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Elementary topology of stratified mappings

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

Let T be a triangulation of a manifold N and T^1 the first barycentric subdivision of T. Stiefel [29] conjectured that the union of i-simplices of T^1 is a (possibly infinite) $\mathbb{Z}_2$-cycle, which is the Poincaré dual of the $(n-i)$-th Stiefel-Whitney class of N ($n = \dim N$). This was proved by Whitney (see [36]), but his proof was not published. Later a proof was sketched by Cheeger [4] and a complete proof was given by Halperin and Toledo [11]. Sullivan showed (Corollary 2 in [30]) that this can be generalized to define a Stiefel-Whitney homology class of singular spaces with the *local mod 2 Euler characteristic condition* (Eu) (for the definition, see §2) such as real algebraic varieties. Sullivan defined that a mapping $f : N \to P$ is *semi-triangulable* if the extended mapping cylinder

$$\text{(M)} \qquad M_f = N \times [0,1] \underset{(x,1)\sim(f(x),1)}{\cup} P \times [1,2]$$

is triangulable, and he proved the formula

$$f_* W_*(N) = W_*(P)$$

for such f on the condition that all fibers of f have odd Euler characteristics.

This was immediately generalized by Grothendieck and Deligne for real algebraic mappings and semialgebraic constructible functions (semialgebraic stratifications with weight in $\mathbb{Z}_2$). Namely, given a semialgebraic constructible function h on a real algebraic variety X (with the local mod 2 Euler characteristic condition (Eu) defined as in §2), a total Stiefel-Whitney homology class $W_*(h) \in H_*(X; \mathbb{Z}_2)$ was defined and the following properties were proved (see cf. [19]):

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(1) $f_*W_*(h) = W_*(f_*h)$ for triangulated mappings $f : X \to Y$, where f_*h is the direct image of h (for the definition, see §2),
(2) $W_*(h+k) = W_*(h) + W_*(k)$,
(3) $W_*(1_X) = W_0(1_X) + W_1(1_X) + \cdots$ is the Poincaré dual of the total Stiefel-Whitney class $W^*(X)$ if X is a manifold.

The local mod 2 Euler characteristic condition was not referred in the paper [19]. The condition was stated by Fulton and MacPherson for triangulated constructible functions in the paper [6], where they generalized the idea for families of fibers of triangulated mappings in their bivariant theory. Recently Fu and McCrory [5] proved the above properties for proper real analytic mappings and subanalytic constructible functions without using the mapping cylinder method. The homology class $W_*(h)$ with these properties was shown to be unique (cf. [6]) by using Thom's representation theorem of $\mathbb{Z}_2$-cycles by images of triangulated maps of manifolds in [31].

Deligne and Grothendieck conjectured the existence of an integral homology class in the complex algebraic category with similar properties. It was proved by MacPherson [20]. Namely he defined the so-called Chern-Schwartz-MacPherson class $C_*(h) \in H_*(X; Z)$ with the following properties for a complex algebraic constructible function h on X.

(1′) $f_*C_*(h) = C_*(f_*h)$ for a proper complex algebraic map f,
(2′) $C_*(h+k) = C_*(h) + C_*(k)$,
(3′) $C_*(1_X) = C_0(1_X) + C_1(1_X) + \cdots$ is the Poincaré dual of the total Chern class $C^*(X)$ if X is smooth.

In proving these properties, MacPherson showed the direct image f_*1_X of the characteristic function of X by an $f : X \to Y$ decomposes naturally into a sum

$$f_*1_X = \Sigma a_i \ \mathrm{Eu}(V_i)$$

with reduced subvarieties V_i of P, where $\mathrm{Eu}(V_i)$ denotes the local Euler obstruction of V_i (see [20] for the definition.) And he proved the image $f_*C_*(X)$ is equal to the weighted sum of the Chern-Mather classes of those V_i (the Poincaré duals of Chern classes of the tautological bundles of Nash blow ups of V_i projected to Y.) These subvarieties V_i are nothing but the projections to Y of the irreducible components of the Nash blow up of the complex mappig cylinder M_f of $f : X \to Y$ at $Y \times 1$. (The coefficient a_i is determined by the multiplicity of the irreducible component and the Euler characteristic of the fiber of the projection.) Recently Kwiecinski [17] generalized the theory for the complex analytic category. It is remarkable that in a different vein this had been studied by Schwartz [2, 27].

In these generalizations, either triangulability or analyticity of mappings was assumed. Now we recall some results on the stratification of mappings. By Thom-Mather theory [21], generic proper C^∞-smooth mappings can be canonically stratified, and the canonical stratifications are A_f-regular. By the theory of subanalytic sets, all proper subanalytic mappings (the graphs are subanalytic) can be also stratified [13]. Before the notion of subanalytic sets was introduced by Hironaka, Sullivan [30] had already suggested that all proper real analytic mappings are semi-triangulable and all proper stratified mappings are also. This seems to be true, but the author does not know any satisfactory reference. The various regularity conditions of the mapping cylinder of stratified mappings were discussed by many authors (c.f. [3], [14]). Cappell and Shaneson suggested, in [3], that the natural stratification of the mapping cylinder of a smooth stratified mapping is not necessarily topologically tame.

In this paper we introduce a new regularity condition, *B_f-regularity condition* of stratified mappings in §1. We prove that Whitney-regular and B_f-regular proper stratified mappings are semi-triangulable (Proposition 2.2). Shiota [28] proved that proper A_f-regular stratified mappings are triangulable and hence semi-triangulable. We apply to those A_f-regular or B_f-regular mappings a weighted version of the mapping cylinder method due to Sullivan in §3 (Theorems 2.3, 2.4). Secondly we apply Theorem 2.4 to some special mappings (Morin mapping) in §5.

The author proved in the paper [23] that for a (locally) subanalytic constructible function h on a C^∞-manifold N, a generic proper smooth mapping $f : N \to P$ admits a canonical A_f-regular, Whitney-regular and B_f-regular stratification compatible with h. This result tells that the direct image of a subanalytic constructible function by a generic proper smooth mapping is constructible. If h satisfies the local mod 2 Euler characteristic condition (Eu), the direct image satisfies also the condition and the above formula (1) holds (Theorem 2.4).

§1. Extended Mapping cylinder and the B_f-regularity.

A *stratification* Σ of a subset K of a manifold N is a set of mutually disjoint locally compact and locally finite submanifolds of N such that $\bigcup_{X\in\Sigma} X = K$. The stratification Σ is *Whitney-regular* if the following condition holds: By using local coordinates assume $N = \mathbb{R}^n$, or N is embedded into an $\mathbb{R}^r$. Let $X, Y \in \Sigma$, and let $x_i \in X, y_i \in Y$ be sequences convergent to a $y \in Y$. Assume that the line $\overline{x_i y_i}$ and the tangent space $T_{x_i}X$ are convergent to a line ℓ and a subspace T respectively in the Grassmann manifolds of lines and dim X-planes. Then $\ell \subset T$ holds.

This condition is independent of the choice of the local coordinates or the embedding, hence well defined.

A *stratification of a mapping* $f : N \to P$ is a pair of stratifications (Σ_N, Σ_P) of the source and target such that f restricts on each $X \in \Sigma_N$ to a submersion to a $Y \in \Sigma_P$.

The stratification Σ_N (or f) is *A_f-regular at a $y \in N$* if the following condition holds: Assume a sequence $x_i \in X, X \in \Sigma_N$, is convergent to $y \in Y, Y \in \Sigma_N$ and $\ker d(f \mid X)_{x_i}$ is convergent to a subspace $K \subset T_yN$. Then $\ker d(f \mid Y)_y \subset K$ holds. We say f is *A_f-regular at* a subset $A \subset N$ if f is A_f-regular at every $y \in A$, and we say simply f is *A_f-regular* if $A = N$. Roughly stating, f is A_f-regular if and only if the fibers are almost parallel to each other (see cf. [21]). If (Σ_N, Σ_P) is Whitney-regular and the restrictions of f to the closures of the strata of Σ_N are proper, the fibers $f^{-1}(y)$ are locally topologically trivial over each stratum of Σ_P by Thom's isotopy theorem [21, 33], and furthermore if Σ_N is A_f-regular, f is locally topologically trivial as a mapping along the strata of Σ_P.

Here we introduce a new regularity condition of stratified mappings.

Definition. A stratum $X \in \Sigma_N$ is *B_f-regular over a stratum $Y \in \Sigma_P$ at a $y \in Y$* if the following condition holds: Assume $P = \mathbb{R}^p$ by using local coordinates. Let $x_i \in X$ be a sequence convergent to an $x \in N$ such that $f(x) = y$, and let $y_i \in Y$ be a sequence convergent to $y \in Y$. Assume that the line $\overline{f(x_i)y_i}$ is convergent to a line ℓ. Then there exists a sequence $v_i \in T_{x_i}X$ such that $\|v_i\| \to 0$ and

$$\frac{df(v_i) - (y_i - f(x_i))}{\min\{\|df(v_i)\|,\ \|y_i - f(x_i)\|\}} \to 0$$

as $i \to \infty$. We say the stratification (Σ_N, Σ_P) (or the mapping f) is *B_f-regular over Y* if all strata of Σ_N are B_f-regular over Y at every point of Y, the stratification is *B_f-regular at* a union K of the strata of Σ_P if it is B_f-regular over all strata in K, and *B_f-regular* if $K = P$.

It is easily seen that the B_f-regularity of f implies the Whitney-regularity of the restriction of Σ_P to the image of f.

Remark. Let (X, Σ) be a smooth complex analytic Whitney-regular stratified space, and $\pi : (\tilde{X}, \tilde{\pi}) \to (X, \pi)$ a strict transformation with a nonsingular closed center $Y \in \Sigma$. In general π does not admit an A_π-regular stratification as it is not flat. (It is well known that non flat mappings of complex analytic varieties are not triangulable.) On the other hald the pair of Σ and its strict transform $\tilde{\Sigma}$ admits a refinement, which is B_π-regular at Y by Theorem 1.3.

The *extended mapping cylinder* of $f : N \to P$ is the topological space defined by (M) in the beginning of the paper. To define a differentiable structure on this space, assume N and P are embedded in $\mathbb{R}^r$ and $\mathbb{R}^s$. Then the extended mapping cylinder is homeomorphic to the subset of $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$ consisting of

$$\begin{array}{ll} ((1-t)x, tf(x), t), & x \in N,\ t \in [0,1], \\ (0, y, t), & y \in P,\ t \in [1,2]. \end{array}$$

We denote this subset also by M_f. When f admits a stratification (Σ_N, Σ_P), the extended mapping cylinder M_f decomposes naturally into the union of the strata of $\Sigma_N \times 0 \times 0$, $0 \times \Sigma_P \times 1$, $0 \times \Sigma_P \times (1,2)$, $\Sigma_P \times 2$, and

$$\{((1-t)x, tf(x), t) \mid x \in X, t \in (0,1)\}, \quad X \in \Sigma_N.$$

Denote this decomposition by ΣM_f. For a union $K \subset N$ of some strata of Σ_N, denote by $M_{f|K}$ the extended mapping cylinder of the restriction $f : K \to P$, and by $\Sigma M_{f|K}$ its natural decomposition.

The next lemma is easily seen.

Lemma 1.1. *Assume the restriction of f to K is proper. Then the extended mapping cylinder $M_{f|K}$ is locally compact and the decomposition $\Sigma M_{f|K}$ is a locally finite (but not necessarily Whitney-regular) stratification. And if the embeddings of N, P into $\mathbb{R}^r, \mathbb{R}^s$ are proper, the extended mapping cylinder $M_{f|K}$ is closed in $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$.*

Proposition 1.2. *Assume the restriction of f to K is proper. Then the stratification $\Sigma M_{f|K}$ is Whitney-regular over a stratum $0 \times Y \times 1$ at a $y_0 = (0, y, 1)$ $(y \in Y, Y \in \Sigma_P)$ if and only if the restriction $f : K \to P$ is B_f-regular over Y at y.*

Proof. First we prove the "if " part. Assume the B_f-regularity. Let

$$F(x,t) = ((1-t)x, tf(x), t) : N \times [0,1] \to \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}.$$

Let $X \in \Sigma_N$ be a stratum in K. It suffices to prove the Whitney-regularity of the stratum $F(X \times (0,1))$ over $0 \times Y \times 1$ at y_0. Let $x_i \in X$, $y_i \in Y$ and $\bar{x}_i = F(x_i, t_i)$, $\bar{y}_i = (0, y_i, 1) \to y_0$. Assume the tangent space $T_i = T_{\bar{x}_i} F(X \times (0,1))$ and the line ℓ_i passing through $\bar{x}_i$, $\bar{y}_i$ are convergent respectively to a subspace T of dimension $\dim X + 1$ and a line ℓ. We show that $\ell \subset T$. Clearly T_i is spanned by the vector

$$\text{(i)} \qquad dF_{(x_i,t_i)}\left(\frac{\partial}{\partial t}\right) = (-x_i, f(x_i), 1)$$

and the subspace

(ii) $$dF_{(x_i,t_i)}(T_{x_i}X \times 0) = \{((1-t_i)v, t_i\, df_{x_i}(v), 0) \mid v \in T_xX\}$$

and ℓ_i is spanned by the vector

$$\left(-x_i, f(x_i) + \frac{y_i - f(x_i)}{1-t_i}, 1\right) = (-x_i, f(x_i), 1) + \left(0, \frac{y_i - f(x_i)}{1-t_i}, 0\right).$$

Since $f \mid K$ is proper, passing to a subsequence, we may assume x_i is convergent to an $x \in \bar{X} \cap f^{-1}(y)$. Clearly $(-x_i, f(x_i), 1) \to (-x, y, 1)$. Since $\ell_i \to \ell$, $w_i = (y_i - f(x_i))/(1-t_i)$ is convergent to a vector $w \in \mathbb{R}^s$ or it is divergent but its linear span is convergent to the line ℓ in $0 \times \mathbb{R}^s \times 0$. If w_i is convergent to 0, then $\ell \subset T$ holds. So assume $w \neq 0$ or w_i is divergent. By the B_f-regularity of the restriction $f \mid K$ over Y at y, there exists a sequence $t_i(1-t_i)v_i \in T_{x_i}X$ such that $t_i(1-t_i)\|v_i\| \to 0$ and

$$\frac{df_{x_i}(t_i(1-t_i)v_i) - (y_i - f(x_i))}{\min\{\|df_{x_i}(t_i(1-t_i)v_i)\|,\ \|y_i - f(x_i)\|\}} \to 0,$$

from which it follows

$$\frac{t_i\, df_{x_i}(v_i) - w_i}{\min\{\|t_i\, df_{x_i}(v_i)\|,\ \|w_i\|\}} \to 0.$$

Since $t_i \to 1$ and $t_i(1-t_i)\|v_i\| \to 0$, it follows $(1-t_i)\|v_i\| \to 0$ and by the above convergence the line generated by the vector

$$((1-t_i)v_i, t_i\, df_{x_i}(v_i), 0) \in dF_{(x_i,t_i)}(T_{x_i}X \times 0)$$

is convergent to that of $(0, w, 0)$ or the line $\ell \subset 0 \times \mathbb{R}^s \times 0$. Therefore $\ell \subset T$ mod $(-x, y, 1)$. Since $(-x, y, 1)$ is contained in the limit T, it follows $\ell \subset T$.

Next consider the "only if " part. So conversely assume $F(X \times (0,1))$ is Whitney-regular over $0 \times Y \times 1$ at y_0. Let $x_i \in X \to x$, $y_i \in Y \to y = f(x)$ and assume the line $\overline{f(x_i)y_i}$ is convergent a line $L \subset \mathbb{R}^s$. Let $1 - t_i = \|y_i - f(x_i)\|$. Then $w_i = (y_i - f(x_i))/(1-t_i)$ is convergent to a unit vector $w \in \mathbb{R}^s$ generating L. Recall that the distance of a line and a linear subspace is defined by the norm of the difference of a unit vector in the line and its orthogonal projection of to the subspace. By the Whitney-regularity, the distance $d(T_i, \ell_i)$ of the tangent space T_i at $\bar{x}_i = F(x_i, t_i)$ and the line ℓ_i passing through $\bar{x}_i$ and $\bar{y}_i = (0, y_i, 0)$ tends to 0 as $i \to \infty$, (since if the distance is not convergent, then there is a subsequence T_{a_j} convergent to a subspace

T such that $d(T_{a_j}, \ell_{a_j}) \to \epsilon \neq 0$ by the compactness of Grassmann manifolds and the boundedness of the distance of subspaces). Since $(0, w, 0)$, $(-x, y, 1)$ are linearly independent and $(-x_i, f(x_i), 1) \in T_i$, the distance of the subspace in (ii) and the line generated by w_i tends to 0 as $i \to \infty$. Define a tangent vector v_i of X at x_i so that

$$((1 - t_i)v_i, t_i\, df_{x_i}(v_i), 0)$$

is the orthogonal projection of $(0, w_i, 0)$ to the subspace in (ii). Then $(1 - t_i)\ \|v_i\| \to 0$ and

$$\|t_i\, df_{x_i}(v_i) - w_i\| \to 0.$$

Since $t_i \to 1$,

$$\frac{df_{x_i}((1 - t_i)v_i) - (y_i - f(x_i))}{\min\{\|df_{x_i}((1 - t_i)v_i)\|,\ \|y_i - f(x_i)\|\}} \to 0.$$

Therefore X is B_f-regular over Y at y. This completes the proof of Proposition 1.2. Q.E.D.

Theorem 1.3. *All complex analytic or subanalytic proper mappings admit Whitney-regular and B_f-regular stratifications.*

Proof. A Whitney-regular stratification of an analytic or subanalytic mapping is constructed by the induction on the codimension of the strata in the target space. Namely assuming that there exist a closed (complex analytic or subanalytic) subset $K \subset P$ of dimension i and a Whitney-regular and B_f-regular stratification (Σ_N, Σ_P) of the restriction $f_K : N - f^{-1}(K) \to P - K$, we construct a (possibly disconnected) stratum Y of dimension i in K and a stratification of $f^{-1}(Y)$ such that $K - Y$ is of dimension $\leq i-1$, Σ_N, Σ_P are Whitney-regular over the strata of $f^{-1}(Y)$ and Y respectively and the restriction $f : f^{-1}(Y) \to Y$ is a stratified submersion. To construct a B_f-regular stratification, delete from the stratum Y the closure of the set of those $y \in Y$ such that the extension ΣM_{f_K} is not Whitney-regular over $0 \times Y \times 1$ at $(0, y, 1)$. Then go to the next step to define a stratum of dimension $\leq i - 1$ in the complement $K - Y$. By the construction, the resulting stratification is B_f-regular. Q.E.D.

Remark. By Thom-Mather theory [21], a generic C^∞-smooth mapping f admits a canonical A_f-regular and Whitney-regular stratification. All map germs f_x of such an f at x admit versal unfoldings F_x, which are smoothly conjugate with polynomial map germs. Versal unfoldings admit good representatives such that their restrictions to the singular point sets are proper and finite-to-one. The mapping cylinder of such good representatives admit canonical B_{F_x}-regular stratifications refining the Thom-Mather canonical stratifications of F_x by Proposition 1.2 and the theory of semialgebraic sets. It is seen that the germs at $f(x)$ of such B_{F_x}-regular stratifications of the target are determined by the germs f_x choosing sufficiently small domains of definition for good representatives of F_x. Imposing a generic condition on f, we may assume the natural inclusions of the germs f_x into their versal unfoldings F_x are transverse to the canonical B_{F_x}-regular stratifications. By the transverse pull back define the stratifications of the germs f_x. By genericity we may assume also the stratifications of the targets of those germs are in general position. By their transverse refinement, define a germ of stratification at each point in the target space. Those germs are characterized by the restriction of f to the singular point set $\Sigma(f)$, which is proper and finite-to-one by genericity. Now by an argument similar to the construction of the canonical stratification of a generic smooth mapping in [21], it is seen that those germs glue together to form a stratification Σ_P of P. Similarly define the stratification Σ_N of the source N by gluing the germs of stratifications defined by the intersection refinements of the B_{f_x}-regular stratifications of the germs f_x induced from that of F_x and the pull back of Σ_P by f_x. Then the resulting stratification (Σ_N, Σ_P) is B_f-regular and a refinement of the canonical stratification due to Mather.

§2. A generalization of Sullivan's result

From now on in this section manifolds and mappings are C^2-smooth unless otherwise stated. Most statements dealing with constructible functions in this section remain valid if $\mathbb{Z}_2$ is changed to $\mathbb{Z}$.

A $\mathbb{Z}_2$-valued function h on N is a *constructible function* if there exists a Whitney-regular stratification Σ of N such that the level sets of h are unions of some strata of Σ. Then we say Σ is *compatible with h*. Write h in two ways with $\mathbb{Z}_2$-coefficients n_X, m_X as follows:

$$h = \sum_{X \in \Sigma} m_X 1_X = \sum_{X \in \Sigma} n_X 1_{\bar{X}},$$

where 1_X, $1_{\bar{X}}$ denote the characteristic functions of the stratum X of Σ and its closure, respectively. The *integration of h over N* is defined by

$$\chi(h) = \int_N h = \sum_{X \in \Sigma} m_X \, \chi(\bar{X}, \partial X) = \sum_{X \in \Sigma} n_X \, \chi(\bar{X}) \in \mathbb{Z}_2.$$

Here χ stands for the mod 2 Euler characteristic define by the infinite homology, and $\chi(\bar{X}, \partial X) = \chi(\bar{X}) - \chi(\partial X)$ is equal to the Euler characteristic $\chi(X)$ of X.

It is not difficult to see that the integration is independent of the stratification Σ compatible with h. Assume a mapping $f : N \to P$ admits a Whitney-regular stratification (Σ_N, Σ_P) such that Σ_N is compatible with h. The *direct image* f_*h of h is then defined by

$$f_*h(y) = \int_N h \cdot 1_{f^{-1}(y)},$$

where the integrand is constructible as f restricts to a submersion on each stratum in N to some stratum in P. By Thom's isotopy theorem [21, 33], f is locally topologically trivial over the strata of Σ_P and the direct image f_*h is a constructible function constant on the strata of Σ_P. In general the direct image is not necessarily constructible even for C^∞-smooth mappings f.

In the real analytic case, if the restriction of f to the support of h is proper, the direct image f_*h is constructible by the stratification theory of subanalytic sets. Let $g : P \to Q$ be a real analytic mapping and assume the restriction of g to the support of f_*h is proper. Then the direct image g_*f_*h is also constructible and we obtain the following functoriality

$$(g \circ f)_*h = g_*\, f_*\, h.$$

In the smooth case this holds when the composite (f, g) admits a triple of Whitney-regular stratifications $(\Sigma_N, \Sigma_P, \Sigma_Q)$ such that f, g are stratified mappings and Σ_N is compatible with h. This is the case when g is generic with respect to the stratification of f compatible with h and f_*h [23].

Now assume N and P are compact, f_*h is constructible and let g be a mapping of P to a point. The composite (f, g) is naturally stratified by the stratification of f compatible with the h, f_*h and the trivial stratification of the target of g, and then the direct image $(g \circ f)_*h$ is nothing but the constant function which asigns the integration of h over N to the point. In particular we obtain the following well known result (c.f. [15, 34]).

Corollary 2.1. *Let N, P be compact and let $f : (N, \Sigma_N) \to (P, \Sigma_P)$ be a Whitney-regular stratified mapping. Let $h : N \to \mathbb{Z}_2$ be a constructible function constant on each stratum of Σ_N. Then*

$$\int_N h = \int_P f_*h.$$

Remark. The statement holds also for non compact N, P if the restriction of f to the support of h is proper and the integration is finite.

Let Σ be a Whitney-regular stratification of N compatible with h. By the Whitney-regularity, a sphere $S_p \subset N$ of codimension 1 centered at a $p \in X$, $X \in \Sigma$ with a sufficiently small radius (in a Riemannian metric) is transverse to Σ. By Thom's isotopy theorem the germ of Σ at p is homeomorphic to the cone of $\Sigma \cap S_p$.

The following condition was first stated by Fulton and MacPherson [6] for triangulated constructible functions.

Definition. The *local mod 2 Euler characteristic condition* of h at a stratum X of Σ is

$$(\mathrm{Eu}_X) \qquad \int_{S_p} h \equiv 0 \mod 2 \qquad (p \in X).$$

This condition is independent of the choice of $p \in X$, and equivalent to the condition

$$\int_L h \equiv 0 \mod 2$$

for the link L of $X \in \Sigma$. We say that *Condition* (Eu) holds if Condition (Eu_X) holds for all strata $X \in \Sigma$.

It is known that Whitney-regular stratified sets are triangulable [8, 16]. So we may present h as a sum of characteristic functions of (closed) simplices of a triangulation T compatible with h,

$$h = \sum_{X \in T} n_X 1_X.$$

Denote the union of all j-simplices of the first barycentric subdivision of a simplex $X \in T$ by $W_j(X)$. The following definition is seen in the paper by Fulton and MacPherson [6].

Definition. Let

$$W_j(h) := \sum_{X \in T} n_X W_j(X).$$

By Lemma 3.2, if h satisfies Condition (Eu), then this is an infinite $\mathbb{Z}_2$-cycle of dimension j. The $\mathbb{Z}_2$-homology class defined by this cycle is called *the Stiefel-Whitney homology class* of h and denoted also by $W_j(h) \in H_j(N; \mathbb{Z}_2)$. The *total Stiefel-Whitney homology class of* h is

$$W_*(h) = W_0(h) + W_1(h) + \cdots .$$

Let k be another constructible function on N and Σ' a Whitney-regular stratification compatible with k. If Σ and Σ' are transverse, the intersection refinement $\Sigma \cap \Sigma'$ is Whitney-regular, compatible with $h+k$, and $h+k$ satisfies Condition (Eu). In this situation

$$W_*(h+k) = W_*(h) + W_*(k)$$

holds. However in general the theory of Stiefel-Whitney homology class for smooth constructible functions does not fit the categorical frameworks such as in [6].

Definition. For a function h on N, the *natural extension* h_f on M_f is defined by: $h_f(x,t) = h(x)$ for $(x,t) \in N \times [0,1)$ and $h_f(y,t) = f_*h(y)$ for $(y,t) \in P \times [1,2]$.

Proposition 2.2. *Assume the restriction of a stratified mapping f to the support of h is proper and B_f-regular at the support of f_*h. Then the extended mapping cylinder $M_{f|\operatorname{supp} h}$ admits a triangulation compatible with h_f.*

Proof. It is known that a Whitney-regular stratified closed subset of a smooth manifold admits a triangulation such that all strata are unions of simplices [8, 16]. Assume N, P are properly embedded respectively in some Euclidean spaces $\mathbb{R}^r$, $\mathbb{R}^s$. By Lemma 1.1 and Proposition 1.2, the extended mapping cylinder $M_{f|\operatorname{supp} h}$ is a closed Whitney-regular stratified subset of $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$. Thus it admits a triangulation. Q.E.D.

Theorem 2.3. *Let $f : N \to P$ be a continuous mapping and h a $\mathbb{Z}_2$-valued function on N. Assume the restriction of f to the support of h is proper and the extended mapping cylinder $M_{f|\operatorname{supp} h}$ of the restriction admits a triangulation compatible with h_f, and h_f satisfies Condition* (Eu) *in the interior $M_f - (N \times 0 \cup P \times 2)$. Then h and f_*h on P satisfy Condition* (Eu)*, and*

$$(*) \qquad f_*W_*(h) = W_*(f_*h).$$

Let $f : (N, \Sigma_N) \to (P, \Sigma_P)$ be a proper stratified mapping and h a constructible function constant on each strata of Σ_N. It is not difficult to see that if h satisfies Condition (Eu), then the direct image f_*h also satisfies Condition (Eu). (This was proved in [5, 6] for triangulated mappings, and in [34] for some more general mappings.)

The above theorem is proved in the next section. The next theorem is a generalization of the formula due to Sullivan, Fulton, MacPherson and McCrory [5, 6, 30].

Theorem 2.4. *Let $f : (N, \Sigma_N) \to (P, \Sigma_P)$ be a Whitney-regular stratified mapping, and let h be a $\mathbb{Z}_2$-valued constructible function on N such that Σ_N is compatible with h. Assume h satisfies Condition* (Eu)*, the restriction of f to the support of h is proper, and Σ_N is A_f-regular at the support of h or Σ_P is B_f-regular at the support of $f_* h$, then the above formula* $(*)$ *holds.*

Proof. First assume $\Sigma_N \mid \operatorname{supp} h$ is A_f-regular. Then by the result of Shiota [28], the restriction of f to $\operatorname{supp} h$ admits a triangulation (S, T) refining (Σ_N, Σ_P). It follows that the extended mapping cylinder of the restriction admits a triangulation compatible with h_f. Secondly assume Σ_P is B_f-regular at the support of $f_* h$. Then the extended mapping cylinder of the restriction of f to the support of h admits a triangulation compatible with h_f by Proposition 2.2. In both cases, the formula $(*)$ holds by Theorem 2.3. Q.E.D.

§3. Mapping cylinder method and the proof of Theorem 2.3.

Proposition 3.1. *Assume the restriction of f to the support of h is proper, h satisfies Condition* (Eu) *and f is B_f-regular at the support of $f_* h$. Then h_f satisfies Condition* (Eu) *at the complement of the boundaries $N \times 0$, $P \times 2 \subset M_f$, and h_f satisfies Condition* (Eu) *at a stratum X on the boundaries if and only if h_f is zero on X.*

Proof. Clearly Condition (Eu) holds off the boundaries and $0 \times P \times 1$. First we prove the condition at a $(0, y, 1) \in 0 \times P \times 1$. Assume N, P are properly embedded into $\mathbb{R}^r$, $\mathbb{R}^s$ respectively. By Lemma 1.1 the mapping cylinder $M_{f|\operatorname{supp} h}$ is closed, and by Proposition 1.2 the natural stratification $\Sigma M_{f|\operatorname{supp} h}$ is Whitney-regular. A transverse intersection of a sphere of codimension 1 in $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$ centered at $(0, y, 1)$ and the stratification $\Sigma M_{f|\operatorname{supp} h}$ is (transversely) isotopic to

$$\begin{aligned} S = {} & f^{-1}(B_y) \times 0 \times 0 \cup (f^{-1}(S_y) \times 0 \times (0,1]) \\ & \cup_f (0 \times S_y \times [1,2)) \cup 0 \times B_y \times 2, \end{aligned}$$

where B_y is a closed ball in P centered at y, S_y is the boundary of the ball transverse to Σ_P, and $\cup_f$ denotes the identifying space by the restriction $f : f^{-1}(S_y) \times 0 \times 1 \to 0 \times S_y \times 1$. By Thom's isotopy theorem, the integration of h_f over the intersection is equal to

$$\begin{aligned} \int_S h_f = {} & \int_{f^{-1}(B_y)} h + \int_{f^{-1}(S_y) \times 0 \times (0,1)} h \\ & + \int_{S_y} f_* h + \int_{0 \times S_y \times (1,2)} f_* h + \int_{B_y} f_* h. \end{aligned}$$

Since $\int_{f^{-1}(S_y)} h \equiv 0$ and $\int_{S_y} f_* h \equiv 0 \mod 2$ as the direct image satisfies Condition (Eu) ([6]),

$$\equiv \int_{f^{-1}(B_y)} h + \int_{B_y} f_* h$$

and by Corollary 2.1

$$\equiv \quad 0 \quad \mod 2.$$

Next we consider Condition ($\mathrm{Eu}_{X\times 0\times 0}$) for an $X \in \Sigma_N$. Let B_x be a small ball in N centered at an $x \in X$ such that the boundary S_x is transverse to Σ_N. An intersection of the stratification $\Sigma M_{f|\operatorname{supp} h}$ with a small transverse sphere of codimension 1 in $\mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}$ centered at $(x, 0, 0)$ is homeomorphic to the stratified set

$$S = S_x \times 0 \cup S_x \times (0,1) \cup B_x \times 1 \subset N \times [0,1].$$

By Thom's isotopy theorem, the integration of h_f over the transverse sphere is equal to

$$\int_S h_f = \int_{S_x} h + \int_{S_x \times (0,1)} h + \int_{B_x} h \equiv h(x) \mod 2.$$

This tells that Condition ($\mathrm{Eu}_{X\times 0\times 0}$) holds if and only if $h = 0$ on X. A similar argument holds for the strata in $0 \times P \times 2$. Q.E.D.

Lemma 3.2. *Let T be a triangulation of the extended mapping cylinder M_f compatible with h_f. The coefficient in $\partial W_j(h)$ of a $(j-1)$-simplex $\Sigma = (\tau_0 \subset \tau_1 \subset \cdots \subset X)$ of the first barycentric subdivision of a simplex $X \in T$ (with the barycenter of X as a vertex of Σ) is* 0 *if and only if Condition* (Eu_X) *of h holds.*

Proof. Write $h_f = \sum_{X \in T} n_X 1_{\bar{X}}$. Let Y be a simplex of T of dimension $\geq j$. Then

$$\partial W_j(Y) = W_{j-1}(\partial Y) = W_{j-1}(1_{\partial Y}) = W_{j-1}\left(\sum_{Z \in T,\, Z \subset \partial Y} 1_{\bar{Z}}\right),$$

which is the $(j-1)$-th skeleton of the first barycentric subdivision of the boundary ∂Y. This formula tells that the coefficient of Σ in $\partial W_j(Y)$ is 1 (mod 2) if and only if $X \subset \partial Y$. Therefore the coefficient of Σ in $\partial W_j(h_f)$ is the number (with weight n_Y) of those closed simplices $Y \in T$ containing X in their boundaries. This number is nothing but the integration of h_f over the link of the stratum X. Q.E.D.

Now we prove Theorem 2.3. It is easily seen that Condition (Eu) of h_f on the interior of $M_{f|\mathrm{supp}\, h}$ implies the condition of h, f_*h. By Proposition 2.2, the extended mapping cylinder $M_{f|\mathrm{supp}\, h}$ admits a triangulation T compatible with h_f, and all simplices of the first barycentric subdivision of T are of the form Σ in Lemma 3.2. By Proposition 3.1 and Lemma 3.2, we obtain the boundary formula

$$\partial W_j(h_f) = W_{j-1}(h) + W_{j-1}(f_*h),$$

from which follows the formula $(*)$ of Theorem 2.3.

§4. Canonical decomposition of the direct image.

It would be worth to apply the theory to the following elementary case. Let $f : (N^n, \Sigma_N) \to (P^p, \Sigma_P)$, $n = p$, be a proper and finite-to-one stratified mapping of manifolds. The local mod 2 mapping degree D_{loc} is locally constant on the strata of Σ_N by Thom's isotopy theorem, hence it is a constructible function on N. By definition

$$(**) \qquad f_*D_{\mathrm{loc}} = D\,1_P,$$

D being the mod 2 mapping degree of f.

Proposition 4.1. *Let f be a proper and finite-to-one stratified mapping of manifolds. The local mapping degree D_{loc} satisfies Condition (Eu).*

Proof. Let $f(X) \subset Y$, $X \in \Sigma_N$, $Y \in \Sigma_P$. Let K denote the link of Y. Then $f^{-1}(K)$ is the link of X. By Corollary 2.1,

$$\int_{f^{-1}(K)} D_{\mathrm{loc}} = \int_K f_*D_{\mathrm{loc}} = \int_K D \cdot 1_P = 0.$$

Q.E.D.

Since f is finite-to-one, f is A_f-regular, and by the proposition and Theorem 2.4, we obtain

Proposition 4.2. *Let $f : (N^n, \Sigma_N) \to (P^n, \Sigma_P)$ be a proper and finite-to-one Whitney-regular stratified mapping. Then*

$$f_*W_*(D_{\mathrm{loc}}) = D\,W_*(P),$$

where D is the mod 2 mapping degree of f.

We generalize this in Theorem 4.7 for smooth Thom-Mather stratified mappings $f : N^n \to P^p$ with $n \geq p$.

Now let $f : N^n \to P^p$, $n \geq p$, be a complex analytic mapping with isolated singularities. The local Milnor fiber $f^{-1}(y')$ of a germ of a complex analytic mapping $f : N \to P$ at a singular point $x \in N$ is homotopic to the bouquet of μ spheres S^{n-p}, all of which vanish as y' tends to $y = f(x)$ [12, 22]. In other words, Euler characteristic of the fiber decreases $(-1)^{n-p}\mu$ from that of a generic local fiber when $y' = y$. It is known that the Milnor number μ of the germ f_x is algebraically determined by the finite jet of f at x. So $\mu = \mu(x)$ is a complex analytic constructible function on N. The difference of Euler characteristic of a (global) singular fiber $f^{-1}(y)$ and that of a non singular fiber is the sum of the Milnor number $(-1)^{n-p}\mu(x)$ at all singular points x of the singular fiber. Therefore we obtain the following generalization of the formula by Yomdin [38].

Proposition 4.3. *Assume $f : N \to P$ is a proper complex analytic mapping with isolated singularities. Then*

$$f_* \left(1_N + (-1)^{n-p}\mu\right) = D\, 1_P,$$

where D is the Euler characteristic of a generic fiber of f and μ is Milnor number.

Remark. $D_{\text{loc}} = \mu + 1$ in the case when $n = p$ and f is complex analytic. The mod 2 Milnor number μ is defined for generic C^∞-smooth map germs later in this section, and it is seen that D_{loc} has the same parity as $\mu + 1$.

Mather [21] proved a generic proper C^∞-smooth mapping admits a canonical Whitney-regular and A_f-regular stratification. (It is known the complement of the set of those generic mappings has positive codimension in the proper mapping space with Whitney topology.) For those generic mappings f, germs at all $x \in N$ are conjugate with polynomial map germs by local continuous coordinate change of the source and target, and the singularities of the fibers are isolated. The mod 2 Milnor number $\mu(x)$ of the germ of f at x is well defined: $\mu(x) = \chi(f^{-1}f(x') \cap B_\epsilon) - 1$ with a small ball B_ϵ centered at x such that the boundary is transverse to the fiber passing through x and an x' sufficiently close to x such that $f^{-1}f(x')$ is also transverse to the boundary. We call those generic mappings *smooth Thom-Mather stratified mappings*.

Remark. In the following we present the various results on smooth Thom-Mather stratified mappings. The condition of being a smooth

Thom-Mather stratified mapping seems to be too strong for those statements. However we postpone to state those results in optimal manner, since their proofs are more technical.

Proposition 4.4. *Let f be a proper smooth Thom-Mather stratified mapping. Then*

$$(***) \qquad f_*(1_N + \mu) \equiv D\,1_P \mod 2,$$

where D is the mod 2 Euler characteristic of a generic fiber of f.

Proof. By Corollary 2.1

$$\begin{aligned} f_*(1_N + \mu)(y) &= \int_{f^{-1}(y)} (1_N + \mu) \\ &= \int_{f^{-1}(y)} 1_N + \int_{f^{-1}(y)} \mu \\ &= \chi(f^{-1}(y)) + \sum_{x \in \operatorname{Sing} f^{-1}(y)} \mu(x) \end{aligned}$$

since $f^{-1}(y)$ is locally contractible at all singular points x

$$\equiv \quad D \mod 2.$$

Q.E.D.

Proposition 4.5. *Let f be a proper smooth Thom-Mather stratified mapping. Then $1 + \mu$ satisfies Condition* (Eu).

Proof. This follows from Proposition 4.6. Here we give a direct proof by integrating $1 + \mu$ on a small traisverse sphere of codimension 1. Let D° be a small open ball centered at $f(x)$ such that the boundary $S' = \partial D^\circ$ is transverse to Σ_P. Then f is transverse to S' hence $f^{-1}(S')$ is smooth and transverse to Σ_N. Let D be a small closed ball centered at x. Since f has isolated singularities, we may assume the boundary $S = \partial D$ is transverse to the fibers over D'. The integration of $1 + \mu$ on a transverse sphere of codimension 1 centered at x is equal to

$$\int_{f^{-1}(\bar{D}^\circ) \cap S} (1 + \mu) + \int_{f^{-1}(S') \cap D} (1 + \mu).$$

Since $f^{-1}(\bar{D}^\circ) \cap S$ is a fiber bundle over the link $K = f^{-1}f(x) \cap S$ with the fiber $\bar{D}^\circ$ and $\mu = 0$ on the fiber bundle, we see the first term is

$$\int_K \left(\int_{\bar{D}^\circ} 1 \right) = \int_K 1.$$

Since the germ of f at x is topologically conjugate with a polynomial map germ, the link K has even Euler characteristic by a result of Sullivan [30]. Therefore the first term is 0 mod 2. By Corollary 2.1 and Proposition 4.4, the second term is

$$\int_{S'} d' \equiv 0 \mod 2,$$

where d' is the mod 2 Euler characteristic of a generic fiber of the restriction $f : f^{-1}(S') \cap D \to S'$. This completes the proof. Q.E.D.

Let $M_1 \subset N$ be the support of μ, and define $M_i \subset M_{i-1}$, $i = 2, 3, \dots$ inductively by the support of the restriction of $\mu - i + 1$ to M_{i-1}. By definition, M_i is closed and

$$\mu = \sum 1_{M_i}$$

in the complex analytic case, and

$$(****) \qquad \mu \equiv \sum 1_{M_i} \mod 2$$

in the smooth case. For polynomial map germs f, all M_i are algebraic [38], hence their characteristic functions satisfy Condition (Eu). By a result of Wall [35], the mod 2 Milnor number is invariant under topological conjugacy of map germs. Therefore all germs of smooth Thom-Mather stratified mappings possess the same properties. We proved

Proposition 4.6. *Condition* (Eu) *of* 1_{M_i} *holds for proper smooth Thom-Mather stratified mappings.*

By Proposition 4.4 and Proposition 4.6, we obtain

Proposition 4.7. *For a proper smooth Thom-Mather stratified mapping* $f : N \to P$,

$$f_*W_*(N) + f_*W_*(\mu) = f_*W_*(N) + \sum_{1 \le i} f_*W_*(M_i) = DW_*(P).$$

Thom [32] proved that for a generic and proper $f : N \to P$, the critical point set $\Sigma(f)$ carries a $\mathbb{Z}_2$-fundamental class, and its Poincaré dual can be written in a polynomial of the Stiefel-Whitney (cohomology) classes of the difference bundle $TN - f^*TP$: the virtual tangent bundle of the fibers of f. The polynomial was explicitly calculated by Porteous (see c.f. [7]). Passing to the target space via Gysin homomorphism $f_!$, we obtain the following relation

$$f_! W^{n-p+1}(N) + f_!(\mathrm{Dual}[\Sigma(f)]) = DW^1(P),$$

where W^* stands for the Stiefel-Whitney cohomology class (for the proof, see [24]). This formula is the dimension $(p-1)$ part of the formula in Theorem 4.7, as $M_1 = \Sigma(f)$ for a generic f. Thom showed also, for a singularity type I (of the contact equivalence relation in [21]), the set of the closure of the set $\Sigma^I(f)$ of those $x \in N$ where the germ of f is contact equivalent to I carries a $\mathbb{Z}_2$-fundamental class. And its Poincaré dual is written in a polynomial of Stiefel-Whitney classes of the difference bundle. The polynomial is called *Thom polynomial*, but it is explicitly calculated only for very special singularity types.

§5. The canonical stratification of Morin mappings.

Let $p \leq n$. We say a C^∞-smooth mapping $f : N^n \to P^p$ is a *Morin mapping* if the following two conditions hold:
(1) f is locally equivalent to the following normal form A_k (or Thom-Boardman $\Sigma^{n-p+1,\overset{k-1}{1,\ldots,1}}$ singularity [1, 21, 32]) for some $k = 1, \ldots, p$:

$$(f_u(x,y), u) = \left(x^{k+1} + Q(y) + \sum_{i=1}^{k-1} u_i x^i, u_1, \ldots, u_{p-1} \right) : \mathbb{R}^n \to \mathbb{R}^p,$$

$x \in \mathbb{R}$, $y \in \mathbb{R}^{n-p}$, $Q(y)$ being a non degenerate quadratic form of y. Here two map germs $f_i : (N_i, x_i) \to (P_i, y_i)$, $i = 1, 2$ are *equivalent* if there exist germs of C^∞-diffeomorphisms $\phi : (N_1, x_1) \to (N_2, x_2)$, $\psi : (P_1, y_1) \to (P_2, y_2)$ such that $f_2 \circ \phi = \psi \circ f_1$ holds.
(2) Let $A_k(f)$ denote the set of those $x \in N$ where the germ of f is equivalent to the above normal form. (By the above normal form, $A_k(f)$ is smooth of dimension $p-k$.) The restrictions of f to $A_k(f)$, $k = 1, \ldots, p$, are multi transverse: the germs of $f \mid A_k(f)$, $k = 1, \ldots, p$ at all finite point sets of N are in general position in P.

Denote by Σ_f the stratification of N consisting of $A_1(f), A_2(f), \ldots, A_p(f)$ and their complement $N - \Sigma(f)$.

In the complex analytic case we define Morin mappings in a similar manner. (See [1, 37] for alternative definitions.)

The *canonical stratification* $f_*\Sigma_f$ (denoted Σ_P) of the discriminant of a Morin mapping f is defined as follows. A stratum of $f_*\Sigma_f$ consists of those y such that the fiber $f^{-1}(y)$ has a singularity type

$$A_{k_1} + A_{k_2} + \cdots + A_{k_\ell},$$

which is smooth of codimension $k_1 + \cdots + k_\ell \leq p$ in P. The *canonical stratification* $\Sigma_f \cap f^{-1} f_* \Sigma_f$ $(= \Sigma_N)$ of the source space N is defined by the intersection refinement of Σ_f and $f^{-1} f_* \Sigma_f$. By the normal form of

Morin singularity it is seen that if f is multi transverse with respect to Σ_f, then

$$f : (N, \Sigma_f \cap f^{-1}f_*\Sigma_f) \to (P, f_*\Sigma_f)$$

is an A_f-regular and Whitney-regular stratified mapping. The closure of the critical set $A_k(f)$ is also a smooth submanifold of N of dimension $p-k$. Let $1_{\bar{A}_k(f)}$ denote the characteristic function of the closure.

The following proposition is a spacial case of $(* * **)$ and can be verified by using the local normal form of Morin singularities.

Proposition 5.1. *Let $f : N^n \to P^p$, $p \le n$, be a proper Morin mapping. Then*

$$\mu = \sum_{k=1}^{p} 1_{\bar{A}_k(f)}$$

in the complex analytic case, and

$$\mu \equiv \sum_{k=1}^{p} 1_{\bar{A}_k(f)} \mod 2$$

in the smooth case.

§6. Application to Morin mappings.

For complex analytic mappings, we obtain the following generalization of a result of Levine [18] by the properties of the Chern-Schwartz-MacPherson class and Proposition 5.1.

Theorem 6.1. *Let $f : N^n \to P^p$, $p \le n$, be a proper complex analytic Morin mapping. Then*

$$f_*C_*(N) + (-1)^{n-p}\{f_*C_*(\bar{A}_1(f)) + \cdots + f_*C_*(\bar{A}_p(f))\} = D\,C_*(P).$$

In the paper [24] the author proved

Proposition 6.2. *Let f be as above. Then*

$$\mathrm{Eu}(D(f)) = f_*\mu,$$

where $D(f)$ denotes the discriminant (critical value set) of f.

From this formula and the definition of the total Chern-Schwartz-MacPherson class of $f_*\mu$ as in the introduction, we obtain the following interpretation of the formula in Proposition 4.4.

Theorem 6.3. *Let $f : N^n \to P^p$, $p \leq n$, be a proper complex analytic Morin mapping. Then*

$$f_* C_*(N) + (-1)^{n-p} C_M(D(f)) = D C_*(P),$$

where C_M stands for the Chern-Mather class.

From Theorem 6.1 and Theorem 6.3 it follows

Theorem 6.4. *Let f be as above. Then*

$$\frac{\mathrm{Dual}(C_M(D(f)))}{C^*(P)} = \frac{f_! C^*(\bar{A}_1(f))}{C^*(P)} + \cdots + \frac{f_! C^*(\bar{A}_p(f))}{C^*(P)},$$

where C^* stands for the total Chern class.

The left hand side of the above equality is nothing but the total Chern class of the "normal bundle" of the Nash blow up of the discriminant set $D(f)$. Since the discriminant is of codimension one, the "normal bundle" is a rank one vector bundle over the Nash blow up. Therefore we obtain

Proposition 6.5. *Let f be as above. then*

$$\frac{\mathrm{Dual}(C_M(D(f)))}{C^*(P)} = \mathrm{Dual}(D(f)) + C^1(N(f)),$$

where $C^1(N(f))$ denotes the first Chern class of the "normal bundle" of the Nash blow up of the discriminant.

This tells certain formulas of the discriminant and the image of the cusp point set $f(\bar{A}_1(f))$ as in the end of §4, and also

Theorem 6.6. *For a proper complex analytic Morin mapping f the cohomology class*

$$\frac{f_! C^*(\bar{A}_1(f))}{C^*(P)} + \cdots + \frac{f_! C^*(\bar{A}_p(f))}{C^*(P)}$$

vanishes in dimension ≥ 3.

This can be generalized to complex analytic mappings with arbitrary generic singularities as follows. If a proper mapping $f : N^n \to P^p$ has only generic singularities of corank $\leq c$, then the cohomology class

$$\sum_{1 \leq i} \frac{\mathrm{Dual}(f_* C_*(M_i(f)))}{C^*(P)}$$

vanishes in certain dimensions depending on c. Full account of the general theory will appear elsewhere.

In the real smooth case we obtain from Theorem 2.4 and Proposition 5.1, the following theorem.

Theorem 6.7. *Let $f : N^n \to P^p$, $p \leq n$, be a proper smooth Morin mapping. Then*

$$f_* W_*(N) + \{f_* W_*(\bar{A}_1(f)) + \cdots + f_* W_*(\bar{A}_p(f))\} = D\,W_*(P),$$

where W_* denotes the total Stiefel-Whitney homology class: Poincaré dual of the Stiefel-Whitney cohomology class.

This theorem was proved by the author in [24] with purely geometric argument based on the definition of Thom-Boardman singularities.

By the dimension $(p - i)$ part of the formula in the theorem, we obtain

Corollary 6.8. *Let $f : N^n \to P^p$, $p \leq n$, be a proper smooth Morin mapping. Then*

$$\begin{aligned} f_* W_{p-i}(N) + f_* W_{p-i}(\bar{A}_1(f)) + \cdots + f_* W_{p-i}(\bar{A}_{i-1}(f)) + f_*[\bar{A}_i(f)] \\ = D\,W_{p-i}(P). \end{aligned}$$

In the real case the mod 2 reductions of Theorem 6.4, Proposition 6.5 and Theorem 6.6 seem to remain valid, while the notion of Nash blow up is not defined yet for constructible functions as well as triangulated subsets.

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