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Assigned base conditions and geometry of foliations on the projective plane

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§0. Introduction

This paper deals with algebro-geometric properties of foliations on the projective plane. Our motivation comes from the study of some linear systems with assigned base conditions which are associated to the geometry of a foliation $\mathcal{F}$ or to its invariant curves.

First, one has the linear system of polars relative to $\mathcal{F}$ introduced in [8]. This is a net of curves (one for each $q \in \mathbf{P}^2$ passing through it) of degree $r+1$, where r is the degree of $\mathcal{F}$. The base points of the net are the singularities of $\mathcal{F}$. One also has infinitely near base points, namely, those necessary to eliminate the indeterminacy of the rational map $\Phi : \mathbf{P}^2 \longrightarrow \check{\mathbf{P}}^2$ given by the net.

In this paper we study the local and global geometry of the polar net. For a non-base point q, we prove that the pencil of polars through q has very special properties: It contains as a special member the polar P_q through q and all its members, except at most one, are smooth. The polar P_q itself is smooth and it is tangent to the line L_q assigned by $\mathcal{F}$ at q. Thus, the local invariant given by the intersection number

$$\kappa_q = I_q(P_q, L_q) \tag{0.1}$$

is such that $\kappa_q \geq 2$. The locus of points q such that $\kappa_q \geq 3$ or q is a base point is shown to be the so-called jacobian curve of the net ([12], p.115). The jacobian is a curve of degree $3r$ which is associated to $\mathcal{F}$ in this way. Finally, we study the behaviour of both the polars and the

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jacobian at the base points. This study will be useful since these curves pass through all the singularities of $\mathcal{F}$.

As an example of such an use, we give an application to the Poincaré problem of bounding the degree d of a curve Q invariant by $\mathcal{F}$, in terms of r (or other invariants of $\mathcal{F}$). Several results around this problem have been given recently: Carnicer [4] and Cerveau-Lins Neto [6] have shown that $d \leq r+2$ if $\mathcal{F}$ is non-dicritical or if Q has only nodes as singularities, respectively. In [1], both the diciritical and arbitrary singularities on Q cases have been considered, and it has been shown that the problem can be reduced to the one consisting in finding a value a, such that $d \leq r+2+a$. Such an integer a is the degree of a curve which should satisfy certain assigned conditions at the base points.

One may use the jacobian and curves derived from the net of polars to compute nice values (linear in r) of the integer a described above, when one limitates the sizes of the singularity types of Q. In the paper, we develope the case of invariant curves having at most simple singularities. We show that if one fixes an integer $c \geq 1$ and if we assume that the curve Q has simple singularities whose associated Dynkin diagram has v vertices, with $[v/2] \leq c-1$, then one has that

$$d \leq cr + (c+1)$$

(nodes are A_1-singularities, so they fit in the case $c = 1$ in the formula above). If we fix the foliation $\mathcal{F}$, one can also find concrete values of c satisfying an inequality as above for every Q (Theorem 4.6). The jacobian is used then to improve the bound in some cases. For instance, improvements appear sometimes in the case when the algebraic multiplicities of $\mathcal{F}$ at its singularities are at least equal to 2.

§1. The polar net relative to a foliation

Consider the projective plane $\mathbf{P}^2$ over an algebraically closed field k. For an algebraic foliation $\mathcal{F}$ (or simply a foliation in the sequel), we mean a nontrivial map of vector bundles $\mathcal{F} : H_{-r+1} \longrightarrow T\mathbf{P}^2$, where H_j denotes the j−th twist of the hyperplane line bundle and $T\mathbf{P}^2$ the tangent bundle of $\mathbf{P}^2$. The integer r is assumed to be non-negative and it is called the degree of the foliation.

A foliation may be seen as an algebraic assignment of a tangent direction at each point $q \in \mathbf{P}^2$, except for finitely many of them, called the singularities of $\mathcal{F}$. The set $\mathrm{Sing}(\mathcal{F})$ of singularities of $\mathcal{F}$ is nothing but the zero set of $\mathcal{F}$, when viewed as a bundle map. Observe now that to give a tangent direction at q amounts the same data as to give a projective line through q; thus, a foliation on $\mathbf{P}^2$ consists of an assignment of a line $L_q = L_q(\mathcal{F})$ to each $q \in \mathbf{P}^2 \setminus \mathrm{Sing}(\mathcal{F})$.

The rational map $\Phi = \Phi_{\mathcal{F}} : \mathbf{P}^2 \longrightarrow \check{\mathbf{P}}^2$ taking a point $q \notin \mathrm{Sing}(\mathcal{F})$ to the point in $\check{\mathbf{P}}^2$ corresponding to L_q is called the polarity map (see [8]) of $\mathcal{F}$. The scheme-theoretic fibres $\Phi^*\ell$ of the lines in $\check{\mathbf{P}}^2$ are degree$-(r+1)$ curves on $\mathbf{P}^2$. Since a line on $\check{\mathbf{P}}^2$ is nothing but a point in $\mathbf{P}^2$, what one has then is an assignment of a degree $r+1$ curve $P_q = P_q(\mathcal{F})$ to each point $q \in \mathbf{P}^2$ (even for the singular points of $\mathcal{F}$). The (perhaps non-reduced) curve P_q is called the polar of q relative to $\mathcal{F}$ and one always has that q is in the support of P_q (since $q \in L_q$). The assignment

$$q \mapsto P_q$$

is linear in the sense that, since the P_q's constitute a 2-dimensional projective linear system of curves of degree $r+1$ (i.e., a net), it is a projective isomorphism. The net of the polar curves $(P_q)_{q \in \mathbf{P}^2}$ relative to $\mathcal{F}$ will be called the polar net relative to $\mathcal{F}$ and will be denoted by $\Delta = \Delta(\mathcal{F})$.

Observe that for each $q \in \mathbf{P}^2$, the support of the polar P_q consists of the singular points of $\mathcal{F}$ (since $\mathrm{Sing}(\mathcal{F})$ is exactly the base point set of the net), plus the points $q' \notin \mathrm{Sing}(\mathcal{F})$ such that $q \in L_{q'}$. This fact justifies the use of the terminology *polar* that we adopted.

If the foliation $\mathcal{F}$ is radial, that is, if there exists some $q_0 \in \mathbf{P}^2$ such that L_q is the line joining q to q_0, for every $q \neq q_0$, then the polar net is degenerated in the sense that the image of the polarity map Φ consists only of a line (the dual of the point q_0) and therefore, the polarity map is not dominant. The radial foliations are the only ones of degree equal to 0 and also the only ones for which Φ is not dominant (see [8], 3.4, recalling that the argument therein works in any characteristic as well). Assume that the degree r of $\mathcal{F}$ is ≥ 1; then since Φ is dominant, the field extension $K(\check{\mathbf{P}}^2) \subset K(\mathbf{P}^2)$ given by Φ is finite and, furthermore, one has that $r = [K(\mathbf{P}^2) : K(\check{\mathbf{P}}^2)]$ (see [8], 3.6). This allows us to give the degree r the geometrical interpretation of being the number of points in a general fiber of Φ, each one counted with multiplicity equal to the inseparability degree of the field extension, and hence, that r is nothing but the number of points (times the inseparability degree) at which the foliation becomes tangent to a general line in $\mathbf{P}^2$. Thus in the separable case the r points are different for a general line.

Now take two points $q \neq q'$ in $\mathbf{P}^2$ and let L be the line joining them. Two possibilities may occur: If L is an integral line, then the polars of all the points in L contain L itself and Φ collapses the line L to a point. If L is not an integral line, then the polars P_q and $P_{q'}$ meet at r points, different from the base points, namely the inverse images under Φ of the dual point of the line L in $\check{\mathbf{P}}^2$. If L is generic, the aforesaid points of intersection are different and, counted with multiplicity equal to the inseparability degree, equal to r. This means that the remaining intersection multiplicity of P_q and $P_{q'}$ is concentrated at the base point set, and is equal to $(r+1)^2 - r = r^2 + r + 1$. On the other hand, if L is not generic, then the intersection outside the base locus is, properly counted, smaller than or equal to r, if $L \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$, and it is strictly smaller than r, if $L \cap \mathrm{Sing}(\mathcal{F}) \neq \emptyset$. In any, case, the intersection $P_q \cap P_{q'}$ consists of points in L.

Now let us take homogeneous coordinates $[X, Y, Z]$ on $\mathbf{P}^2$. Up to a scalar factor, there are two equivalent ways to define a foliation $\mathcal{F}$ of degree r in analytic terms (see [9]):

(1) By means of a reduced homogeneous vector field, that is, by a vector field

$$T = U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} + W\frac{\partial}{\partial Z},$$

such that U, V and W are homogeneous polynomials of degree r without common factors. It is well defined up to a multiple of the radial vector field, in the sense that $\mathcal{F}$ is defined by both T and $T + G \cdot (X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z})$, for any degree $r-1$ homogeneous polynomial G, or

(2) By means of a reduced homogeneous $1-$ form, which means a $1-$ form

$$\Omega = AdX + BdY + CdZ$$

such that A, B and C are homogeneous polynomials of degree $r+1$, without common factors, and satisfying the so-called Euler's condition

$$XA + YB + ZC = 0. \tag{1.1}$$

The equivalence between (1) and (2) is analytically realized by the fact that

$$\Omega = \det \begin{pmatrix} dX & dY & dZ \\ X & Y & Z \\ U & V & W \end{pmatrix}. \tag{1.2}$$

We shall use the presentation (2) for the rest of the paper.

From the local data Ω it is easy to handle the foliation in local terms: Take any standard affine chart $\mathbf{A}^2$, for instance $Z \neq 0$. Then, if one writes $x = \frac{X}{Z}, y = \frac{Y}{Z}$ and $a(x,y) = A(x,y,1), b(x,y) = B(x,y,1)$ the foliation is defined by the $1-$ form ω (resp. by the vector field δ) given by

$$\omega = a(x,y)dx + b(x,y)dy \quad (\text{resp.}\, \delta = -b(x,y)\frac{\partial}{\partial x} + a(x,y)\frac{\partial}{\partial y}).$$

The equallity (1.1) allows us to recover Ω from the local data ω (or δ), taking into account that $\max\{\deg(a), \deg(b)\} = r$ or $r+1$, according to the case when the line $Z = 0$ is or it is not invariant by $\mathcal{F}$.

Finally, notice that a point $q \in \mathbf{A}^2$ is a singularity of $\mathcal{F}$ if and only if $a(q) = b(q) = 0$. To each singularity q, one attaches two invariants: the Milnor number $\mu_q = \mu_q(\mathcal{F})$ and the algebraic multiplicity $\nu_q = \nu_q(\mathcal{F})$, which in local terms are given respectively by

$$\begin{aligned} \mu_q &= \dim_k\left(\frac{\mathcal{O}_{\mathbf{P}^2,q}}{(a,b)\cdot\mathcal{O}_{\mathbf{P}^2,q}}\right) \\ \nu_q &= \min\{\mathrm{ord}_q(a), \mathrm{ord}_q(b)\}, \end{aligned}$$

where ord_q means the $\mathbf{m}_q$-adic order, $\mathbf{m}_q$ being the maximal ideal of the local ring $\mathcal{O}_{\mathbf{P}^2,q}$.

Note that we can also write $\mu_q = \nu_q = 0$, for $q \notin \mathrm{Sing}(\mathcal{F})$.

Theorem 1.1. *Let $\mathcal{F}$ be a foliation on $\mathbf{P}^2$ of degree r. Then, with the notations as above, one has*

(i) *If $q = [\alpha, \beta, \gamma] \in \mathbf{P}^2$, then the polar P_q is the curve given by*

$$\alpha A + \beta B + \gamma C = 0$$

(ii) *The polars of equations A, B, C generate the polar net.*

(iii) *For any $q \in \mathrm{Sing}(\mathcal{F})$, let $\mathcal{I}_q$ denote the ideal of $\mathcal{O}_{\mathbf{P}^2,q}$ generated by the equations of the germs at q of every polar P in Δ. Then*

$$\begin{aligned} \mu_q &= \dim_k\left(\frac{\mathcal{O}_{\mathbf{P}^2,q}}{\mathcal{I}_q}\right) \\ \nu_q &= \mathrm{ord}_q(\mathcal{I}_q) = \min\{\mathrm{ord}_q(f) : f \in \mathcal{I}_q\}. \end{aligned}$$

(iv) *One has $\sum_{q \in \mathrm{Sing}(\mathcal{F})} \mu_q = r^2 + r + 1$.*

Proof. The line $L_{q'}$ associated to $q' = [\alpha', \beta', \gamma']$ is given by

$$XA(\alpha', \beta', \gamma') + YB(\alpha', \beta', \gamma') + ZC(\alpha', \beta', \gamma') = 0;$$

therefore, if ℓ is the dual line of $q = [\alpha, \beta, \gamma]$, the divisor $\Phi^*\ell$ is given by the equation $\alpha A + \beta B + \gamma C = 0$, which proves (i). In particular, since A, B, C give the polars of the reference points, the fact that $q \mapsto P_q$

is a projective isomorphism proves (ii). Using local coordinates, the relation (1.1) shows that $\mathcal{I}_q$ is the ideal generated by the germs of a, b and $-xa-yb$; thus $\mathcal{I}_q = (a,b)\cdot\mathcal{O}_{\mathbf{P}^2,q}$, which shows (iii). Finally, to prove (iv), take generic coordinates in such a way that Sing($\mathcal{F}$) is contained in the standard chart $\mathbf{A}^2$; then each μ_q is equal to the intersection multiplicity at q of those polars that correspond to the two points at infinity (that is, A and B) and since the intersection divisor on the line at infinity has degree exactly equal to r, it follows from Bèzout's theorem that $\sum_{q\in\mathrm{Sing}(\mathcal{F})}\mu_q = (r+1)^2 - r = r^2+r+1$. Q.E.D.

§2. Polars at non-base points. The jacobian of the polar net

Consider a point $q \notin \mathrm{Sing}(\mathcal{F})$. The polars $P \in \Delta$ passing through q form a pencil $\Gamma_q \subset \Delta$ which contains P_q as a distinguished member. This section is devoted to the study of such pencils.

First of all, notice that the members of Γ_q are exactly the polars of the points lying on L_q. If L_q is an invariant line of $\mathcal{F}$, then Γ_q has L_q as a base line. Otherwise, Γ_q has only finitely many base points (that properly counted amount to r, if L_q does not meet the base locus).

Consider the integer κ_q defined in (0.1). If L_q is invariant by $\mathcal{F}$, one has that $\kappa_q = \infty$, otherwise, κ_q is finite.

Proposition 2.1. *Let $q \notin \mathrm{Sing}(\mathcal{F})$. Then, with the notations as above, one has*

(i) *The polar P_q is non-singular at q and its tangent there is given by the line L_q. In particular $\kappa_q \geq 2$.*

(ii) *If L_q is not invariant by $\mathcal{F}$, then for every $Q_q \in \Gamma_q \setminus \{P_q\}$, one has that $I_q(Q_q, L_q) = \kappa_q - 1$. If $\kappa_q = 2$, then every Q_q is also smooth at q. If $\kappa_q \geq 3$, then there exists a unique polar $Q_q \in \Gamma_q \setminus \{P_q\}$ which is singular at q.*

Proof. By taking coordinates, we can assume that $q = [1,0,0]$ and therefore, that P_q is given by $A = 0$. Since $q \notin \mathrm{Sing}(\mathcal{F})$, one of the three values $A(q), B(q), C(q)$ is non-zero and L_q is given by $XA(q) + YB(q) + ZC(q) = 0$. Taking derivatives in the Euler condition (1.1), one gets $A_X(q) = -A(q), A_Y(q) = -B(q), A_Z(q) = -C(q)$. It follows that P_q is smooth at q with tangent line equal to L_q , which proves (i). To prove (ii), we will also use (1.1). Taking coordinates as before, we may assume furthermore that L_q is given by the line $Z = 0$, and that $Q_q = P_{q'}$, with $q' = [0,1,0]$. Thus Q_q is given by the equation B. From (1.1) it follows that $I_q(P_q, L_q) = 1 + I_q(Q_p, L_q)$, which proves the first part of (ii). For the second part, notice that those germs at q of elements in Γ_q also constitute a pencil (of germs). If $\kappa_q = 2$, every Q_q is smooth and transversal to L_q . If $\kappa_q \geq 3$, since the pencil of germs has one smooth member, all members except at most one must be smooth also (the fact that they are also tangent to L_q -being $\kappa_q - 1 \geq 2$- implies that there is exactly one singular member). Q.E.D.

According to classical geometry (see for instance [12], p.115), one can associate to a net of projective curves a new curve, called its jacobian. It is a possibly non-reduced curve J whose equation is given by the following determinant:

$$D = \begin{vmatrix} A_X & A_Y & A_Z \\ B_X & B_Y & B_Z \\ C_X & C_Y & C_Z \end{vmatrix},$$

where A, B, C are any vector basis of the 3-dimensional vector space of the equations of the members of the net. It is clear that J does not depend on the choice of the vector basis nor on the choice of the coordinates $[X, Y, Z]$.

For the case of the polar net, we can take for A, B, C the forms that appear in Euler's condition (1.1). Hence, we can write the equation of J in the following way:

Lemma 2.2. *For the polar net of a foliation, the equation of its jacobian is given by*

$$D = \frac{(r+1)}{Z^2} \begin{vmatrix} A_X & A_Y & A \\ B_X & B_Y & B \\ A & B & 0 \end{vmatrix}.$$

In particular, if the characteristic of the ground field k is both positive and divides $r+1$, then $D = 0$.

Proof. It is enough to use the Euler equality for the homogeneous polynomials A, B, C, to change the last column in the determinant defining J. Then one uses the equalities obtained by taking partial derivatives in (1.1) to change the last row. Up to a sign, one obtains the expression for the equation D of J appearing in the Lemma. Q.E.D.

Next, we will study the geometry of J. In order to do so, assume that $D \neq 0$ (in particular $r + 1 \neq 0$ in k). First, there are some special components in J. In fact, notice that, although Δ has no base components, it may well happen that some concrete polar $P \in \Delta$ has a multiple component. On the other hand, a pencil $\Gamma \subset \Delta$ could also have base components. However, observe that if this is the case, then such a component should be an invariant line of $\mathcal{F}$.

Proposition 2.3. *Assume $D \neq 0$, then one has*

(i) *If F is a component of multiplicity $l \geq 2$ of some polar $P \in \Delta$, then F is also a component of J, of multiplicity $l-1$.*

(ii) *If L is an invariant line of $\mathcal{F}$, then L is a component of J.*

Proof. Using the determinant D that defines J, it is clear that the $(l-1)$–th power of an equation of F divides D. This proves (i). To see (ii), take coordinates such that L is the line $Z=0$. Then $A = Z\bar{A}, B = Z\bar{B}$, since every polar of a point lying on L contains L. Using Lemma 2.2 one gets

$$D = (r+1)Z \begin{vmatrix} \bar{A}_X & \bar{A}_Y & \bar{A} \\ \bar{B}_X & \bar{B}_Y & \bar{B} \\ \bar{A} & \bar{B} & 0 \end{vmatrix}, \tag{2.1}$$

which shows that L is a component of J, as required in (ii). Q.E.D.

Corollary 2.4. *Assume $D \neq 0$. Then the total degree of the multiple components of the polars of Δ counted with multiplicity one less, plus the total degree of the invariant lines is less than or equal to $3r$.*

Proof. This follows from Proposition 2.1 and Proposition 2.3, (i). In fact, a common component of a pair of different polars must be an integral line of $\mathcal{F}$, and, in view of (i) from Proposition 2.1, no invariant line is a multiple component. Q.E.D.

Proposition 2.5. *Assume $D \neq 0$ and let $q \notin \operatorname{Sing}(\mathcal{F})$, not lying on an invariant line. One has then that:*

(i) *The relation $I_q(J, L_q) \geq \kappa_q - 2$ holds, and the equality is valid if and only if $\kappa_q - 1 \neq 0$ in the field k.*

(ii) *The relation $I_q(J, P_q) \geq \kappa_q - 2$ holds, and the equality is valid if and only if $\kappa_q - 1 \neq 0$ in the field k.*

Proof. Take coordinates such that $q = [1,0,0]$ and L_q is the line $Z = 0$. Thus A is an equation for P_q and B is such for Q_q, as in Proposition 2.1. Computing the expression (2.1) of D, and expressing it in affine coordinates (y,z), one obtains, taking into account part (i) of Proposition 2.1, that

$$D(1,y,0) = \lambda(r+1)(\kappa_q - 1)y^{\kappa_q - 2} + \text{higher order terms}, \quad 0 \neq \lambda \in k.$$

On the other hand, since P_q has a local parametrization around q of the form $y = t$, $z = t^{\kappa_q}$ + higher order terms, one also gets

$$D(1,t,t^{\kappa_q}+\dots) = \mu(r+1)(\kappa_q - 1)t^{\kappa_q - 2} + \text{higher order terms}, \quad 0 \neq \mu \in k,$$

which proves (i) and (ii). Notice that $r+1 \neq 0$ in k, as $D \neq 0$. Q.E.D.

Theorem 2.6. *Let J be the jacobian of the polar net relative to a foliation $\mathcal{F}$ of degree r, then*

(1) *If $r+1 \neq 0$ in k, then the support of J is the set*

$$\mathrm{Sing}(\mathcal{F}) \cup \{q \notin \mathrm{Sing}(\mathcal{F}) : \kappa_q \geq 3\}$$

(2) *One has that $D \neq 0$ if and only if $r+1 \neq 0$ in k and the polarity map Φ is separable.*

Proof. From (i) in Proposition 2.5, if $r+1 \neq 0$ in k one has that $I_q(J_q, L_q) \geq 1$ if and only if $\kappa_q \geq 3$, for $q \notin \mathrm{Sing}(\mathcal{F})$. Now, from the determinant expression (2.1) in Lemma 2.2 it is clear that the base points are also in the support of J, which shows part (1).

Now, if $r+1 \neq 0$ in k and Φ is separable, take a generic line L in $\mathbf{P}^2$ and consider the r points of L at which L is tangent to $\mathcal{F}$. We claim that at each one of these points q, one has that $\kappa_q = 2$: For, in general, by part (ii) in Proposition 2.1, the number $\kappa_q - 1$ is, for a non-singular point $q \notin \mathrm{Sing}(\mathcal{F})$, the counting index of q as an inverse image of $\Phi(q)$. By part (1) above, the existence of points q with $\kappa_q = 2$ shows that $D \neq 0$. On the other hand, if Φ is inseparable and $r_i > 1$ is the inseparability degree, then for a generic line L in $\mathbf{P}^2$ as above, the $\frac{r}{r_i}$ points of tangency with $\mathcal{F}$ have $r_i + 1$ as value of κ_q. Thus $\kappa_q \geq 3$ holds for an open and dense set of points q. Again by part (1) above, one concludes that $D = 0$. Q.E.D.

Example 2.7. Consider the foliations in characteristic 2 given respectively by

$$\Omega_1 = \alpha YZdX + \beta XZdY + \gamma XYdZ, \quad \alpha+\beta+\gamma=0, \quad \alpha\beta\gamma \neq 0.$$
$$\Omega_2 = Z^3dX + ZX^2dY - (Z^2X + YX^2)dZ.$$

In both cases one has that $D = 0$. In the first case, $r = 1$, so $r+1=0$ in k. In the second case, $r = 2$ and Φ is inseparable.

§3. The base points of the polar net

The base points of a linear system include the infinitely near ones. Let us take a moment to explain briefly this statement.

An infinitely near point q is a point obtained by a sequence of point blow ups, starting from $\mathbf{P}^2$. For a configuration on $\mathbf{P}^2$ is meant a set $\mathcal{C}$ of infinitely near points, such that if $q \in \mathcal{C}$, then every point needed to produce q also belongs to $\mathcal{C}$. Notice that the points of $\mathcal{C}$ belong to different surfaces and that they are related by two different kind of relations. First, one can write $q > q'$, if q' is necessary to create q, and $q \to q'$, if both $q > q'$ and q belongs to the strict transform (at the surface containing q) of the exceptional divisor $B_{q'}$ of the blow up of q'. The relation $\to$ is called proximity and if $q \to q'$, we say that q is proximate to q'.

By blowing up successively the points of $\mathcal{C}$ (compatible with the relation $>$), one gets a surface $S = S_{\mathcal{C}}$ and a morphism $\pi : S \longrightarrow \mathbf{P}^2$, which is the composition of all the blow ups. The exceptional set has as components the curves E_q which are the strict transforms of the exceptional divisors B_q. If E_q^* denotes the total transform of B_q at S, one has

$$E_q = E_q^* - \sum_{q' \to q} E_{q'}^*,$$

for every $q \in \mathcal{C}$. This means that every divisor E with exceptional support (i.e., a relative to π divisor) can be written in the form $E = \sum_{q \in \mathcal{C}} l_q E_q^*$, with $l_q \in \mathbf{Z}$. A (not necessarily reduced) curve H in $\mathbf{P}^2$ is said to pass through $\mathcal{C}$ with assigned (or virtual) multiplicities $\{l_q\}$, if the relation $\pi^* H \geq E := \sum_{q \in \mathcal{C}} l_q E_q$ holds (see [3]). This means that for q in $\mathcal{C} \cap \mathbf{P}^2$, the multiplicity of H at q is greater than or equal to l_q; that by blowing up at such points q and by taking the virtual transform of H (that is, taking off l_q copies of B_q), the new divisor has multiplicity at least equal to l_q at the new points q, and so on.

Configurations appear in connection with several phenomena. For instance, for a reduced curve Q in $\mathbf{P}^2$, one has its resolution configuration $\mathcal{C}_Q$, that is, the configuration consisting of the points that are necessary to produce the minimal embedded resolution of Q. If $\{e_q = e_q(Q)\}$ is the set of multiplicities of the strict transforms of Q at the points $q \in \mathcal{C}$, one also has an associated relative divisor $E_Q := \sum_{q \in \mathcal{C}} e_q E_q^*$.

For a polar net one also has an associated configuration $\mathcal{C}_\Delta$ and a relative divisor E_Δ, which are nothing but the resolution configuration and half the relative divisor, respectively, of the union of two generic polars of the net. To wit, if $\pi : S_\Delta \longrightarrow \mathbf{P}^2$ is the corresponding morphism, then Φ can be extended to a morphism $\hat{\Phi} : S_\Delta \longrightarrow \check{\mathbf{P}}^2$, and one has that $\pi^* P \geq E_\Delta$, for every polar P (and so, all of them pass virtually through $\mathcal{C}_\Delta$), and for a general enough polar P , one has that $\pi^* P - E_\Delta$ has no exceptional components. Note that if $E_\Delta = \sum_{q \in \mathcal{C}} m_q E_q^*$, then $m_q = \nu_q$ for all points $q \in \mathcal{C} \cap \mathbf{P}^2$, and for $q \notin \mathbf{P}^2$, the integer m_q has a similar meaning with respect to the transformed net. The points q of $\mathcal{C}_\Delta$ are also said to be base points of the polar net and are said to have multiplicity equal to m_q.

We shall now study the way in which the jacobian J passes through the configuration $\mathcal{C}_\Delta$.

Theorem 3.1. *Assume that the jacobian J is a curve (i.e., $D \neq 0$). Then, for the polar configuration one has that*

$$\pi^* J \geq \sum_{q \in C} (3m_q - 1)E_q^* = 3E_\Delta - K_{S_\Delta/\mathbf{P}^2},$$

where $K_{S_\Delta/\mathbf{P}^2}$ denotes the relative canonical divisor. In particular, the jacobian J passes through the points $q \in \operatorname{Sing}(\mathcal{F})$ with multiplicities greater than or equal to $3m_q - 1 = 3\nu_q - 1$.

Proof. Let $q \in \operatorname{Sing}(\mathcal{F})$ and take coordinates such that $q = [1, 0, 0]$. Then by (2.1), one has that the germ of D at q is, up to a unit, the same as that of $AB(A_Y + B_X) - A^2 B_Y - B^2 A_X$. Now, the curves given by A and B pass through $\mathcal{C}_\Delta$ with virtual multiplicities $\{m_q\}$, and those given by A_X, A_Y, B_X, B_Y, with virtual multiplicities $\{m_q - 1\}$, since they are different polar curves (see [5]), and thus, D passes with multiplicities $\{3m_q - 1\}$ as required. On the other hand, it is easy to see that the relative canonical divisor is given by $\sum_{q \in C} E_q^*$. This completes the proof of the theorem. Q.E.D.

Remark 3.2. (a) Let $q \in \operatorname{Sing}(\mathcal{F})$. Choose coordinates such that $q = [0, 0, 1]$ and let L_A and L_B be the leading terms in the power series $A(X, Y, 1)$ and $B(X, Y, 1)$, respectively. Then it is not difficult to see that the multiplicity of J at q is exactly equal to $3\nu_q - 1$ if and only if L_A and L_B are linearly independent, and one has that $XL_A + YL_B \neq 0$. To see this, write down the determinant (2.1), look for the leading terms and its derivatives, and impose the condition that the resulting determinant is non-zero. Geometrically speaking, the condition that L_A and L_B are linearly independent means that the exceptional divisor B_q is dicritical for the net Δ (i.e., that it is not contracted by $\hat{\Phi}$, see [8] and [7]). On the other hand, the condition $XL_A + YL_B \neq 0$ means that B_q is non-dicritical for $\mathcal{F}$, i.e., that B_q is invariant by $\mathcal{F}$.

(b) The determinant (2.1), when viewed as a germ at $q = [0, 0, 1]$, behaves well under blowing ups. This means that the Remark (a) above can be extended to a similar one, at every point $q \in \mathcal{C}$, by taking the transforms of the net and the values $3m_q - 1$.

(c) In characteristic zero, one also has a resolution configuration $\mathcal{C}_\mathcal{F}$ for the foliation $\mathcal{F}$. This configuration is different to $\mathcal{C}_\Delta$ since, for instance, it follows from Remark (a) above that being dicritical for $\mathcal{F}$ is not the same as being it for Δ. However, $\mathcal{C}_{\mathcal{F}}$ contains the configurations $\mathcal{C}_Q$ for every reduced curve Q invariant by $\mathcal{F}$.

§4. Applications to the Poincaré problem

Consider a reduced curve $Q \subset \mathbf{P}^2$, invariant by a foliation $\mathcal{F}$. Let $\mathcal{C} = \mathcal{C}_Q$ be the resolution configuration for Q. We shall consider several values associated to each $q \in \mathcal{C}$. Namely, let $e_q = e_q(Q)$ be the multiplicity of the strict transform of Q at q; let $\nu_q = \nu_q(\mathcal{F})$ be the multiplicity of the strict transform of the foliation $\mathcal{F}$ at q, and let $s_q = s_q(\mathcal{F})$ be the number of components of the exceptional divisor passing through q (in the surface containing q), that are invariant by $\mathcal{F}$.

Now, for each $q \in \mathcal{C}$, consider an integer l_q, such that

$$e_q + s_q \leq \nu_q + 1 + l_q. \tag{4.1}$$

The main result in [1] provides a method to bound the degree of Q, which consists in solving a problem on assigned conditions. It is the following:

Proposition 4.1 ([1]). *With the notations as above, if H is a (non necessarily reduced) plane curve passing through the points of $\mathcal{C}$ with multiplicities $\{l_q\}$, then one has that*

$$d \leq r + 2 + a,$$

where d and a are the degrees of Q and H, respectively, and r is the degree of $\mathcal{F}$.

Although the proof in [1] is carried over the field $\mathbf{C}$ of complex numbers, the result is also true in any characteristic, as is shown in [2].

In order to apply the Proposition above, assume that $\{l_q\}$ is given. For each $q \in \mathrm{Sing}(\mathcal{F})$ and every maximal chain b, relative to the relation $>$, starting at q, consider the value $n_{q,b} = \sum_{q' \in b} l_{q'}$. Now let n_q be the maximum of the values $n_{q,b}$, for all such maximal chains b. One has then the following result:

Lemma 4.2. *With the notations as above, if H is a curve in $\mathbf{P}^2$ passing through the points $q \in \mathrm{Sing}(\mathcal{F})$ with multiplicities at least equal to n_q, then H passes through $\mathcal{C}$ with assigned multiplicities $\{l_q\}$.*

Proof. One can check that the conditions on virtual passage are satisfied for each maximal chain by using recurrence on its lenght. The inductive step is based on the fact that, if a germ has multiplicity $n > l$, then the virtual transform with respect to l has multiplicity greater than or equal to $n-l$, at every point q' in the exceptional divisor of the blow up of the origin q of the germ. In fact, if $f(x,y) = f_n(x,y) + f_{n+1}(x,y) + \ldots$ is the Taylor expansion of the germ equation with respect to local coordinates x, y at q, such that $x = 0$ is transversal to the direction corresponding to q, then the virtual transform germ at q' is given by the equation $x^{n-l} \cdot f'(x,y')$, where x, y' are local coordinates at q', and f' is an equation for the strict transform germ at q'. Q.E.D.

Now we will restrict our attention to the case where the curve Q has only simple singularities. Such simple curve singularities correspond to the Dynkin diagrams A_k, $k \geq 1$; D_k, $k \geq 4$; E_6, E_7 and E_8 (see [11], [10]). The resolution configurations will be shown below. In the following Theorem we will consider the case of characteristic zero, but the positive characteristic case may be treated with few changes. However, the main remark in positive characteristic is the fact that there exist regular foliations having singular invariant germs.

Theorem 4.3. *Let $\mathcal{F}$ be a foliation of degree r on the projective plane $\mathbf{P}^2$, over a field of characteristic zero. Let Q be a reduced curve of degree d, invariant by $\mathcal{F}$, and having only simple singularities in a set $S \subset \mathrm{Sing}(\mathcal{F})$. Assume that for each $q \in S$, the Dynkin diagram of the singularity has v_q vertices. Let H be a curve of degree a passing through each point $q \in S$, with multiplicity greater than or equal to $[v_q/2]$. Then one has*

$$d \leq r + 2 + a.$$

Proof. We begin by recalling the list of simple singularities (in characteristic zero):

Dynkin diagram	Same resolution as the curve singularity
A_k	$x^2 + y^{k+1}, \quad k \geq 1$
D_k	$x^2y + y^{k-1}, \quad k \geq 4$
E_6	$x^3 + y^4$
E_7	$x^3 + xy^3$
E_8	$x^3 + y^5$

For the singularities A_{2h} the resolution configuration consists of a chain $q_0 < q_1 < \ldots < q_{h+1}$, with $q_{h+1} \to q_{h-1}$ as nontrivial proximity relation. One has $s_0 = 0, s_i \leq 1$, for $1 \leq i \leq h$, $s_{h+1} \leq 2$; $e_i = 2$, for $0 \leq i \leq h-1$, $e_h = e_{h+1} = 1$. If $s_i = 1$ then $\nu_i \geq 1$ and if $s_{h+1} = 2$ then $\nu_{h+1} \geq 1$. This means that we can take $l_0 = l_h = 0, l_i = 1$, $1 \leq i \leq h-1$, $l_{h+1} = 1$.

For the singularities A_{2h+1} the resolution configuration consists of a chain $q_0 < q_1 < \ldots < q_h$, without nontrivial proximities. One has $s_0 = 0$, and $s_i \leq 1$, for $1 \leq i \leq h$, $e_i = 2$, for $1 \leq i \leq h$. If $s_i = 1$ then $\nu_i \geq 1$. Thus we can take $l_0 = 0, l_i = 1$, $1 \leq i \leq h$.

For the singularities D_{2h} the resolution graph is also a chain $q_0 < q_1 < \ldots < q_{h-2}$, without nontrivial proximities. One has $s_0 = 0$, $s_i \leq 1$, for $1 \leq i \leq h-2$, $e_0 = 3$, $e_i = 2$, for $1 \leq i \leq h-2$. One has $\nu_0 \geq 1$ and $\nu_i \geq 1$ if $s_i = 1$. Thus we can take $l_i = 1$ for $0 \leq i \leq h-2$.

For the singularities D_{2h+1} the resolution configuration consists of a chain $q_0 < q_1 < \ldots < q_h$, with $q_h \to q_{h-2}$ as nontrivial proximity relation. One has $s_0 = 0, s_i \leq 1$, for $1 \leq i \leq h-1$, $s_h \leq 2$; $e_0 = 3$, $e_i = 2$, for $1 \leq i \leq h-1$, $e_{h-1} = e_h = 1$. Again, $\nu_0 \geq 1$ and $\nu_i \geq 1$, if $s_i = 1$ or $\nu_h \geq 1$, if $s_h = 2$. We can take $l_0 = \ldots = l_{h-2} = l_h = 1$, $l_{h-1} = 0$.

For the singularity E_6 the resolution configuration consists of a chain $q_0 < q_1 < q_2 < q_3$, with $q_2 \to q_0$ and $q_3 \to q_0$ as nontrivial proximity relations. One has $s_0 = 1, s_1 \leq 1$, $s_2 \leq 2$, $s_3 \leq 2$, $e_0 = 3$, $e_i = 1$, for $1 \leq i \leq 3$. One has $\nu_1 \geq 1$ if $s_i = 1$; $\nu_2 \geq 1$ if $s_2 = 2$; $\nu_3 \geq 1$ if $s_3 = 2$. Thus one can take $l_0 = l_2 = l_3 = 1$, $l_1 = 0$.

For the singularity E_7 the resolution configuration consists of a chain $q_0 < q_1 < q_2$, with $q_2 \to q_0$ as nontrivial proximity relation. One has $s_0 = 0, s_1 \leq 1$, $s_2 \leq 2$, $e_0 = 3$, $e_1 = 2$, $e_2 = 1$. One has $\nu_0 \geq 1$, $\nu_1 \geq 1$ if $s_1 = 1$; $\nu_2 \geq 1$ if $s_2 = 2$. One can take $l_0 = l_1 = l_2 = 1$.

For the singularity E_8 the resolution configuration consists of a chain $q_0 < q_1 < q_2 < q_3$, with $q_2 \to q_0$ and $q_3 \to q_1$ as nontrivial proximity relations. One has $s_0 = 0, s_1 \leq 1$, $s_2 \leq 2$, $s_3 \leq 2$, $e_0 = 3$, $e_1 = 2$, $e_2 = e_3 = 1$. One has $\nu_0 \geq 1$, $\nu_1 \geq 1$, and $\nu_2 \geq 1$, $\nu_3 \geq 1$ if, respectively, $s_2 = 2$ or $s_3 = 2$. Thus one can take $l_0 = l_1 = l_2 = l_3 = 1$.

Thus, according to the choice of $\{l_i\}$, it follows that for this choice one has that $n_q \leq [v_q/2]$ and the Theorem follows from Lemma 4.2. Q.E.D.

Remark 4.4. (a) In the Theorem above, in fact one has that $n_q \leq [v_q/2]$, except for the singularity D_{2h}, for which $n_q = (v_q - 1)/2$. Also notice from the proof that, for simple singularities, one can take values 0 and 1 for the integers l_q, for $q \in \mathcal{C}$.

(b) Nodes are A_1 singularities. Since $[v_q/2] = 0$ for them, the Theorem (and Remark (a) above) can be seen to be an extension of the result in [6]. If one allows further simple singularities one can get concrete bounds for d. Thus, for instance A_1, A_2 and A_3 singularities, if one chooses as H to be any polar, one obtains $d \leq 2r + 3$.

A precise extension of the result in [6] is the following:

Corollary 4.5. *Let Q be a reduced curve of degree d, which is invariant by a foliation $\mathcal{F}$ of degree r, and let $c \geq 1$ be an integer. Assume that Q has also simple singularities $\{q\}$, whose Dynkin diagrams have v_q vertices, with $[v_q/2] \leq c-1$. Then one has*

$$d \leq cr + c + 1$$

Proof. Take as H any union of $c-1$ polars. Q.E.D.

Now, suppose that the foliation $\mathcal{F}$ is fixed. For each $q \in \mathrm{Sing}(\mathcal{F})$, let t_q be the largest length of a maximal chain of infinitely near points $q = q_1 < q_2 < \ldots < q_t$, without non-trivial proximity relations existing in the resolution configuration of $\mathcal{F}$. Set

$$u_q = \max(t_q, 4), \quad c_q = 1 - [-\frac{u_q}{\nu_q}], \quad c'_q = -[-\frac{u_q}{e_q(J)}],$$

where $e_q(J)$ denotes the multiplicity at q of the jacobian. Now define the polar and jacobian complexity of $\mathcal{F}$ respectively by

$$c = \max_{q \in \mathrm{Sing}(\mathcal{F})} (c_q) \quad \text{and} \quad c' = \max_{q \in \mathrm{Sing}(\mathcal{F})} (c'_q).$$

Theorem 4.6. *With the notations as above, let $\mathcal{F}$ be a foliation of degree r, with polar and jacobian complexities given respectively by c and c'. Let Q be a reduced curve of degree d, with at most simple singularities, which is invariant by $\mathcal{F}$. Then*

$$d \leq cr + (c+1) \quad \textit{and} \quad d \leq (3c'+1)r + 2.$$

Proof. It is enough to take for the curve H in Theorem 4.3 a union of $c-1$ polars, for the first inequality above, and c' times the jacobian for the second. Q.E.D.

Remark 4.7. Since $e_q(J) \geq 3\nu_q - 1$ (by Theorem 3.1), one of the bounds above can be better than the other, depending on the values of the invariants. The first one is usually better and the second improves the first one in some cases. Such improvements appear sometimes in the case when the algebraic multiplicities of $\mathcal{F}$ at its singularities are at least 2.

Example 4.8. The foliations in $\mathbf{A}^2$ given respectively by

$$2xdy - 5ydx, \quad xdy - 2ydx, \quad 2xdy - 3ydx$$

have for invariants $r = 1$, $c = 3$ and $c' = 2$, so the bounds that Theorem 4.6 brings for these cases are $d \leq 7$ and $d \leq 9$. The curve $Y^2Z^3 - X^5 = 0$ has A_4 and E_8 singularities and is integral for the first example. Similarily, the second example has $X(Y^2Z^2 - X^4) = 0$, with two D_6 singularities; the third example has $XZ(Y^2Z - X^3) = 0$, with D_5 and D_8 singularities, and $YZ(Y^2Z - X^3) = 0$, with A_1, A_5 and E_7 singularities. All these integral curves have degree $5 < 7$.

References

[1] A. Campillo, M. Carnicer, *Proximity inequalities and bounds for the degree of invariant curves by foliations in* $\mathbf{P}^2_{\mathbf{C}}$. Trans. Am. Math Soc. Vol. 349, No. 6. (1997), p. 2211-2228.

[2] A. Campillo, J. García, M. Carnicer. *On invariant curves by vector fields on algebraic varieties.* Preprint U. Valladolid (1997).

[3] A. Campillo, G. González-Sprinberg, M. Lejeune-Jalabert, *Clusters of Infinitely near points.* Math. Ann. 306 (1), (1996), p. 169-194.

[4] M. Carnicer *The Poincaré problem in the non-dicrital case.* Ann. of Math. 140, (1994), p. 289-294.

[5] E. Casas Alvero. *Infinitely near imposed singularities of polar curves.* Math. Ann. 287, (1990), p. 429-454.

[6] D. Cerveau, A. Lins-Neto *Holomorphic foliations in* $\mathbf{CP}^2$ *having an invariant algebraic curve.* Ann. Inst. Fourier 41, Fas. 4, (1991), p. 883-903.

[7] J. García, *Systemes linéaires de séries de puissances, pinceaux et feuilletage associés.* C. R. Ac. Sci. Paris, t. 315, Série I, (1992), p. 703-708.

[8] J. García, A. Reguera. *Polarity with respect to a foliation.* Algebraic geometry and singularities (Campillo, Narváez eds.). Progress in Math., Birkhäuser (1992), p. 157-170.

[9] X. Gómez-Mont, L. Ortiz, *Sistemas Dinámicos Holomorfos en Superficies.* Aportaciones Matemáticas 3, (1989). Sociedad Matemática Mexicana.

[10] G.M. Greuel, H. Kroening, *Simple singularities in positive characteristics.* Math. Z. 203 (1990), p. 339-374.

[11] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization.* Publ. IHES 36, (1969), p. 195-279.

[12] I.G. Semple, L. Roth, *Introduction to algebraic Geometry.* Clarendon Press, Oxford.

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