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A Chern-Weil theory for Milnor classes

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Abstract.

Dans un travail antérieur ([BLSS]), en collaboration avec Brasselet, Suwa et Seade, nous avons presenté une théorie des classes de Milnor pour les ensembles analytiques complexes V qui sont localement des intersections complètes dans une variété holomorphe ambiante M sans singularité. Le principe consistait à comparer, dans l'homologie $H_{2*}(V)$, deux théories des classes de Chern de V, les classes de Schwartz-MacPherson $c_*^{\mathrm{SMP}}(V)$ et les classes virtuelles $c_*^{\mathrm{vir}}(V)$ (encore appelées de Fulton-Johnson): ces deux théories sont égales lorsque V est lisse, et coïncident alors avec l'image des classes de Chern usuelles par la dualité de Poincaré. Dans le cas général, leur différence se "localise" près de la partie singulière S de V: il existe un élément $\mu_*(V,S) \in H_{2*}(S)$, défini naturellement, dont l'image dans $H_{2*}(V)$ est égale à $(-1)^n\left[c_*^{\mathrm{Vir}}(V) - c_*^{\mathrm{SMP}}(V)\right]$. En outre, si $(S_\alpha)_\alpha$ désigne la famille des composantes connexes de S, la composante $\mu_0(V,S_\alpha)$ de $\mu_0(V,S)$ sur $H_0(S_\alpha)$ est égale au nombre de Milnor de S_α dans tous les cas où celui-ci a déjà été défini.

Dans [BLSS], nous utilisions à la fois des méthodes de Topologie et de Géométrie différentielle. Nous proposons ici une version de pure Géométrie différentielle.

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§1. Introduction

In a joint work with J.P. Brasselet, J. Seade and T. Suwa, we presented in [BLSS] a theory of Milnor classes for singular compact subvarieties V which are locally complete intersections in an analytic complex manifold M. The principle was to compare, in the homology $H_{2*}(V)$ of V, two different theories for Chern classes of V, namely the Schwartz-MacPherson classes $c_*^{\mathrm{SMP}}(V)$ and the virtual classes $c_*^{\mathrm{Vir}}(V)$, both of

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them coinciding with the Poincaré dual of usual Chern classes $c^{n-*}(V)$ in cohomology, when V is non-singular of complex dimension n. The difference $c_*^{\mathrm{Vir}}(V)-c_*^{\mathrm{SMP}}(V)$ of these two Chern classes is in fact localized near the singular part S of V, i.e. there exists a well defined element $\mu_*(V,S)$ in $H_{2*}(S)$, whose image in $H_{2*}(V)$ is equal to $(-1)^n\left[c_*^{\mathrm{Vir}}(V)-c_*^{\mathrm{SMP}}(V)\right]$. Furthermore, denoting by $(S_\alpha)_\alpha$ the family of connected components of S, the component $\mu_0(V,S_\alpha)$ of $\mu_0(V,S)$ on $H_0(S_\alpha)$ is equal to the Milnor number of S_α any time this one has already been defined (i.e. for S_α being an isolated point by Milnor ([Mi]) in case of hypersurfaces and Hamm ([H]) in any codimension, and for V being a hypersurface with general compact S_α by Parusiński ([P])). Notice also that such a theory for Milnor classes has been suggested by Yokura ([Y]), and given for complex hypersurfaces by Aluffi ([A2]) and Parusiński-Pragacz ([PP3]).

Both methods of topology and differential geometry were mixed in [BLSS]. In this paper, we wish to present the theory from a unified point of view, only in differential geometry. [This implies in particular that we use real coefficients in cohomology and homology, in fact as in [BLSS] while the theory with integral coefficients could have been defined there].

Most of the ideas in this paper are already in [BLSS], to which we refer also for examples. The main novelty is the explicit and systematical use of the Čech-de Rham complex with three kinds of open sets: the "ambiant" open set $\tilde{U}_A = M - V$, a tubular neighborhood $\tilde{U}_0$ of the regular part V_0 of V, and regular neighborhoods $\tilde{U}_\alpha$ of the S_α's. In fact, because it may happen that the differential forms that we are going to consider have the required properties only near some skeleton of a convenient cellular structure of M, we preferably use the image by integration of this Čech-de Rham complex into the cellular cochains (see [Le]). Furthermore, at least in a first step, instead of comparing the two theories of SMP and virtual classes in the homology $H_{2*}(V)$, it seems to us more natural to work in $H^{2(m-*)}(M,M-V)$ where $2m = dim_{\mathbb{R}}\, M$ (as originally in fact for SMP classes in [MHS]), Alexander duality $A : H^{2(m-*)}(M,M-V) \to H_{2*}(V)$ being an isomorphism when V is compact. There are two reasons for this: first it makes sense even if V or S is not compact, and secondly we do it implicitely in any case, because of the factorization $P_V = A\circ\tau$ at the chain and cochain level for V and S compact, where $P_V : H^{2n-*}(V) \to H_*(V)$ denotes the Poincaré homomorphism ($2n = dim_{\mathbb{R}}\, V$), and $\tau : H^{2n-*}(V) \to H^{2m-*}(M,M-V)$ the Thom-Gysin homomorphism (see [Br]). It turns out that this Thom-Gysin homomorphism is very easy to write down and compute in our framework, V being not necessarily compact.

Therefore, the organisation of the paper is the following: we describe in section 2 the geometrical situation that we are going to study. Main tools, such as the integration over suitable subcomplexes of the Čech-de Rham complex, or the Chern-Weil theory, are recalled in section 3. Section 4 is devoted to the computation of the Thom-Gysin homomorphism using differential geometry, section 5 to that of virtual classes, and section 6 to that of SMP classes. The Milnor classes are defined in section 7, using only radial frame fields such as in [Sc1] for the original definition of SMP classes. In section 8, finally, we sketch a transcription of the point of view adopted in [BLSS], using more general frame fields.

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§2. Locally complete intersections.

Let $E \to M$ be a holomorphic vector bundle of rank k on acomplex manifold M of complex dimension $m = n + k$. Let s be a holomorphic section of E, and V be the zero set of s. If we assume furthermore s to be generically transverse to the zero section, the section s is then automatically regular, and the components of s with respect to a local trivialization generate the ideal of (local) holomorphic functions vanishing on V (after [T]). Thus, V is a **locally complete intersection in** M. The restriction of E to the regular part V_0 of V may be canonically identified with the normal bundle of V_0 in M. Thus $E|_V$ is an extension to all of V of this normal bundle . We still call it **normal bundle** to V as in the non singular case. The bundle $E|_V$ depends only on V and not on (E, s).

The natural projection $\pi_0 : TM|_{V_0} \to E|_{V_0}$ may be extended as a (smooth) projection $\pi : TM|_{\tilde{U}_0} \to E|_{\tilde{U}_0}$ (no more unique but it does not matter) on any tubular neighborhood $\tilde{U}_0$ of V_0, the kernel H of π being a smooth bundle on $\tilde{U}_0$ extending TV_0.

Let Σ be an analytic subset of V containing the singular part of V. After Lojasiewicz, there exists a smooth triangulation (K) of M adapted to V and Σ, (i.e. having V and Σ as subcomplexes). Denote respectively by (K') and $(K")$ the first and the second barycentric subdivision of K, and by (D) a smooth cellular structure dual to $(K")$.

Denoting by $(S_\alpha)_\alpha$ the set of connected components of Σ, we shall make the following assumption: each S_α is either included in the regular part V_0 of V or is a connected component of the singular part $Sing(V)$, but none of them intersects simultaneously V_0 and $Sing(V)$. In fact, once fixed the homological dimension $*$ in which we wish to compute $\mu_*(V,S)$ it is sufficient to assume that the intersection $S_\alpha \cap (D)^{2(m-*)}$ of S_α with the $2(m-*)$ skeleton of (D) does not intersect simultaneously V_0 and $Sing(V)$.

Let $\tilde{U}_A = M - V$ (the index "A" meaning "ambiant"). Let $\tilde{U}_\alpha$ be the interior of the link of S_α for (K'), and $\tilde{U}_1 = \bigcup_\alpha \tilde{U}_\alpha$. From now on, choose for tubular neighborhood $\tilde{U}_0$ of V_0 the interior of the link of V_0 for (K'). Then, $\tilde{\mathcal{U}} = (\tilde{U}_A, \tilde{U}_0, \tilde{U}_1)$ is a covering of M by open sets, such that $\tilde{U}_\alpha$ is a regular neighborhood of S_α, $\tilde{U}_0$ is a tubular neighborhood of V_0, and $U(V) = \tilde{U}_0 \cup \tilde{U}_1$ is a regular neighborhood of V, which is covered by $\mathcal{U} = (\tilde{U}_0, \tilde{U}_1)$. Furthermore, $\tilde{U}_\alpha \cap \tilde{U}_\beta = \emptyset$ for $\alpha \neq \beta$.

We have $V_0 = \tilde{U}_0 \cap V$. Let $U_\alpha = \tilde{U}_\alpha \cap V$ and $U_1 = \tilde{U}_1 \cap V$.

We define now a honeycomb system of cells $(\tilde{R}_A, \tilde{R}_0, \tilde{R}_1 = \bigcup_\alpha \tilde{R}_\alpha)$ (see the definition in [Le]) adapted to the open covering $\tilde{\mathcal{U}}$ of M, in the following way:

Let $\tilde{R}_A$ be the union of the $(K")$-simplices which do not intersect V.
Let $\tilde{R}_0$ be the union of the $(K")$-simplices which intersect V_0 but not Σ.
Let $\tilde{R}_\alpha$ be the union of the $(K")$-simplices which intersect S_α.

As usually, we denote by $\tilde{R}_{A0}$, $\tilde{R}_{A1} = \bigcup_\alpha \tilde{R}_{A\alpha}$, $\tilde{R}_{01} = \bigcup_\alpha \tilde{R}_{0\alpha}$, and $\tilde{R}_{A01} = \bigcup_\alpha \tilde{R}_{A0\alpha}$ the intersections of the above honeycombs, with suitable orientations. In fact, we shall often omit the *tilde* any time that the given set does not intersect V (i.e. when A does not occur in the indices). If it does, the omission of the *tilde* means that we take the intersection with V: for instance, $R_A = \tilde{R}_A$, $R_{A0} = \tilde{R}_{A0}$ and $R_{A\alpha} = \tilde{R}_{A\alpha}$, while $R_0 = \tilde{R}_0 \cap V$, $R_\alpha = \tilde{R}_\alpha \cap V$ and $R_{0\alpha} = \tilde{R}_{0\alpha} \cap V$...

We also write $\tilde{R} = \tilde{R}_0 \cup \tilde{R}_1$, with $\partial\tilde{R} = \tilde{R}_{A0} \cup \tilde{R}_{A1}$. Let $b\tilde{R} = \partial\tilde{R} \cup \tilde{R}_{01} = \tilde{R}_{A0} \cup \tilde{R}_{A1} \cup \tilde{R}_{01}$.

For any $(K")$-subcomplex X of M, we denote by $\mathcal{T}_D(X)$ the union of the (D) cells intersecting X. If Y is a subcomplex of X, $\mathcal{T}_D(X-Y)$ denotes the union of the (D) cells intersecting X but not Y.

For instance:
$\mathcal{T}_D(V)$ has V for deformation retract,
$\mathcal{T}_D(\Sigma)$ (resp. $\mathcal{T}_D(S_\alpha)$) has Σ (resp. S_α) for deformation retract,
$\mathcal{T}_D(M-V)$ is a deformation retract of $M-V$,
$\mathcal{T}_D(M-\Sigma)$ is a deformation retract of $M-\Sigma$,
$\mathcal{T}_D(V-\Sigma)$ has the homotopy type of $V-\Sigma$,
and $\mathcal{T}_D(b\tilde{R})$ has $b\tilde{R}$ for deformation retract.

We shall respectively denote by $C_D^*(M), C_D^*(V), C_D^*(\Sigma), C_D^*(M-V)$, $C_D^*(M-\Sigma)$ and $C_D^*(V-\Sigma)$ the complexes of cellular cochains with coefficients in $\mathbb{C}$ for (D)-cells respectively in M, $\mathcal{T}_D(V)$, $\mathcal{T}_D(\Sigma)$, $\mathcal{T}_D(M-V)$, $\mathcal{T}_D(M-\Sigma)$, and $\mathcal{T}_D(V-\Sigma)$. The corresponding cohomology algebras are respectively canonically isomorphic to $H^*(M), H^*(V), H^*(\Sigma), H^*(M-V)$, $H^*(M-\Sigma)$ and $H^*(V-\Sigma)$.

Denote also respectively by $C_D^*(M, M-V)$, $C_D^*(M, M-\Sigma)$ and $C_D^*(V, V-\Sigma)$ the kernels of the surjections $C_D^*(M) \to C_D^*(M-V)$, $C_D^*(M) \to C_D^*(M-\Sigma)$ and
$C_D^*(V) \to C_D^*(V-\Sigma)$. Their cohomology are respectively canonically isomorphic to $H^*(M, M-V)$, $H^*(M, M-\Sigma)$ and $H^*(V, V-\Sigma)$.

Notice that V, (resp. ∂R_A, $\partial \tilde{R}_0$ and $\partial \tilde{R}_\alpha$) is a subcomplex of $(K")$. Thus, (D)-cells of dimension j are transversal to them, and intersect them therefore in dimension $j-2k$ (resp. $2m-1$).

Frame fields and radial frame fields

Let r be an integer $(1 \leq r \leq n)$. We set $p = n-r+1$, and $q = p+k = m-r+1$. We shall denote by $\tilde{F}^{(r)} = (\tilde{F}^{(r-1)}, \tilde{v}_r)$ a field of smooth non singular r frames tangent to M near $\mathcal{T}_D(b\tilde{R}) \cap (D)^{2q}$, ($\tilde{F}^{(r-1)}$ denoting the $r-1$ frame generated by the $r-1$ first vectors, and $\tilde{v}_r$ denoting the last vector field of the frame), and having the following properties:

(i) Its restriction $F^{(r)} = (F^{(r-1)}, v_r)$ to V_0 is tangent to V_0. More generally $\tilde{F}^{(r)}$ remains in H over $\mathcal{T}_D(\partial\tilde{R}_0) \cap (D)^{2q}$.

(ii) A smooth non singular extension of $\tilde{F}^{(r-1)}$ is given in $\mathcal{T}_D(\tilde{R}_0) \cap (D)^{2q}$, still in H.

After usual obstruction theory, there always exists such frame fields: in fact, $b\tilde{R}$ is a deformation retract of $\mathcal{T}_D(b\tilde{R})$, and $b\tilde{R} \cap (D)^{2q}$ is $2q-1$ dimensional.

Among all frame fields having the above properties, there are in particular after [MHS] **radial** frame fields, denoted by $\tilde{F}_0^{(r)}$ in the sequel. (For a precise definition of a radial frame field, see [MHS] or [BS]). Notice that the properties (i) (ii) are far from characterizing the frame fields which are radial. For instance, in the case $r=1$, if $\tilde{v}_1$ is radial, it is possible to choose the honeycombs such that $\tilde{v}_1$ be transversal to bR. After [MHS] all radial frame fields are homotopic.

Particular connections:

We shall call <u>*s connection*</u> every connection $\nabla^{s,E}$ on E over M, which is s trivial ($\nabla^{s,E}s = 0$) off $\mathcal{T}_D(V)$ and in particular over ∂R_A.

For any frame field $\tilde{F}^{(r)}$ satisfying the properties (i) and (ii) above, we shall call <u>$\tilde{F}^{(r)}$ *connection*</u> every connection ∇_F^M on TM over M, preserving the subbundle H of TM over $\mathcal{T}_D(\tilde{R}_0)$, which is $\tilde{F}^{(r)}$ trivial over $\mathcal{T}_D(b\tilde{R}) \cap (D)^{2q}$, the induced connection ∇^H over H being $\tilde{F}^{(r-1)}$ trivial over $\mathcal{T}_D(\tilde{R}_0) \cap (D)^{2q}$. Notice that the connection ∇^M_F, while having particular properties only over some subspace of M, has been extended over all of M.

Lemma 1. *There always exists a pair of connections $(\nabla^M_F, \nabla^{s,E})$, compatible with the projection $\pi : TM \to E$ over $\mathcal{T}_D(\tilde{R}_0)$, where ∇^M_F is an $\tilde{F}^{(r)}$ connection, and $\nabla^{s,E}$ an s connection.*

Such a pair will be called a **compatible** $(\tilde{F}^{(r)}, s)$ **pair.**

Proof. Obvious, using partition of unity. Q.E.D.

§3. Backgrounds and notations

A) Recall that the Čech de Rham complex $CDR^*(\tilde{\mathcal{U}})$ is the differential graded algebra of elements

$$\omega = \begin{pmatrix} \omega_A & \omega_0 & \omega_1 = (\omega_\alpha) \\ \omega_{A0} & \omega_{A1} = (\omega_{A\alpha}) & \omega_{01} = (\omega_{0\alpha}) \\ & \omega_{A01} = (\omega_{A0\alpha}) & \end{pmatrix},$$

(where ω_A, ω_0, ω_α, ω_{A0}, $\omega_{A\alpha}$, $\omega_{0\alpha}$, $\omega_{A0\alpha}$ denote respectively de Rham forms on the open sets $\tilde{U}_A$, $\tilde{U}_0$, $\tilde{U}_\alpha$, $\tilde{U}_{A0} = \tilde{U}_A \cap \tilde{U}_0$, $U_{A\alpha} = \tilde{U}_A \cap \tilde{U}_\alpha$, $\tilde{U}_{0\alpha} = \tilde{U}_0 \cap \tilde{U}_\alpha$, $\tilde{U}_{A0\alpha} = \tilde{U}_A \cap \tilde{U}_0 \cap \tilde{U}_\alpha$, and the parenthesis denote families of forms indexed by α),
with the differential

$$D\omega = \begin{pmatrix} d\omega_A & d\omega_0 & (d\omega_\alpha) \\ -d\omega_{A0} + \omega_0 - \omega_A & (-d\omega_{A\alpha} + \omega_\alpha - \omega_A) & (-d\omega_{0\alpha} + \omega_\alpha - \omega_0) \\ & (d\omega_{A0\alpha} + \omega_{0\alpha} - \omega_{A\alpha} + \omega_{A0}) & \end{pmatrix}.$$

This differential is a derivation

$$D(\omega \smile \eta) = D\omega \smile \eta + (-1)^{\dim \omega} \omega \smile D\eta$$

for the following product (which is not graded commutative):

$$\begin{pmatrix} \omega_A & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix} \smile \begin{pmatrix} \eta_A & \eta_0 & (\eta_\alpha) \\ \eta_{A0} & (\eta_{A\alpha}) & (\eta_{0\alpha}) \\ & (\eta_{A0\alpha}) & \end{pmatrix} =$$

$$\begin{pmatrix} \omega_A \wedge \eta_A & \omega_0 \wedge \eta_0 & (\omega_\alpha \wedge \eta_\alpha) \\ (-1)^p \omega_A \wedge \eta_{A0} + \omega_{A0} \wedge \eta_0 & ((-1)^p \omega_A \wedge \eta_{A\alpha} + \omega_{A\alpha} \wedge \eta_\alpha) & ((-1)^p \omega_0 \wedge \eta_{0\alpha} + \omega_{0\alpha} \wedge \eta_\alpha) \\ (\omega_A \wedge \eta_{A0\alpha} & + \quad (-1)^{p-1} \omega_{A0} \wedge \eta_{0\alpha} & + \quad \omega_{A0\alpha} \wedge \eta_\alpha) \end{pmatrix}:$$

The cohomology algebra of $CDR^*(\tilde{\mathcal{U}})$ is naturally isomorphic to the de Rham cohomology of M (with complex coefficients), while the differential subalgebras $CDR^*(\tilde{\mathcal{U}}, M-V)$ (resp. $CDR^*(\tilde{\mathcal{U}}, M-\Sigma)$, resp. $CDR^*(\tilde{\mathcal{U}}, \tilde{U}_A \cup (\cup_\alpha \tilde{U}_\alpha))$ of elements ω such that $\omega_A = 0$ (resp. $\omega_A = 0$, $\omega_0 = 0$, and $\omega_{A0} = 0$), (resp. $\omega_A = 0$, $\omega_0 = 0$, and $\omega_{A0} = 0$) provide respectively the relative cohomology $H^*(M, M-V)$, $H^*(M, M-\Sigma)$ and $H^*(M, \tilde{U}_A \cup (\cup_\alpha \tilde{U}_\alpha))$ with complex coefficients. We shall write [0] instead of 0 when we wish to insist that some ω is taken in the subalgebra $CDR^*(\tilde{\mathcal{U}}, M-V)$, $CDR^*(\tilde{\mathcal{U}}, M-\Sigma)$ or $CDR^*(\tilde{\mathcal{U}}, \tilde{U}_A \cup \tilde{U}_1)$, and not in $CDR^*(\tilde{\mathcal{U}})$ itself, writing respectively such elements

$$\begin{pmatrix} [0] & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}, \begin{pmatrix} [0] & [0] & (\omega_\alpha) \\ [0] & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}$$

or

$$\begin{pmatrix} [0] & \omega_0 & [0] \\ \omega_{A0} & [0] & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}.$$

Since the honeycombs R_A, $\tilde{R}_0$, and $\tilde{R}_\alpha$ are subcomplexes of $(K")$, the cells of (D) are transversal to these honeycombs, so that we may integrate elements $\omega \in CDR^j(\tilde{\mathcal{U}})$ along j cells γ of (D) (cf. [Le]): recall that $\int_\gamma \omega$ is equal to

$$\int_{\gamma \cap R_A} \omega_A + \int_{\gamma \cap \tilde{R}_0} \omega_0 + \int_{\gamma \cap\ R_{A0}} \omega_{A0}$$
$$+ \sum_\alpha \left[\int_{\gamma \cap \tilde{R}_\alpha} \omega_\alpha + \int_{\gamma \cap R_{A\alpha}} \omega_{A\alpha} + \int_{\gamma \cap \tilde{R}_{0\alpha}} \omega_{0\alpha} + \int_{\gamma \cap R_{A0\alpha}} \omega_{A0\alpha} \right]$$

with suitable orientations of the domains $R_A \cdots R_{A0} \cdots R_{A0\alpha}$.

The integration defines therefore a morphism from $CDR^*(\tilde{\mathcal{U}})$ into the cellular cochains $C^*_{(D)}(M)$, which commutes with the differentials and induces an algebra isomorphism in cohomology (see [Le]). We shall denote by

$$\left(\!\!\begin{pmatrix} \omega_A & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

the image of

$$\begin{pmatrix} \omega_A & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix} \text{ in } C^*_{(D)}(M).$$

Similarly

$$\left(\!\!\begin{pmatrix} [0] & \omega_0 & (\omega_\alpha) \\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right), \left(\!\!\begin{pmatrix} [0] & [0] & (\omega_\alpha) \\ [0] & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

or

$$\left(\!\!\begin{pmatrix} [0] & \omega_0 & [0] \\ \omega_{A0} & [0] & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

will denote elements in $C^j_{(D)}(M, M-V)$, in $C^j_{(D)}(M, M-\Sigma)$ or in $C^j_{(D)}(M, \mathcal{T}_D(M-V) \cup \mathcal{T}_D(\Sigma))$.

The notation

$$\left(\!\!\begin{pmatrix} [0] & [0] & (\omega_\alpha) \\ [0] & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right) + \left(\!\!\begin{pmatrix} [0] & \omega'_0 & [0] \\ \omega'_{A0} & [0] & (\omega'_{0\alpha}) \\ & (\omega'_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

will denote in fact the sum

$$\left(\!\!\begin{pmatrix} [0] & 0 & (\omega_\alpha) \\ 0 & (\omega_{A\alpha}) & (\omega_{0\alpha}) \\ & (\omega_{A0\alpha}) & \end{pmatrix}\!\!\right) + \left(\!\!\begin{pmatrix} [0] & \omega'_0 & 0 \\ \omega'_{A0} & 0 & (\omega'_{0\alpha}) \\ & (\omega'_{A0\alpha}) & \end{pmatrix}\!\!\right)$$

of the images in $C^*_{(D)}(M, M-V)$.

The Čech-de Rham complex $CDR^*(\mathcal{U})$ is the differential graded algebra of elements

$$\omega = \big(\omega_0, \omega_1 = (\omega_\alpha), \omega_{01} = (\omega_{0\alpha})\big)$$

(where ω_0, ω_α, $\omega_{0\alpha}$ denote respectively de Rham forms on the open sets $\tilde{U}_0$, $\tilde{U}_\alpha$, $\tilde{U}_{0\alpha}$, and the parenthesis denote families of forms indexed by α), with the differential

$$D\omega = \big(d\omega_0\ ,\ (d\omega_\alpha)\ , (-d\omega_{0\alpha} + \omega_\alpha - \omega_0)\big),$$

which is a derivation with respect to the (non graded commutative) product

$$\begin{aligned}&\big(\omega_0\ ,\ (\omega_\alpha)\ ,\ (\omega_{0\alpha})\big)\smile\big(\eta_0\ ,\ (\eta_\alpha)\ ,\ (\eta_{0\alpha})\big)\\ &=\ \big(\omega_0\wedge\eta_0\ ,\ (\omega_\alpha\wedge\eta_\alpha)\ ,\ ((-1)^p\omega_0\wedge\eta_{0\alpha}+\omega_{0\alpha}\wedge\eta_\alpha)\big).\end{aligned}$$

The cohomology algebra of $CDR^*(\mathcal{U})$ is naturally isomorphic to the de Rham cohomology of V (with complex coefficients), while the differential subalgebra $CDR^*(\mathcal{U},V-\Sigma)$ of elements ω such that $\omega_0=0$ provide the relative cohomology $H^*(V,V-\Sigma)$. We shall write $\big([0],(\omega_\alpha),(\omega_{0\alpha})\big)$ the elements of $CDR^*(\mathcal{U},V-\Sigma)$, and $\big(\omega_A,[0],(\omega_{O\alpha})\big)$ those of $CDR^*(\mathcal{U},\Sigma)$.

We may integrate elements $\omega\in CDR^j(\mathcal{U})$ along j cells γ of $\mathcal{T}_D(V)$, and define $\int_\gamma\omega$ as being equal to

$$\int_{\gamma\cap R_0}\omega_0+\sum_\alpha\left[\int_{\gamma\cap R_\alpha}\omega_\alpha+\int_{\gamma\cap R_{0\alpha}}\omega_{0\alpha}\right].$$

The integration defines therefore a morphism from $CDR^*(\mathcal{U})$ into the cellular cochains $C^*_{(D)}(V)$ on V with complex coefficients, which commutes with the differentials and induces an algebra isomorphism in cohomology. We shall denote by $\big(\!\!\big(\omega_0,(\omega_\alpha),(\omega_{0\alpha})\big)\!\!\big)$ the image of $\big(\omega_0,(\omega_\alpha),(\omega_{0\alpha})\big)$ in $C^*_{(D)}(V)$.

Similarly $\big(\!\!\big([0],(\omega_\alpha),(\omega_{0\alpha})\big)\!\!\big)$ will denote elements in $C^j_{(D)}(V,V-\Sigma)$, and $\big(\!\!\big(\omega_0,[0],(\omega_{0\alpha})\big)\!\!\big)$ elements in $C^j_{(D)}(V,\Sigma)$.

The notation $\big(\!\!\big([0],(\omega_\alpha),(\omega_{0\alpha})\big)\!\!\big)+\big(\!\!\big(\omega_0',[0],(\omega_{0\alpha}')\big)\!\!\big)$ will denote in fact the sum $\big(\!\!\big(0,(\omega_\alpha),(\omega_{0\alpha})\big)\!\!\big)+\big(\!\!\big(\omega_0',0,(\omega_{0\alpha}')\big)\!\!\big)$ in $C^j_{(D)}(V)$.

Remark. When ω and γ are j dimensional, $\int_\gamma\omega$ depends only on the behaviour of ω near the j skeleton $(D)^j$ of (D), (i.e the behaviour of ω_A near $R_A\cap(D)^j$, etc....). Thus, it is sufficient that

$$\omega=\begin{pmatrix}\omega_A & \omega_0 & (\omega_\alpha)\\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha})\\ & (\omega_{A0\alpha}) & \end{pmatrix}$$

be defined near $(D)^j$, for

$$\big(\!\!\big(\omega\big)\!\!\big)=\left(\!\!\left(\begin{matrix}\omega_A & \omega_0 & (\omega_\alpha)\\ \omega_{A0} & (\omega_{A\alpha}) & (\omega_{0\alpha})\\ & (\omega_{A0\alpha}) & \end{matrix}\right)\!\!\right)$$

to make sense. However, be careful to the fact that, in this case, the Stokes formula $d\big(\!\!\big(\omega\big)\!\!\big)=\big(\!\!\big(D\omega\big)\!\!\big)$ does not hold any more necessarily.

A similar remark holds for $\left(\!\left(\omega_0, (\omega_\alpha), (\omega_{0\alpha})\right)\!\right)$.

B) In general, for a Chern polynomial φ (i.e., a polynomial of the Chern classes), and a connection ∇ on a complex C^∞ vector bundle, $C \to X$, we denote by $\varphi(\nabla)$ the cocycle on the base which is the image of φ by the Chern-Weil homomorphism asoociated to ∇. Thus it is a closed form whose cohomology class in the de Rham cohomology is the (real) characteristic class $\varphi(C)$ of the bundle associated to φ. In particular, the class of $c^i(\nabla)$ is the real i^{th} Chern class of A. If $(\nabla_0, \nabla_1, \dots, \nabla_r)$ is a family of $r+1$ connections on a same vector bundle C, $\varphi(\nabla_0, \nabla_1, \dots, \nabla_r)$ will denote more generally the Bott difference operator ([B]), so that

$$d\varphi(\nabla_0, \nabla_1, \dots, \nabla_r) = \sum_{i=0}^{r} (-1)^i \varphi(\nabla_0, \nabla_1, \dots, \widehat{\nabla}_i, \cdots, \nabla_r).$$

In particular, for $r = 1$, $d\varphi(\nabla_0, \nabla_1) = \varphi(\nabla_1) - \varphi(\nabla_0)$.

Denoting by c^i and by c'^j the Chern classes of some smooth complex bundles C and C', of ranks $n+q$ and q respectively, over a same manifold X, recall that the h-th Chern class $c''^h = c^h([C - C'])$ of the virtual bundle $[C - C'] \in KU(X)$ is a polynomial with respect to the c^i's and the c'^j's, defined as the coefficient of t^h in the expansion of the expression $(1 + \sum_i t^i c^i) \cdot (1 + \sum_j t^j c'^j)^{-1}$. This polynomial may be written as a finite sum

$$c''^h = \sum_\ell \varphi_\ell(c^1, \dots, c^{n+q}) \cdot \psi_\ell(c'^1, \dots, c'^q),$$

for some polynomials φ_ℓ and ψ_ℓ.

Let ∇ and ∇' be connections on C and C' respectively. Denoting by $\nabla^\bullet$ the pair (∇, ∇'), we set

$$c^h(\nabla^\bullet) = \sum_\ell \varphi_\ell(\nabla) \wedge \psi_\ell(\nabla') .$$

Then $c^h(\nabla^\bullet)$ is a closed $2h$-form on X which defines the class $c^h([C-C'])$.

If $\nabla_1^\bullet = (\nabla_1, \nabla_1')$ and $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ are two such pairs, we set

$$c^h(\nabla_1^\bullet, \nabla_2^\bullet) = \sum_\ell \left(\psi_\ell(\nabla_1') \cdot \varphi_\ell(\nabla_1, \nabla_2) + \psi_\ell(\nabla_1', \nabla_2') \cdot \varphi_\ell(\nabla_2)\right).$$

Then we have:

$$d\,c^h(\nabla_1^\bullet, \nabla_2^\bullet) = c^h(\nabla_2^\bullet) - c^h(\nabla_1^\bullet).$$

If $\nabla_1^\bullet = (\nabla_1, \nabla_1')$, $\nabla_2^\bullet = (\nabla_2, \nabla_2')$ and $\nabla_3^\bullet = (\nabla_3, \nabla_3')$ are three such pairs, we denote by $c^h(\nabla_1^\bullet, \nabla_2^\bullet, \nabla_3^\bullet)$ the form

$$\sum_\ell \big(\psi_\ell(\nabla_1') \cdot \varphi_\ell(\nabla_1, \nabla_2, \nabla_3) \\ + \psi_\ell(\nabla_1', \nabla_2') \cdot \varphi_\ell(\nabla_2, \nabla_3) + \psi_\ell(\nabla_1', \nabla_2', \nabla_3') \cdot \varphi_\ell(\nabla_3)\big).$$

Then we have

$$d\,c^h(\nabla_1^\bullet, \nabla_2^\bullet, \nabla_3^\bullet) = c^h(\nabla_2^\bullet, \nabla_3^\bullet) - c^h(\nabla_1^\bullet, \nabla_3^\bullet) + c^h(\nabla_1^\bullet, \nabla_2^\bullet).$$

§4. Thom-Gysin homomorphism

The complex $CDR^*(\mathcal{U})$ is a quotient of $CDR^*(\tilde{\mathcal{U}})$, and we already observed in [Le] that the cup product

$$CDR^*(\tilde{\mathcal{U}}, M - V) \smile \big[\ker : CDR^*(\tilde{\mathcal{U}}) \to CDR^*(\mathcal{U})\big]$$

is identically zero, defining therefore a multiplication

$$CDR^*(\tilde{\mathcal{U}}, M - V) \times CDR^*(\mathcal{U}) \xrightarrow{\smile} CDR^*(\tilde{\mathcal{U}}, M - V),$$

which induces the product $H^*(M, M - V) \times H^*(V) \to H^*(M, M - V)$. Similarly, we get multiplications

$$CDR^*(\tilde{\mathcal{U}}, M - V) \times CDR^*(\mathcal{U}, V - \Sigma) \xrightarrow{\smile} CDR^*(\tilde{\mathcal{U}}, M - \Sigma),$$

and $CDR^*(\tilde{\mathcal{U}}, M - V) \times CDR^*(\mathcal{U}, \bigcup_\alpha U_\alpha) \xrightarrow{\smile} CDR^*(\tilde{\mathcal{U}}, \tilde{U}_A \cup (\bigcup_\alpha \tilde{U}_\alpha))$,

inducing respectively the products $H^*(M, M - V) \times H^*(V, V - \Sigma) \to H^*(M, M - \Sigma)$, and $H^*(M, M - V) \times H^*(V, \Sigma) \to H^*(M, M - \tilde{R}_0)$.

For $V = s^{-1}(0)$ as in section 2, the data of the section s, non vanishing on $M - V$, defines a natural lift $c^k(E, s)$ of the Chern class $c^k(E)$ by the morphism $H^{2k}(M, M - V) \to H^{2k}(M)$. It is proved in [Su2] that $c^k(E, s)$ corresponds to the fundamental class $[V]$ by the Alexander duality. Therefore, the cup product by the so-called "Thom class" $c^k(E, s)$ induces in cohomology the Thom-Gysin morphism τ such that $A \circ \tau = P_V$.

Let ∇^E be any C^∞ connection on E, and ∇'^E any s trivial connection on $E|_{M-V}$ (s trivial means : $\nabla'^E s = 0$). Then the Thom class $c^k(E, s)$ of E is represented by the cocycle

$$\begin{pmatrix} [0] & c^k(\nabla_E) & (c^k(\nabla_E)) \\ c^k(\nabla'^E, \nabla_E) & (c^k(\nabla'^E, \nabla_E)) & 0 \\ & 0 & \end{pmatrix} \in CDR^*(\tilde{\mathcal{U}}, M - V),$$

and the Thom Gysin morphism is induced by the map
$\tau : CDR^*(\mathcal{U}) \longrightarrow CDR^*(\tilde{\mathcal{U}}, M-V)$ such that

$$\tau\big(\eta_0\ ,\ (\eta_\alpha)\ ,\ (\eta_{0\alpha})\big)$$
$$= \begin{pmatrix} & [0] & c^k(\nabla^E)\wedge\eta_0 & \big(c^k(\nabla^E)\wedge\eta_\alpha\big) \\ & c^k({\nabla'}^E,\nabla^E)\wedge\eta_0 & \big(c^k({\nabla'}^E,\nabla^E)\wedge\eta_\alpha\big) & \big(c^k(\nabla^E)\wedge\eta_{0\alpha}\big) \\ & & \big(-c^k({\nabla'}^E,\nabla^E)\wedge\eta_{0\alpha}\big) & \end{pmatrix}.$$

This Thom-Gysin morphism may then be refined by the maps

$$\tau_\Sigma : CDR^*(\mathcal{U}, V-\Sigma) \longrightarrow CDR^*(\tilde{\mathcal{U}}, M-\Sigma)$$
$$\text{and}\quad \tau_0 : CDR^*(\mathcal{U}, (\bigcup_\alpha U_\alpha)) \longrightarrow CDR^*(\tilde{\mathcal{U}}, U_A \cup (\bigcup_\alpha \tilde{U}_\alpha)),$$

respectively defined by the formulas

$$\tau_\Sigma\big([0]\ ,\ (\eta_\alpha)\ ,\ (\eta_{0\alpha})\big)$$
$$= \begin{pmatrix} [0] & [0] & \big(c^k(\nabla^E)\wedge\eta_\alpha\big) \\ [0] & \big(c^k({\nabla'}^E,\nabla^E)\wedge\eta_\alpha\big) & \big(c^k(\nabla^E)\wedge\eta_{0\alpha}\big) \\ & \big(-c^k({\nabla'}^E,\nabla^E)\wedge\eta_{0\alpha}\big) & \end{pmatrix},$$

and

$$\tau_0\big(\eta_0\ , [0],\ (\eta_{0\alpha})\big)$$
$$= \begin{pmatrix} [0] & c^k(\nabla^E)\wedge\eta_0 & [0] \\ c^k({\nabla'}^E,\nabla^E)\wedge\eta_0 & [0] & \big(c^k(\nabla^E)\wedge\eta_{0\alpha}\big) \\ & \big(-c^k({\nabla'}^E,\nabla^E)\wedge\eta_{0\alpha}\big) & \end{pmatrix}.$$

These maps do not depend in cohomology on the choices of ∇^E and ${\nabla'}^E$. In fact, if ∇_1^E and ∇_2^E denote two connections on E, then

$$\begin{pmatrix} [0] & c^k(\nabla_2^E) & \big(c^k(\nabla_2^E)\big) \\ c^k({\nabla'}^E,\nabla_2^E) & \big(c^k({\nabla'}^E,\nabla_2^E)\big) & 0 \\ & 0 & \end{pmatrix}$$
$$- \begin{pmatrix} [0] & c^k(\nabla_1^E) & \big(c^k(\nabla_1^E)\big) \\ c^k({\nabla'}^E,\nabla_1^E) & \big(c^k({\nabla'}^E,\nabla_1^E)\big) & 0 \\ & 0 & \end{pmatrix}$$
$$= D\begin{pmatrix} [0] & c^k(\nabla_1^E,\nabla_2^E) & \big(c^k(\nabla_1^E,\nabla_2^E)\big) \\ c^k({\nabla'}^E,\nabla_1^E,\nabla_2^E) & \big(c^k({\nabla'}^E,\nabla_1^E,\nabla_2^E)\big) & 0 \\ & 0 & \end{pmatrix},$$

i.e. is a coboundary in $CDR^*(\tilde{\mathcal{U}}, M-V)$.

Similarly, if ∇'^E_1 and ∇'^E_2 denote two connections on $E|_{M-V}$, both preserving s, then

$$\begin{pmatrix} [0] & c^k(\nabla^E) & \big(c^k(\nabla^E)\big) \\ c^k(\nabla'^E_2, \nabla^E) & \big(c^k(\nabla'^E_2, \nabla^E)\big) & 0 \\ & 0 & \end{pmatrix} - \begin{pmatrix} [0] & c^k(\nabla^E) & \big(c^k(\nabla^E)\big) \\ c^k(\nabla'^E_1, \nabla^E) & \big(c^k(\nabla'^E_1, \nabla^E)\big) & 0 \\ & 0 & \end{pmatrix}$$
$$= D \begin{pmatrix} [0] & 0 & 0 \\ c^k(\nabla'^E_1, \nabla'^E_2, \nabla^E) & \big(c^k(\nabla'^E_1, \nabla'^E_2, \nabla^E)\big) & 0 \\ & 0 & \end{pmatrix},$$

(because $c^k(\nabla'^E_1, \nabla'^E_2) = 0$, since ∇'^E_1 and ∇'^E_2 are both preserving the same s): we still get a coboundary in $CDR^*(\tilde{\mathcal{U}}, M - V)$.

Remark. If we take for ∇^E a s connection, then $c^k(\nabla'^E, \nabla^E) = 0$ off $\mathcal{T}_D(V)$, and in particular over R_{A0} and $R_{A\alpha}$.

§5. Virtual classes

They are characteristic classes of the virtual tangent bundle $TV = [TM - E]|_V$ in $KU(V)$. Let $\nabla^\bullet = (\nabla^M, \nabla^E)$ be a pair of connections ∇^M on TM and ∇^E on E. Then the p^{th} Chern class $c^p_{\mathrm{vir}}(V)$ of the above virtual TV may be represented, in the Chern-Weil theory by the de Rham form $c^p(\nabla^\bullet) = \big[c(\nabla^M)/c(\nabla^E)\big]_p$ on $U(V) = \tilde{U}_0 \cup \tilde{U}_1$, (where $[...]_p$ denotes the homogeneous component of dimension $2p$), or equivalently by the element $\big(c^p(\nabla^\bullet), (c^p(\nabla^\bullet)), 0\big)$ in $CDR^{2p}(\mathcal{U})$. It does not depend on $\nabla^\bullet$ since, for two choices $\nabla^\bullet$ and $\bar{\nabla}^\bullet$ of the pair of connections, we have: $c^p(\bar{\nabla}^\bullet) - c^p(\nabla^\bullet) = dc^p(\nabla^\bullet, \bar{\nabla}^\bullet)$.

Let $\tilde{F}_0^{(r)}$ be a **radial** frame field. Let $\nabla^M_{F_o}$ be any $\tilde{F}_0^{(r)}$ connection on TM, and denote by ∇^H the induced connection on H over $\tilde{R}_0$. Set $\nabla^\bullet_{F_o} = (\nabla^M_{F_o}, \nabla^E)$, and define

$$\mathrm{Vir}^p_0 = \big(\!\big(c^p(\nabla^\bullet_{F_o}), [0], 0\big)\!\big) \text{ and } \mathrm{Vir}^p_\Sigma = \big(\!\big([0], c^p(\nabla^\bullet_{F_o}), 0\big)\!\big).$$

Proposition 1. (i) Vir^p_0 *and* Vir^p_Σ *are relative cocycles modulo* $\mathcal{T}_D(\Sigma)$ *and* $\mathcal{T}_D(R_0)$ *respectively.*

(ii) *Their cohomology class* $c^p_{0,\mathrm{vir}}(V, \tilde{F}_o^{(r)}) \in H^{2p}(\mathcal{T}_D(V), \mathcal{T}_D(\Sigma)) \cong H^{2p}(V, \Sigma)$, *and* $c^p_{\Sigma,\mathrm{vir}}(V, \tilde{F}_o^{(r)}) \in H^{2p}(\mathcal{T}_D(V), \mathcal{T}_D(R_0)) \cong H^{2p}(V, V - \Sigma)$.

[Notice that $\big(c^p(\nabla^\bullet_{F_o}), [0], 0\big)$ and $\big([0], c^p(\nabla^\bullet_{F_o}), 0\big)$ might not be cocycles!]

For any $2p$ dimensional (D)-cell σ in $\mathcal{T}_D(V)$, $\langle \mathrm{Vir}^p_\Sigma, \sigma\rangle$ is equal to $\int_{\tilde{R}_1\cap\sigma} c^p(\nabla^\bullet_{F_o})$.
If σ is in $\mathcal{T}_D(R_0)$, then $\tilde{R}_1\cap\sigma$ is empty or is in $\mathcal{T}_D(\tilde{R}_{01})\cap(D)^{2p}$ where $c^p(\nabla^\bullet_{F_o}) = c^p(\nabla^H) = 0$. Thus, $\langle \mathrm{Vir}^p_\Sigma, \sigma\rangle = 0$, which proves that Vir^p_Σ vanishes on $\mathcal{T}_D(R_0)$. Similarly, Vir^p_0 vanishes on $\mathcal{T}_D(\Sigma)$.

On the other hand, for any $2p+1$ dimensional (D)-cell τ, $\langle d\,\mathrm{Vir}^p_\Sigma, \tau\rangle = \langle \mathrm{Vir}^p_\Sigma, \partial\tau\rangle$ is equal to $\int_{\tilde{R}_1\cap\partial\tau} c^p(\nabla^\bullet_{F_0})$, that is $\int_{\tilde{R}_{01}\cap\tau} c^p(\nabla^\bullet_{F_0})$ after Stokes formula. If $\tilde{R}_{01}\cap\tau$ is not empty, it is included in $\mathcal{T}_D(\tilde{R}_{01})\cap(D)^{2p}$, where $c^p(\nabla^\bullet_{F_o}) = 0$: thus Vir^p_Σ is a cocycle. A similar proof works for Vir^p_0.

For two different $\tilde{F}_0^{(r)}$ connections $\nabla^\bullet_{1,F_o}$ and $\nabla^\bullet_{2,F_o}$, we have:

$$\big(\!\big([0], c^p(\nabla^\bullet_{2,F_o}), 0\big)\!\big) - \big(\!\big([0], c^p(\nabla^\bullet_{1,F_o}), 0\big)\!\big) = d\big(\!\big([0], 0, c^p(\nabla_{1,F_o}, \nabla_{2,F_o}\big)\!\big),$$

since $c^p(\nabla_{1,F_o}, \nabla_{2,F_o}) = 0$ near $\mathcal{T}_D R_{01}\cap(D)^{2q}$, both connections ∇_{1,F_o} and ∇_{2,F_o} preserving there a same $\tilde{F}_0^{(r)}$. Since two radial frame fields are always homotopic, these classes do not depend neither of the choice of the radial frame field, as far as it is radial.

After section 4, if we assume furthermore that $\nabla^{s,E}$ is a s connection, this decomposition has for image by the Thom-Gysin homomorphism

$$\left(\!\!\left(\begin{matrix} [0] & c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o}) & [0] \\ 0 & [0] & 0 \\ & 0 & \end{matrix}\right)\!\!\right) + \left(\!\!\left(\begin{matrix} [0] & [0] & \big(c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o})\big) \\ [0] & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right).$$

§6. SMP classes

Let r, p and q be as above.

Proposition 2. *Let $\nabla^M_{F_o}$ denote some $\tilde{F}_0^{(r)}$ connection on TM, for a* **radial** *frame field $\tilde{F}_0^{(r)}$.*
(i) *Then*

$$\mathrm{SMP}^{2q} = \left(\!\!\left(\begin{matrix} [0] & c^q(\nabla^M_{F_o}) & \big(c^q(\nabla^M_{F_o})\big) \\ 0 & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right),$$

is a cocycle in $C^{2q}\big(M, \mathcal{T}_D(R_A)\big)$.
(ii) *Its cohomology class is well defined in*

$$H^{2q}\big(M, \mathcal{T}_D(R_A)\big) \cong H^{2q}(M, M-V).$$

(iii) *This cohomology class $c^{2q}_{\mathrm{SMP}}(V)$ is equal to the image in the cohomology with real coefficients of the* SMP *class defined in* [MHS] *and* [BS] *with integral coefficients.*

In fact, for any $2q$ (D)-cell σ, we have:
$\langle \mathrm{SMP}^{2q},\sigma\rangle = \int_{\tilde R\cap\sigma} c^q(\nabla^M_{F_o})$.

For σ in $\mathcal{T}_D(R_A)$, we get $\langle \mathrm{SMP}^{2q},\sigma\rangle = 0$ because $c^q(\nabla^M_{F_o}) = 0$ if σ intersects ∂R_A, and $\tilde R\cap\sigma = \emptyset$ if it doesn't. Thus SMP^{2q} is a relative cochain modulo $\mathcal{T}_D(R_A)$. On the other hand, for any $2q+1$ dimensional (D)-cell τ, $\langle d\,\mathrm{SMP}^{2q},\tau\rangle$ is equal to $\int_{\tilde R\cap\partial\tau} c^q(\nabla^M_{F_o})$. If τ intersects $\partial\tilde R$, then $c^q(\nabla^M_{F_o}) = 0$. If not, then $\tilde R\cap\partial\tau = \partial\tau$ and $\int_{\tilde R\cap\partial\tau} c^q(\nabla^M_{F_o}) = 0$ after Stokes formula. Thus $d\,\mathrm{SMP}^{2q} = 0$, and we get part (i) of the proposition.

For two different $\tilde F_0^{(r)}$ connections ∇^M_{1,F_o} and ∇^M_{2,F_o}, we have:

$$\left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{2,F_o}) & (c^q(\nabla^M_{2,F_o}))\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right) - \left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{1,F_o}) & (c^q(\nabla^M_{1,F_o}))\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right)$$
$$= d\left(\!\!\left(\begin{matrix}[0] & c^q(\nabla^M_{1,F_o},\nabla^M_{2,F_o}) & (c^q(\nabla^M_{1,F_o},\nabla^M_{2,F_o}))\\ 0 & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right),$$

since $c^p(\nabla^M_{1,F_o},\nabla^M_{2,F_o}) = 0$ near $\mathcal{T}_D(\partial\tilde R)\cap(D)^{2q}$, both connections ∇^M_{1,F_o} and ∇^M_{2,F_o} preserving there a same $\tilde F_0^{(r)}$. Two radial frame fields being always homotopic, these classes do not depend neither on the choice of the frame field, as far as it is radial, hence part (ii) of the proposition.

Part (iii) results that the above definition is just a differential geometric transcription of the definition given in [MHS] and [BS].

Remarks. (i) For the moment, as far that we wish only define $c^{2q}_{\mathrm{SMP}}(V)$, we do not need the covering $\tilde{\mathcal{U}}$ with 3 open sets $M-V$, $\tilde U_0$ and $\tilde U_1$: we could as well work with the 2 open sets $M-V$ and $\tilde U_0\cup\tilde U_1$. But we shall need it soon, when decomposing $c^{2q}_{\mathrm{SMP}}(V)$ into the contributions $c^{2q}_{\Sigma,\mathrm{SMP}}(V)$ and $c^{2q}_{0,\mathrm{SMP}}(V)$ of the regular and the singular part of V.

(ii) Notice that

$$\left(\begin{matrix}[0] & c^q(\nabla^M) & (c^q(\nabla^M))\\ c^q(\nabla^M_{F_o},\nabla^M) & (c^q(\nabla^M_{F_o},\nabla^M)) & 0\\ & 0 & \end{matrix}\right)$$

might not be a cocycle, because $c^q(\nabla^M_{F_o})$ vanishes only over $\mathcal{T}_D(\partial R_A)\cap(D)^{2q}$, and may be not on all of U_{A0} and $U_{A\alpha}$.

(iii) The SMP class is an obstruction for the radial frame field $\tilde F_0^{(r)}$ to be extended to all of $U(V)\cap(D)^{2q}$. In fact, if such an extension exists, then $c^q(\nabla^M_{F_o}) = 0$ on all of the above domain, so that the previous cocycle is equal to

$$d\left(\!\!\left(\begin{array}{ccc}[0] & c^q(\nabla^M_{F_o},\nabla^M) & (c^q(\nabla^M_{F_o},\nabla^M))\\ 0 & 0 & 0\\ & 0 & \end{array}\right)\!\!\right).$$

In the definition above of the SMP class, we used only that $\nabla^M_{F_o}$ preserves $\tilde{F}_0^{(r)}$ over $\mathcal{T}_D(\partial R_A)\cap(D)^{2q}$. If we remember that it is still true over $\mathcal{T}_D(\tilde{R}_{01}\cap(D)^{2q}$, the above cocycle providing $c^q_{\mathrm{SMP}}(V)$ may be decomposed into

$$\left(\!\!\left(\begin{array}{ccc}[0] & c^q(\nabla^M_{F_o}) & [0]\\ 0 & [0] & 0\\ & 0 & \end{array}\right)\!\!\right)+\left(\!\!\left(\begin{array}{ccc}[0] & [0] & (c^q(\nabla^M_{F_o}))\\ [0] & 0 & 0\\ & 0 & \end{array}\right)\!\!\right),$$

which are still relative cocycles respectively in $C^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_1)$ and $C^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_0)$, whose relative cohomology classes, respectively denoted by $c^q_{0,\mathrm{SMP}}(V,\tilde{F}_o^{(r)})$ and $c^q_{\Sigma,\mathrm{SMP}}(V,\tilde{F}_o^{(r)})$ in $H^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_1)$ and $H^{2q}(M,\mathcal{T}_D(\tilde{R}_A\cup\tilde{R}_0)\cong H^{2q}(M,M-\Sigma)$ still do not depend on the choices of the various connections (similar proof).

§7. Milnor classes

Lemma 2. *The relative cohomology classes $\tau_0(c^p_{0,\mathrm{vir}}(V,\tilde{F}_o^{(r)}))$ and $c^q_{0,\mathrm{SMP}}(V,\tilde{F}_o^{(r)})$ are equal in $H^{2q}(M\ ,\ \mathcal{T}_D(R_A\cup\bigcup_\alpha\tilde{R}_\alpha)$.*

Proof. Choose a compatible $(\tilde{F}_0^{(r)},s)$ pair $(\nabla^M_{F_o},\nabla^{s,E})$ of connections, and let ∇^H be the connection induced by $\nabla^M_{F_o}$ on H over $\tilde{R}_0$.

$$\left(\!\!\left(\begin{array}{ccc}[0] & c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o}) & [0]\\ 0 & [0] & 0\\ & 0 & \end{array}\right)\!\!\right)=\left(\!\!\left(\begin{array}{ccc}[0] & c^q(\nabla^M_{F_o}) & [0]\\ 0 & [0] & 0\\ & 0 & \end{array}\right)\!\!\right).$$

In fact, $c^q(\nabla^M_{F_o})=c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o})+\sum_{j>0}[c^{k-j}(\nabla^{s,E})\wedge c^{p+j}(\nabla^\bullet_{F_o})]$. But $c^{p+j}(\nabla^\bullet_{F_o})=0$ over $\mathcal{T}_D(\tilde{R}_0)\cap(D)^{2q}$, since it is equal to $c^{p+j}(\nabla^H)$ (because of the compatibility of $(\nabla^H,\nabla^M_{F_o},\nabla^{s,E})$ with the exact sequence), and since $c^{p+j}(\nabla^H)=0$ over $\mathcal{T}_D(\tilde{R}_0)\cap(D)^{2q}$ for $j>0$ (because ∇^H preserves there the $r-1$ frame $\tilde{F}^{(r-1)}$). Q.E.D.

We deduce the

Theorem. (i) *The cohomology class*
$\tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)}_o)$ *of the cocycle*

$$\left(\!\!\left(\begin{matrix}[0] & [0] & \left(c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o}) - c^q(\nabla^M_{F_o})\right)\\ [0] & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right)$$

is well defined in $H^{2q}(M,M-\Sigma)$, i.e. does not depend on the choice of the $(\tilde F^{(r)}_0, s)$ pair $(\nabla^M_{F_o},\nabla^{s,E})$ of connections.
(ii) *It is a "localization" of $\tau(c^p_{\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\mathrm{SMP}}(V,\tilde F^{(r)}_o)$, which means:*

$$\tau(c^p_{\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\mathrm{SMP}}(V,\tilde F^{(r)}_o) = \beta\left[\tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)}_o)\right],$$

where $\beta : H^{2q}(M,M-\Sigma)\to H^{2q}(M,M-V)$ denotes the natural map.
(iii) *The α component $\mu^q(V,S_\alpha)$ of*
$(-1)^n\left[\tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V,\tilde F^{(r)}_o)) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde F^{(r)}_o)\right]$ *in $H^{2q}(M,M-S_\alpha)$, defined by the cocycle*

$$(-1)^n\left(\!\!\left(\begin{matrix}[0] & [0] & \left(c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o}) - c^q(\nabla^M_{F_o})\right)_\alpha\\ [0] & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right),$$

corresponds by Alexander duality to the homological Milnor class $\mu_{m-q}(V,S_\alpha)\in H_{2(m-q)}(S_\alpha)$ defined in [BLSS].

Proof. The parts (i) and (ii) have already been proved, part (ii) resulting from Lemma 2. On the other hand, the image of

$$\left(\!\!\left(\begin{matrix}[0] & [0] & \left(c^k(\nabla^{s,E})\wedge c^p(\nabla^\bullet_{F_o})\right)_\alpha\\ [0] & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right)$$

by Alexander duality $A : H^{2q}(M,M-S_\alpha)\to H_{2(m-q}(S_\alpha)$ is still equal to the image of $\left(\!\left(c^p(\nabla^\bullet_{F_o}),[0],0\right)\!\right)$ by the Poincaré morphism $P_V : H^{2p}(V,V-S_\alpha)\to H_{2(m-q}(S_\alpha)$: this is exactly the definition given in [BLSS] for the virtual index $\mathrm{Vir}(\tilde F^{(r)}_0,S_\alpha)$ of $\tilde F^{(r)}_0$ at S_α. Similarly,

$$(-1)^n\left(\!\!\left(\begin{matrix}[0] & [0] & \left(c^q(\nabla^M_{F_o})\right)_\alpha\\ [0] & 0 & 0\\ & 0 & \end{matrix}\right)\!\!\right)$$

has for image by A the Schwartz index $\mathrm{Sch}(\tilde F^{(r)}_0,S_\alpha)$ of $\tilde F^{(r)}_0$ at S_α. Thus, $A(\mu^q(V,S_\alpha) = (-1)^n(\mathrm{Vir}(\tilde F^{(r)}_0,S_\alpha) - \mathrm{Sch}(\tilde F^{(r)}_0,S_\alpha)$: this corresponds to the definition of the homological Milnor class $\mu_{2(m-q)}(V,S_\alpha)$ given in [BLSS]. Q.E.D.

Remarks. 1) The Milnor class $\mu^q(V, S_\alpha)$ vanishes for any α such that $S_\alpha \cap (D)^{2q}$ is included in V_0: in fact, for such α's, the definition of H over $\tilde{R}_0$ may be extended to $\tilde{R}_\alpha$, so that we can add $\tilde{R}_\alpha$ to $\tilde{R}_0$ in Lemma 2.

Therefore, $\mu^q(V)$ arises in fact from a well defined element of $H^{2q}(M, M - Sing(V))$.

2) For $r = 1$, it results from the theorem that the Milnor number of V at S_α, such as defined in [BLSS] (the usual Milnor number if S_α is an isolated point ([M], [H]), or such as defined in [P] when V is an hypersurface in M), is given by:

$$\mu_0(V, S_\alpha) = \int_{\tilde{R}_\alpha} \left[c^k(\nabla^{s,E}) \wedge c^n(\nabla^\bullet_{F_o}) - c^m(\nabla^M_{F_o})\right].$$

§8. Virtual and Schwartz indices

Let more generally $\tilde{F}^{(r)}$ be a frame field satisfying properties (i) and (ii) of the end of section 2, but not necessarily radial.

Replacing the $(\tilde{F}_0^{(r)}, s)$ pair of connections $(\nabla^M_{F_o}, \nabla^{s,E})$ by a $(\tilde{F}^{(r)}, s)$ pair $(\nabla^M_F, \nabla^{s,E})$, we can even take the same $\nabla^{s,E}$ in both pairs. Then everything works in the same way as in section 5, for the definitions of $c^p_{0,\mathrm{vir}}(V, \tilde{F}^{(r)}))$ and $c^p_{\Sigma,\mathrm{vir}}(V, \tilde{F}^{(r)}))$. We get the following decomposition of $\tau c^p_{\mathrm{vir}}(V)$:

$$\tau_0(c^p_{0,\mathrm{vir}}(V, \tilde{F}^{(r)})) = \left(\!\!\left(\begin{matrix} [0] & c^k(\nabla^{s,E}) \wedge c^p(\nabla^\bullet_F) & [0] \\ 0 & [0] & 0 \\ & 0 & \end{matrix}\right)\!\!\right)$$

$$\text{and} \quad \tau_\Sigma(c^p_{\Sigma,\mathrm{vir}}(V, \tilde{F}^{(r)})) = \left(\!\!\left(\begin{matrix} [0] & [0] & \left(c^k(\nabla^{s,E}) \wedge c^p(\nabla^\bullet_F)\right) \\ [0] & 0 & 0 \\ & 0 & \end{matrix}\right)\!\!\right).$$

However, as we already mentionned, we would not get the SMP classes if we just replace $\tilde{F}_0^{(r)}$ by $\tilde{F}^{(r)}$ in sections 6 and 7. Thus, we still define $c^p_{0,\mathrm{SMP}}(V, \tilde{F}^{(r)}))$ by the similar procedure:

$$c^p_{0,\mathrm{SMP}}(V, \tilde{F}^{(r)})) = \left(\!\!\left(\begin{matrix} [0] & c^q(\nabla^M_F) & [0] \\ 0 & [0] & 0 \\ & 0 & \end{matrix}\right)\!\!\right).$$

Therefore, we still have, as in Lemma 2 (similar proof):

$$c^p_{0,\mathrm{vir}}(V, \tilde{F}^{(r)})) = c^p_{0,\mathrm{SMP}}(V, \tilde{F}^{(r)})).$$

But, now, as a transcription of what we did in [BLSS], we define

$$c^p_{\Sigma,\mathrm{SMP}}(V,\tilde{F}^{(r)})) = c^p_{\Sigma,\mathrm{SMP}}(V,\tilde{F}^{(r)}_0) + \tau_\Sigma\big[(c^p_{\Sigma,\mathrm{vir}}(V,\tilde{F}^{(r)}) - c^p_{\Sigma,\mathrm{vir}}(V,\tilde{F}^{(r)}_0)\big].$$

More precisely, we define the "difference" of the two frames, as the cohomology class $\delta^p(\tilde{F}^{(r)}_0,\tilde{F}^{(r)})$ of $\big(\!\big(0,0,\big(c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)\big)\!\big) \in H^{2p}(V)$. Since $\big(\!\big(c^p(\nabla^\bullet_F) - c^p(\nabla^\bullet_{F_o}),[0],-c^p(\nabla^\bullet_{F_0},\nabla^\bullet_F)\big)\!\big) = D\big(\!\big(c^p(\nabla^\bullet_{F_0},\nabla^\bullet_F),[0],0\big)\!\big)$, then $\big[(c^p_{0,\mathrm{vir}}(V,\tilde{F}^{(r)}) - c^p_{0,\mathrm{vir}}(V,\tilde{F}^{(r)}_0)\big]$ and $\big(\!\big(0,[0],\big(c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)\big)\!\big)$ are equal in $H^{2p}(V,\Sigma)$.
Similarly $\big[(c^p_{\Sigma,\mathrm{vir}}(V,\tilde{F}^{(r)}) - c^p_{\Sigma,\mathrm{vir}}(V,\tilde{F}^{(r)}_0)\big]$ and $\big(\!\big([0],0,\big(-c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)\big)\!\big)$ are equal in $H^{2p}(V,V-\Sigma)$.

By the Thom-Gysin homomorphism, we get:

$$\tau\delta^p(\tilde{F}^{(r)}_0,\tilde{F}^{(r)}) = \left(\!\!\left(\begin{matrix} [0] & 0 & 0 \\ 0 & 0 & c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F) \\ & 0 & \end{matrix}\right)\!\!\right),$$

whose cohomology class is defined as well in $H^{2q}(M\ ,\ \mathcal{T}_D(R_A\cup\bigcup_\alpha \tilde{R}_\alpha)$ as in $H^{2q}(M,M-\Sigma)$. Thus, we get:

$$c^q_{\Sigma,\mathrm{SMP}}(V,\tilde{F}^{(r)}) = \left(\!\!\left(\begin{matrix} [0] & 0 & c^q(\nabla^M_{F_o}) \\ 0 & 0 & -c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F) \\ & 0 & \end{matrix}\right)\!\!\right).$$

Of course, we have done what we needed for still guetting

$$\mu^q(V) = (-1)^n\big[c^q_{\Sigma,\mathrm{vir}}(V,\tilde{F}^{(r)}) - c^q_{\Sigma,\mathrm{SMP}}(V,\tilde{F}^{(r)})\big],$$

which does not depend on the frame field $\tilde{F}^{(r)}$. In particular, we have:

$$\mu^q(V,S_\alpha) = (-1)^n\left(\!\!\left(\begin{matrix} [0] & [0] & \big(c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_F) - c^q(\nabla^M_{F_o})\big)_\alpha \\ [0] & 0 & \big(c^k(\nabla^{\,s,E})\wedge c^p(\nabla^\bullet_{F_o},\nabla^\bullet_F)\big)_\alpha \\ & 0 & \end{matrix}\right)\!\!\right).$$

On the other hand, as for $\tilde{F}^{(r)}_0$, $c^q_{\mathrm{SMP}}(V)$ is still equal to the sum of the images of $c^q_{0,\mathrm{SMP}}(V,\tilde{F}^{(r)})$ and $c^q_{\Sigma,\mathrm{SMP}}(V,\tilde{F}^{(r)})$ in $H^{2q}(M,M-V)$. This is an obvious corollary of the generalization of Lemma 2 to $\tilde{F}^{(r)}$.

The virtual index (resp. the Schwartz index) of $\tilde{F}^{(r)}$ at S_α such as defined in [BLSS] is nothing else but the image by the Alexander duality of the α component $c^q_{S_\alpha,\mathrm{vir}}(V,\tilde{F}^{(r)})$ (resp. $c^q_{S_\alpha,\mathrm{SMP}}(V,\tilde{F}^{(r)})$) of $c^q_{\Sigma,\mathrm{vir}}(V,\tilde{F}^{(r)})$ (resp. $c^q_{\Sigma,\mathrm{SMP}}(V,\tilde{F}^{(r)})$).

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