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Finiteness of a certain Motivic Cohomology Group of Varieties over Local and Global Fields

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INTRODUCTION

In this paper, I would like to survey my recent research [22]. I would like to express gratitude to the organizers for giving me this opportunity to write this manuscript.

Let k be a global field, i.e., an algebraic number field (case (N)) or a function field in one variable over a finite field (case (F)). Let X be a projective smooth geometrically connected k-variety. Let l be a prime number invertible in k. The l-adic regulator map of Soulé [24]

$$r_l^{i,n} : \mathrm{H}^i_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Q}_l} \to \mathrm{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)).$$

is a central topic in the arithmetic geometry. Here $\mathrm{H}^i_{\mathcal{M}}(X, \mathbb{Q}(n))$ denotes the motivic cohomology and is defined by the n-th Adams eigenspace of the algebraic K-group $K_{2n-i}(X)_{\mathbb{Q}}$ ([17] and [25]), and the right hand side is the continuous etale cohomology group (cf. Jannsen [9]). The coefficient $\mathbb{Q}_l(n)$ in the right hand side means the n-th Tate twist of $\mathbb{Q}_l$. In the case $i = 2n$, it is known that this map coincides with the cycle map for the Chow group of algebraic cycles of codimension n modulo rational equivalence ([9] 6.14):

$$\mathrm{cl}_l : \mathrm{CH}^n(X)_{\mathbb{Q}_l} \to \mathrm{H}^{2n}_{\mathrm{cont}}(X, \mathbb{Q}_l(n)).$$

We write $\mathrm{F}^\bullet$ for the Hochschild–Serre filtration on the continuous etale cohomology group w.r.t. the covering $X \otimes_k k^{\mathrm{sep}} \to X$. For instance, F^2 of $\mathrm{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$ is defined by the image of the Hochschild–Serre mapping

$$\mathrm{H}^2_{\mathrm{cont}}(G_k, \mathrm{H}^{i-2}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l(n))) \to \mathrm{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)),$$

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and it is trivial if $i < 2$. In this manuscript, we start from the following conjecture, which is based on the philosophy of mixed motives of Beilinson [2] and Bloch [3] (cf. Jannsen [11] 11.6) and the Beilinson–Deligne–Jannsen conjecture (cf. [11] 11.4, 12.18).

Conjecture 1. *For arbitrary integers i and n satisfying $0 \leq n \leq d+1$ and $0 \leq i \leq 2n$ ($d := \dim X$), the following map induced by $r_l^{i,n}$,*

$$\overline{r}_l^{i,n} : \mathrm{H}^i_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Q}_l} \to \mathrm{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n))/\mathrm{F}^2\mathrm{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)),$$

is injective.

The cases $(n,i) = (0,0)$ and $(d+1, 2d+2)$ are trivial. It is also the case, when $(n,i) = (1,2)$, by the fact that the Picard group of X is a finitely generated $\mathbb{Z}$-module and by the Kummer theory for the Picard variety (cf. Reskind [19] Appendix). As for the conjecture on the image of $\overline{r}_l^{i,n}$, see [11], 12.18, and Bloch [4], §5. Conjecture 1 at least implies the following:

Conjecture 2. *For integers i and n satisfying $1 \leq n \leq d+1$ and $2 \leq i \leq 2n$, the image of the l-adic regulator map $r_l^{i,n}$ intersects with F^2 trivially:*

$$\mathrm{Im}(r_l^{i,n}) \cap \mathrm{F}^2\mathrm{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n)) = 0.$$

This is clearly true in the case $(n,i) = (1,2)$ by the above remark. In this manuscript, we are concerned with Conjecture 2. The main result is

Theorem 3 ([22] **Corollary 5.4**). *Let k be the case* (F), *and X be a proper smooth variety over k. Then Conjecture* 2 *is true in the case $i = n+1$ with n at least* 2.

In the proof, the finiteness result stated below (§2, Theorem 6) will play an important role (see §2). By the Merkur'ev–Suslin theorem ([15], §18), a result of Soulé ([25], Théorème 4 (iv)), and Theorem 3, we can show the following:

Corollary 4 ([22] **Theorem 0.2**). *Let k, X, and l be as in Theorem* 3. *Then we have*

$$\ker(\overline{r}_l^{3,2}) \simeq (K_1(X)^{(2)} \otimes \mathbb{Z}_l)_{l\text{-div}} \otimes \mathbb{Q}.$$

Here the superscript (2) *means the second Adams eigenspace, and the subscript l-*div *means the maximal l-divisible subgroup.*

We will not prove this corollary here (cf. [22], §6). Accoding to Bass' general conjecture [1], §9 (predicting that algebraic K-groups of a regular scheme of finite type over Spec $\mathbb{Z}$ should be finitely generated $\mathbb{Z}$-modules), the right hand side in Corollary 4 would be trivial. In other words, the injectivity problem of $\overline{r}_l^{3,2}$ is reduced to the Bass conjecture by this corollary.

If k is a function field, Conjecture 2 is true in several cases. We will review them in the first section. On the other hand, if k is a number field, there are only a few known cases (cf. Langer and Raskind [14], Theorem 0.2). One of the difficulties lies in the point that one needs, in a step of proofs, some local-global principle (cf. (1.2) below), which is known to hold in the function field case, but have not been proven yet in general in the number field case (cf. [14] Theorem 5.5).

§1. Review of known results

Throughout this section, k, X and l are as in Theorem 3. We write d for the dimension of X. Then Conjecture 2 is known to be true in the following cases (Figure 1).

(0) $(n, i) = (1, 2)$.
(1) X has potentially good reduction everywhere.
(2) $i \leq n$.
(3) $(n, i) = (d + 1, 2d + 1)$.
(4) $(n, i) = (d, 2d)$.
(5) $i = 2n$, $2 \leq n \leq d - 1$ (with an additional geometrical assumption).

Theorem 3 corresponds to the line (6) in Figure 1.

We shall review the local-global argument of Raskind briefly ([19] Proposition 3.6; see also [22] Theorem 5.1), which is a key step in the proof of the cases (1)–(5). We will also use this argument in our proof of Theorem 3 (cf. §2). For a place $\mathfrak{p}$ of k, we write $k_\mathfrak{p}$ for the completion of k at $\mathfrak{p}$, and write $r_{l,\mathfrak{p}}^{i,n}$ for the regulator map for $X_{k_\mathfrak{p}}$. We fix a finite set S of places of k containing all the places where X does not have good reduction.

First, in the cases (1)–(5), we have

$$\text{(1.1)} \qquad \operatorname{Im}(r_{l,\mathfrak{p}}^{i,n}) \cap \mathrm{F}^2\mathrm{H}^i_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l(n)) = 0$$

for any place $\mathfrak{p}$ of k (Case (1): Deligne [6] Corollaire 3.3.9, and Nekovář [16] Theorem D (i). Case (2): Remark 5 below. Case (3): Saito [20] p.64, Theorem 4.1. Case (4): Raskind [19] the earlier part of the proof

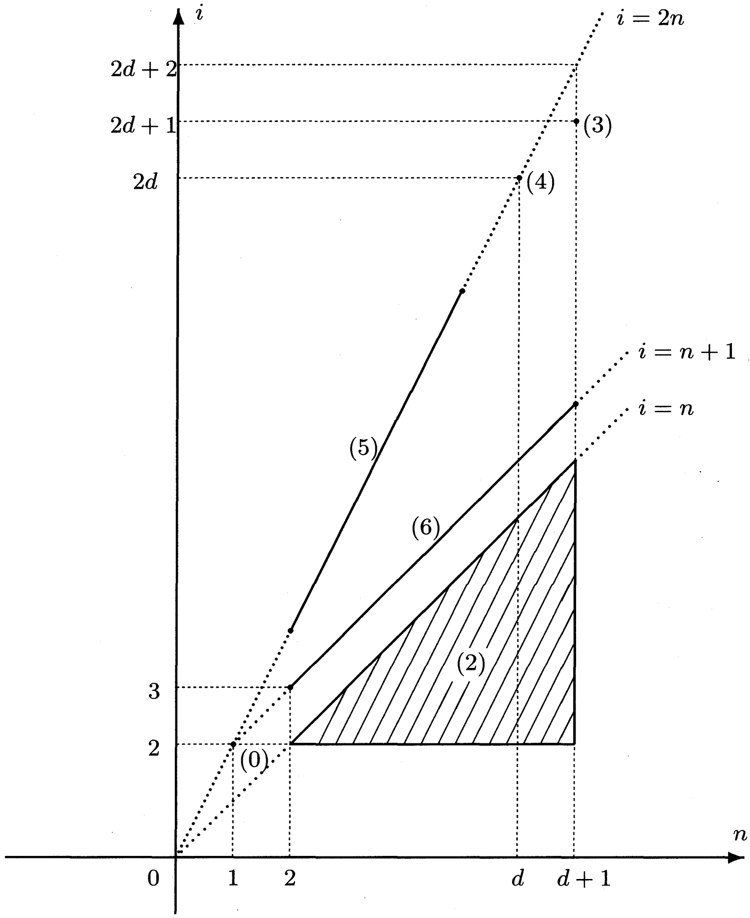

Fig. 1. Table of the known cases

of Proposition 3.2. Case (5): [16] Theorem D (ii). See also Corollary 7 below). Then by a diagram chase which is not so difficult, we can see that $\mathrm{Im}(r_l^{i,n}) \cap \mathrm{F}^2\mathrm{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_l(n))$ is contained in the image of the following $\mathbb{Q}_l$-vector space:

(1.2)

$$\ker\left(a_S^{i,n} : \mathrm{H}^2_{\mathrm{Gal}}(G_S, \mathrm{H}^{i-2}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l(n))) \to \bigoplus_{\mathfrak{p}\in S} \mathrm{H}^2_{\mathrm{Gal}}(k_{\mathfrak{p}}, \mathrm{H}^{i-2}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l(n)))\right).$$

Here $G_S := \mathrm{Gal}(k_S/k)$, and k_S denotes the maximal galois extension of k unramified outside of S. Finally, the map $a_S^{i,n}$ is injective by results of Jannsen since $i \le 2n$ ([10] §6 Theorem 4, [19] Theorem 4.1).

Remark 5. In the case $i \le n$, one can show that for every place $\mathfrak{p}$ of k,

$$\mathrm{H}^2_{\mathrm{Gal}}(k_{\mathfrak{p}}, \mathrm{H}^{i-2}_{\mathrm{et}}(\overline{X}, \mathbb{Q}_l(n))) = 0. \tag{1.3}$$

Therefore $\mathrm{F}^2\mathrm{H}^i_{\mathrm{et}}(X_{k_{\mathfrak{p}}}, \mathbb{Q}_l(n)) = 0$ for any place $\mathfrak{p}$ of k. If X has potentially good reduction at $\mathfrak{p}$, (1.3) immediately follows from Deligne's proof of the Weil conjecture [6] 3.3.9. If X does not have potentially good reduction at $\mathfrak{p}$, (1.3) follows from the alteration theorem of de Jong [12], the Rapoport–Zink theorems [18] Satz 2.21, 2.23, and the Weil conjecture.

§2. Finieness theorem

In this section, we will prove the vanishing (1.1) for the case $i = n+1$. We call the completion of a global field at a non-archimedean place a local field. The essential result is the following:

Theorem 6 ([22] Theorem 2.1). *Let K be a local field, and X a proper smooth variety over K. Let l be a prime number different from the characteristic of K, and n an arbitrary integer at least 2. Then the group*

$$\mathrm{N}^1\mathrm{H}^{n+1}_{\mathrm{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cap \mathrm{F}^2\mathrm{H}^{n+1}_{\mathrm{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$$

is finite. Here $\mathbb{Q}_l/\mathbb{Z}_l(n) := \varinjlim_{\nu} (\mu_{l^\nu})^{\otimes n}$, and μ_{l^ν} denotes the etale sheaf of l^ν-th roots of unity. $\mathrm{N}^\bullet$ denotes the coniveau filtration and $\mathrm{F}^\bullet$ denotes the Hochschild–Serre filtration (see the map α_l^n below).

In the case $n = 2$, this finiteness was originally proved by Salberger [21]. We will give a rough proof of Theorem 6 later. Admitting Theorem 6, we prove

Corollary 7. *Let k be a global field, and X be a proper smooth variety over k. Let l be a prime number which is different from the characteristic of k, and n be an arbitrary integer at least 3. Then for every non-archimedean place $\mathfrak{p}$ of k, we have*

$$\mathrm{Im}(r_{l,\mathfrak{p}}^{n+1,n}) \cap \mathrm{F}^2\mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_{\mathfrak{p}}}, \mathbb{Q}_l(n)) = 0.$$

Here $r_{l,\mathfrak{p}}^{n+1,n}$ denotes the regulator map for $X_{k_{\mathfrak{p}}}$.

This corollary and the argument in §1 imply Theorem 3. The condition that "k is a function field" in Theorem 3 was used to control the $\mathbb{Q}_l$-vector space (1.2).

Proof of "Theorem 6 $\Longrightarrow$ Corollary 7". Let $\mathfrak{p}$ be an arbitrary place of k. Consider the image of the group

$$I := \mathrm{Im}(r_{l,\mathfrak{p}}^{n+1,n}) \cap \mathrm{F}^2\mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l(n))$$

under the canonical map

$$\pi : \mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l(n)) \to \mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(n)). \tag{2.1}$$

Note that I is a divisible group. By a result of Soulé [25] 2.1 Théorème 1, the image of I is contained in the subgroup

$$\mathrm{N}^1\mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(n)) \cap \mathrm{F}^2\mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(n)),$$

which is finite by Theorem 6. Therefore I has trivial image in this group and is contained in $\ker(\pi)$. On the other hand, $\ker(\pi)$ is finitely generated as a $\mathbb{Z}_l$-module by the exact sequence

$$\mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Z}_l(n)) \to \mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l(n)) \xrightarrow{\pi} \mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Q}_l/\mathbb{Z}_l(n))$$

and the fact that $\mathrm{H}^{n+1}_{\mathrm{et}}(X_{k_\mathfrak{p}}, \mathbb{Z}_l(n))$ is a finitely-generated $\mathbb{Z}_l$-module. Hence $\ker(\pi)$ contains no non-trivial divisible subgroup, and I is trivial. Q.E.D.

Finally, we state the outline of a proof of Theorem 6. In the following, cohomology groups of a scheme are taken over the etale topology. Cohomology groups of a field mean etale cohomology groups of the spectrum, or equivalently, Galois cohomology groups of the absolute Galois group. We consider the following composite map:

$$\begin{aligned}\alpha_l^n : \mathrm{H}^2(K, \mathrm{H}^{n-1}(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) &\to \mathrm{H}^{n+1}(X, \mathbb{Q}_l/\mathbb{Z}_l(n)) \\ &\to \mathrm{H}^{n+1}(K(X), \mathbb{Q}_l/\mathbb{Z}_l(n)).\end{aligned}$$

Here the first arrow is the Hochschild–Serre mapping, and the subgroup F^2 of $\mathrm{H}^{n+1}_{\mathrm{et}}(X, \mathbb{Q}_l/\mathbb{Z}_l(n))$ is defined by the image. On the other hand, the subgroup N^1 is defined by the kernel of the second map. Therefore, our task is to prove that α_l^n has finite kernel.

We write O_K for the ring of integers of K, and write $\mathbb{F}$ for the residue field of K. Thanks to the alteration theorem of de Jong [12], the problem is reduced to the case X has a regular model proper flat over O_K with strict semi-stable reduction (cf. [22] (2.1)). In the following, we assume that X has regular model $\mathfrak{X}/O_K$ as above, and that $l \neq \mathrm{ch}(\mathbb{F})$ (see [22] §4 for the ch($\mathbb{F}$)-primary case). We write $R^*\Psi\mathbb{Q}_l/\mathbb{Z}_l$ for the sheaf of vanishing cycles, and J^n (resp. $\overline{J}^n$) for the set of the generic points of the intersections of n irreducible components of $Y := \mathfrak{X} \otimes_{O_K} \mathbb{F}$ (resp. $Y \otimes_{\mathbb{F}} \overline{\mathbb{F}}$).

Intuitively, we compute the quotient of weight -2 of $\mathrm{H}^{n-1}(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))$ by the Rapoport–Zink theorems [18] Satz 2.21, 2.23 and the Weil conjecture [6], and prove the finiteness of $\ker(\alpha_l^n)$. Precisely, we can construct the following commutative diagram (cf. [22] (2.2)):

$$\begin{array}{ccc} \mathrm{H}^2(K, \mathrm{H}^{n-1}(\overline{X}, \mathbb{Q}_l/\mathbb{Z}_l(n))) & \xrightarrow{\alpha_l^n} & \mathrm{H}^{n+1}(K(X), \mathbb{Q}_l/\mathbb{Z}_l(n)) \\ \gamma_l^n \downarrow & & \downarrow d_l^n \\ \mathrm{H}^2\left(K, \bigoplus_{z \in \overline{J}^n} R^{n-1}\Psi\mathbb{Q}_l/\mathbb{Z}_l(n)_{\overline{z}}\right) & \xrightarrow{\beta_l^n} & \bigoplus_{x \in J^n} \mathrm{H}^1(x, \mathbb{Q}_l/\mathbb{Z}_l), \end{array} \tag{2.2}$$

and prove that the map β_l^n is injective (loc. cit. (2.3)–(2.4)), and that the map γ_l^n has finite kernel (loc. cit. Lemma 2.6). Moreover, we can show that γ_l^n is injective for almost all primes ($\neq \mathrm{ch}(\mathbb{F})$) by a theorem of Gabber [7], and hence that the group in Theorem 6 is trivial for almost all l ([22], Lemma 3.2).

Remark 8. The local-global principle of Jannsen ([10], Theorem 3) and Theorem 6 imply that for a proper smooth variety over a number field and for an arbitrary integer $n \geq 2$, the group

$$\mathrm{N}^1\mathrm{H}^{n+1}_{\mathrm{et}}(X, \mathbb{Q}/\mathbb{Z}(n)) \cap \left(\mathrm{F}^2\mathrm{H}^{n+1}_{\mathrm{et}}(X, \mathbb{Q}/\mathbb{Z}(n))\right)_{\mathrm{div}}$$

is finite [22], §4.

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