Advanced Studies in Pure Mathematics 34, 2002
Minimal Surfaces, Geometric Analysis and Symplectic Geometry
pp. 143–154

On 4-dimensional CR-Submanifolds of a 6-dimensional Sphere

Hideya Hashimoto, Katsuya Mashimo and Kouei Sekigawa

Abstract.

We prove several fundamental properties of 4-dimensional CR-submanifolds of a nearly Kähler 6-dimensional sphere and construct explicit examples of such submanifolds.

§1. Introduction

Let S^6 be the 6-dimensional unit sphere centered at the origin of a 7-dimensional Euclidean space $\mathbf{R}^7$. We denote by $\mathbf{O}$ the normed algebra of octonions (or Cayley algebra) and identify the set of pure imaginary octonions $\operatorname{Im}\mathbf{O}$ with $\mathbf{R}^7$. An almost complex structure on S^6 is defined as follows:

$$JX = X \times x, \quad x \in S^6, \ X \in T_x(S^6),$$

where $\times$ denotes the cross product of octonions. The almost complex structure J is compatible with the canonical metric $\langle\ ,\ \rangle$ and the almost Hermitian structure $(J, \langle , \rangle)$ on S^6 is nearly Kähler ([F-I]).

In this paper, we shall study 4-dimensional CR-submanifolds of the nearly Kähler manifold $(S^6, J, \langle , \rangle)$. Let M be a submanifold of S^6. We put $\mathcal{H}_x = T_xM \cap J(T_xM)$ for $x \in M$ and denote by $\mathcal{H}_x^\perp$ the orthogonal complement of $\mathcal{H}_x$ in T_xM. If the dimension of $\mathcal{H}_x$ is constant and $J(\mathcal{H}_x^\perp) \subset T_x^\perp M$ for any $x \in M$, the submanifold M is called a *CR submanifold.*

Concerning the existence of almost complex submanifolds and totally real submanifolds of $(S^6, J, \langle , \rangle)$, many results have been obtained (see, [Gr], [Se]). On the other hand, about the existence of CR-submanifolds, only a result by Sekigawa was known before our previous paper ([H-M]), in which the first and the second authors proved that there exist many 3-dimensional CR-submanifolds.

One aim of this paper is to give some topological restrictions on the existence of compact 4-dimensional CR-submanifolds of S^6. For

2000 *Mathematics Subject Classification.* Primary 53B25; Secondary 53C15.

example, we prove that the Euler number of a compact 4-dimensional CR-submanifold is equal to zero. We also consider the integrability of the distributions $\mathcal{H}$ and $\mathcal{H}^{\perp}$. Many examples of 4-dimensional CR-submanifolds of S^6 will be given in the last section.

The authors wish to express their gratitude to Professor Yasuo Matsushita for his many valuable comments on characteristic classes.

§2. Preliminaries

Let $\mathbf{Q}$ be the skew field of all quaternions. The algebra of octonions $\mathbf{O}$ is the direct sum $\mathbf{O} = \mathbf{Q} \oplus \mathbf{Q}$ with the following multiplication:

$$(q, r) \cdot (s, t) = (qs - t^{\iota}\, r, tq + r\, s^{\iota}), \quad q, r, s, t \in \mathbf{Q},$$

where $^{\iota}$ means the conjugation in $\mathbf{Q}$. We define a conjugation in $\mathbf{O}$ by $(q, r)^{\iota} = (q^{\iota}, -r)$, $q, r \in \mathbf{Q}$, and an inner product $\langle , \rangle$ by

$$\langle x, y \rangle = \frac{(x \cdot y^{\iota} + y \cdot x^{\iota})}{2}, \quad x, y \in \mathbf{O}.$$

We denote by $\mathbf{G}_2$ the group of automorphisms of $\mathbf{O}$, that is,

$$\mathbf{G}_2 = \{g \in \mathbf{GL}(8, \mathbf{R});\ g(uv) = g(u)g(v) \text{ for any } u,\ v\ \in \mathbf{O}\}.$$

Each element of $\mathbf{G}_2$ leaves invariant the identity element $(1, 0)$ and its orthogonal complement $\operatorname{Im}\mathbf{O}$. Thus we may regard $\mathbf{G}_2$ as a subgroup of $\mathbf{GL}(7, \mathbf{R}) = \mathbf{GL}(\operatorname{Im}\mathbf{O})$.

Now, we define a basis of $\mathbf{C} \otimes \operatorname{Im}\mathbf{O}$,

$$(\varepsilon, E, \overline{E}) = (\varepsilon, E_1, E_2, E_3, \overline{E}_1, \overline{E}_2, \overline{E}_3)$$

as follows:

$$\varepsilon = (0, 1) \in \mathbf{Q} \oplus \mathbf{Q},$$

$$E_1 = iN,\ E_2 = jN,\ E_3 = -kN,$$

$$\overline{E}_1 = i\overline{N},\ \overline{E}_2 = j\overline{N},\ \overline{E}_3 = -k\overline{N},$$

where $N = (1 - \sqrt{-1}\varepsilon)/2$, $\overline{N} = (1 + \sqrt{-1}\varepsilon)/2 \in \mathbf{C} \otimes \mathbf{O}$. We denote also by g the complex linear extension of $g \in \mathbf{G}_2$. A basis $(u,\ f,\ \overline{f})$ of $\mathbf{C} \otimes \operatorname{Im}\mathbf{O}$ is said to be *admissible*, if there exists an element g of $\mathbf{G}_2$ such that $(u,\ f,\ \overline{f}) = (\varepsilon,\ E,\ \overline{E})g$. We identify an element of $\mathbf{G}_2$ with an admissible basis by the injection

$$\iota\ :\ \mathbf{G}_2 \to \mathbf{GL}(7, \mathbf{C})\ ;\ g \mapsto (\varepsilon,\ E,\ \overline{E})g.$$

We denote by $M_{p\times q}(\mathbf{C})$ the set of $p \times q$ complex matrices. Let $[a]$ be the element given by

$$[a] = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} \in M_{3\times 3}(\mathbf{C})$$

for $a = {}^t(a_1\ a_2\ a_3) \in M_{3\times 1}(\mathbf{C})$. Then we have

$$[a]b + [b]a = 0,$$

where $a, b \in M_{3\times 1}(\mathbf{C})$. We adopt the matrix representation of elements of $\mathbf{GL}(7, \mathbf{C})$ with respect to $(\varepsilon,\ E,\ \overline{E})$.

Proposition 2.1 (cf. Bryant [Br]). *The pull-back Φ of the Maurer-Cartan form of $\mathbf{GL}(7, \mathbf{C})$ is of the form*

$$\Phi = \begin{pmatrix} 0 & -\sqrt{-1}\ {}^t\overline{\theta} & \sqrt{-1}\ {}^t\theta \\ -2\sqrt{-1}\ \theta & \kappa & [\overline{\theta}] \\ 2\sqrt{-1}\ \overline{\theta} & [\theta] & \overline{\kappa} \end{pmatrix} \tag{2.1}$$

where $\kappa = (\kappa_j{}^i)$ $(1 \leq i, j \leq 3)$ (resp. $\theta = {}^t(\theta^1\ \theta^2\ \theta^3)$) is an $\mathfrak{su}(3)$-valued (resp. $M_{3\times 1}(\mathbf{C})$-valued) left invariant 1-forms. The Maurer-Cartan equation $d\Phi = -\Phi \wedge \Phi$ reduces to

$$d\theta = -\kappa \wedge \theta + [\overline{\theta}] \wedge \overline{\theta}, \tag{2.2}$$
$$d\kappa = -\kappa \wedge \kappa + 3\theta \wedge {}^t\overline{\theta} - ({}^t\theta \wedge \overline{\theta})\, I_3. \tag{2.3}$$

§3. Structure equations

Let $\varphi : M \to S^6$ be a 4-dimensional submanifold of S^6. We denote by ∇ (resp. D) the Levi Civita connection of M (resp. S^6) and by $\nabla^\perp$ the induced connection on the normal bundle of M in S^6. We denote by σ the second fundamental form and A_ν the shape operator in the direction of ν. The Gauss and the Weingarten formulas are given respectively by

$$\begin{aligned} D_X(\varphi_*(Y)) &= \varphi_*(\nabla_X Y) + \sigma(X, Y), \\ D_X\nu &= -\varphi_*(A_\nu(X)) + \nabla^\perp{}_X\nu, \end{aligned}$$

where X, Y are tangent vector fields and ν is a normal vector field.

Let $\varphi : M \to S^6$ be an oriented 4-dimensional CR-submanifold of S^6. Define an orientation on $\mathcal{H}^\perp$ in such a way that an orthonormal base $\{\xi_1, \xi_2\}$ of $\mathcal{H}_p^\perp$ for $p \in M$ is oriented if and only if $\{v,\ J(v),\ \xi_1,\ \xi_2\}$ is oriented for some unit vector $v \in \mathcal{H}_p$.

Lemma 3.1. *Take an oriented orthonormal base* $\{\xi_1, \xi_2\}$ *of* $\mathcal{H}_p^{\perp}$ *for* $p \in M$. *The vector* $\xi_1 \times \xi_2$ *is an element of* $\mathcal{H}_p$ *and is independent of the choice of the base.*

We denote by $\mathcal{F}$ the bundle of unit vectors of $\mathcal{H}^{\perp}$. For a vector $\xi \in \mathcal{F}$ we denote by ξ' the vector such that $\{\xi, \xi'\}$ is an oriented orthonormal frame of $\mathcal{F}$. We define a mapping $\psi : \mathcal{F} \to \mathbf{GL}(7, \mathbf{C})$ by

$$\psi(\xi) = (\varphi \circ \pi(\xi), f, \overline{f})$$

where

$$\begin{aligned} f_1 &= \frac{1}{2}(\xi - \sqrt{-1}J\xi), \\ f_2 &= \frac{1}{2}(\xi' - \sqrt{-1}J\xi'), \\ f_3 &= -\overline{f_1 \times f_2} = -\frac{1}{2}\left(\xi \times \xi' - \sqrt{-1}J(\xi \times \xi')\right). \end{aligned}$$

Define $\mathbf{C} \otimes \operatorname{Im} \mathbf{O}$-valued functions f_3, Ξ_1 and Ξ_2 on $\mathcal{F}$ as follows:

$$\begin{aligned} \mathbf{f}_3\left((\varphi \circ \pi(\xi), f, \overline{f})\right) &= f_3, \\ \Xi_1\left((\varphi \circ \pi(\xi), f, \overline{f})\right) &= \xi, \\ \Xi_2\left((\varphi \circ \pi(\xi), f, \overline{f})\right) &= \xi'. \end{aligned}$$

Note that the image of the mapping ψ is contained in $\iota(\mathbf{G}_2)$. Also any element of the fibre is expressed as $\cos(\theta)\ \xi + \sin(\theta)\ \xi'$.

Proposition 3.2. *Restricting the* 1*-forms* ${\kappa_i}^j$ *and* θ^i *given in Proposition* 2.1 *to* $\mathcal{F}$, *we have the following*:

$$\begin{aligned} (3.1)\quad d\varphi \circ \pi_* &= \mathbf{f}_3 \otimes (-2\sqrt{-1}\ \theta^3) + \overline{\mathbf{f}_3} \otimes (2\sqrt{-1}\ \overline{\theta^3}) \\ &\quad + \Xi_2 \otimes \mu_2 + \Xi_1 \otimes \mu_1, \\ (3.2)\quad \theta^3(\tilde{X}) &= \sqrt{-1}\left\langle \pi^* d\varphi(\tilde{X}), \overline{\mathbf{f}_3} \right\rangle, \\ \theta^1(\tilde{X}) &= \frac{\sqrt{-1}}{2}\left\langle \pi^* d\varphi(\tilde{X}), \Xi_1 \right\rangle = \frac{\sqrt{-1}}{2}\mu_1(\tilde{X}), \\ (3.3)\quad \theta^2(\tilde{X}) &= \frac{\sqrt{-1}}{2}\left\langle \pi^* d\varphi(\tilde{X}), \Xi_2 \right\rangle = \frac{\sqrt{-1}}{2}\mu_2(\tilde{X}), \\ (3.4)\qquad d\mathbf{f}_3 &= \pi \circ \psi \otimes (-\sqrt{-1}\ \overline{\theta^3}) + \mathbf{f}_3 \otimes {\kappa_3}^3 \\ &\quad + \Xi_2 \otimes \frac{1}{2}\left(\frac{\sqrt{-1}}{2}\mu_1 + {\kappa_3}^2\right) \end{aligned}$$

$$
\begin{aligned}
& - \Xi_1 \otimes \frac{1}{2}\left(\frac{\sqrt{-1}}{2}\mu_2 - {\kappa_3}^1\right) \\
& - J\Xi_2 \otimes \frac{1}{2}\left(\frac{1}{2}\mu_1 + \sqrt{-1}{\kappa_3}^2\right) \\
& + J\Xi_1 \otimes \frac{1}{2}\left(\frac{1}{2}\mu_2 - \sqrt{-1}{\kappa_3}^1\right),
\end{aligned}
$$

$$
\begin{aligned}
(3.5) \qquad d\Xi_2 \ = \ & \pi \circ \psi \otimes (-\mu_2) + \mathbf{f}_3 \otimes \left({\kappa_2}^3 + \frac{\sqrt{-1}}{2}\mu_1\right) \\
& + \overline{\mathbf{f}_3} \otimes \left(\overline{{\kappa_2}^3} - \frac{\sqrt{-1}}{2}\mu_1\right) \\
& + \Xi_1 \otimes \frac{1}{2}({\kappa_2}^1 + \overline{{\kappa_2}^1} + \theta^3 + \overline{\theta^3}) \\
& - J\Xi_2 \otimes (\sqrt{-1}{\kappa_2}^2) \\
& + J\Xi_1 \otimes \frac{\sqrt{-1}}{2}(-{\kappa_2}^1 + \overline{{\kappa_2}^1} + \theta^3 - \overline{\theta^3}),
\end{aligned}
$$

$$
\begin{aligned}
(3.6) \qquad d\Xi_1 \ = \ & \pi \circ \psi \otimes (-\mu_1) + \mathbf{f}_3 \otimes \left({\kappa_1}^3 - \frac{\sqrt{-1}}{2}\mu_2\right) \\
& + \overline{\mathbf{f}_3} \otimes \left(\overline{{\kappa_1}^3} + \frac{\sqrt{-1}}{2}\mu_2\right) \\
& + \Xi_2 \otimes \frac{1}{2}({\kappa_1}^2 + \overline{{\kappa_1}^2} - \theta^3 - \overline{\theta^3}) \\
& + J\Xi_2 \otimes \frac{\sqrt{-1}}{2}(-{\kappa_1}^2 + \overline{{\kappa_1}^2} - \theta^3 + \overline{\theta^3}) \\
& + J\Xi_1 \otimes (-\sqrt{-1}{\kappa_1}^1),
\end{aligned}
$$

$$
\begin{aligned}
(3.7) \ d(J\Xi_2) \ = \ & \mathbf{f}_3 \otimes \sqrt{-1}\left({\kappa_2}^3 - \frac{\sqrt{-1}}{2}\mu_1\right) \\
& - \overline{\mathbf{f}_3} \otimes \sqrt{-1}\left(\overline{{\kappa_2}^3} + \frac{\sqrt{-1}}{2}\mu_1\right) + \Xi_2 \otimes \sqrt{-1}{\kappa_2}^2 \\
& + \Xi_1 \otimes \frac{\sqrt{-1}}{2}({\kappa_2}^1 - \overline{{\kappa_2}^1} + \theta^3 - \overline{\theta^3}) \\
& + J\Xi_1 \otimes \frac{1}{2}({\kappa_2}^1 + \overline{{\kappa_2}^1} - \theta^3 - \overline{\theta^3}),
\end{aligned}
$$

$$
\begin{aligned}
(3.8) \ d(J\Xi_1) \ = \ & \mathbf{f}_3 \otimes \sqrt{-1}\left({\kappa_1}^3 + \frac{\sqrt{-1}}{2}\mu_2\right) \\
& - \overline{\mathbf{f}_3} \otimes \sqrt{-1}\left(\overline{{\kappa_1}^3} - \frac{\sqrt{-1}}{2}\mu_2\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Xi_2 \otimes \frac{\sqrt{-1}}{2}({\kappa_1}^2 - \overline{{\kappa_1}^2} - \theta^3 + \overline{\theta^3}) \\
&+ \Xi_1 \otimes \sqrt{-1}{\kappa_1}^1 \\
&+ J\Xi_2 \otimes \frac{1}{2}({\kappa_1}^2 + \overline{{\kappa_1}^2} + \theta^3 + \overline{\theta^3}).
\end{aligned}
$$

Remark 3.3. From Lemma 3.1, there exists a complex valued global 1-form Θ on M^4 such that $\pi^*\Theta = \theta^3$.

Next we give the explicit expression of the integarability conditions (2.2) and (2.3).

Lemma 3.4. *On $\mathcal{F}$, we have the following:*

$$
\begin{aligned}
&(3.9) & d\mu^1 &= -{\kappa_1}^1 \wedge \mu^1 - {\kappa_2}^1 \wedge \mu^2 \\
&& &\quad - {\kappa_3}^1 \wedge (-2\sqrt{-1}\,\theta^3) + 2\mu^2 \wedge \overline{\theta^3}, \\
&(3.10) & d\mu^2 &= -{\kappa_1}^2 \wedge \mu^1 - {\kappa_2}^2 \wedge \mu^2 \\
&& &\quad - {\kappa_3}^2 \wedge (-2\sqrt{-1}\,\theta^3) - 2\mu^1 \wedge \overline{\theta^3}, \\
&(3.11) & d\theta^3 &= -\frac{\sqrt{-1}}{2}({\kappa_1}^3 \wedge \mu^1 + {\kappa_2}^3 \wedge \mu^2) \\
&& &\quad - {\kappa_3}^3 \wedge \theta^3 + \frac{1}{2}\mu^1 \wedge \mu^2, \\
&(3.12) & d{\kappa_3}^3 &= -\sum_{j=1}^{3} {\kappa_j}^3 \wedge {\kappa_3}^j + 2\theta^3 \wedge \overline{\theta^3}, \\
&(3.13) & d{\kappa_i}^i &= -\sum_{j=1}^{3} {\kappa_j}^i \wedge {\kappa_i}^j - \theta^3 \wedge \overline{\theta^3} \quad (i = 1, 2), \\
&(3.14) & d{\kappa_2}^1 &= -\sum_{j=1}^{3} {\kappa_j}^1 \wedge {\kappa_2}^j + \frac{4}{3}\mu^1 \wedge \mu^2, \\
&(3.15) & d{\kappa_3}^1 &= -\sum_{j=1}^{3} {\kappa_j}^1 \wedge {\kappa_3}^j + \frac{3\sqrt{-1}}{2}\mu^1 \wedge \overline{\theta^3}, \\
&(3.16) & d{\kappa_3}^2 &= -\sum_{j=1}^{3} {\kappa_j}^2 \wedge {\kappa_3}^j + \frac{3\sqrt{-1}}{2}\mu^2 \wedge \overline{\theta^3}.
\end{aligned}
$$

Finally we shall represent the connection 1-form $\left\langle (d\Xi_1)(\tilde{X}), \Xi_2 \right\rangle$ of the S^1 bundle $\mathcal{F}$ explicitely, in terms of the local data. We put

$$\partial_\theta = \frac{d}{d\theta}\bigg|_{\theta=0} (\cos(\theta)\xi + \sin(\theta)\xi') = \xi',$$

and denote by $d\theta$ its dual 1-form. By (3.6), we obtain

$$\left\langle (d\Xi_1)(\tilde{X}), \Xi_2 \right\rangle = -\frac{1}{2}({\kappa_2}^1 + \overline{{\kappa_2}^1} + \theta^3 + \overline{\theta^3})(\tilde{X}) = \left\langle \nabla_{d\pi(\tilde{X})}\xi_1, \xi_2 \right\rangle + d\theta(\tilde{X}).$$

In particular, we have $(1/2)({\kappa_2}^1 + \overline{{\kappa_2}^1})(\partial_\theta) = 1$.

§4. Topological restrictions

In this section we prove several topological properties of 4-dimensional CR-submanifolds of S^6. From Lemma 3.1 and Hopf's Index theorem, we immediately obtain the following

Proposition 4.1. *Let $\varphi : M^4 \to S^6$ be an oriented 4-dimensional CR-submanifold of S^6. Then both of the Euler class of M^4 and the Euler class of the complex subbundle $\mathcal{H}$ over M vanish. If M^4 is compact, then the Euler number $\chi(M^4)$ is equal to zero. In particular, $S^4, S^2 \times S^2$ and $\mathbf{C}P^2$ can not be immersed into S^6 as a CR-submanifold.*

Next we shall establish the relations of the various characteristic classes of the bundles $\mathcal{H}$, $\mathcal{H}^\perp$ and $T^\perp M^4$ over M^4. We denote by $J_\mathcal{H}$ the restriction to $\mathcal{H}$ of the almost complex structure of S^6, and J' the almost complex structure on $\mathcal{H}^\perp$ such that the orientaion determined by the almost complex structure $J_1 = J_\mathcal{H} \oplus J'$ on M coincides with that given on M. We denote by J_2 the opposite almost complex structure: $J_2 = J_\mathcal{H} \oplus (-J')$. We also denote by $J^\perp$ the almost complex structure of $T^\perp M^4$ which is compatible with the orientation of $T^\perp M^4$. Recall that

$$\varphi^*(TS^6)|_{M^4} = \mathcal{H} \oplus \mathcal{H}^\perp \oplus T^\perp M^4. \tag{4.1}$$

Let V be the direct sum $V = \mathcal{H}^\perp \oplus T^\perp M^4$. We denote by J_V the restriction to V of the almost complex structure J of S^6. We denote by $V^{(1,0)}$ (resp. $V^{(0,1)}$) the set of vectors of type (1,0) (resp. (0,1)) in the complexification $V \otimes \mathbf{C}$.

Proposition 4.2. *Let $\varphi : M \to S^6$ be an oriented 4-dimensional CR-submanifold of S^6. Then we have in $H^*(M; \mathbf{Z})$*

(1) $e(\mathcal{H}) = c_1(\mathcal{H}^{(1,0)}) (\equiv c_1(\mathcal{H}^{(1,0)}, J_\mathcal{H})) = 0,$

(2) $p_1(TM^4) = \{c_1(\mathcal{H}^{\perp(1,0)}, J')\}^2 = -\{c_1(T^{\perp(1,0)}M^4, J^\perp)\}^2$,
(3) $p_1(V) = 0$,
(4) $c_1(V^{(1,0)}) = 0$,

where we denote by $p_1(\ )$ (*resp.* $c_1(\ )$) *the first Pontrjagin* (*resp. Chern*) *class and by* $e(\ )$ *the Euler class of the respective bundles.*

Proof. By Lemma 3.1, we get (1) immediately. For (2), we calculate the second Chern class of the complexified tangent bundle $TM^4 \otimes \mathbf{C}$ by making use of the above decomposition. Then, we have

$$\begin{aligned} c(TM^4 \otimes \mathbf{C}) &= c(\mathcal{H}^{(1,0)} \oplus \mathcal{H}^{(0,1)} \oplus \mathcal{H}^{\perp(1,0)} \oplus \mathcal{H}^{\perp(0,1)}) \\ &= (1 - \{c_1(\mathcal{H}^{(1,0)})\}^2)(1 - \{c_1(\mathcal{H}^{\perp(1,0)})\}^2). \end{aligned}$$

Therefore we have $c_2(TM^4 \otimes \mathbf{C}) = -\{c_1(\mathcal{H}^{(1,0)})\}^2 - \{c_1(\mathcal{H}^{\perp(1,0)})\}^2$, from which we get $p_1(TM^4) = \{c_1(\mathcal{H}^{(1,0)})\}^2 + \{c_1(\mathcal{H}^{\perp(1,0)})\}^2$. Hence we have (2).

Next, we prove (3) and (4). From the decomposition $\varphi^*(T^{(1,0)}S^6)|_{M^4} = \mathcal{H}^{(1,0)} \oplus V^{(1,0)}$ and $c(T^{(1,0)}S^6) = 1$, we have

$$\begin{aligned} 1 &= 1 + c_1(\mathcal{H}^{(1,0)}) + c_1(V^{(1,0)}) \\ &\quad + c_1(\mathcal{H}^{(1,0)})c_1(V^{(1,0)}) + c_2(V^{(1,0)}) + c_1(\mathcal{H}^{(1,0)})c_2(V^{(1,0)}). \end{aligned}$$

Thus we obtain (4). Since $c_2(V^{(1,0)}) = 0$, we have $p_1(V) = -c_2(V \otimes C) = c_1(V^{(1,0)})^2 - 2c_2(V^{(1,0)}) = 0$. □

Theorem 4.3. *Let* $\varphi : M^4 \to S^6$ *be an oriented* 4*-dimensional CR-submanifold of* S^6*. Then the first Portrjagin class of* M^4 *vanishes. In particular, if* M^4 *is compact, its Hirzebruch signature is equal to zero.*

Proof. First we can show that the structure group of the vector bundle V reduces to $Sp(1) \simeq SU(2)$. The vector bundle $V = \mathcal{H}^\perp \oplus T^\perp M^4$ admits two different orthogonal almost complex structures $J' \oplus J^\perp$ and J_V. We may easily check that the composition $(J' \oplus J^\perp) \circ J_V$ is also an orthogonal almost complex structure on V. Furthermore, these three orthogonal almost complex structures satisfy the quaternionic relations. Thus we get $c_1(V, (J' \oplus J^\perp)) = c_1(V, -(J' \oplus J^\perp)) = -c_1(V, (J' \oplus J^\perp))$ (see [p.46; Theorem (5.11); Kob]). Therefore, we have

$$c_1(V, (J' \oplus J^\perp) = c_1(\mathcal{H}^\perp, J') + c_1(T^\perp M^4, J^\perp) = 0,$$

from which we get immediately $c_1(\mathcal{H}^{\perp(1,0)}) + c_1(T^{\perp(1,0)}M^4) = 0$. Therefore, by Proposition 4.2 (2), we obtain the desired result. □

§5. Distributions $\mathcal{H}$ and $\mathcal{H}^\perp$

Proposition 5.1. *The totally real distribution $\mathcal{H}^\perp$ of an oriented 4-dimensional CR-submanifold $\varphi : M \to S^6$ is not involutive.*

Proof. By Frobenius' theorem, $\mathcal{H}^\perp$ is involutive if and only if

$$d\theta^3 \equiv 0 \mod \{\theta^3, \overline{\theta^3}, d\theta\}. \tag{5.1}$$

From 3.12, we have

$$d\theta^3 \equiv \frac{\sqrt{-1}}{2}\left(-\sqrt{-1} + {\kappa_1}^3(E_2) - {\kappa_2}^3(E_1)\right)\mu_1 \wedge \mu_2 \mod \{\theta^3, \overline{\theta^3}, d\theta\},$$

where $\{E_1, E_2\}$ is the dual basis of $\{\mu_1, \mu_2\}$. Thus (5.1) is equivalent to

$$-\sqrt{-1} + {\kappa_1}^3(E_2) - {\kappa_2}^3(E_1) = 0.$$

On the other hand, taking account of (3.5), (3.6) and $\pi^* d\varphi(E_i) = \Xi_i$ for $i = 1, 2$, we get

$$\begin{aligned} {\kappa_1}^3(E_2) &= \sqrt{-1}\left(2\left\langle \sigma(\Xi_2, \overline{f_3}), J\Xi_1\right\rangle - \frac{1}{2}\right), \\ {\kappa_2}^3(E_1) &= \sqrt{-1}\left(2\left\langle \sigma(\Xi_1, \overline{f_3}), J\Xi_2\right\rangle + \frac{1}{2}\right). \end{aligned}$$

Finally, by (3.6) and (3.7), we have

$$\begin{aligned} &-\sqrt{-1} + {\kappa_1}^3(E_2) - {\kappa_2}^3(E_1) \\ &\quad= -2\sqrt{-1} + 2\sqrt{-1}\left(\left\langle \sigma(\Xi_2, \overline{f_3}), J\Xi_1\right\rangle - \left\langle \sigma(\Xi_1, \overline{f_3}), J\Xi_2\right\rangle\right) \\ &\quad= -2\sqrt{-1} + 2\sqrt{-1}\left(\left\langle d\Xi_2(\overline{\mathbf{f}_3}), J\Xi_1\right\rangle - \left\langle d\Xi_1(\overline{\mathbf{f}_3}), J\Xi_2\right\rangle\right) \\ &\quad= -2\sqrt{-1} - 2\overline{\theta^3}(\overline{\mathbf{f}_3}) \\ &\quad= -3\sqrt{-1}, \end{aligned}$$

which is a contradaiction. □

As an immediate consequence of Proposition 4.2 (1), we have the following lemma on the involutivity of the distibution $\mathcal{H}$.

Lemma 5.2. *Let $\varphi : M^4 \to S^6$ be an oriented 4-dimensional CR-submanifold of S^6. If the distribution $\mathcal{H}$ is involutive, then each compact leaf of $\mathcal{H}$ is homeomorphic to a torus.*

Let $\varphi : M \to S^6$ be an oriented 4-dimensional CR-submanifold of S^6. Take a (locally defined) oriented orthonormal frame $\{\xi_1, \xi_2\}$ of $\mathcal{H}^\perp$. We put $e_1 = \xi_1 \times \xi_2$, $e_2 = J(e_1)$ and denote by ω_1, ω_2, ω_3, ω_4 the dual 1-forms of e_1, e_2, ξ_1, ξ_2, respectively. From Lemma 3.1, ω_1, ω_2 are independent of the choice of the frame, and it is easily seen that so is the 2-form $\omega_3 \wedge \omega_4$.

Proposition 5.3. *Let $\varphi : M \to S^6$ be an oriented 4-dimensional CR-submanifold of S^6. The pull-back by $\pi : \mathcal{F} \to M$ of the complex valued 3-form*

$$(\omega_1 + \sqrt{-1}\omega_2) \wedge \omega_3 \wedge \omega_4$$

is equal to $2\sqrt{-1}\theta^3 \wedge \mu_1 \wedge \mu_2$ and is a closed form.

Proof. By (3.10), (3.11) and (3.12), we have

$$d(\theta^3 \wedge \mu_1 \wedge \mu_2) = -({\kappa_3}^3 + {\kappa_2}^2 + {\kappa_1}^1) \wedge \theta^3 \wedge \mu_1 \wedge \mu_2 = 0.$$

□

Remark 5.4. The proposition 5.3 is equivalent to the fact that $\mathrm{div}(e_1) = \mathrm{div}(J(e_1)) = 0$.

§6. Examples

In this section, we give two kinds of 4-dimensional CR-submanifolds of S^6. A 4-dimensional submanifold M of S^6 is a CR-submanifold if and only if the normal bundle $T^{\perp}M$ of M is a totally real subbundle (namely, $\Omega(T^{\perp}M) = \Omega \wedge \Omega(TM) = 0$, where Ω is the fundamental 2-form of S^6 defined by $\Omega(X, Y) = \langle JX, Y\rangle$ for $X, Y \in \mathfrak{X}(S^6)$).

Proposition 6.1. *Let $\gamma : I \to S^2 \subset \mathrm{Im}\,\mathbf{Q}$ be a regular curve in the unit 2-sphere. Then the following immersion $\psi : I \times Sp(1) \to S^6$ is a 4-dimensional CR-submanifold of S^6:*

$$\psi(t, q) = a\gamma(t) + bq^{\iota}\varepsilon,$$

where a, b are positive real numbers satisfying $a^2 + b^2 = 1$.

Proof. It is easy to verify that the vector fields

$$\nu_1 = \dot{\gamma}(t) \times \gamma(t), \quad \nu_2 = b\gamma(t) - aq^{\iota}\varepsilon$$

form an orthonormal frame field of the normal bundle and satisfy $\langle \nu_1, J(\nu_2)\rangle = 0$. □

For an element (z, q) of $U(1) \times Sp(1)$, we have an automorphism $\tau(z, q)$ of the Cayley algebra defined by

$$(6.1) \qquad (\tau(z, q))(r + s\varepsilon) = (qrq^{\iota}) + (zsq^{\iota})\varepsilon, \quad r, s \in \mathbf{Q}, r + r^{\iota} = 0.$$

We denote by L the image of the Lie group homomorphism $\tau : U(1) \times Sp(1) \to \mathrm{Aut}(\mathbf{O}) = \mathbf{G}_2$.

It is easily verified that on each orbit of the action of L on S^6, there exists a point of the form $ai + (b + cj)\varepsilon$ with $a \geq 0,\ b \geq 0,\ c \geq 0$ and $a^2 + b^2 + c^2 = 1$.

Proposition 6.2. *For any positive numbers a, b, c satisfying $a^2 + b^2 + c^2 = 1$, the orbit*

$$a(qiq^\iota) + (z(b + cj)q^\iota)\,\varepsilon, \quad z \in U(1),\ q \in Sp(1),$$

is a 4-dimensional CR-submanifold of S^6.

Proof. We denote by X^* a Killing vector field on S^6 induced by $X \in T_1(U(1) \times Sp(1))$. If we denote by X_0, X_1, X_2, X_3 the vectors $(i,0), (0,i), (0,j), (0,k)$ of $T_1(U(1)\times Sp(1))$ respectively, then the tangent space $T_{p_0}(L(p_0))$ of the orbit $L(p_0)$ through the point $p_0 = ai + (b + cj)\varepsilon$ is spanned by the vectors

$$\begin{aligned} {X_0}^*(p_0) &= (bi + ck)\varepsilon, & {X_1}^*(p_0) &= (-bi + ck)\varepsilon, \\ {X_2}^*(p_0) &= -2ak + (c - bj)\varepsilon, & {X_3}^*(p_0) &= 2aj - (ci + bk)\varepsilon. \end{aligned}$$

From

$$\Omega(X_i^*(p_0), X_j^*(p_0)) = \begin{cases} 6abc, & \text{if } i = 0,\ j = 2, \\ a(5 - 9a^2), & \text{if } i = 2,\ j = 3, \\ 0, & \text{otherwise}, \end{cases}$$

we easily obtain

$$\Omega \wedge \Omega({X_0}^*(p_0), {X_1}^*(p_0), {X_2}^*(p_0), {X_3}^*(p_0)) = 0.$$

□

Proposition 6.3. *The orbit of L through the point $p = ai+(b+cj)\varepsilon$ $(a, b, c \geq 0,\ a^2 + b^2 + c^2 = 1)$ is a minimal submanifold of S^6 if and only if*

$$a = \sqrt{\frac{3 + \sqrt{57}}{24}}, \quad b = c = \sqrt{\frac{21 - \sqrt{57}}{48}}.$$

Proof. With respect to the basis $\{X_0(p_0), X_1(p_0), X_2(p_0), X_3(p_0)\}$, the induced metric g is represented as follows:

$$g = \begin{pmatrix} b^2 + c^2 & c^2 - b^2 & 0 & -2bc \\ c^2 - b^2 & b^2 + c^2 & 0 & 0 \\ 0 & 0 & 3a^2 + 1 & 0 \\ -2bc & 0 & 0 & 3a^2 + 1 \end{pmatrix}.$$

Since the orbit of the action (6.1) through a point $p = (ai) + (b + cj)\varepsilon$ $(a,\ b,\ c\ > 0)$ is diffeomorphic to $U(2)$, the volume of the orbit is equal to

$$\text{const.} \times \det(g) = \text{const.} \times 4abc\sqrt{1+3a^2}.$$

Considering the extremal of the volume under the condition $a^2+b^2+c^2 = 1$, we obtain the result. □

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Hideya Hashimoto
Nippon Institute of Technology
4-1, Gakuendai, Miyashiro
Minami-Saitama Gun, Saitama, 345-8501
Japan.
hideya@nit.ac.jp

Katsuya Mashimo
Department of Mathematics
Tokyo University of Agriculture and Technology
Koganei, Tokyo 184-8588
Japan
mashimo@cc.tuat.ac.jp

Kouei Sekigawa
Department of Mathematics
Niigata University
Niigata 950-2181
Japan