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On the Lattice of all Subgroups of a Finite Noncyclic Simple Group

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In some of his earliest work Suzuki studied the lattice $L(G)$ of all subgroups of a group G. Amongst other results he showed that if G is a finite noncyclic simple group and $L(H) \approx L(G \times G)$ then $H \approx G \times G$ [2, Theorem 19, p.53]. (Here of course $\approx$ denotes either a lattice isomorphism or a group isomorphism as appropriate.) In particular this implies that if G and H are finite simple groups with $L(G \times G) \approx L(H \times H)$ then $G \approx H$. (The case that G is cyclic of prime order is clear.) If G is cyclic of prime order then $L(G)$ has just 2 elements so that $L(G) \approx L(H)$ implies only that H is cyclic of some prime order and so need not be isomorphic to G. However this leaves open the natural question of whether a finite noncyclic simple group is characterized by its lattice of subgroups. The purpose of this note is to show that by using the classification of the finite simple groups and further results from [2] that this is the case. More precisely:

Theorem. *Let G and H be finite noncyclic simple groups. Then $G \approx H$ if and only if $L(G) \approx L(H)$.*

It is clear that if $G \approx H$ then $L(G) \approx L(H)$. The proof of the converse needs some deep results. If G is a finite group and p is a prime let $n_p(G)$ denote the number of elements of order p in G. Then G contains exactly $n_p(G)/(p-1)$ subgroups of order p. The following will be needed:

(I) [2, Theorem 15, p.51] *If G is a finite noncyclic simple group and H is a finite group with $L(G) \approx L(H)$ then H is a finite noncyclic simple group of the same order as G.*

(II) [2, Theorem 8, p.45] *Let G be a finite noncyclic simple group and let p be a prime. If φ is a lattice isomorphism of $L(G)$*

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onto $L(H)$ then φ maps every subgroup of G of order p onto a subgroup of H of order p. In particular $n_p(G) = n_p(H)$.

(III) [A consequence of the classification of the finite simple groups] *The only pairs of nonisomorphic simple groups of the same order are the following*:

(i) $A_8, PSL_3(4)$

(ii) $PSp_{2m}(q), SO_{2m+1}(q)$ *for* $m > 2$ *and* q *an odd prime power.*

Proof of the Theorem. Suppose that $L(G) \approx L(H)$. By (I) $G$and H have the same order and so must be one of the pairs in (III). In Case (i) the character tables in the ATLAS imply that $n_5(PSL_3(4)) = 3n_5(A_8) \neq 0$ and so by (II) these groups do not have isomorphic subgroup lattices. In Case (ii) $n_2(PSp_{2m}(q)) \neq n_2((SO_{2m+1}(q))$ by [1, Lemma 2.5] and once again these groups do not have isomorphic subgroup lattices.

Added in Proof. Roland Schmidt has informed us that while the result of this paper has not appeared in a Journal, it is in his book "Subgroup Lattices of Groups" p. 439.

References

[1] W. Kimmerle, R. Lyons, R. Sandling and D.N. Teague, Composition factors from the group ring and Artin's Theorem on orders of simple groups, P.L.M.S., **60** (1990), 89–122.

[2] M. Suzuki, Structure of a group and the structure of its lattice of subgroups, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. **10** (1956), Springer Verlag, Berlin, Göttingen, Heidelberg.

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