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Invariants for Representations of Weyl Groups, Two-sided Cells, and Modular Representations of Iwahori-Hecke Algebras

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§1. Introduction

1.1. q-Series identity.

Let $s_\lambda(x)$ be the Schur function in infinite variables $x = (x_1, x_2, \dots)$ corresponding to a Young diagram λ. For each node v in the diagram λ, $h(v)$ denotes the hook length of λ at v. Cf. [9] for the Young diagrams and related notions. In a recent work [7], Kawanaka obtained a q-series identity

$$\sum_\lambda I_\lambda(q) s_\lambda(x) = \prod_i \prod_{r=0}^{\infty} \frac{1 + x_i q^{r+1}}{1 - x_i q^r} \prod_{i<j} \frac{1}{1 - x_i x_j}, \tag{1}$$

where

$$I_\lambda(q) = \prod_{v\in\lambda} \frac{1 + q^{h(v)}}{1 - q^{h(v)}}, \tag{2}$$

and the sum on the left hand side of (1) is taken over all Young diagrams λ. If $q = 0$, then (1) reduces to the Schur-Littlewood identity.

Using (1), Kawanaka showed that for a Youndg diagram λ with n nodes, (2) is expressed as

$$I_\lambda(q) = |\mathfrak{S}_n|^{-1} \sum_{s\in\mathfrak{S}_n} \chi_\lambda(s^2) \frac{\det(1 + q\rho(s))}{\det(1 - q\rho(s))}, \tag{3}$$

where χ_λ is the irreducible character of the symmetric group $\mathfrak{S}_n$ corresponding to λ and $\rho : \mathfrak{S}_n \to GL_n(\mathbb{Z})$ is the representation of $\mathfrak{S}_n$ by permutation matrices.

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Since (3) is expressed in terms of the symmetric group and its representation, we can generalize such rational function for characters of other Weyl groups.

Definition 1.1. Let W be a Weyl group acting on a complex vector space $\mathfrak{h}$ faithfully as a reflection group. For a character χ of a finite dimensional representation π of W, we define a rational function of an indeterminate q by

$$I_W(\chi; q) = |W|^{-1} \sum_{w \in W} \chi(w^2) \frac{\det(1 + qw|_{\mathfrak{h}})}{\det(1 - qw|_{\mathfrak{h}})},$$

and we call it the *Kawanaka invariant* of π.

The main object of this paper is the expression for the Kawanaka invariants. We have obtained it in the B_l-case, which is stated in §2 (Theorem 2.1). This is not an immediate corollary of Kawanaka's result; in fact, we need a non-trivial argument. If we proceed to the D_l-case, the situation becomes much more difficult. We succeeded in expressing it by means of the Littlewood-Richardson coefficients (Theorem 2.2) and we obtained a conjectural formula for it (Conjecture 3.2). These are included in §2 and §3.

1.2. Invariants for cells.

The Kawanaka invariant plays a role as an invariant for two-sided cells.

In [4], a polynomial invariant

$$\tau^*(\chi; t) := \chi(e)^{-1} \sum_{w \in W} \chi(w) t^{\dim \mathfrak{h}^w}$$

is defined for a character χ of a finite dimensional representation π of W. Here, $\mathfrak{h}^w$ is the subset of w-fixed vectors in $\mathfrak{h}$. It is observed that, if W is of type A_l or B_l, then τ^* characterizes the two-sided cells. If W is not of these types, some deviation occurs. Trying to save this defect, a modified invariant

$$\tilde{\tau}(\chi; q, y) := |W|^{-1} \sum_{w \in W} \chi(w) \frac{\det(1 + yw|_{\mathfrak{h}})}{\det(1 - qw|_{\mathfrak{h}})},$$

motivated by [1]Chap. V, §5, Ex. 3, is introduced, and the relationship between $\tilde{\tau}$ and the two-sided cells is studied in [4]. Note that

$$|W|^{-1} \tau^*(\chi; t) = \lim_{q \to 1} \tilde{\tau}(\chi; q, -1 + t(1 - q)).$$

Hence, in principle, we can extract information on τ^* from $\tilde{\tau}$. In other words, $\tilde{\tau}$ is a refinement of the invariant τ^*.

Because of the resemblance between the definition of $\tilde{\tau}$ and I_W, we expect that the Kawanaka invariant is also related to the two-sided cells. Detailed discussion on the two-sided cells and invariants τ^*, $\tilde{\tau}$, I_W is contained in §4.

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After completing the first draft, the authors learned from Kawanaka his recent result, which incidentally implies our Conjecture 3.2. Thus our conjecture is affirmatively settled.

§2. Expression of the Kawanaka invariant

In this section, we present closed expression for Kawanaka invariants.

2.1. A_l-case.

As is explained in §1, the Kawanaka invariant for representations of symmetric group $\mathfrak{S}_l$ is given by

$$I_{\mathfrak{S}_l}(\chi_\lambda; q) = \prod_{v \in \lambda} \frac{1 + q^{h(v)}}{1 - q^{h(v)}}.$$

2.2. B_l-case.

In the B_l-case, we have similar expression. The irreducible representation of $W = W(B_l) \simeq \mathfrak{S}_l \ltimes \mathbb{Z}_2^l$ is parametrized by the ordered pair (λ', λ'') of Young diagrams (cf. [8]). Let $\chi_{\lambda',\lambda''}$ be the corresponding irreducible character.

Theorem 2.1 ([5]). *We have*

$$\begin{aligned} I_{W(B_l)}(\chi_{\lambda',\lambda''}; q) &= \prod_{v' \in \lambda'} \frac{1 + q^{2h(v')}}{1 - q^{2h(v')}} \prod_{v'' \in \lambda''} \frac{1 + q^{2h(v'')}}{1 - q^{2h(v'')}} \\ &= I_{\mathfrak{S}_{l'}}(\chi_{\lambda'}; q^2) I_{\mathfrak{S}_{l''}}(\chi_{\lambda''}; q^2), \end{aligned}$$

where $l' = |\lambda'|$ and $l'' = |\lambda''|$.

2.3. D_l-case.

Let us denote the restriction of $\chi_{\lambda',\lambda''}$ of $W(B_l)$ to $W(D_l) \simeq \mathfrak{S}_l \ltimes \mathbb{Z}_2^{l-1}$ by the same symbol $\chi_{\lambda',\lambda''}$. If $\lambda' \neq \lambda''$, then $\chi_{\lambda',\lambda''}$ is an irreducible character. If $\lambda = \lambda' = \lambda''$, then $\chi_{\lambda,\lambda}$ decomposes into two inequivalent irreducible characters χ_λ^I and χ_λ^{II}, which are interchanged by the outer automorphism induced from the conjugation by the non-unit element of $W(B_l)/W(D_l)$. So we have $I_{W(D_l)}(\chi_\lambda^I;q) = I_{W(D_l)}(\chi_\lambda^{II};q) = I_{W(D_l)}(\chi_{\lambda,\lambda};q)/2$. Therefore, it is enough to compute $I_{W(D_l)}(\chi_{\lambda',\lambda''};q)$ for obtaining Kawanaka invariants in the D_l-case.

Denote by ε the one dimensional representation of $W(B_l)$, induced from $W(B_l) \twoheadrightarrow W(B_l)/W(D_l) \simeq \{0,1\} \ni \epsilon \mapsto (-1)^\epsilon$. Since $W(D_l) = \operatorname{Ker}\varepsilon$ and $|W(B_l)| = 2|W(D_l)|$, we have

$$I_{W(D_l)}(\chi_{\lambda',\lambda''};q) = I_{W(B_l)}(\chi_{\lambda',\lambda''};q) + I^*(\chi_{\lambda',\lambda''};q),$$

where

$$I^*(\chi_{\lambda',\lambda''};q) = |W(B_l)|^{-1} \sum_{w\in W(B_l)} \chi(w^2)\varepsilon(w)\frac{\det(1+qw|_{\mathfrak{h}})}{\det(1-qw|_{\mathfrak{h}})}.$$

Since the explicit form of $I_{W(B_l)}(\chi_{\lambda',\lambda''};q)$ is known (Theorem 2.1), in order to determine the explicit form of $I_{W(D_l)}(\chi_{\lambda',\lambda''};q)$ it is enough to determine $I^*(\chi_{\lambda',\lambda''};q)$.

Unfortunately, we have not obtained a closed formula of I^*. The next theorem is the expression by means of the Littlewood-Richardson coefficients.

Theorem 2.2 ([5]). *Denote by $c_{\nu,\mu}^\lambda$ the Littlewood-Richardson coefficient. Then $I^*(\chi_{\lambda',\lambda''};q)$ is given by*

(4)

$$I^*(\chi_{\lambda',\lambda''};q) = \sum_{N=0}^{\min\{|\lambda'|,|\lambda''|\}} q^{l-2N} \times \sum_{\nu',\nu''}\left(\sum_{|\mu|=N} c_{\nu',\mu}^{\lambda'} c_{\nu'',\mu}^{\lambda''}\right) G(\chi_{\nu'};q^2)G(\chi_{\nu''};q^2),$$

where

$$G(\chi_\lambda;q) = q^{n(\lambda)}\prod_{v\in\lambda}\frac{1+q^{c(v)}}{1-q^{h(v)}}.$$

2.4. Other cases.

For Weyl groups of exceptional types and for dihedral groups, we have calculated the Kawanaka invariants of all the irreducible representations explicitly.

§3. Conjectures on Kawanaka invariants of type D_l

In this section, we give two conjectures, which are formulated in [5]. The first one follows from the second one. The second one is of purely combinatorial nature, which involves only an identity of polynomial functions.

3.1. Conjectural formula for I^*.

For partitions λ' and λ'' with $l(\lambda') \leq 3$ and $|\lambda''| \leq 3$, we calculated (4) explicitly with the help of *Mathematica*, and we obtained a conjectural formula of $I^*(\chi_{\lambda',\lambda''};q)$.

Definition 3.1 (The rational function $T_{\lambda',\lambda''}(q)$). If $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_n \geq 0)$ and $\lambda'' = (\lambda''_1 \geq \lambda''_2 \geq \cdots \geq \lambda''_n \geq 0)$ are a pair of partitions, put $\mu'_i := \lambda'_i + n - i$, $\mu''_i := \lambda''_i + n - i$, and define new partitions by $\mu' := (\mu'_1, \mu'_2, \cdots)$ and $\mu'' := (\mu''_1, \mu''_2, \cdots)$. Put

$$T_{\lambda',\lambda''}(q) := 2^n q^{|\mu'|+|\mu''|} \prod_{v'\in\lambda'} \frac{1+q^{2h(v')}}{1-q^{2h(v')}} \prod_{v''\in\lambda''} \frac{1+q^{2h(v'')}}{1-q^{2h(v'')}}$$
$$\times \frac{\prod_{1\leq i<j\leq n}(q^{2\mu'_j}+q^{2\mu'_i})(q^{2\mu''_j}+q^{2\mu''_i})}{\prod_{1\leq i,j\leq n}(q^{2\mu'_i}+q^{2\mu''_j})}.$$

Our first conjecture is as follows.

Conjecture 3.2 (A closed formula for $I^*(\chi_{\lambda',\lambda''};q)$).

$$I^*(\chi_{\lambda',\lambda''};q) = T_{\lambda',\lambda''}(q).$$

Example 3.3.

1. If $\lambda'' = \emptyset$, we get $I^*(\chi_{\lambda',\emptyset};q) = T_{\lambda',\emptyset}(q)$ from (4).
2. If λ' and λ'' correspond to trivial representations, i.e. $\lambda' = [l']$, $\lambda'' = [l'']$, we can prove $I^*(\chi_{[l'],[l'']};q) = T_{[l'],[l'']}(q)$ by induction on $\min\{l', l''\}$.
3. As is written at the beginning of this subsection, if $l(\lambda') \leq 3$ and $|\lambda''| \leq 3$, our conjecture is true. We check it by the aid of *Mathematica*.

Remark 3.4. If $\lambda = \lambda' = \lambda''$, it is not difficult to see

$$T_{\lambda,\lambda}(q) = \left(\prod_{v\in\lambda} \frac{1+q^{2h(v)}}{1-q^{2h(v)}}\right)^2 = I_{\mathfrak{S}_{|\lambda|}}(\chi_\lambda;q^2)^2.$$

3.2. A recursive formula for I^*.

Toward the proof of Conjecture 3.2, we exploited a recursive formula for $I^*(\chi_{\lambda',\lambda''};q)$.

Define an inner product on the space of symmetric functions with n variables $y = (y_1, \cdots, y_n)$ by $\langle s_{\lambda'}(y), s_{\lambda''}(y)\rangle_{GL_n(y)} := \delta_{\lambda',\lambda''}$, where $s_\lambda(y)$'s are the Schur functions. For infinitely many variables $x = (x_1, x_2, \cdots)$, consider $s_\lambda(x,y)$'s as symmetric functions in y, and put $\tilde{I}(\chi_{\lambda';\lambda''}, x) := \langle s_{\lambda'}(x,y), s_{\lambda''}(x,y)\rangle_{GL_n(y)}$. Consider the specialization

$$\text{elementary symmetric function } e_r(x) \mapsto q^r \prod_{i=1}^{r} \frac{1+q^{2i-2}}{1-q^{2i}}.$$

By this specialization, $s_\lambda(x)$ becomes $q^{|\lambda|}G(\lambda;q^2)$, and $I^*(\chi_{\lambda',\lambda''};q)$ is the result coming out from $\tilde{I}(\chi_{\lambda',\lambda''};x) = \sum_\mu s_{\lambda'/\mu}(x)s_{\lambda''/\mu}(x)$.

Theorem 3.5 (A recursive formula for I^*). *Fix partitions λ', λ'' and a positive integer r. Denote by $V(r)$ the set of all vertical r-strips, i.e., the skew diagrams which have at most one square in each row. Then*

$$\sum_{\substack{\mu' \\ \mu'-\lambda' \in V(r)}} I^*(\chi_{\mu',\lambda''}, q) = \sum_{\substack{i,j\geq 0 \\ i+j=r}} e_i \sum_{\substack{\mu'' \\ \lambda''-\mu'' \in V(j)}} I^*(\chi_{\lambda',\mu''}, q).$$

Thanks to this theorem, our first conjecture reduces to the following second conjecture.

Conjecture 3.6. *$T_{\lambda',\lambda''}$ satisfies the same recursive formula.*

§4. Application – Invariants for two-sided cells

In this section, we discuss the two-sided cells and the invariants τ^*, $\tilde{\tau}$, I_W.

Here we do not reproduce the definition of the two-sided cell [8] §4.2, but we note that this concept is important in the representation theory, e.g., in the work of A. Joseph [6] on the classification of primitive ideals of the enveloping algebras of complex semisimple Lie algebras, and in the work of G.Lusztig [8] on the classification and the description of irreducible characters of finite Chevalley groups.

4.1. Invariant τ^*.

Let us recall the definition of τ^* and $\tilde{\tau}$. We assume the same notation as in Definition 1.1.

Definition 4.1. For a character χ of a finite dimensional representation of a Weyl group W, we define

$$\tau^*(\chi;t) = \chi(e)^{-1}\sum_{w\in W}\chi(w)t^{\dim \mathfrak{h}^w} \quad \text{and}$$
$$\tilde{\tau}(\chi;q,y) = |W|^{-1}\sum_{w\in W}\chi(w)\frac{\det(1+yw|_{\mathfrak{h}})}{\det(1-qw|_{\mathfrak{h}})}.$$

Example 4.2. Let χ_λ be the irreducible character of $\mathfrak{S}_l$, associated to the Young diagram λ. Then we have

$$\tau^*(\chi_\lambda;t) = \prod_{v\in\lambda}(t+c(v)) \text{ and}$$
$$\tilde{\tau}(\chi_\lambda;q,y) = q^{n(\lambda)}\prod_{v\in\lambda}\frac{1+yq^{c(v)}}{1-q^{h(v)}},$$

where $c(v)$'s are the contents, and

$$n(\lambda) := \sum_{i>0}(i-1)\lambda_i \text{ where } \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots).$$

For the Weyl group of type B_l, we also have a similar formula for τ^* and $\tilde{\tau}$ (Cf. [4]). Especially, they are factorized analogously.

Looking over these results, we can observe a curious phenomenon.

Observation 4.3 ([4]). *Let W be the Weyl group of type A_l or B_l ($l > 2$), then for two irreducible character χ and χ' of W, the two invariants $\tau^*(\chi;t)$ and $\tau^*(\chi';t)$ coincide if and only if χ and χ' belong to the same two-sided cell.*

The arguments used in the theory of two-sided cells is sometimes very deep, based on IC-complexes, D-modules, and so on. Sometimes it is very ad hoc. Therefore it is surprising that such an easy invariant like τ^* characterizes two-sided cells. However such a heavenly simple picture is not true in general. Even if we replace τ^* by $\tilde{\tau}$ in the Observation 4.3, we can not extend the simple picture Observation 4.3 for general W. Therefore we want to understand the deviation itself.

4.2. Refined two-sided cells.

For the above purpose, we introduce a certain refinement of the two-sided cells.

Definition 4.4 (Iwahori-Hecke algebra). For an irreducible Weyl group W, let S be the set of simple reflections. Let $\{q_s\}_{s\in S}$ be a set of indeterminates such that $q_s = q_{s'}$ if and only if s and s' are W-conjugate and such that the different q_s's are algebraically independent. Put $R := \mathbb{Z}[q_s^{1/2}, q_s^{-1/2}]_{s\in S}$. Let K be the fractional field $\mathrm{Frac}(R)$ of R, and $H(W)_R = \bigoplus_{w\in W} RT_w$ the free R-module generated by the formal basis parametrized by W. Then an associative R-algebra structure of $H(W)_R$ is given by

$$T_w T_{w'} = T_{ww'} \text{ if } l(w) + l(w') = l(ww'), \text{ and}$$
$$(T_s + 1)(T_s - q_s) = 0 \text{ for } s \in S.$$

Now consider the specialization

$$R \stackrel{\mathrm{mod} p}{\longrightarrow} \mathrm{Frac}(R \otimes \mathbb{Z}/p\mathbb{Z}), \tag{5}$$

and consider the modular representation theory of $H(W)_K := H(W)_R \otimes K$ with respect to this specialization; in particular, consider the blocks of $H(W)_K^\vee$. Here $H(W)_K^\vee$ is the set of irreducible characters of $H(W)_K$, or equivalently, the set of irreducible representations modulo isomorphism.

Recall that $H(W)_K^\vee$ can be identified with $W^\vee$:

$$H(W)_K^\vee = W^\vee.$$

Definition 4.5 (The equivalence relation $\underset{*}{\sim}$). For two characters $\chi, \chi' \in H(W)_K^\vee = W^\vee$, and for a prime number p, define equivalence relations $\underset{p}{\sim}$ and $\underset{*}{\sim}$ by

1. $\chi \underset{p}{\sim} \chi'$ if and only if χ and χ' belong to the same block of $H(W)_K^\vee$ with respect to the specialization (5).
2. $\chi \underset{*}{\sim} \chi'$ if and only if there exist prime numbers $p_1, \ldots, p_n$ and irreducible characters $\chi_1, \ldots, \chi_{n-1}$ such that

$$\chi \underset{p_1}{\sim} \chi_1 \underset{p_2}{\sim} \cdots \underset{p_{n-1}}{\sim} \chi_{n-1} \underset{p_n}{\sim} \chi'.$$

Theorem 4.6 ([3], [4] § 4.2). *Assume that W is of type A_l, D_l or E_l. Then $\chi \underset{*}{\sim} \chi'$ if and only if χ and χ' belong to the same two-sided cell. In general, the implication 'only if' holds.*

In the sequel, let us call *refined two-sided cells*, the equivalence classes in $W^\vee$ with respect to the equivalence relation $\underset{*}{\sim}$.

4.3. Invariants $\tilde{\tau}$ and I_W.

We have calculated $\tilde{\tau}$'s and the Kawanaka invariants systematically using *Mathematica* and MAPLE in [4] and [5]. Looking over the results of the calculation, we have made some observations. For the statement of our observation, we need the following definition.

Definition 4.7 (Modified exceptional representations). Put

$$W^\vee_{\text{ex.m}} = \begin{cases} \{\chi \in W^\vee \mid \dim\chi = 2\}, & \text{if } W = W(G_2), \\ \{\chi \in W^\vee \mid \dim\chi = 512\}, & \text{if } W = W(E_7), \\ \{\chi \in W^\vee \mid \dim\chi = 4096\}, & \text{if } W = W(E_8), \\ \phi, & \text{otherwise.} \end{cases}$$

Observation 4.8. 1. *An irreducible character* $\chi \in W^\vee \setminus W^\vee_{\text{ex.m}}$ *forms a refined two-sided cell by itself if and only if*

$$\tilde{\tau}(\chi; q, y) = q^n \prod_{i=1}^{l} \frac{1 + yq^{c_i}}{1 - q^{h_i}}, \quad l = \dim \mathfrak{h}$$

with some integers n, $\{c_i\}_{1 \le i \le l}$ *and* $\{h_i\}_{1 \le i \le l}$, *which are uniquely determined by* χ.

2. *If* $\chi \in W^\vee$ *forms a refined two-sided cell by itself, then*

$$I_W(\chi; q) = \prod_{i=1}^{l} \frac{1 + q^{h_i}}{1 - q^{h_i}}, \quad l = \dim \mathfrak{h}$$

with the same integers $\{h_i\}_i$ *as above.*

Note that, in the A_l or B_l-case, every irreducible character $\chi \in W^\vee$ forms a refined two-sided cell by itself and $\tilde{\tau}$ is factorized as above. See Example 4.2.

In this way, we observed that the invariants $\tilde{\tau}$ and the Kawanaka invariants I are related to the two-sided cells and the refined two-sided cells.

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