$ is adapted forms a system of fundamental neighbourhoods of zero in $\mathbb{R}^n$.*

This theorem enables us to find a formal solution satisfying a Gevrey estimate.

It then naturally arises the question what the formal Gevrey functions are.

Let us first examine it in the simplest case of three-dimensional Heisenberg group (Example 3.2). A formal Gevrey function on $H$ at 0 is a formal function $F$ at 0 satisfying:

$$|(X^iY^jZ^kF)(0)| \le C i!\, j!\, (2k)!\, \rho^{i+j+2k}.$$

Let $D$ be the contact distribution generated by $X$ and $Y$, that is,

$$D_p = \{v \in T_p\mathbb{R}^3 \mid \langle \omega, v\rangle = 0\},$$

where

$$\omega = dz - \frac{1}{2}x\,dy + \frac{1}{2}y\,dx.$$

Roughly speaking, a formal Gevrey function is analytic along the contact distribution, or a little more precisely, if $\gamma(t)$ is an analytic integral curve of $D$ with $\gamma(0) = 0$ and if $F$ is formal Gevrey at 0 then $F \circ \gamma$ is analytic since every higher order derivatives can be expressed in terms of those of $\gamma$ and $(X^iY^jF)(0)$ which satisfy analytic estimates. On the other hand, Chow's theorem [Cho40] implies that, since $X$, $Y$ and their bracket generate the tangent space at all point, any point $p$ can be joined to 0 by an integral curve of $D$. This suggests that a formal Gevrey function is already something which looks like a "real" function; we might attempt to define the value $F(p)$ to be $F(\gamma(t_1))$ by taking an integral curve $\gamma$ with $\gamma(0) = 0$, $\gamma(t_1) = p$. However, the value might depend on the curve chosen.

To choose "nice" integral curves we will take curves of minimal length by making use of sub-riemannian geometry.

We define an inner product $g$ on the subbundle $D$ by $g = (dx)^2 + (dy)^2$. Then the length $L$ of an integral curve $\gamma(t) (a \le t \le b)$ is given by:

$$\int_a^b g(\dot\gamma, \dot\gamma)^{1/2}\, dt.$$

If $(x(t), y(t), z(t))$, $a \le t \le b$, is an integral curve joining 0 and $(x_0, y_0, z_0)$ then

$$z_0 = \int_a^b dz = \frac{1}{2}\int_a^b x\,dy - y\,dx$$

$$L(\gamma) = \int_a^b (\dot x^2 + \dot y^2)^{1/2}\, dt.$$

Now in this case we can easily solve the variation problem: If a curve $\gamma(t) = (x(t), y(t), z(t))$, $a \le t \le b$, is of minimal length among the integral curves with the fixed endpoints then the projection $\bar\gamma(t) = (x(t), y(t))$ must be an arc of a circle.

This in mind, we define the exponential mapping:

$$\psi\colon \mathbb{R}^3 \ni (\lambda, \theta, t) \to (x, y, z) \in \mathbb{R}^3$$

by

$$(x, y) = \frac{1}{\lambda} e^{i(\theta+\pi/2)}(1 - e^{i\lambda t}),$$
$$z = \frac{1}{2}\int x\,dy - y\,dx.$$

Then for each fixed $\lambda$, $\theta$, the curve $\psi(\lambda, \theta, t)$ gives a geodesic of this subriemannian metric.

Now let $F$ be a formal Gevrey function at 0. Then, as noted in the above, $F \circ \psi(\lambda, \theta, t)$ is analytic in $t$ in a neighbourhood of 0. But thanks to the good parametrization of integral curves, we see moreover that $F \circ \psi$ is analytic in $\lambda$, $\theta$, $t$ at $t = 0$.

Here we recall Gabrièlov's theorem ([Gab73], [Izu89], [Tou90]) which just applies to our situation:

**Theorem 3.5** (Gabrièlov). *If $\Psi\colon X \to Y$ is an analytic map of generically maximal rank of analytic spaces. Then a formal function $F$ at $y \in Y$ is convergent if the pull-back $F \circ \Psi$ is convergent at a point $x \in \Psi^{-1}(y)$*

Hence we see that a formal Gevrey function on the Heisenberg Lie group turns out to be an analytic function.

The consideration above generalizes to:

**Theorem 3.6.** *Let $N$ be a Lie group with Lie algebra $\mathfrak{n} = \bigoplus_{i=1}^{\mu} \mathfrak{n}_i$. If $\mathfrak{n}$ is generated by $\mathfrak{n}_1$, that is, $\mathfrak{n}_{i+1} = [\mathfrak{n}_1, \mathfrak{n}_i]$ for $i > 0$ (Hörmander condition), then the formal Gevrey functions on $N$ are analytic.*

The outline of a proof is as follows. Let $\theta$ be the Maurer-Cartan form of $N$ taking values in $\mathfrak{n}$, which decomposes as: $\theta = \sum \theta_i$ with $\theta_i$ taking values in $\mathfrak{n}_i$. We identify the cotangent bundle $T^*N$ with $N \times \mathfrak{n}^*$ by assigning $(x, \lambda) \in N \times \mathfrak{n}^*$ to $\lambda \circ \theta_x$. Then the Liouville form is given by $\Theta = \lambda \circ \theta = \sum \lambda_i \circ \theta^i$, where $\lambda = \sum \lambda_i$ with $\lambda_i \in \mathfrak{n}^*$. The symplectic form is given by:

$$\Omega = d\Theta = \sum d\lambda_i \wedge \theta^i + \sum \lambda_i \wedge d\theta^i.$$

Choosing an inner product $(\ ,\ )$ on $\mathfrak{n}_1^*$, we set

$$H = \frac{1}{2}(\lambda_1, \lambda_1)$$

and let $X_H$ be the hamiltonian vector field associated with the energy function $H$, i.e., the vector field determined by $X_H \rfloor d\Theta = -dH$. Let $\varphi_t$ be the flow generated by $X_H$ and

$$\Phi\colon N \times \mathfrak{n}^* \times \mathbb{R} \to N \times \mathfrak{n}^*$$

the map defined by $\Phi(x, \lambda, t) = \varphi_t(x, \lambda)$ on a neighbourhood $U$ of $N \times \mathfrak{n}^* \times \{0\}$ Let $\iota\colon \mathfrak{n}^* \times \mathbb{R} \to \{o\} \times \mathfrak{n}^* \times \mathbb{R} \subset N \times \mathfrak{n}^* \times \mathbb{R}$ be the canonical injection and $\pi\colon N \times \mathfrak{n}^* \to N$ the canonical projection and put $\Psi = \pi \circ \Phi \circ \iota$. This map $\Psi\colon \mathfrak{n}^* \times \mathbb{R} \to N$ is the polar form of the exponential map associated to the energy $H$ and $\Psi(\lambda, t)$ gives an extremal for each fixed $\lambda$. Then we have

**Proposition 3.2.** *If $F$ is a formal Gevrey function at $o$ of $N$ the pull-back $\Psi^* F$ is convergent at every point $(\lambda, 0) \in \mathfrak{n}^* \times \{0\}$*

We will also have:

**Proposition 3.3.** *If $\mathfrak{n}$ is generated by $\mathfrak{n}_1$ then $\Psi$ is generically of maximal rank.*

Then Theorem 3.6 follows from above two propositions and Gabrièlov's theorem.

We notice however that our original proof of Proposition 3.3 is not complete since it uses a delicate theorem of Strichard ([Str86] and [Str89]). Nevertheless, it is quite plausible that Proposition 3.3 can be verified concretely from the structure of a nilpotent graded Lie group.

While B. Jakubczyk has communicated to the author a simpler proof of Theorem 3.6 [Jak00]. His idea is as follows: Let $X_1, \ldots, X_{n_1}$ be a basis of $\mathfrak{n}_1$. In view of Chow's theorem and Sard's theorem we see that there exist $i_1, \ldots, i_s \in \{1, \ldots, n_1\}$ such that the map $\varphi\colon \mathbb{R}^s \to N$ given by

$$\varphi(t_1, \ldots, t_s) = (\exp t_1 X_{i_1} \cdots \exp t_s X_{i_s})(o)$$

is generically of maximal rank and it is easy to see that the properties of Proposition 3.2 and 3.3 are satisfied for this map $\varphi$. Hence the theorem follows from Gabrièlov's theorem.

Finally we have established the following:

**Theorem 3.7.** *Let* (E) *be an analytic system of non-linear partial differential equations of weighted order $k$ on a graded nilpotent Lie group $N$ with a Lie algebra $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$. Assume that $\mathfrak{n}$ is generated by $\mathfrak{n}_1$. If $F^k \in \mathfrak{J}_o^k(N \times V)$ is a weighted $k$-jet solution of* (E) *at $o \in N$ and strongly prolongeable, then there exists an analytic solution $F$ of* (E) *defined in a neighbourhood of $o$ such that $\mathfrak{j}_o^k F = F^k$.*

**Theorem 3.8.** *Let* (E) *be an analytic system of non-linear partial differential equations of weighted order $k$ on a graded nilpotent Lie group $N$ with a Lie algebra $\mathfrak{n} = \bigoplus_{p=1}^{\mu} \mathfrak{n}_p$. Assume that $\mathfrak{n}$ is generated by $\mathfrak{n}_1$. If* (E) *is weightedly involutive, then there exists an analytic solution for any prescribed weighted $k$-jet solution.*

It should be remarked that the above theorems apply to a wide class of systems of non-linear partial differential equations with singularities.

Thus we are led to a non-trivial generalization of the Cartan-Kähler theorem by the nilpotent analysis.

## References


[CS00] A. Čap and H. Schichl, *Parabolic geometries and canonical Cartan connections*, Hokkaido Math. J., **29** (2000), 453–505.

[Car04] É. Cartan, *Sur la structure des groupes infinis de transformations*, Ann. École Norm. Sup., **21** (1904), 153–206.

[Car05] É. Cartan, *Sur la structure des groupes infinis de transformations*, Ann. École Norm. Sup., **22** (1905), 219–308.

[Car08] É. Cartan, *Les sous-groupes des groupes continus de transformations*, Ann. École Norm. Sup., **25** (1908), 57–194.

[Car09] É. Cartan, *Les groupes de transformations continus, infinis, simples*, Ann. École Norm. Sup., **26** (1909), 93–161.

[Car10] É. Cartan, *Les systèmes de Pfaff à cinq variables et les équations aux*, dérivées partielles du second ordre Ann. École Norm. Sup., **27** (1910), 109–192.

[Car31] É. Cartan, *Notice sur les travaux scientifiques*, Élie Cartan Oeuvres complètes, Partie 1, Volume 1, 1952.

[Che54] S. S. Chern, *Pseudo-groupes continus infinis*, Colloque de Geom. Diff., Strasbourg, (1954), 119–136.

[Cho40] W. L. Chow, *Über Systeme Von Linearen Partial Differentialgleichungen*, Erster Ordnung Math. Ann., **117** (1940–1941), 98–105.

[DKM99] B. Doubrov, B. Komrakov and T. Morimoto, *Equivalence of holonomic differential equations*, Lobachevskii Journal of Mathematics, **3** (1999), 39–71.

[Ehr53] C. Ehresmann, *Introduction à la théorie des structures infinitésimal et des pseudo-groupes de Lie*, Colloque International de CNRS, Strasbourg, 1953.

[Gab73] A. M. Gabrièlov, *Formal relations between analytic functions*, Ivz. Akad. Nauk SSSR, **37** (1973), 1056–1088.

[Gol67a] H. Goldschmidt, *Existence theorems for analytic linear partial differential equations*, Ann. of Math., **86** (1967), 246–270.

[Gol67b] H. Goldschmidt, *Integrability criteria for systems of non-linear partial differential equations*, J. Differential Geom., **1** (1967), 269–307.

[GS64] V. W. Guillemin and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc., **70** (1964), 16–47.

[Hay70] I. Hayashi, *Embedding and existence theorems of infinite Lie algebras*, J. Math. Soc. Japan, **22** (1970), 1–14.

[Kis79] K. Kiso, *Local properties of intransitive infinite Lie algebra sheaves*, Japan. J. Math., **5** (1979), 101–155.

[Izu89] S. Izumi, *The rank condition and convergence of formal functions*, Duke Math. J., **59** (1989), 241–264.

[Jak00] B. Jakubczyk, *Convergence of formal power series along vector fields and their commutators*, Ann. Polonici Math., **74** (2000), 117–132.

[Käh34] E. Kähler, Einfürhung in die Theorie der Systeme von Differentialgleichungen, Teubner, Leipzig, 1934.

[Kob72] S. Kobayashi, Transformation groups in differential geometry, Springer-Verlag, 1972.

[KN66] S. Kobayashi and T. Nagano, *On filtered Lie algebras and geometric structures,* IV, J. of Math. and Mech., **15** (1966), 163–175.

[Kur57] M. Kuranishi, *On E. Cartan's prolongtion theorem of exterior differential systems*, Amer. J. Math., **79** (1957), 1–47.

[Kur62] M. Kuranishi, Lectures on exterior differential systems, Tata Institute of Fundamental Research, Bombay, 1962.

[Mal72] B. Malgrange, *Equations de Lie* II, J. of Differential Geometry, **7** (1972), 117–141.

[Mal77] B. Malgrange, *Frobenius avec singularités* (2. *Le cas général*), Inventiones Math., **39** (1977), 67–89.

[Mor83] T. Morimoto, *Sur le problème d'équivalence des structures géométriques*, Japan. J. Math., **9-2** (1983), 293–372.

[Mor88] T. Morimoto, *Transitive Lie algebras admitting differential systems*, Hokkaido Math. J., **17** (1988), 45–81.

[Mor90] T. Morimoto, *Théorème de Cartan-Kähler dans une classe de fonctions formelles Gevrey*, C. R. Acad. Sci. Paris, **311** (1990), 433–436.

[Mor91] T. Morimoto, *Generalized Spencer cohomology groups and quasi-regular bases*, Tokyo J. Math., **14** (1991), 165–179.

[Mor93] T. Morimoto, *Geometric structures on filtered manifolds*, Hokkaido Math. J., **22** (1993), 263–347.

[Mor95] T. Morimoto, *Théorème d'existence de solutions analytiques pour des systèmes d'équations aux dérivées partielles non-linéaires avec singularités*, C. R. Acad. Sci. Paris, **321** (1995), 1491–1496.

[Mor97] T. Morimoto, *Monge-Ampère equations viewed from contact geometry*, Banach Center Publications "Symplectic singularities and geometry of gauge fields", **39** (1997), 105–120.

[Mor0x] T. Morimoto, *Differential equations on filtered manifolds*, in preparation.

[Qui64] D. Quillen, *Formal properties of over-determined systems of linear partial differential equations*, Ph. D. thesis, Harvard University, 1964.

[Sha97] R. W. Sharpe, Differential geometry, Cartan's generalization of Klein's Erlangen Program, Springer, 1997.

[SS65] I. M. Singer and S. Sternberg, *On the infinite groups of Lie and Cartan,* I, J. Analyse Math., **15** (1965), 1–114.

[Spe69] D. C. Spencer, *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc., **75** (1969), 179–239.

[Ste64] S. Sternberg, Lectures on differential geometry, Prentice-Hall, 1964.

[Str86] R. S. Strichartz, *Sub-riemannian geometry*, J. Differential Geometry, **24** (1986), 221–263.

[Str89] R. S. Strichartz, *Corrections to "Sub-riemannian geometry"*, J. Differential Geometry, **30** (1989), 595–596.

[SY98] H. Sato and A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connections*, J. Math. Soc. Japan, **50** (1998), 993–1013.

[Tan62] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of $n$ complex variables*, J. Math. Soc. Japan, **14** (1962), 397–429.

[Tan70] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ., **10** (1970), 1–82.

[Tan76] N. Tanaka, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japanese J. Math., **2** (1976), 131–190.

[Tan79] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J., **8** (1979), 23–84.

[Tan82] N. Tanaka, *On geometry and integration of systems of second order ordinary differential equations*, (in Japanese), Proc. Symposium on Differential Geometry, (1982), 194–205.

[Tan89] N. Tanaka, *Geometric theory of ordinary differential equations*, Report of Grant-in-Aid for Science Research MESC Japan (1989).

[Tou90] J. C. Tougeron, *Sur les racines d'une polynome à coéfficients séries formelles*, in Real analytic and algebraic geometry (ed. by M. Galbiati and A. Tognoli), Lecture Note in Math. (Springer), **1420** (1990), 325–363.

[Yam82] K. Yamaguchi, *Contact geometry of higher order*, Japan. J. Math., **8** (1982), 109–176.

[Yam93] K. Yamaguchi, *Differential systems associated with simple graded Lie algebras*, Adv. Studies in Pure Math., **22** (1993), 413–494.

[Yam99] K. Yamaguchi, *$G_2$-geometry of overdetermined systems of second order*, Trends in Math. (Analysis and geometry in several complex variables) (1999), 289–314.

[Yat92] T. Yatsui, *On completely reducible transitive graded*, Lie algebras of finite depth, Japan. J. Math., **18** (1992), 291–329.

*Department of Mathematics*
*Nara Women's University*
*Nara 630-8506*
*Japan*
*morimoto@cc.nara-wu.ac.jp*